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量子环面上一类导子李代数的结构和自同构群***

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提 要

本文研究量子环面上的一类导子李代数，它包含了 Virasoro-Like 代数及其 q 类似. 首先证明了这类导子李代数之间的同构一定是分次同构，并进一步给出了代数同构的充要条件及同构映射的具体表达式，最后确定了该类李代数的自同构群.

关键词 李代数，导子，同构，量子环面
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§1. 引 言

Virasoro 代数是一类非常重要的无穷维李代数，它与理论物理及数学的其它分支有着紧密的联系. 事实上， Virasoro 代数不外是单变量罗朗多项式环的导子所构成的李代数 (Witt 代数) 的泛中心扩张. 人们自然希望把相关的研究推广到多变量的情形，然而知道多变量 Witt 代数不存在非平凡的中心扩张. E.Kirkman 等人 [3] 在研究量子环面李代数的结构时，研究了两个变量的罗朗多项式环上的部分导子所构成的李子代数，并称之为 Virasoro-Like 代数，同时还指出 Virasoro-Like 代数与一类量子环面上的导子李代数 Virasoro-Like 代数的 q 类似之间的关系，并证明了两者都存在非平凡的泛中心扩张. 文 [1, 2] 研究了上述两类李代数的自同构群及导子的结构. 文 [4] 给出了量子环面上 Skew 导子李代数的一类不可约表示. 文 [5] 研究了 Skew 导子李代数的自同构群. 本文研究 Skew 导子李代数的导出子代数 L_q, 它的两种极限情形恰好就是 Virasoro-Like 代数和 Virasoro-Like 代数的 q 类似，推广了文 [1, 2, 5] 的相关结论. 下面先引入本文要研究的导子李代数 L_q:

设 Γ 是二维平面上的整格点所组成的集合，即 $\Gamma=\{(x,y)|x\in\mathbf{Z},y\in\mathbf{Z}\}$. 为了表达的方便，记 $e_1=(1,0),e_2=(0,1)$, 则 $\Gamma=\mathbf{Z}e_1+\mathbf{Z}e_2$. 设 q 是 p 次本原单位根.

定义 1.1 记 $\Gamma_p=p\Gamma$, 定义李代数 L_q 为由 $\{D_q(\vec{m}),\vec{m}\in\Gamma\backslash\{\mathbf{0}\}\}$ 张成的向量空间. L_q 上的李运算定义为 $[D_q(\vec{m}),D_q(\vec{n})]=g_q(\vec{m},\vec{n})D_q(\vec{m}+\vec{n})$, 其中 $\vec{m}=m_1e_1+m_2e_2,\vec{n}=n_1e_1+n_2e_2\in\Gamma\setminus\{0\}$,

$$g_q(\vec{m},\vec{n})=\begin{cases}q^{m_2n_1}-q^{m_1n_2}, & \vec{m},\vec{n}\notin\Gamma_p,\\ m_2n_1-m_1n_2, & \text{其它情形}.\end{cases}$$

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注 1.1 约定 $D_q(0)=0$, 并称 L_q 为量子环面上的导子李代数. 显然 L_q 完全由非零常量 q 所决定. 为了表达方便, 有时将 $\vec{m}=m_1e_1+m_2e_2$ 记为 (m_1,m_2).

第 2 节给出李代数 L_{q_1} 与 L_{q_2} 同构的充要条件; 第 3 节给出了同构映射的具体表达式; 第 4 节研究了李代数 L_q 的自同构群的结构.

§2. L_{q_1} 和 L_{q_2} 同构的充要条件

本节设 q_1 和 q_2 分别是 p_1 和 p_2 次本原单位根, 其中 p_1,p_2 为正整数. 设 $\Gamma_{p_1}=p_1\Gamma,\Gamma_{p_2}=p_2\Gamma$, 相应的两个李代数为 L_{q_1} 和 L_{q_2}.

定义 2.1 设 G 为 Abel 群, $g=\bigoplus\limits_{m\in G}g_m$, $g'=\bigoplus\limits_{m\in G}g'_m$ 为 G-分次代数, 对 g 到 g' 的代数同态 σ, 若存在一个群 G 的同态 φ 满足 $\sigma(g_m)\subseteq g'_{\varphi(m)}$, 则称 σ 为 g 到 g' 的分次同态. 进一步, 若 σ 为同构映射, 则称 σ 为分次同构.

由于 L_{q_1} 和 L_{q_2} 的分次空间都是一维的, 故若 L_{q_1} 与 L_{q_2} 分次同构, 则对任意分次同构映射 $\sigma:L_{q_1}\to L_{q_2}$ 及 $\vec{m}\in\Gamma\setminus\{0\}$, 存在 $\lambda\neq 0$, 使得 $\sigma(D_{q_1}(\vec{m}))=\lambda D_{q_2}(\vec{n})$. 下面将证明, L_{q_1} 到 L_{q_2} 的代数同构一定是分次同构.

在 Γ 中定义字典序, 即 $(m_1,n_1)<(m_2,n_2)$ 当且仅当

$$m_1<m_2 \text{ 或 } m_1=m_2 \text{ 且 } n_1<n_2.$$

为了方便叙述, 假定以后出现的求和式 $\sum\limits_{i=1}^{n}\lambda_iD_{q_2}(\vec{m_i})$ 已按 $\vec{m_i}$ 的分量进行了字典排序且若 $\sigma\Big(\sum\limits_{i=1}^{a}\lambda_iD_{q_1}(\vec{m_i})\Big)=f_1D_{q_2}(\vec{n_1})+\cdots+f_bD_{q_2}(\vec{n_b})$, 称 $D_{q_2}(\vec{n_1})$ 和 $D_{q_2}(\vec{n_b})$ 分别是 $\sigma\Big(\sum\limits_{i=1}^{a}\lambda_iD_{q_1}(\vec{m_i})\Big)$ 的首项和尾项.

引理 2.1 设 $\sigma:L_{q_1}\longrightarrow L_{q_2}$ 是一个代数同态, $\forall\vec{v}\in\Gamma$, 及 $\sum\limits_{i=1}^{a}\lambda_iD_{q_1}(\vec{m_i})$, 设 $D_{q_2}(\vec{r_1}),D_{q_2}(\vec{r_t})$ 分别是 $\sigma(D_{q_1}(\vec{v}))$ 的首项和尾项, $D_{q_2}(\vec{s_1}),D_{q_2}(\vec{s_l})$ 分别是 $\sigma(D_{q_1}(-\vec{v}))$ 的首项和尾项, $D_{q_2}(\vec{n_1}),D_{q_2}(\vec{n_b})$ 分别是 $\sigma\Big(\sum\limits_{i=1}^{a}\lambda_iD_{q_1}(\vec{m_i})\Big)$ 的首项和尾项. 则

(A) 若 $\vec{r_1}+\vec{s_1}<(0,0)$, 有 $[[D_{q_2}(\vec{n_1}),D_{q_2}(\vec{r_1})],D_{q_2}(\vec{s_1})]=0$ 或 $\vec{n_1}\in\Gamma_{p_2}$.

(B) 若 $\vec{r_t}+\vec{s_l}>(0,0)$, 有 $[[D_{q_2}(\vec{n_b}),D_{q_2}(\vec{r_t})],D_{q_2}(\vec{s_l})]=0$ 或 $\vec{n_b}\in\Gamma_{p_2}$.

证 先证 (A). 关于项数 a 做归纳. 当 a=1 时, 设 $\sigma(D_{q_1}(\vec{m}))=f_1D_{q_2}(\vec{n_1})+\cdots$. 因为 σ 是同态, 有

$$\sigma[[D_{q_1}(\vec{m}),D_{q_1}(\vec{v})],D_{q_1}(-\vec{v})]=[[\sigma(D_{q_1}(\vec{m})),\sigma(D_{q_1}(\vec{v}))],\sigma(D_{q_1}(-\vec{v}))],$$

上式左边若为 0, 则有 $[[D_{q_2}(\vec{n_1}),D_{q_2}(\vec{r_1})],D_{q_2}(\vec{s_1})]=0$. 若上式左边不为 0, 则左边首项为 $D_{q_2}(\vec{n_1})$.

再考虑右边, 此时若 $[[D_{q_2}(\vec{n_1}),D_{q_2}(\vec{r_1})],D_{q_2}(\vec{s_1})]\neq 0$, 右边首项为 $D_{q_2}(\vec{n_1}+\vec{r_1}+\vec{s_1})=D_{q_2}(\vec{n_1})$, 与 $\vec{r_1}+\vec{s_1}<(0,0)$ 矛盾. 因此 $[[D_{q_2}(\vec{n_1}),D_{q_2}(\vec{r_1})],D_{q_2}(\vec{s_1})]=0$. 这就证明了当 $a=1$ 时, (A) 成立.

设当 $a\leq k$ 时, (A) 成立, 而对 $a=k+1$ 时不成立, 即

$$\sigma\Big(\sum_{i=1}^{k+1}\lambda_iD_{q_1}(\vec{m_i})\Big)=f_1D_{q_2}(\vec{n_1})+\cdots,$$

其中 $\overrightarrow{n_1}\notin\Gamma_{p_2}$ 且 $[[D_{q_2}(\overrightarrow{n_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]\neq 0$. 现在分析如下等式:

$$\sigma\Big[\Big[\sum_{i=1}^{k+1}\lambda_i D_{q_1}(\overrightarrow{m_i}),D_{q_1}(\overrightarrow{v})\Big],D_{q_1}(-\overrightarrow{v})\Big]=\Big[\Big[\sigma\Big(\sum_{i=1}^{k+1}\lambda_i D_{q_1}(\overrightarrow{m_i})\Big),\sigma(D_{q_1}(-\overrightarrow{v}))\Big],\sigma(D_{q_1}(-\overrightarrow{v}))\Big].$$

上式右边首项为 $D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1})$, 而左边为

$$\sigma\Big(\sum_{i=1}^{k+1}l_iD_{q_1}(\overrightarrow{m_i})\Big)=\frac{l_1}{\lambda_1}\sigma\Big(\sum_{i=1}^{k+1}\lambda_iD_{q_1}(\overrightarrow{m_i})\Big)+\sigma\Big(\sum_{i=2}^{k+1}\Big(l_i-\lambda_i\frac{l_1}{\lambda_1}\Big)D_{q_1}(\overrightarrow{m_i})\Big).$$

因为 $\overrightarrow{r_1}+\overrightarrow{s_1}<0$, 有 $\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}<\overrightarrow{n_1}$, 故 $D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1})$ 是 $\sigma\Big(\sum\limits_{i=2}^{k+1}(l_i-\lambda_i\frac{l_1}{\lambda_1})D_{q_1}(\overrightarrow{m_i})\Big)$ 的首项，而 $\sum\limits_{i=2}^{k+1}(l_i-\lambda_i\frac{l_1}{\lambda_1})D_{q_1}(\overrightarrow{m_i})$ 至多只有 k 项，由归纳假设，有

$$[[D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]=0$$

或 $\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}\in\Gamma_{p_2}$. 但是 $\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}\notin\Gamma_{p_2}$ 且 $[[D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]\neq 0$.

事实上，设 $\overrightarrow{n_1}=(n_{11},n_{12}),\overrightarrow{r_1}=(r_{11},r_{12}),\overrightarrow{s_1}=(s_{11},s_{12})$, 因为 $\overrightarrow{r_1}$ 和 $\overrightarrow{s_1}$ 分别是 $\sigma(D_{q_1}(\overrightarrow{v}))$ 和 $\sigma(D_{q_1}(\overrightarrow{-v}))$ 的首项，所以 $[D_{q_2}(\overrightarrow{r_1}),D_{q_2}(\overrightarrow{s_1})]=0$.

下面以 $\overrightarrow{r_1},\overrightarrow{s_1}\notin\Gamma_{p_2}$ 为例证明: $[[D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]\neq 0$. 因为 $[[D_{q_2}(\overrightarrow{n_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]\neq 0$, 所以 $[D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}),D_{q_2}(\overrightarrow{s_1})]\neq 0$, 由此得

$$p_2\nmid(n_{12}r_{11}-n_{11}r_{12}),\quad p_2\nmid((n_{12}+r_{12})s_{11}-(n_{11}+r_{11})s_{12}).\tag{2.1}$$

又因为 $[D_{q_2}(\overrightarrow{r_1}),D_{q_2}(\overrightarrow{s_1})]=0$, 所以

$$p_2\mid r_{12}s_{11}-r_{11}s_{12}.\tag{2.2}$$

联立 (2.1), (2.2) 得 $p_2\nmid n_{12}s_{11}-n_{11}s_{12}$. 所以有

$$[[D_{q_2}(\overrightarrow{n_1}+\overrightarrow{r_1}+\overrightarrow{s_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]\neq 0.$$

矛盾. 因此，当 $s=k+1$ 时成立. (A) 得证. 对尾项考虑，同理可得 (B).

根据这一结论，就可以得出

引理 2.2 若 $\sigma:L_{q_1}\longrightarrow L_{q_2}$ 是一个满的代数同态，则 σ 是一个分次同态且对任给的 $\overrightarrow{v}\in\Gamma$, 若 $\sigma(D_{q_1}(\overrightarrow{v}))=\lambda D_{q_2}(\overrightarrow{n})$, 则 $\sigma(D_{q_1}(-\overrightarrow{v}))=\mu D_{q_2}(-\overrightarrow{n})$.

证 任取 $\overrightarrow{v}\in\Gamma$, 设 $D_{q_2}(\overrightarrow{r_1})$, $D_{q_2}(\overrightarrow{r_t})$ 分别是 $\sigma(D_{q_1}(\overrightarrow{v}))$ 的首项和尾项， $D_{q_2}(\overrightarrow{s_1})$, $D_{q_2}(\overrightarrow{s_l})$ 分别是 $\sigma(D_{q_1}(-\overrightarrow{v}))$ 的首项和尾项. 若能证明: 存在 $\overrightarrow{u_1}\notin\Gamma_{p_2}$, 使得

$$[[D_{q_2}(\overrightarrow{u_1}),D_{q_2}(\overrightarrow{r_1})],D_{q_2}(\overrightarrow{s_1})]\neq 0\tag{2.3}$$

且存在 $\overrightarrow{u_2}\notin\Gamma_{p_2}$, 使得

$$[[D_{q_2}(\overrightarrow{u_2}),D_{q_2}(\overrightarrow{r_t})],D_{q_2}(\overrightarrow{s_l})]\neq 0,\tag{2.4}$$

则由引理 2.1 可知 $\overrightarrow{r_1}+\overrightarrow{s_1}\geq(0,0)$ 且 $\overrightarrow{r_t}+\overrightarrow{s_l}\leq(0,0)$. 由定义有 $\overrightarrow{r_1}\leq\overrightarrow{r_t}$, $\overrightarrow{s_1}\leq\overrightarrow{s_l}$. 因此 $\overrightarrow{r_1}=\overrightarrow{r_t}=-\overrightarrow{s_1}=-\overrightarrow{s_l}$, 这样就得到

$$\sigma(D_{q_1}(\overrightarrow{v}))=\lambda D_{q_2}(\overrightarrow{n}),\quad\sigma(D_{q_1}(-\overrightarrow{v}))=\mu D_{q_2}(-\overrightarrow{n}).$$

下面先证明 (2.3) 成立. 当 $\overrightarrow{r_1}\notin\Gamma_{p_2},\overrightarrow{s_1}\notin\Gamma_{p_2}$ 时，此时 $r_{11},r_{12},s_{11},s_{12}$ 至多存在两个数能被 p_2 整除，以 $p_2\mid r_{11},p_2|s_{12}$, $p_2\nmid r_{12}$, $p_2\nmid s_{11}$ 为例，取 $\overrightarrow{u_1}=(1,1)$, 则 $(1,1)\notin\Gamma_{p_2}$ 且 $[[D_{q_2}(1,1),D_{q_2}(\overrightarrow{r_1})]D_{q_2}(\overrightarrow{s_1})]\neq 0$. 对于 $\overrightarrow{r_1}\notin\Gamma_{p_2},\overrightarrow{s_1}\notin\Gamma_{p_2}$ 的其他情形，可以类似地给出 $\overrightarrow{u_1}$. 当 $\overrightarrow{r_1}\notin\Gamma_{p_2},\overrightarrow{s_1}\in\Gamma_{p_2}$ 或 $\overrightarrow{r_1}\in\Gamma_{p_2}$ 时，也可以类似地给出 $\overrightarrow{u_1}$. 类似地可以证明 (2.4).

由引理 2.2 立即可得

命题 2.1 L_{q_1} 到 L_{q_2} 的代数同构是分次同构且对任给的 $\vec{v}\in\Gamma$, 若 σ 是分次同构映射，满足 $\sigma(D_{q_1}(\vec{v}))=\lambda D_{q_2}(\vec{n})$, 则 $\sigma(D_{q_1}(\overrightarrow{-v}))=\mu D_{q_2}(\overrightarrow{-n})$.

引理 2.3 设 q_1,q_2 分别为 p_1,p_2 次本原单位根，若 $\sigma:L_{q_1}\to L_{q_2}$ 是同构映射，则存在映射 $f:\Gamma\to\mathbf{C}\setminus\{0\},\varphi:\Gamma\setminus\{0\}\to\Gamma\setminus\{0\}$ 和整数 $\alpha_1,\beta_1,\alpha_2,\beta_2\in\mathbf{Z}$, 使得

$$\sigma D_{q_1}(k_1,k_2)=f(k_1,k_2)D_{q_2}(k_1\alpha_1+k_2\alpha_2,k_1\beta_1+k_2\beta_2) \tag{2.5}$$

且 $\alpha_1\beta_2-\alpha_2\beta_1=\pm1$.

证 由命题 2.1 可知，存在 $f:\Gamma\to\mathbf{C}\setminus\{0\},\varphi:\Gamma\setminus\{0\}\to\Gamma\setminus\{0\}$, 使 $\sigma D_{q_1}(k_1,k_2)=f(k_1,k_2)D_{q_2}(\varphi(k_1,k_2))$. 记 $\varphi(1,0)=(\alpha_1,\beta_1)$, $\varphi(0,1)=(\alpha_2,\beta_2)$, 由命题 2.1 知 $\varphi(0,-1)=(-\alpha_2,-\beta_2)$. 下面先证明 $\forall k\in\mathbf{Z}$, $\sigma D_{q_1}(k,0)=f(k,0)D_{q_2}(k\alpha_1,k\beta_1)$.

用归纳法证明 $k\in\mathbf{Z}_+$ 的情形. 当 $k=1$ 时，$\sigma D_{q_1}(1,0)=f(1,0)D_{q_2}(\alpha_1,\beta_1)$, 结论成立. 设当 $k\le n-1$ 时，结论成立. 当 $k=n$ 时，若 $p_1\nmid n$,

$$\sigma[[[D_{q_1}(n-1,0),D_{q_1}(0,1)]D_{q_1}(1,0)]D_{q_1}(0,-1)]=a_n\sigma D_{q_1}(n,0),$$

$$[[[\sigma D_{q_1}(n-1,0),\sigma D_{q_1}(0,1)]\sigma D_{q_1}(1,0)]\sigma D_{q_1}(0,-1)]=b_nD_{q_2}(n\alpha_1,n\beta_1),$$

其中 $a_n,b_n\in\mathbf{C}^*$. 比较上面两式，得 $\sigma D_{q_1}(n,0)=f(n,0)D_{q_2}(n\alpha_1,n\beta_1)$. 类似地可给出 $p_1|n$ 的证明. 所以 $\sigma D_{q_1}(n,0)=f(n,0)D_{q_2}(n\alpha_1,n\beta_1)$. 由归纳法得，当 $k\in\mathbf{Z}_+$ 时，有 $\sigma D_{q_1}(k,0)=f(k,0)D_{q_2}(k\alpha_1,k\beta_1)$. 进一步由命题 2.1, 当 $k\in\mathbf{Z}_-$ 时，有

$$\sigma D_{q_1}(k,0)=f(k,0)D_{q_2}(k\alpha_1,k\beta_1).$$

所以 $\forall k\in\mathbf{Z}$, 有 $\sigma D_{q_1}(k,0)=f(k,0)D_{q_2}(k\alpha_1,k\beta_1)$. 类似地可证 $\forall k\in\mathbf{Z}$,

$$\sigma D_{q_1}(0,k)=f(0,k)D_{q_2}(k\alpha_2,k\beta_2).$$

接着证明 $\sigma D_{q_1}(k_1,k_2)=f(k_1,k_2)D_{q_2}(k_1\alpha_1+k_2\alpha_2,k_1\beta_1+k_2\beta_2)$. 若 $p_1|k_1$ 或 $p_1|k_2$, 则

$$\sigma[D_{q_1}(k_1,0),D_{q_1}(0,k_2)]=g_1\sigma D_{q_1}(k_1,k_2),$$

$$[\sigma D_{q_1}(k_1,0),\sigma D_{q_1}(0,k_2)]=g_2D_{q_2}(k_1\alpha_1+k_2\alpha_2,k_1\beta_1+k_2\beta_2),$$

其中 $g_1,g_2\in\mathbf{C}^*$, 比较上面两式，得

$$\sigma D_{q_1}(k_1,k_2)=f(k_1,k_2)D_{q_2}(k_1\alpha_1+k_2\alpha_2,k_1\beta_1+k_2\beta_2).$$

类似地可给出其他情形的证明. 最后证明 $\alpha_1\beta_2-\alpha_2\beta_1=\pm1$. 为此先证 $\alpha_1\beta_2-\alpha_2\beta_1\neq0$. 考虑 $\sigma[D_{q_1}(1,0),D_{q_1}(0,1)]=[\sigma D_{q_1}(1,0),\sigma D_{q_1}(0,1)]$. 上式左边等于 $(1-q_1)\sigma D_{q_1}(1,1)$, 而右边等于 $f(1,0)f(0,1)[D_{q_2}(\alpha_1,\beta_1),D_{q_2}(\alpha_2,\beta_2)]$. 若 $\alpha_1\beta_2-\alpha_2\beta_1=0$, 则 $\sigma D_{q_1}(1,1)=0$, 与 σ 为同构映射矛盾. 所以 $\alpha_1\beta_2-\alpha_2\beta_1\neq0$. 再由 σ 为同构映射知，存在 $(r_1,s_1),(r_2,s_2)\in\Gamma\setminus0$, 使得

$$D(r_1,s_1)=f(r_1,s_1)L(1,0),\quad \sigma D(r_2,s_2)=f(r_2,s_2)L(0,1).$$

则由 (2.5) 得

$$r_1\alpha_1+s_1\alpha_2=1,\quad r_1\beta_1+s_1\beta_2=0,\quad r_2\alpha_1+s_2\alpha_2=0,\quad r_2\beta_1+s_2\beta_2=1.$$

求解上方程组得

$$r_1=\frac{\beta_2}{\alpha_1\beta_2-\alpha_2\beta_1},\quad s_1=\frac{-\beta_1}{\alpha_1\beta_2-\alpha_2\beta_1},\quad r_2=\frac{-\alpha_2}{\alpha_1\beta_2-\alpha_2\beta_1},\quad s_2=\frac{\alpha_1}{\alpha_1\beta_2-\alpha_2\beta_1}.$$

所以

$$\begin{vmatrix} \dfrac{\beta_2}{\alpha_1\beta_2-\alpha_2\beta_1} & \dfrac{-\beta_1}{\alpha_1\beta_2-\alpha_2\beta_1} \\ \dfrac{-\alpha_2}{\alpha_1\beta_2-\alpha_2\beta_1} & \dfrac{\alpha_1}{\alpha_1\beta_2-\alpha_2\beta_1} \end{vmatrix} = \frac{1}{\alpha_1\beta_2-\alpha_2\beta_1}.$$

因此 $\alpha_1\beta_2-\alpha_2\beta_1=\pm1$.

引理 2.4 若 q_1,q_2 分别为 p_1,p_2 次本原单位根且 L_{q_1} 与 L_{q_2} 同构，则 $p_1=p_2$.

证 由引理 2.3, 可设 $\sigma D_{q_1}(k_1,k_2)=f(k_1,k_2)D_{q_2}(k_1\alpha_1+k_2\alpha_2,k_1\beta_1+k_2\beta_2)$. 则

$$\begin{aligned}[\sigma D_{q_1}(p_1,-1),\sigma D_{q_1}(0,1)] &= f(p_1,-1)f(0,1)[D_{q_2}(p_1\alpha_1-\alpha_2,p_1\beta_1-\beta_2),D_{q_2}(\alpha_2,\beta_2)]\\ &= \sigma[D_{q_1}(p_1,-1),D_{q_1}(0,1)]=0.\end{aligned}$$

于是， $p_2|(p_1\beta_1\alpha_2-\alpha_2\beta_2)-(p_1\alpha_1\beta_2-\alpha_2\beta_2)$, 即 $p_2|p_1(\beta_1\alpha_2-\alpha_1\beta_2)$, 故 $p_2|p_1$. 同理可证 $p_1|p_2$, 又 p_1,p_2 都是正整数，故 $p_1=p_2$.

根据引理 2.3 和引理 2.4 结论，为了方便，下面总设 q_1 和 q_2 都是 p 次本原单位根.

引理 2.5 若 σ 为 L_{q_1} 到 L_{q_2} 的同构映射且 $\sigma D_{q_1}(\overrightarrow{m})=f(\overrightarrow{m})D_{q_2}(\varphi(\overrightarrow{m}))$, 则 $g_{q_1}(\overrightarrow{m},\overrightarrow{n})=0$ 当且仅当 $g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))=0$.

证 为表述方便， $\forall\overrightarrow{m}=(m_1,m_2),\overrightarrow{n}=(n_1,n_2)$, 记 $\varphi(m_1,m_2)=(m_1',m_2'),\varphi(1,0)=(\alpha_1,\beta_1),\varphi(0,1)=(\alpha_2,\beta_2)$. 则由引理 2.3 及引理 2.4 知

$$m_2'n_1'-m_1'n_2'=(\alpha_1\beta_2-\alpha_2\beta_1)(m_2n_1-m_1n_2). \tag{2.6}$$

下面先证明

$$\overrightarrow{m}\in\Gamma_p\Leftrightarrow\varphi(\overrightarrow{m})\in\Gamma_p. \tag{2.7}$$

若 $\overrightarrow{m}\in\Gamma_p$, 由引理 2.3, 显然有 $\varphi(\overrightarrow{m})\in\Gamma_p$. 由于 σ 是同构映射，故对 σ^{-1}, 利用引理 2.3 同理可得，若 $\varphi(\overrightarrow{m})\in\Gamma_p$, 则 $\overrightarrow{m}\in\Gamma_p$.

最后证明

$$g_{q_1}(\overrightarrow{m},\overrightarrow{n})=0\Leftrightarrow g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))=0.$$

分两种情形来证明.

情形 1 当 $\overrightarrow{m}\in\Gamma_p$ 或 $\overrightarrow{n}\in\Gamma_p$ 时，

$$\begin{aligned}&g_{q_1}(\overrightarrow{m},\overrightarrow{n})=0\Leftrightarrow m_2n_1-m_1n_2=0\\ &\Leftrightarrow m_2'n_1'-m_1'n_2'=0\Leftrightarrow g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))=0,\end{aligned}$$

其中第二个等价关系利用了 (2.6), 第三个等价关系利用了 (2.7).

情形 2 当 $\overrightarrow{m},\overrightarrow{n}\notin\Gamma_p$ 时，

$$g_{q_1}(\overrightarrow{m},\overrightarrow{n})=0\Leftrightarrow p|(m_1n_2-m_2n_1)\Leftrightarrow g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))=0,$$

其中最后一个等价关系利用了 (2.6), (2.7). 所以引理 2.5 成立.

引理 2.6 若 L_{q_1} 与 L_{q_2} 同构，则 $q_2=q_1$ 或 $q_2=q_1^{-1}$.

证 因 L_{q_1} 与 L_{q_2} 同构, 设 σ 为同构映射. 由命题 2.1, 设 $\sigma(D_{q_1}(\overrightarrow{m}))=f(\overrightarrow{m})D_{q_2}(\varphi(\overrightarrow{m}))$. 记 $\varphi(1,0)=(\alpha_1,\beta_1),\varphi(0,1)=(\alpha_2,\beta_2)$, 则 $\alpha_1\beta_2-\alpha_2\beta_1=\pm1$. 为简便起见, 设 $\varphi(m_1,m_2)=(m_1',m_2')$, 由 (2.6) 有 $m_2'n_1'-m_1'n_2'=(\alpha_1\beta_2-\alpha_2\beta_1)(m_2n_1-m_1n_2)$. 下面讨论 $\alpha_1\beta_2-\alpha_2\beta_1=-1$ 的情形.

若 $\overrightarrow{m},\overrightarrow{n}\notin\Gamma_p$ 且 $p\nmid(m_1n_2-m_2n_1)$, 则 $g_{q_1}(\overrightarrow{m},\overrightarrow{n})\neq 0$. 由引理 2.5 可知 $\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n})\notin\Gamma_p$. 故由引理 2.3 得

$$\begin{aligned}\frac{g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))}{g_{q_1}(\overrightarrow{m},\overrightarrow{n})}&=\frac{q_2^{m_2'n_1'}-q_2^{m_1'n_2'}}{q_1^{m_2n_1}-q_1^{m_1n_2}}=\frac{q_2^{m_2'n_1'}(1-q_2^{m_1'n_2'-m_2'n_1'})}{q_1^{m_1n_2}(q_1^{m_2n_1-m_1n_2}-1)}\\&=\frac{q_2^{(m_1\beta_1+m_2\beta_2)(n_1\alpha_1+n_2\alpha_2)}(1-q_2^{((\alpha_1\beta_2-\alpha_2\beta_1)(m_1n_2-m_2n_1))})}{q_1^{m_1n_2}(q_1^{m_2n_1-m_1n_2}-1)}\\&=\frac{q_2^{(m_1\beta_1+m_2\beta_2)(n_1\alpha_1+n_2\alpha_2)}}{q_1^{m_1n_2}}\frac{1-q_2^{m_2n_1-m_1n_2}}{q_1^{m_2n_1-m_1n_2}-1}.\end{aligned}\tag{2.8}$$

由 $\sigma[D_{q_1}(\overrightarrow{m}),D_{q_1}(\overrightarrow{n})]=[\sigma D_{q_1}(\overrightarrow{m}),\sigma D_{q_1}(\overrightarrow{n})]$ 得

$$g_{q_1}(\overrightarrow{m},\overrightarrow{n})f(\overrightarrow{m+n})=g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))f(\overrightarrow{m})f(\overrightarrow{n}).\tag{2.9}$$

在上式中将 $\overrightarrow{n}$ 换成 $\overrightarrow{n}+\overrightarrow{r}$, 得

$$g_{q_1}(\overrightarrow{m},\overrightarrow{n}+\overrightarrow{r})f(\overrightarrow{m}+\overrightarrow{n}+\overrightarrow{r})=g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}+\overrightarrow{r}))f(\overrightarrow{m})f(\overrightarrow{n}+\overrightarrow{r}).\tag{2.10}$$

又由 $\sigma[[D_{q_1}(\overrightarrow{m}),D_{q_1}(\overrightarrow{n})]D_{q_1}(\overrightarrow{r})]=[[\sigma D_{q_1}(\overrightarrow{m}),\sigma D_{q_1}(\overrightarrow{n})]\sigma D_{q_1}(\overrightarrow{r})]$, 得

$$\begin{aligned}&g_{q_1}(\overrightarrow{m},\overrightarrow{n})g_{q_1}(\overrightarrow{m}+\overrightarrow{n},\overrightarrow{r})f(\overrightarrow{m}+\overrightarrow{n}+\overrightarrow{r})\\&=g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))g_{q_2}(\varphi(\overrightarrow{m}+\overrightarrow{n}),\varphi(\overrightarrow{r}))f(\overrightarrow{m})f(\overrightarrow{n})f(\overrightarrow{r}).\end{aligned}\tag{2.11}$$

当 $g_{q_1}(\overrightarrow{m},\overrightarrow{n})g_{q_1}(\overrightarrow{m},\overrightarrow{n}+\overrightarrow{r})g_{q_1}(\overrightarrow{m}+\overrightarrow{n},\overrightarrow{r})\neq 0$ 时, 联立 (2.10), (2.11) 得

$$f(\overrightarrow{n}+\overrightarrow{r})=\frac{g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}))g_{q_2}(\varphi(\overrightarrow{m}+\overrightarrow{n}),\varphi(\overrightarrow{r}))g_{q_1}(\overrightarrow{m},\overrightarrow{n}+\overrightarrow{r})}{g_{q_1}(\overrightarrow{m},\overrightarrow{n})g_{q_1}(\overrightarrow{m}+\overrightarrow{n},\overrightarrow{r})g_{q_2}(\varphi(\overrightarrow{m}),\varphi(\overrightarrow{n}+\overrightarrow{r}))}f(\overrightarrow{n})f(\overrightarrow{r}).\tag{2.12}$$

当 $p>4$ 时, 联立 (2.8), (2.12), 并取 $\overrightarrow{m}=(1,0),\overrightarrow{n}=(0,1),\overrightarrow{r}=(0,1)$, 得

$$\begin{aligned}f(0,2)&=\frac{g_{q_2}(\varphi(1,0),\varphi(0,1))g_{q_2}(\varphi(1,1),\varphi(0,1))g_{q_1}((1,0),(0,2))}{g_{q_1}((1,0),(0,1))g_{q_1}((1,1),(0,1))g_{q_2}(\varphi(1,0),\varphi(0,2))}f^2(0,1)\\&=\frac{q_2^{\beta_1\alpha_2+(\beta_1+\beta_2)\alpha_2-2\beta_1\alpha_2}}{q_1^{2-2}}\Big(\frac{1-q_2^{-1}}{q_1^{-1}-1}\Big)^2\frac{1-q_1^{-2}}{q_2^{-2}-1}f^2(0,1)\\&=q_2^{\beta_2\alpha_2}\Big(\frac{1-q_2^{-1}}{q_1^{-1}-1}\Big)^2\frac{1-q_1^{-2}}{q_2^{-2}-1}f^2(0,1).\end{aligned}\tag{2.13}$$

联立 (2.8), (2.12), 并取 $\overrightarrow{m}=(2,0),\overrightarrow{n}=(0,1),\overrightarrow{r}=(0,1)$, 解得

$$\begin{aligned}f(0,2)&=\frac{g_{q_2}(\varphi(2,0),\varphi(0,1))g_{q_2}(\varphi(2,1),\varphi(0,1))g_{q_1}((2,0),(0,2))}{g_{q_1}((2,0),(0,1))g_{q_1}((1,1),(0,1))g_{q_2}(\varphi(2,0),\varphi(0,2))}f^2(0,1)\\&=\frac{q_2^{2\beta_1\alpha_2+(2\beta_1+\beta_2)\alpha_2-4\beta_1\alpha_2}}{q_1^{2+2-4}}\Big(\frac{1-q_2^{-2}}{q_1^{-2}-1}\Big)^2\frac{1-q_1^{-4}}{q_2^{-4}-1}f^2(0,1)\\&=q_2^{\beta_2\alpha_2}\Big(\frac{1-q_2^{-2}}{q_1^{-2}-1}\Big)^2\frac{1-q_1^{-4}}{q_2^{-4}-1}f^2(0,1).\end{aligned}\tag{2.14}$$

比较 (2.13), (2.14) 两式得

$$\Big(\frac{q_2^{-1}+1}{q_1^{-1}+1}\Big)^2\frac{(q_1^{-2}+1)}{(q_2^{-2}+1)}=1.$$

由此可得 $q_2=q_1$ 或 $q_2=q_1^{-1}$. 此外, 当 $p\leq 4$ 时, p 次本原单位根显然相等或互为倒数. 综上所述, 若 L_{q_1} 与 L_{q_2} 同构, 则 $q_2=q_1$ 或 $q_2=q_1^{-1}$. 同理可证, 当 $\alpha_1\beta_2-\alpha_2\beta_1=1$ 时, 命题也成立.

定理 2.1 $L_{q_1} \cong L_{q_2}$ 当且仅当 $q_1 = q_2$ 或 $q_1 = q_2^{-1}$.

证 由引理 2.6, 只需证充分性. 当 $q_1 = q_2$ 时，充分性显然成立. 当 $q_1 = q_2^{-1}$ 时，令

$$\sigma(D_{q_1}(l,k)) = \begin{cases} D_{q_2}(l,-k), & (l,k) \notin \Gamma_p, \\ -D_{q_2}(l,-k), & (l,k) \in \Gamma_p, \end{cases}$$

直接验证可得 σ 是一个同构映射，故 $L_{q_1} \cong L_{q_2}$. 定理 2.1 得证.

§3. L_{q_1} 和 L_{q_2} 同构映射的表达式

当 $q_2 = q_1$ 或 $q_2 = q_1^{-1}$ 时，若 σ 是 L_{q_1} 到 L_{q_2} 的同构映射，由命题 2.1 知，σ 是分次同构且存在映射 $f : \Gamma \to \mathbf{C}\setminus\{0\}, \varphi : \Gamma\setminus\{0\} \to \Gamma\setminus\{0\}$, 使得 $\sigma(D_{q_1}(\vec{m})) = f(\vec{m})D_{q_2}(\varphi(\vec{m}))$. 令 $a = f(1,0), b = f(0,1)$ 记 $\sigma D_{q_1}(1,0) = aD_{q_2}(\alpha_1,\beta_1), \sigma D_{q_1}(0,1) = bD_{q_2}(\alpha_2,\beta_2)$, 则由引理 2.3 可知,

$$\varphi(l,k) = (l\alpha_1 + k\alpha_2, l\beta_1 + k\beta_2)$$

且 $\alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1$. 下面进一步求出 $f(\vec{m})$, 从而得到同构映射的完整表达式.

引理 3.1 若 $q_2 = q_1^{-1}$, 则

(A)

$$f(0,n) = \begin{cases} (\alpha_2\beta_1 - \alpha_1\beta_2)^{n-1} q_2^{\frac{n(n-1)\alpha_2\beta_2}{2}} b^n, & p \nmid n, \\ -(\alpha_2\beta_1 - \alpha_1\beta_2)^{n-1} q_2^{\frac{n(n-1)\alpha_2\beta_2}{2}} b^n, & p \mid n; \end{cases} \tag{3.1}$$

(B)

$$f(l,0) = \begin{cases} (\alpha_2\beta_1 - \alpha_1\beta_2)^{l-1} q_2^{\frac{l(l-1)\alpha_1\beta_1}{2}} a^l, & p \nmid l, \\ -(\alpha_2\beta_1 - \alpha_1\beta_2)^{l-1} q_2^{\frac{l(l-1)\alpha_1\beta_1}{2}} a^l, & p \mid l. \end{cases} \tag{3.2}$$

证 先证明当 $\alpha_1\beta_2 - \alpha_2\beta_1 = -1$ 且 $q_2 = q_1^{-1}$ 时,

$$f(0,k_1+k_2) = \begin{cases} q^{k_1k_2\alpha_2\beta_2} f(0,k_1)f(0,k_2), & p \nmid k_1, p \nmid k_2, p \nmid (k_1+k_2), \\ -q^{k_1k_2\alpha_2\beta_2} f(0,k_1)f(0,k_2), & \text{其他情况}. \end{cases} \tag{3.3}$$

设 $q_2 = q_1^{-1}$ 时，若 $\vec{m}, \vec{n} \notin \Gamma_p$ 且 $g_{q_1}(\vec{m},\vec{n}) \neq 0$, 由 (2.8) 可得,

$$\frac{g_{q_2}(\varphi(\vec{m}),\varphi(\vec{n}))}{g_{q_1}(\vec{m},\vec{n})} = q_2^{(m_1\beta_1+m_2\beta_2)(n_1\alpha_1+n_2\alpha_2)+m_2n_1}.$$

当 $\vec{m} \in \Gamma_p$ 或 $\vec{n} \in \Gamma_p$ 且 $g_{q_1}(\vec{m},\vec{n}) \neq 0$ 时，由 (2.6) 得

$$\frac{g_{q_2}(\varphi(\vec{m}),\varphi(\vec{n}))}{g_{q_1}(\vec{m},\vec{n})} = \frac{m_2'n_1' - m_1'n_2'}{m_2n_1 - m_1n_2} = \alpha_1\beta_2 - \alpha_2\beta_1 = -1.$$

由此可得，当 $g_{q_1}(\vec{m},\vec{n}) \neq 0$ 时,

$$\frac{g_{q_2}(\varphi(\vec{m}),\varphi(\vec{n}))}{g_{q_1}(\vec{m},\vec{n})} = \begin{cases} q_2^{(m_1\beta_1+m_2\beta_2)(n_1\alpha_1+n_2\alpha_2)+m_2n_1}, & \vec{m},\vec{n} \notin \Gamma_p, \\ -q_2^{(m_1\beta_1+m_2\beta_2)(n_1\alpha_1+n_2\alpha_2)+m_2n_1}, & \text{其它情况}. \end{cases} \tag{3.4}$$

在 (2.12) 中取 $\vec{m}=(1,0), \vec{n}=(0,k_1), \vec{r}=(0,k_2)$, 使得 $p\nmid k_1, p\nmid k_2, p\nmid(k_1+k_2)$. 联立 (2.12), (3.4) 得

$$\begin{aligned}&f(0,k_1+k_2)\\&=\frac{g_{q_2}(\varphi(1,0),\varphi(0,k_1))}{g_{q_1}((1,0),(0,k_1))}\frac{g_{q_2}(\varphi(1,k_1),\varphi(0,k_2))}{g_{q_1}((1,k_1),(0,k_2))}\frac{g_{q_2}(\varphi(1,0),\varphi(0,k_1+k_2))}{g_{q_1}((1,0),(0,k_1+k_2))}f(0,k_1)f(0,k_2)\\&=q_2^{k_1\alpha_2\beta_1}q_2^{(\beta_1+k_1\beta_2)k_2\alpha_2}q_2^{-(k_1+k_2)\beta_1\alpha_2}f(0,k_1)f(0,k_2)\\&=q_2^{k_1k_2\alpha_2\beta_2}f(0,k_1)f(0,k_2).\end{aligned}$$

同理可证 (3.3) 的其他情况. 下面证明 (3.1). 先用归纳法证明 $n\in\mathbf{Z}_+$ 的情形. 当 $n=1$ 时, (3.1) 显然成立. 设当 $n=k-1$ 时, (3.1) 成立. 下证当 $n=k$ 时, (3.1) 也成立, 注意到此时 $\alpha_1\beta_2-\alpha_2\beta_1=-1$, 若 $p\nmid k, p\nmid k-1$ 且 $p\nmid 2k-1$, 有

$$\begin{aligned}f(0,k)=f(0,k-1+1)&=q_2^{(k-1)\alpha_2\beta_2}f(0,k-1)f(0,1)\\&=q_2^{(k-1)\alpha_2\beta_2}q^{\frac{(k-2)(k-1)\alpha_2\beta_2}{2}}b^{k-1}b^1=q_2^{\frac{k(k-1)\alpha_2\beta_2}{2}}b^k.\end{aligned}$$

对于 $p\nmid k$ 但 $p|k-1$ 或 $p|2k-1$ 或 $p|k$ 的情况, 同理可证 $n\in\mathbf{Z}_+$ 时, (3.1) 成立. 这样就证明了 $n\in\mathbf{Z}_+$ 时, (3.1) 成立. 而当 $n\in\mathbf{Z}_-$ 时, 在 (3.3) 取 $k_1=n$, $k_2=-n+1\in\mathbf{Z}_+$, 即得所要结论.

综上所述, 当 $\alpha_1\beta_2-\alpha_2\beta_1=-1$ 时, (3.1) 成立. 同理可证, 当 $\alpha_1\beta_2-\alpha_2\beta_1=1$ 时, (3.1) 也成立. 引理的第一部分得证. 同理可证 (3.2).

引理 3.2 若 $q_2=q_1^{-1}$, 则有

$$f(l,k)=\begin{cases}(\alpha_2\beta_1-\alpha_1\beta_2)^{l+k-1}q_2^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1+1)+l(l-1)\alpha_1\beta_1+k(k-1)\alpha_2\beta_2}{2}}a^lb^k, & (l,k)\notin\Gamma_p,\\-(\alpha_2\beta_1-\alpha_1\beta_2)^{l+k-1}q_2^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1+1)+l(l-1)\alpha_1\beta_1+k(k-1)\alpha_2\beta_2}{2}}a^lb^k, & (l,k)\in\Gamma_p.\end{cases}$$

证 首先证明 $\alpha_2\beta_1-\alpha_1\beta_2=1$ 时的情形. 设 $p\nmid l, p\nmid k$ 且 $p\nmid kl$. 在 (2.9) 中取 $\vec{m}=(l,0), \vec{n}=(0,k)$, 联立 (2.9), (3.1), (3.2), (3.4) 得

$$\begin{aligned}f(l,k)&=q_2^{kl\alpha_2\beta_1}f(l,0)f(0,k)\\&=q_2^{kl\alpha_2\beta_1}q^{\frac{l(l-1)\alpha_1\beta_1}{2}}a^lq^{\frac{k(k-1)\alpha_2\beta_2}{2}}b^k\\&=q_2^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1+1)+l(l-1)\alpha_1\beta_1+k(k-1)\alpha_2\beta_2}{2}}a^lb^k.\end{aligned}$$

其他情形可类似地给出证明.

类似引理 3.1 和引理 3.2 的证明可证.

引理 3.3 若 $q_1=q_2$,

$$f(l,k)=(\alpha_1\beta_2-\alpha_2\beta_1)^{l+k-1}q_2^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1-1)+l(l-1)\alpha_1\beta_1+k(k-1)\alpha_2\beta_2}{2}}a^lb^k.$$

定理 3.1 若 σ 为 L_{q_1} 到 L_{q_2} 的同构映射, 则 σ 是分次同构, 即存在映射

$$f:\Gamma\to\mathbf{C}\setminus\{0\},\varphi:\Gamma\setminus\{0\}\to\Gamma\setminus\{0\},$$

使 $\sigma D_{q_1}(l,k)=f(l,k)D_{q_2}(\varphi(l,k))$. 同时还存在 $\alpha_1,\beta_1,\alpha_2,\beta_2\in\mathbf{Z}, a,b\in\mathbf{C}^*$, 满足 $\alpha_1\beta_2-\alpha_2\beta_1=\pm1$, 使得 $\varphi(l,k)=(l\alpha_1+k\alpha_2, l\beta_1+k\beta_2)$, 而对映射 f, 则必有下列两种情形之一成立:

(1) 若 $q_2 = q_1^{-1}$,

$$f(l,k) = \begin{cases} \epsilon a^l b^k, & (l,k) \notin \Gamma_p, \\ -\epsilon a^l b^k, & (l,k) \in \Gamma_p, \end{cases}$$

其中

$$\epsilon = (\alpha_2\beta_1 - \alpha_1\beta_2)^{l+k-1} q_2^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1+1)+l(l-1)\alpha_1\beta_1+k(k-1)\alpha_2\beta_2}{2}}.$$

(2) 若 $q_2 = q_1$, 则 $f(l,k) = \epsilon' a^l b^k$, 其中

$$\epsilon' = (\alpha_1\beta_2 - \alpha_2\beta_1)^{l+k-1} q_2^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1-1)+l(l-1)\alpha_1\beta_1+k(k-1)\alpha_2\beta_2}{2}}.$$

证 由引理 2.3, 引理 3.2 和引理 3.3 直接可得定理 3.1.

§4. L_q 的自同构群

定理 4.1 设 $\mathrm{Aut}\, L_q$ 为李代数 L_q 的自同构群, 则

$$\mathrm{Aut}\, L_q \cong G(2,\mathbf{Z}) \propto (\mathbf{C}^* \times \mathbf{C}^*),$$

其中 $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$,

$$G(2,\mathbf{Z}) = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \middle| \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \pm 1,\ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{Z} \right\}.$$

证 在定理 3.1 中令 $a' = (\alpha_1\beta_2 - \alpha_2\beta_1) q^{\frac{\alpha_1\beta_1}{2}} a$, $b' = (\alpha_1\beta_2 - \alpha_2\beta_1) q^{\frac{\alpha_2\beta_2}{2}} b$, 可得 $f(l,k) = (\alpha_1\beta_2 - \alpha_2\beta_1) q^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1-1)+l^2\alpha_1\beta_1+k^2\alpha_2\beta_2}{2}} a'^l b'^k$. 首先定义集合 $(G(2,\mathbf{Z}), \mathbf{C}^*, \mathbf{C}^*)$ 上的乘法运算如下:

$$(B, x_1, y_1) \cdot (A, x, y) = (BA, x x_1^{a_{11}} y_1^{a_{21}}, y x_1^{a_{12}} y_1^{a_{22}}),$$

其中 $A = (a_{ij})_{2\times 2}, B = (b_{ij})_{2\times 2} \in G(2,\mathbf{Z})$, x, y, x_1, y_1 为非零复数. 直接验证可得此运算满足结合律且有单位元 $(I, 1, 1)$, 同时 $(G(2,\mathbf{Z}), \mathbf{C}^*, \mathbf{C}^*)$ 中任一元素 (A, x, y) 有逆元 $(A^{-1}, x^{-\frac{a_{22}}{|A|}} y^{\frac{a_{21}}{|A|}}, x^{\frac{a_{12}}{|A|}} y^{-\frac{a_{11}}{|A|}})$. 因此 $(G(2,\mathbf{Z}), \mathbf{C}^*, \mathbf{C}^*)$ 关于上述乘法构成群.

进一步, 因为 $(A, x_1, y_1)^{-1} \cdot (I, x, y) \cdot (A, x_1, y_1) = (I, x^{a_{11}} y^{a_{21}}, x^{a_{12}} y^{a_{22}})$, 故 $(I, \mathbf{C}^*, \mathbf{C}^*)$ 是群 $(G(2,\mathbf{Z}), \mathbf{C}^*, \mathbf{C}^*)$ 的正规子群, 此外, 显然有

$$(I, \mathbf{C}^*, \mathbf{C}^*) \cong (I, \mathbf{C}^*, 1) \times (I, 1, \mathbf{C}^*) \cong \mathbf{C}^* \times \mathbf{C}^*,$$

因此 $(G(2,\mathbf{Z}), \mathbf{C}^*, \mathbf{C}^*) = G(2,\mathbf{Z}) \propto (\mathbf{C}^* \times \mathbf{C}^*)$.

根据定理 3.1, 可定义一个双射 $\psi : G(2,\mathbf{Z}) \propto (\mathbf{C}^* \times \mathbf{C}^*) \longrightarrow \mathrm{Aut}\, L_q$, 使得

$$\begin{aligned} &\psi(A, x, y)(D_q(l,k)) \\ &= (\alpha_1\beta_2 - \alpha_2\beta_1) q^{\frac{kl(\alpha_1\beta_2+\alpha_2\beta_1-1)+l^2\alpha_1\beta_1+k^2\alpha_2\beta_2}{2}} x^l y^k D_q(l\alpha_1 + k\alpha_2, l\beta_1 + k\beta_2). \end{aligned}$$

由定理 3.1 可知, 对 $\mathrm{Aut}\, L_q$ 中任意元素 σ, 若记 $\sigma(D_q(1,0)) = aD_q(\alpha_1, \beta_1)$, $\sigma(D_q(0,1)) = bD_q(\alpha_2, \beta_2)$, 则同构映射 σ 完全由 $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbf{Z}$ 和 $a, b \in \mathbf{C}^*$ 所确定. 根据定理 3.1, 易验算

$$\begin{aligned} &\psi(B, x_1, y_1)\psi(A, x, y)(D_q(1,0), D_q(0,1)) \\ &= (x x_1^{a_{11}} y_1^{a_{21}} q^{\frac{1}{2}c_{11}c_{21}} D_q(c_{11}, c_{12}), y x_1^{a_{12}} y_1^{a_{22}} q^{\frac{1}{2}c_{12}c_{22}} D_q(c_{21}, c_{22})) \\ &= \psi(BA, x x_1^{a_{11}} y_1^{a_{21}}, y x_1^{a_{12}} y_1^{a_{22}})(D_q(1,0), D_q(0,1)) \\ &= \psi((B, x_1, y_1) \cdot (A, x, y))(D_q(1,0), D_q(0,1)), \end{aligned}$$

其中 $BA = (c_{ij})_{2\times 2}$, 故

$$\psi(B, x_1, y_1)\psi(A, x, y) = \psi((B, x_1, y_1)\cdot(A, x, y)).$$

因此, ψ 是一个群同态. 故 $\mathrm{Aut}\, L \cong G(2, \mathbf{Z}) \propto (\mathbf{C}^* \times \mathbf{C}^*)$.

参 考 文 献

[1] Jiang, C. & Meng, D., The derivation algebra of the associtive algebra $\mathbf{C}_q[x, y, x^{-1}, y^{-1}]$, *Comm. in Algebra*, **26**:6(1998), 1723–1736.

[2] Jiang, C. & Meng, D., The automorphism group of the Derivation algebra of the virasoro-like algebra [J], *Advances in Math. (China)*, **27**:2(1998), 175–183.

[3] Kirkman, E., Procesi, C. & Small, L. A., *q*-analog for the virasoro algebra [J], *Comm. in Algebra*, **22**:10(1994), 3755–3774.

[4] 林卫强、谭绍滨, 量子环面上斜导子李代数的表示 [J], 数学进展, **34**:4(2005), 477–487.

[5] 薛 旻、林卫强, 量子环面上 Skew 导子李代数的自同构群 [J], 厦门大学学报, **44**:1 (2005), 5–10.

[6] Zhang, H. & Zhao, K., Representations of the virasoro-like algebra and its q-analog [J], *Comm. in Algebra*, **24**:14(1996), 4361–4372.

[7] Kaplansky, I. & Santharoubane, L. J., Harish-Chandra Modules over the Virasoro Algebra [M], Infinite-Dimensional Groups with Applications, MSRI series, **4**:10(1985), Springer-Verlag, 217–232.

[8] Kassel, C., Cyclic homology of differential operators, the Virasoro algebra and a q-analogue [J], *Comm. Math. Phys.*, **146**(1992), 343–356.

[9] Mathieu, O., Classification of Harish-Chanda modules over the Virasoro Lie algebra [J], *Invent. Math.*, **107**(1992), 225–234.

STRUCTURE AND AUTOMORPHISM GROUP OF A CLASS OF DERIVATION LIE ALGEBRAS OVER QUANTUM TORUS

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Abstract

This paper studies a class of derivation Lie algebras over quantum torus, which includes the Virasoro-Like algebra and its q-analog, and proves that every isomorphism between such Lie algebras must be a graded isomorphism. Moreover, the authors give a necessary and sufficient condition for the isomorphic Lie algebras, and obtain the explicit expression of the isomorphisms. Finally, the authors give the structure of the automorphism group of the Lie algebra.

Keywords Lie algebra, Derivation, Isomorphism, Quantum torus

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