THE INVERSE EIGENPROBLEM OF CENTROSYMMETRIC MATRICES WITH A SUBMATRIX CONSTRAINT AND ITS APPROXIMATION

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Abstract. In this paper, we first consider the existence of and the general expression for the solution to the constrained inverse eigenproblem defined as follows: given a set of complex n-vectors $\{\mathbf{x}_i\}_{i=1}^m$ and a set of complex numbers $\{\lambda_i\}_{i=1}^m$, and an s-by-s real matrix C_0, find an n-by-n real centrosymmetric matrix C such that the s-by-s leading principal submatrix of C is C_0, and $\{\mathbf{x}_i\}_{i=1}^m$ and $\{\lambda_i\}_{i=1}^m$ are the eigenvectors and eigenvalues of C respectively. We then concerned with the best approximation problem for the constrained inverse problem whose solution set is nonempty. That is, given an arbitrary real n-by-n matrix $\tilde{C}$, find a matrix C which is the solution to the constrained inverse problem such that the distance between C and $\tilde{C}$ is minimized in the Frobenius norm. We give an explicit solution and a numerical algorithm to the best approximation problem. Some illustrative experiments are also presented.

Key words. Inverse problem, centrosymmetric matrix, best approximation.

AMS subject classifications. 65F18, 65F15, 65F35

1. Introduction.

Let E_n be the n-by-n backward identity matrix, i.e, E_n has 1 on the anti-diagonal and zeros elsewhere. An n-by-n real matrix C is said to be *centrosymmetric* if $C = E_nCE_n$. The centrosymmetric matrices have practical applications in many areas such as pattern recognition [10], the numerical solution of certain differential equations [1, 4], Markov processes [22] and various physical and engineering problems [11, 12]. The symmetric Toeplitz matrices, an important subclass of the class of symmetric centrosymmetric matrices, appear naturally in digital signal processing applications and other areas [13].

The inverse eigenproblems play an important role in many applications such as control theory [23], the design of Hopfield neural networks [8, 16], vibration theory [20] and structure mechanics and molecular spectroscopy [14]. For the recent progress, see for instance [7, 25]. The inverse eigenproblem for centrosymmetric matrices has been discussed by Bai and R. Chan [2]. However, the inverse eigenproblem for centrosymmetric matrices with a submatrix constraint has not been discussed. In this paper, we will consider two related problems. The first problem is the constrained inverse eigenproblem for centrosymmetric matrices:

Problem I. Given the eigenpairs $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m] \in \mathbb{C}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^{m\times m}$, and a matrix $C_0 \in \mathbb{R}^{s\times s}$, find a centrosymmetric matrix C in $\mathbb{R}^{n\times n}$ such that $CX = X\Lambda$ and the s-by-s leading principal submatrix of C is C_0.

The prototype of this problem initially arose in the design of Hopfield neural networks [8, 16]. It also occurs in the design of vibration in mechanism, civil engineering and aviation [5]. The problem has been studied for bisymmetric matrices in [18]. The second problem we consider in this paper is the problem of best approximation.

Problem II. Let $\mathcal{L}_S$ be the solution set of Problem I. Given a matrix $\tilde{C} \in \mathbb{R}^{n\times n}$,

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find $C^* \in \mathcal{L}_S$ such that

$$\|C^* - \tilde{C}\| = \min_{C \in \mathcal{L}_S} \|C - \tilde{C}\|,$$

where $\|\cdot\|$ is the Frobenius norm.

The best approximation problem arises frequently in experimental design, see for instance [17, p.123]. Here the matrix $\tilde{C}$ may be a matrix obtained from experiments, but it may not satisfy the structural requirement (centrosymmetric or the submatrix constraint) and/or spectral requirement (having eigenpairs X and Λ). The best estimate C^* is the matrix that satisfies both requirements and is the best approximation of $\tilde{C}$ in the Frobenius norm. In addition, because there are fast algorithms for solving various kinds of centrosymmetric matrices [15], the best approximate C^* of $\tilde{C}$ can also be used as a preconditioner in the preconditioned conjugate gradient method for solving linear systems with coefficient matrix $\tilde{C}$, see for instance [3].

We remark that when $s = 0$, Problem I is reduced to the inverse eigenproblem for centrosymmetric matrices discussed by Bai and R. Chan [2]. When $s = n$, $C^* = C_0$ is the best approximation of the matrix $\tilde{C}$ to Problem II. In this paper, we consider the general case when $0 < s < n$.

In this paper, we use the following notations. We denote the identity matrix of order n by I_n. Let $\mathrm{rank}(A)$ be the rank of a matrix A. Let A^+ and $A(1:s)$ denote the Moore-Penrose generalized inverse and the leading principal submatrix of a matrix A respectively. $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the column space and the null space of A respectively.

The paper is organized as follows. In §2 we first review the structure of centrosymmetric matrices and give some useful lemmas. In §3 we provide the solvability conditions for and the general solutions of Problem I. In §4 we show the existence and uniqueness of the solution to Problem II when the solution set of Problem I is nonempty, and derive a formula for the best approximation of Problem II, and then propose a numerical algorithm for computing the minimizer. In §5 an experiment is presented to illustrate our results. Finally, in §6, we give some conclusions.

2. Preliminary Lemmas. In this section, we will recall the properties of centrosymmetric matrices and give some preliminary lemmas.

Let $k = [n/2]$ denote the largest integer number that is not greater than $n/2$. When $n = 2k$, we define

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ E_k & -E_k \end{pmatrix}$$

and when $n = 2k+1$, let

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & \mathbf{0} & I_k \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ E_k & \mathbf{0} & -E_k \end{pmatrix}.$$

Clearly, K is orthogonal. Then we have the following splitting of centrosymmetric matrices into smaller submatrices using K, see for example [9, 2].

LEMMA 2.1. **[9]** *Denote the set of all n-by-n real centrosymmetric matrices by $\mathcal{C}_n$. Then any $C \in \mathcal{C}_{2k}$ can be written as*

$$C = \begin{pmatrix} A & BE_k \\ E_kB & E_kAE_k \end{pmatrix} = K \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix} K^T, \quad A, B \in \mathbb{R}^{k\times k}.$$

Any $C \in \mathcal{C}_{2k+1}$ can be written as

$$C = \begin{pmatrix} A & \mathbf{p} & BE_k \\ \mathbf{q}^T & c & \mathbf{q}^T E_k \\ E_k B & E_k \mathbf{p} & E_k A E_k \end{pmatrix} = K \begin{pmatrix} A+B & \sqrt{2}\mathbf{p} & 0 \\ \sqrt{2}\mathbf{q}^T & c & 0 \\ 0 & 0 & A-B \end{pmatrix} K^T,$$

where $A, B \in \mathbb{R}^{k\times k}$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}^k$, $c \in \mathbb{R}$. Moreover, for all $n = 2k$ and $2k+1$, any $C \in \mathcal{C}_n$ is of the form:

$$C = K \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} K^T, \quad F_1 \in \mathbb{R}^{(n-k)\times(n-k)}, F_2 \in \mathbb{R}^{k\times k}. \tag{2.1}$$

LEMMA 2.2. *Suppose that $C \in \mathcal{C}_n$ and $C_0 = C(1:s)$. If $s < n-k$, then*

$$F_1(1:s) + F_2(1:s) = 2C_0, \tag{2.2}$$

where F_1 and F_2 are the same as (2.1), and if $s \geq n-k$, then we obtain

$$C = \begin{pmatrix} C_{11} & C_{12} & HE_{n-s} \\ C_{21} & C_{22} & E_{2s-n}C_{21}E_{n-s} \\ E_{n-s}H & E_{n-s}C_{12}E_{2s-n} & E_{n-s}A_{11}E_{n-s} \end{pmatrix}, \tag{2.3}$$

where $H \in \mathbb{R}^{(n-s)\times(n-s)}$ and $C_0 = C(1:s) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ with $C_{11} \in \mathbb{R}^{(n-s)\times(n-s)}$ and $C_{22} \in \mathcal{C}_{2s-n}$.

Proof. If $s < n-k$, we get from Lemma 2.1 that

$$C(1:s) = A(1:s)$$

and

$$F_1(1:s) + F_2(1:s) = (A+B)(1:s) + (A-B)(1:s) = 2A(1:s).$$

Thus (2.2) holds.

If $s \geq n-k$, then since $C(1:s) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, we can partition C into the following form

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}, \tag{2.4}$$

where $C_{11} \in \mathbb{R}^{(n-s)\times(n-s)}$, $C_{22} \in \mathbb{R}^{(2s-n)\times(2s-n)}$, $C_{33} \in \mathbb{R}^{(n-s)\times(n-s)}$. By (2.4) and comparing the two sides of $C = E_n C E_n$, we obtain $C_{13} = E_{n-s}C_{31}E_{n-s}$, $C_{22} = E_{2s-n}C_{22}E_{2s-n}$, $C_{23} = E_{2s-n}C_{21}E_{n-s}$, $C_{32} = E_{n-s}C_{12}E_{2s-n}$ and $C_{33} = E_{n-s}C_{11}E_{n-s}$. Let $H = C_{13}E_{n-s}$, we get $C_{13} = HE_{n-s}$ and $C_{31} = E_{n-s}H$. Substituting C_{13}, C_{23}, C_{31}, C_{32} and C_{33} into (2.4) and noticing that $C_{22} = E_{2s-n}C_{22}E_{2s-n}$, we have (2.3). □

In order to investigate the solvability of Problem I, we need the following lemmas.

LEMMA 2.3. **[21, Lemma 1.3]** *Given $X, G \in \mathbb{R}^{n\times m}$ with* $\operatorname{rank}(X) = l$. *Then $YX = G$ has a solution $Y \in \mathbb{R}^{n\times n}$ if and only if $GX^+X = G$. In this case the general solution is*

$$Y = GX^+ + ZU_2^T,$$

where $U_2 \in \mathbb{R}^{n\times(n-l)}$, $U_2^TU_2 = I_{n-l}$, $\mathcal{R}(U_2) = \mathcal{N}(X^T)$, *and* $Z \in \mathbb{R}^{n\times(n-l)}$ *is arbitrary.*

LEMMA 2.4. **[24, Lemma 3.1]** *Given any* $E, F \in \mathbb{R}^{n\times n}$. *Then there exists a unique* $Y^* \in \mathbb{R}^{n\times n}$ *such that*

$$\|Y^* - E\|^2 + \|Y^* - F\|^2 = \min_{Y\in\mathbb{R}^{n\times n}}\{\|Y - E\|^2 + \|Y - F\|^2\}$$

and

$$Y^* = \frac{E+F}{2}.$$

LEMMA 2.5. *Given any* $E, F \in \mathbb{R}^{u\times v}$, $D = \mathrm{diag}(d_1, \dots, d_v) > 0$ *and* $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_v)$, *where* $\theta_i = 1/(1 + d_i^2)$. *Then there exists a unique* $Y^* \in \mathbb{R}^{u\times v}$ *such that*

$$\|Y^* - E\|^2 + \|Y^*D - F\|^2 = \min_{Y\in\mathbb{R}^{u\times v}}\{\|Y - E\|^2 + \|YD - F\|^2\}$$

and

$$Y^* = (E + FD)\Theta.$$

Proof. Let $Y = (y_{ij}), E = (e_{ij}), F = (f_{ij})$. Since

$$\|Y - E\|^2 + \|YD - F\|^2 = \sum_{i=1}^{u}\sum_{j=1}^{v}(y_{ij} - e_{ij})^2 + \sum_{i=1}^{u}\sum_{j=1}^{v}(y_{ij}d_j - f_{ij})^2$$

$$= \sum_{i=1}^{u}\sum_{j=1}^{v}[y_{ij}^2(1 + d_j^2) - 2y_{ij}(e_{ij} + f_{ij}d_j) + e_{ij}^2 + f_{ij}^2]$$

$$= \sum_{i=1}^{u}\sum_{j=1}^{v}(1 + d_j^2)[(y_{ij} - \frac{e_{ij} + f_{ij}d_j}{1 + d_j^2})^2 + \frac{e_{ij}^2 + f_{ij}^2}{1 + d_j^2} - \frac{(e_{ij} + f_{ij}d_j)^2}{(1 + d_j^2)^2}].$$

Thus there exists $Y \in \mathbb{R}^{u\times v}$ such that $\|Y - E\|^2 + \|YD - F\|^2 = \min$ is equivalent to $y_{ij} = (e_{ij} + f_{ij}d_j)/(1 + d_j^2)$, i.e. $Y^* = (E + FD)\Theta$. □

From Lemma 2.5, we can easily see that Lemma 2.4 is a special case of Lemma 2.5 where $u = v = n$, $D = I_n$, and $\Theta = 1/2I_n$.

3. Solvability Conditions and General Solutions of Problem I. Before we come to Problem I, we first note the following facts: For a real matrix $C \in \mathcal{C}_n$, its complex eigenvectors and eigenvalues are complex conjugate pairs. If $a \pm b\sqrt{-1}$ and $\mathbf{x} \pm \sqrt{-1}\mathbf{y}$ are one of its eigenpairs, then we have $C\mathbf{x} = a\mathbf{x} - b\mathbf{y}$ and $C\mathbf{y} = a\mathbf{y} + b\mathbf{x}$, i.e.

$$C[\mathbf{x}, \mathbf{y}] = [\mathbf{x}, \mathbf{y}]\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Therefore, without loss of generality, we can assume that $X \in \mathbb{R}^{n\times m}$ and

$$\Lambda = \mathrm{diag}(\Psi_1, \Psi_2, \dots, \Psi_l, g_1, \dots, g_{m-2l}) \in \mathbb{R}^{m\times m}, \tag{3.1}$$

where $\Psi_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ with a_i, b_i and g_i are real numbers.

THEOREM 3.1. *Given $X \in \mathbb{R}^{n\times m}$ and Λ as in (3.1), and $C_0 \in \mathbb{R}^{s\times s}$ where $s < n-k$. Partition $K^T X$ as*

$$K^T X = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}, \quad \tilde{X}_2 \in \mathbb{R}^{k\times m}. \tag{3.2}$$

Define

$$M_1 = [I_s, O_1]U_2, \quad M_2 = [I_s, O_2]V_2, \tag{3.3}$$

where $U_2 \in \mathbb{R}^{(n-k)\times(n-k-l_1)}$ and $V_2 \in \mathbb{R}^{k\times(k-l_2)}$ are column orthonormal, $\mathcal{R}(U_2) = \mathcal{N}(\tilde{X}_1^T)$, $\mathcal{R}(V_2) = \mathcal{N}(\tilde{X}_2^T)$, $l_1 = \operatorname{rank}(\tilde{X}_1)$, $l_2 = \operatorname{rank}(\tilde{X}_2)$, and $O_1 \in \mathbb{R}^{s\times(n-k-s)}$ and $O_2 \in \mathbb{R}^{s\times(k-s)}$ are zero matrices. Suppose that the generalized singular value decomposition (GSVD) of the matrix pair M_1^T and M_2^T is

$$M_1^T = P\Sigma_1 S^T, \quad M_2^T = Q\Sigma_2 S^T, \tag{3.4}$$

where S is an s-by-s nonsingular matrix, $P \in \mathbb{R}^{(n-k-l_1)\times(n-k-l_1)}$, $Q \in \mathbb{R}^{(k-l_2)\times(k-l_2)}$ are orthogonal, and

$$\Sigma_1 = \begin{array}{c} \begin{array}{cccc} r & t & h-r-t & s-h \end{array} \\ \begin{pmatrix} I_r & & & \\ & \Gamma_1 & & O \\ & & O_3 & \end{pmatrix} \end{array}, \quad \Sigma_2 = \begin{array}{c} \begin{array}{cccc} r & t & h-r-t & s-h \end{array} \\ \begin{pmatrix} O_4 & & & \\ & \Gamma_2 & & O \\ & & I_{h-r-t} & \end{pmatrix} \end{array} \tag{3.5}$$

with $h = \operatorname{rank}(M) = \operatorname{rank}([M_1, M_2])$, $r = h - \operatorname{rank}(M_2)$, $t = \operatorname{rank}(M_1) + \operatorname{rank}(M_2) - h$, O, O_3 and O_4 are zero matrices of size implied by context, and $\Gamma_1 = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_t)$, $\Gamma_2 = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_t)$ with $1 \geq \gamma_t \geq \cdots \geq \gamma_1 > 0$, $0 < \delta_1 \leq \cdots \leq \delta_t$, $\gamma_i^2 + \delta_i^2 = 1$ for $i = 1, \ldots, t$. Let

$$\tilde{G} = 2C_0 - [I_s, O_1]\tilde{X}_1\Lambda\tilde{X}_1^+[I_s, O_1]^T - [I_s, O_2]\tilde{X}_2\Lambda\tilde{X}_2^+[I_s, O_2]^T \tag{3.6}$$

and partition $\tilde{G}S^{-T}$ into the following form:

$$\tilde{G}S^{-T} = \begin{array}{c} \\ r \\ t \\ s-r-t \end{array} \begin{array}{c} \begin{array}{cccc} r & t & h-r-t & s-h \end{array} \\ \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \end{pmatrix} \end{array}. \tag{3.7}$$

Then there exists a matrix $C \in \mathcal{C}_n$ such that $CX = X\Lambda$ and $C(1:s) = C_0$ if and only if

$$\tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_1\Lambda, \quad \tilde{X}_2\Lambda\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_2\Lambda \quad \text{and} \quad [G_{14}^T, G_{24}^T, G_{34}^T] = 0. \tag{3.8}$$

In this case, the general solution is given by

$$C = K\begin{pmatrix} \tilde{X}_1\Lambda\tilde{X}_1^+ + Z_1U_2^T & 0 \\ 0 & \tilde{X}_2\Lambda\tilde{X}_2^+ + Z_2V_2^T \end{pmatrix}K^T \tag{3.9}$$

with

$$Z_1 = \begin{array}{c} \\ r \\ t \\ s-r-t \\ n-k-s \end{array} \begin{array}{c} \begin{array}{ccc} r & t & n-k-l_1-r-t \end{array} \\ \begin{pmatrix} G_{11} & X_{12} & X_{13} \\ G_{21} & X_{22} & X_{23} \\ G_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{pmatrix} \end{array} P^T,$$

$$Z_2=\begin{array}{c} \\ r\\ t\\ s-r-t\\ k-s\end{array}\!\!\begin{array}{c}\begin{array}{ccc} k-l_2+r-h & t & h-r-t\end{array}\\ \begin{pmatrix} Y_{11} & (G_{12}-X_{12}\Gamma_1)\Gamma_2^{-1} & G_{13}\\ Y_{21} & (G_{22}-X_{22}\Gamma_1)\Gamma_2^{-1} & G_{23}\\ Y_{31} & (G_{32}-X_{32}\Gamma_1)\Gamma_2^{-1} & G_{33}\\ Y_{41} & Y_{42} & Y_{43}\end{pmatrix}\end{array} Q^T,$$

where X_{12}, X_{13}, X_{22}, X_{23}, X_{32}, X_{33}, X_{41}, X_{42}, X_{43}, Y_{11}, Y_{21}, Y_{31}, Y_{41}, Y_{42} *and* Y_{43} *are arbitrary matrices.*

Proof. By Lemmas 2.1 and 2.2, $C\in\mathcal{C}_n$ is a solution to Problem I if and only if there exist $F_1\in\mathbb{R}^{(n-k)\times(n-k)}$ and $F_2\in\mathbb{R}^{k\times k}$ such that

$$C=K\begin{pmatrix}F_1 & 0\\ 0 & F_2\end{pmatrix}K^T,\quad F_1(1:s)+F_2(1:s)=2C_0 \tag{3.10}$$

and

$$K\begin{pmatrix}F_1 & 0\\ 0 & F_2\end{pmatrix}K^TX=X\Lambda. \tag{3.11}$$

Using (3.2), (3.11) is equivalent to

$$F_1\tilde{X}_1=\tilde{X}_1\Lambda\quad\text{and}\quad F_2\tilde{X}_2=\tilde{X}_2\Lambda. \tag{3.12}$$

According to Lemma 2.3, equations (3.12) have solutions if and only if

$$\tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1=\tilde{X}_1\Lambda,\quad \tilde{X}_2\Lambda\tilde{X}_2^+\tilde{X}_2=\tilde{X}_2\Lambda.$$

Moreover in this case, the general solution of (3.12) is given by

$$F_1=\tilde{X}_1\Lambda\tilde{X}_1^++Z_1U_2^T, \tag{3.13}$$
$$F_2=\tilde{X}_2\Lambda\tilde{X}_2^++Z_2V_2^T, \tag{3.14}$$

where $Z_1\in\mathbb{R}^{(n-k)\times(n-k-l_1)}$ and $Z_2\in\mathbb{R}^{k\times(k-l_2)}$ are both arbitrary. Putting (3.13) and (3.14) into $F_1(1:s)+F_2(1:s)=2C_0$, and using the definition of $M_1,M_2,\tilde{G}$ and the GSVD of the matrix pair M_1^T and M_2^T, it is easy to show that Z_1 and Z_2 must satisfy

$$[I_s,O_1]Z_1P\Sigma_1+[I_s,O_2]Z_2Q\Sigma_2=\tilde{G}S^{-T}. \tag{3.15}$$

Partition Z_1P and Z_2Q into the following form:

$$Z_1P=\begin{pmatrix}X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\\ X_{41} & X_{42} & X_{43}\end{pmatrix},\quad Z_2Q=\begin{pmatrix}Y_{11} & Y_{12} & Y_{13}\\ Y_{21} & Y_{22} & Y_{23}\\ Y_{31} & Y_{32} & Y_{33}\\ Y_{41} & Y_{42} & Y_{43}\end{pmatrix}. \tag{3.16}$$

Substituting (3.5), (3.7) and (3.16) into (3.15) yields

$$\begin{pmatrix}X_{11} & X_{12}\Gamma_1+Y_{12}\Gamma_2 & Y_{13} & 0\\ X_{21} & X_{22}\Gamma_1+Y_{22}\Gamma_2 & Y_{23} & 0\\ X_{31} & X_{32}\Gamma_1+Y_{32}\Gamma_2 & Y_{33} & 0\end{pmatrix}=\begin{pmatrix}G_{11} & G_{12} & G_{13} & G_{14}\\ G_{21} & G_{22} & G_{23} & G_{24}\\ G_{31} & G_{32} & G_{33} & G_{34}\end{pmatrix}. \tag{3.17}$$

Thus (3.17), and hence (3.15) holds if and only if

$$[G_{14}^T,G_{24}^T,G_{34}^T]=0, \tag{3.18}$$

$$X_{11} = G_{11}, \quad X_{21} = G_{21}, \quad X_{31} = G_{31}, \quad Y_{13} = G_{13}, \quad Y_{23} = G_{23}, \quad Y_{33} = G_{33}, \tag{3.19}$$

$$Y_{12} = (G_{12} - X_{12}\Gamma_1)\Gamma_2^{-1}, \quad Y_{22} = (G_{22} - X_{22}\Gamma_1)\Gamma_2^{-1}, \quad Y_{32} = (G_{32} - X_{32}\Gamma_1)\Gamma_2^{-1}. \tag{3.20}$$

Therefore, the solvability conditions for Problem I and the general expression of the solution of Problem I are obtained by (3.10), (3.12)–(3.14), (3.16), and (3.18)–(3.20). □

THEOREM 3.2. *Given $X \in \mathbb{R}^{n\times m}$ and Λ as in (3.1), and $C_0 \in \mathbb{R}^{s\times s}$ where $s \geq n-k$. Partition C_0 and X as*

$$C_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{3.21}$$

where $C_{11} \in \mathbb{R}^{(n-s)\times(n-s)}$, $C_{22} \in \mathbb{R}^{(2s-n)\times(2s-n)}$, $X_1, X_3 \in \mathbb{R}^{(n-s)\times m}$ and $X_2 \in \mathbb{R}^{(2s-n)\times m}$. Let

$$U = [X_1, E_{n-s}X_3] \tag{3.22}$$

and

$$V = [E_{n-s}X_3\Lambda - C_{12}E_{2s-n}X_2 - C_{11}E_{n-s}X_3, X_1\Lambda - C_{11}X_1 - C_{12}X_2]. \tag{3.23}$$

Then Problem I is solvable if and only if

$$VU^+U = V, \quad C_{21}X_1 + C_{22}X_2 + E_{2s-n}C_{21}E_{n-s}X_3 = X_2\Lambda, \quad C_{22} \in \mathcal{C}_{2s-n}. \tag{3.24}$$

In this case, the general solution to Problem I can be expressed as

$$C = \begin{pmatrix} C_{11} & C_{12} & HE_{n-s} \\ C_{21} & C_{22} & E_{2s-n}C_{21}E_{n-s} \\ E_{n-s}H & E_{n-s}C_{12}E_{2s-n} & E_{n-s}C_{11}E_{n-s} \end{pmatrix}, \tag{3.25}$$

where $H = VU^+ + WQ_2^T$, where $Q_2 \in \mathbb{R}^{(n-s)\times(n-s-l_3)}$ is orthogonal, $\mathcal{R}(Q_2) = \mathcal{N}(U^T)$, $l_3 = \mathrm{rank}(U)$ and $W \in \mathbb{R}^{(n-s)\times(n-s-l_3)}$ is arbitrary.

Proof. By Lemma 2.2, there exists $C \in \mathcal{C}_n$ such that $CX = X\Lambda$ and $C_0 = C(1:s)$ if and only if there exists $H \in \mathbb{R}^{(n-s)\times(n-s)}$ such that

$$C = \begin{pmatrix} C_{11} & C_{12} & HE_{n-s} \\ C_{21} & C_{22} & E_{2s-n}C_{21}E_{n-s} \\ E_{n-s}H & E_{n-s}C_{12}E_{2s-n} & E_{n-s}C_{11}E_{n-s} \end{pmatrix}, \quad CX = X\Lambda.$$

Equivalently,

$$C_{21}X_1 + C_{22}X_2 + E_{2s-n}C_{21}E_{n-s}X_3 = X_2\Lambda, \quad C_{22} \in \mathcal{C}_{2s-n}$$

and

$$HU = V. \tag{3.26}$$

From Lemma 2.3, (3.26) holds if and only if

$$VU^+U = V, \tag{3.27}$$

and when (3.27) holds, H can be expressed as

$$H = VU^+ + WQ_2^T.$$

Thus Problem I is solvable if and only if the conditions in (3.24) hold, and the general solution can be expressed as (3.25). □

4. The Solution of Problem II. In this section, we solve Problem II over $\mathcal{L}_S$ when $\mathcal{L}_S$ is nonempty.

THEOREM 4.1. *Given $X \in \mathbb{R}^{n\times m}$ and Λ as in (3.1), and $C_0 \in \mathbb{R}^{s\times s}$ where $s < n-k$. Suppose the solution set $\mathcal{L}_S$ of Problem I be nonempty. Let*

$$K^T \tilde{C} K = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix}, \tag{4.1}$$

$$(\tilde{C}_{11} - \tilde{X}_1 \Lambda \tilde{X}_1^+) U_2 P = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \\ E_{41} & E_{42} & E_{43} \end{pmatrix}, \tag{4.2}$$

$$(\tilde{C}_{22} - \tilde{X}_2 \Lambda \tilde{X}_2^+) V_2 Q = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \\ F_{41} & F_{42} & F_{43} \end{pmatrix}, \tag{4.3}$$

where $\tilde{X}_1, \tilde{X}_2$ are the same as (3.2), the size of matrices $\tilde{C}_{11}$ and $\tilde{C}_{22}$ are the same as F_1 and F_2 in (3.10) respectively, the partition form of (4.2) and (4.3) are the same as (3.16). Then Problem II has a unique solution and the solution is given by

$$C^* = K \begin{pmatrix} \tilde{X}_1 \Lambda \tilde{X}_1^+ + Z_1 U_2^T & 0 \\ 0 & \tilde{X}_2 \Lambda \tilde{X}_2^+ + Z_2 V_2^T \end{pmatrix} K^T, \tag{4.4}$$

where

$$Z_1 = \begin{pmatrix} G_{11} & \hat{X}_{12} & E_{13} \\ G_{21} & \hat{X}_{22} & E_{23} \\ G_{31} & \hat{X}_{32} & E_{33} \\ E_{41} & E_{42} & E_{43} \end{pmatrix} P^T, \quad Z_2 = \begin{pmatrix} F_{11} & (G_{12} - \hat{X}_{12}\Gamma_1)\Gamma_2^{-1} & G_{13} \\ F_{21} & (G_{22} - \hat{X}_{22}\Gamma_1)\Gamma_2^{-1} & G_{23} \\ F_{31} & (G_{32} - \hat{X}_{32}\Gamma_1)\Gamma_2^{-1} & G_{33} \\ F_{41} & F_{42} & F_{43} \end{pmatrix} Q^T.$$

$$\hat{X}_{12} = (G_{12}\Gamma_1\Gamma_2^{-2} + E_{12} - F_{12}\Gamma_1\Gamma_2^{-1})\Theta,$$

$$\hat{X}_{22} = (G_{22}\Gamma_1\Gamma_2^{-2} + E_{22} - F_{22}\Gamma_1\Gamma_2^{-1})\Theta,$$

$$\hat{X}_{32} = (G_{32}\Gamma_1\Gamma_2^{-2} + E_{32} - F_{32}\Gamma_1\Gamma_2^{-1})\Theta,$$

$$\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_t), \quad \theta_i = \frac{\delta_i^2}{\delta_i^2 + \gamma_i^2}.$$

Proof. When $\mathcal{L}_S$ is nonempty, it is easy to verify from (3.9) that $\mathcal{L}_S$ is a closed convex set. Since $\mathbb{R}^{n\times n}$ is a uniformly convex Banach space under the Frobenius norm, there exists a unique solution for Problem II [6, p. 22]. Moreover, because the Frobenius norm is unitary invariant, Problem II is equivalent to

$$\min_{C \in \mathcal{L}_S} \|K^T \tilde{C} K - K^T C K\|^2. \tag{4.5}$$

By (3.9) and (4.1)–(4.3), (4.5) is equivalent to

$$\min_{Z_1\in\mathbb{R}^{(n-k)\times(n-k-l_1)}}\|\tilde{X}_1\Lambda\tilde{X}_1^{+}+Z_1U_2^T-\tilde{C}_{11}\|^2+\min_{Z_2\in\mathbb{R}^{k\times(k-l_2)}}\|\tilde{X}_2\Lambda\tilde{X}_2^{+}+Z_2V_2^T-\tilde{C}_{22}\|^2.$$

Equivalently,

$$\min_{Z_1\in\mathbb{R}^{(n-k)\times(n-k-l_1)}}\|Z_1-(\tilde{C}_{11}-\tilde{X}_1\Lambda\tilde{X}_1^{+})U_2\|^2+\min_{Z_2\in\mathbb{R}^{k\times(k-l_2)}}\|Z_2-(\tilde{C}_{22}-\tilde{X}_2\Lambda\tilde{X}_2^{+})V_2\|^2.$$

Clearly, the solution is given by X_{12}, X_{13}, X_{22}, X_{23}, X_{32}, X_{33}, X_{41},X_{42}, X_{43} and Y_{11}, Y_{21}, Y_{31}, Y_{41}, Y_{42}, Y_{43} such that

$$\|X_{13}-E_{13}\|=\min,\quad \|X_{23}-E_{23}\|=\min,\quad \|X_{33}-E_{33}\|=\min, \tag{4.6}$$

$$\|X_{41}-E_{41}\|=\min,\quad \|X_{42}-E_{42}\|=\min,\quad \|X_{43}-E_{43}\|=\min, \tag{4.7}$$

$$\|Y_{11}-F_{11}\|=\min,\quad \|Y_{21}-F_{21}\|=\min,\quad \|Y_{31}-F_{31}\|=\min, \tag{4.8}$$

$$\|Y_{41}-F_{41}\|=\min,\quad \|Y_{42}-F_{42}\|=\min,\quad \|Y_{43}-F_{43}\|=\min, \tag{4.9}$$

$$\|X_{12}-E_{12}\|^2+\|X_{12}\Gamma_1\Gamma_2^{-1}-(G_{12}\Gamma_2^{-1}-F_{12})\|^2=\min, \tag{4.10}$$

$$\|X_{22}-E_{22}\|^2+\|X_{22}\Gamma_1\Gamma_2^{-1}-(G_{22}\Gamma_2^{-1}-F_{22})\|^2=\min, \tag{4.11}$$

$$\|X_{32}-E_{32}\|^2+\|X_{32}\Gamma_1\Gamma_2^{-1}-(G_{32}\Gamma_2^{-1}-F_{32})\|^2=\min\,. \tag{4.12}$$

By (4.6)–(4.9), we get

$$X_{13}=E_{13},\quad X_{23}=E_{23},\quad X_{33}=E_{33},\quad X_{41}=E_{41},\quad X_{42}=E_{42},\quad X_{43}=E_{43}, \tag{4.13}$$

$$Y_{11}=F_{11},\quad Y_{21}=F_{21},\quad Y_{31}=F_{31},\quad Y_{41}=F_{41},\quad Y_{42}=F_{42},\quad Y_{42}=F_{42},\quad Y_{43}=F_{43}. \tag{4.14}$$

Applying Lemma 2.5 to (4.10)–(4.12), we obtain

$$X_{12}=(G_{12}\Gamma_1\Gamma_2^{-2}+E_{12}-F_{12}\Gamma_1\Gamma_2^{-1})\Theta,\quad X_{22}=(G_{22}\Gamma_1\Gamma_2^{-2}+E_{22}-F_{22}\Gamma_1\Gamma_2^{-1})\Theta, \tag{4.15}$$

$$X_{32}=(G_{32}\Gamma_1\Gamma_2^{-2}+E_{32}-F_{32}\Gamma_1\Gamma_2^{-1})\Theta. \tag{4.16}$$

By (3.9) and (4.13)–(4.16), we have the unique solution of Problem II is given by (4.4). □

THEOREM 4.2. *Given $X\in\mathbb{R}^{n\times m}$ and Λ as in (3.1), and $C_0\in\mathbb{R}^{s\times s}$ where $s\geq n-k$. Suppose the solution set $\mathcal{L}_S$ of Problem I is nonempty. Let*

$$\tilde{C}=\begin{pmatrix}W_{11}&W_{12}&W_{13}\\W_{21}&W_{22}&W_{23}\\W_{31}&W_{32}&W_{33}\end{pmatrix}, \tag{4.17}$$

where $W_{11} \in \mathbb{R}^{(n-s)\times(n-s)}$, $W_{22} \in \mathbb{R}^{(2s-n)\times(2s-n)}$, $W_{33} \in \mathbb{R}^{(n-s)\times(n-s)}$. *Then Problem II has a unique solution and it can be expressed as*

$$C^* = \begin{pmatrix} C_{11} & C_{12} & \hat{H}E_{n-s} \\ C_{21} & C_{22} & E_{2s-n}C_{21}E_{n-s} \\ E_{n-s}\hat{H} & E_{n-s}C_{12}E_{2s-n} & E_{n-s}C_{11}E_{n-s} \end{pmatrix}, \tag{4.18}$$

where

$$\hat{H} = VU^+ + \hat{W}Q_2^T, \quad \hat{W} = \frac{1}{2}(W_{13}E_{n-s} + E_{n-s}W_{31})Q_2.$$

Proof. As in the proof of Theorem 4.1, we can show that Problem II has a unique solution in $\mathcal{L}_S$. By (3.25) and (4.17), we know that Problem II is equivalent to

$$\min_{H\in\mathbb{R}^{(n-s)\times(n-s)}} (\|HE_{n-s} - W_{13}\|^2 + \|E_{n-s}H - W_{31}\|^2).$$

Equivalently,

$$\min_{H\in\mathbb{R}^{(n-s)\times(n-s)}} (\|H - W_{13}E_{n-s}\|^2 + \|H - E_{n-s}W_{31}\|^2).$$

By Lemma 2.4, it is in turn equivalent to

$$\min_{H\in\mathbb{R}^{(n-s)\times(n-s)}} \|H - \frac{1}{2}(W_{13}E_{n-s} + E_{n-s}W_{31})\|.$$

That is,

$$\min_{W\in\mathbb{R}^{(n-s)\times(n-k-l_3)}} \|VU^+ + WQ_2^T - \frac{1}{2}(W_{13}E_{n-s} + E_{n-s}W_{31})\|.$$

Since Q_2 is orthogonal and $U^+Q_2 = 0$, we have

$$W = \frac{1}{2}(W_{13}E_{n-s} + E_{n-s}W_{31})Q_2.$$

Therefore, the solution of Problem II can be expressed as (4.18). □

Based on the above discussion, we give the following algorithm for solving Problem II.

ALGORITHM I

Given $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m] \in \mathbb{R}^{n\times m}$ and Λ as in (3.1), $C_0 \in \mathbb{R}^{s\times s}$, and $\tilde{C} \in \mathbb{R}^{n\times n}$.

1. Calculate $k = [n/2]$.
2. If $s < n - k$, then
   - (a) Compute $\tilde{X}_1$ and $\tilde{X}_2$ by (3.2) and then compute $\tilde{X}_1^+$ and $\tilde{X}_2^+$.
   - (b) If $\tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_1\Lambda$ and $\tilde{X}_2\Lambda\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_2\Lambda$, then we continue. Otherwise we stop.
   - (c) Calculate M_1 and M_2 as in (3.3).
   - (d) Construct the GSVD of the matrix pair $[M_1^T, M_2^T]$ by (3.4).
   - (e) Compute $\tilde{G}$ as in (3.6) and then calculate $\tilde{G}S^{-T}$.
   - (f) Partition $\tilde{G}S^{-T}$ as in (3.7).

(g) If G_{14}, G_{24} and G_{34} are zero matrices, then calculate C^* as in (4.4). Otherwise we stop.

3. else
   (a) Partition X and C_0 as in (3.21), and calculate U and V as in (3.22) and (3.23).
   (b) If the conditions of (3.24) are satisfied, then compute C^* as in (4.18). Otherwise we stop.

Now we consider the computational complexity of our algorithm. We first consider the cost of Step 2. For Substep (a), since K has only 2 nonzero entries per row, it requires $O(nm)$ operations to compute $\tilde{X}_1$ and $\tilde{X}_2$. Then using singular value decomposition (SVD) to compute $\tilde{X}_1^+$ and $\tilde{X}_2^+$ requires $O(n^2m + m^3)$ operations. Substep (b) obviously requires $O(n^2m)$ operations. For Substep (c), because U_2 and V_2 can be obtained by SVD of $\tilde{X}_1$ and $\tilde{X}_2$ in Substep (a) respectively, it requires no operations to compute M_1 and M_2. For Substep (d), if we use Paige's algorithm [19] to compute the GSVD of the matrix pair $[M_1^T, M_2^T]$, then the cost will be of $O(s^2(n-l_1-l_2-s/3))$ operations if $n-l_1-l_2 \geq s$ ($O((n-l_1-l_2)^2(s-(n-l_1-l_2)/3))$ operations if $n-l_1-l_2 \leq s$). Substep (e) requires $O(n^2m+s^3)$ operations. Substep (f) requires no operations. Finally, because of the sparsity of K again, Step (g) requires $O(n^2(n-k-l_1)+n(n-k-l_1)^2+n^2(k-l_2)+n(k-l_2)^2)$ operations. Thus the total complexity of Step 2 is $O(n^2(n-l_1-l_2)+s^2(n-l_1-l_2-s/3)+s^3+n^2m+m^3)$ if $n-l_1-l_2 \geq s$ ($O(n^2(n-l_1-l_2)+s^2(s-(n-l_1-l_2)/3)+s^3+n^2m+m^3)$ if $n-l_1-l_2 \leq s$).

Next, we consider the cost of Step 3. For Substep (a), since E_n is a backward identity matrix, it requires $O((n-s)^2m+(n-s)(2s-n)m)$ operations to form U and V. For Substep (b), using SVD to compute U^+ requires $O((n-s)^2m+m^3)$ operations. If we compute VU^+U as $[V(U^+U)]$, then the cost will only be of $O(m^2(n-s))$ operations. Thus the cost for Substep (b) is $O((n-s)^2m+m^3+m^2(n-s)+(n-s)^3)$. Therefore, the total cost of Step 3 is $O((n-s)^3+(n-s)^2m+(n-s)(2s-n)m+m^2(n-s)+m^3)$.

From above, we know that the total cost of the algorithm is the cost required by Step 2 if $s < n-k$ or by Step 3 if $s \geq n-k$. We remark that in practice, $m \ll n$.

5. Numerical Experiments. In this section, we will demonstrate the algorithm using Matlab.

Example 1. We consider the following Hopfield neural network system

$$\frac{d\mathbf{u}}{dt} = T^{-1}(-\mathbf{u} + \Omega\mathbf{f}(\mathbf{u})), \tag{5.1}$$

where $T = \mathrm{diag}(\tau_1, \ldots, \tau_n)$, $\Omega = [\omega_{ij}]$, and $\mathbf{f} = [f_1(u_1), \ldots, f_n(u_n)]^T$ with $f_i(u_i)$ are squashing functions, see [8] for detail.

In this example, we design a neural network such that $\mathbf{u}^*$ is a stable equilibrium, with $f_i(u_i^*) = 1/(1+e^{-u_i^*}) \neq 0$. It is known that $\mathbf{u}^*$ is an equilibrium only if

$$\mathbf{u}^* = \Omega\mathbf{f}. \tag{5.2}$$

It implies that

$$\Omega = TCG_d^{-1} + G_d^{-1}, \tag{5.3}$$

where C satisfies that

$$CG_d^{-1}\mathbf{f} = T^{-1}(\mathbf{u}^* - G_d^{-1}\mathbf{f}). \tag{5.4}$$

Here, $G_d = \mathrm{diag}(f_1^{(1)}(u_1^*), \ldots, f_n^{(1)}(u_n^*))$, where $(\cdot)^{(1)}$ denotes the 1th derivatives.

For any given T, the design problem is reduced to finding a stable matrix C that maps $G_d^{-1}\mathbf{f}$ to $T^{-1}(\mathbf{u}^* - G_d^{-1}\mathbf{f})$. Moreover, we know that if $T^{-1}(\mathbf{u}^* - G_d^{-1}\mathbf{f}) = \lambda G_d^{-1}\mathbf{f}$ for some real negative number λ, then there exists a stable matrix C such that (5.4) holds, see [8, Theorem 4.1].

In practice, we may be interested in that the matrix C is a centrosymmetric matrix and its s-by-s leading principal submatrix is the given matrix C_0. Moreover, we can obtain an experimental matrix $\tilde{C}$ which may not satisfy the structural requirement (centrosymmetric or the submatrix constraint) and/or spectral requirement (having eigenpairs $G_d^{-1}\mathbf{f}$ and λ). We want to find such structural stable matrix C^* which maps $G_d^{-1}\mathbf{f}$ to $T^{-1}(\mathbf{u}^* - G_d^{-1}\mathbf{f}) = \lambda G_d^{-1}\mathbf{f}$ $(\lambda < 0)$ and is the best approximation of $\tilde{C}$ in Frobenius norm. Therefore the design problem turn into Problems I and II proposed in this paper.

For demonstration purpose, we let $n = 8$, $m = 1$, $s = 5$. Given $\mathbf{u}^* = \mathbf{0}$. Then we have $f_i(u_i^*) = 1/2$ and $f_i^{(1)}(u_i^*) = 1/4$ for $i = 1, \ldots, n$. Thus $G_d = 1/4 I_n$ and $\mathbf{f} = 1/2\mathbf{e}$ where $\mathbf{e}$ denotes the n-vector of all ones. Therefore, the given eigenvector $G_d^{-1}\mathbf{f} = 2\mathbf{e}$. For this example, we chose $T = 0.4938 I_n$ so that one eigenvalue of C is $\lambda = -1/0.4938 = -2.0251$.

Given $X = G_d^{-1}\mathbf{f} = 2\mathbf{e}$, $\Lambda = \lambda = -2.0251$ and

$$C_0 = \begin{pmatrix} 1.0134 & -0.6262 & -0.6091 & 0.2024 & 0.8464 \\ 0.3118 & 0.1653 & 1.1857 & 0.8940 & 0.0265 \\ 0.1912 & 0.6515 & -0.9667 & 1.0504 & -0.5886 \\ -0.7399 & 0.4515 & -0.6165 & -0.5674 & -0.9952 \\ -0.0169 & -0.8830 & -0.2698 & -0.9952 & -0.5674 \end{pmatrix}.$$

Assume that from the experiment, we get the following matrix $\tilde{C} \notin \mathcal{C}_8$:

$$\tilde{C} = \begin{pmatrix} 3.6448 & -1.5739 & 0.5661 & 1.2763 & 0.5473 & 0.5312 & 0.2992 & -1.2917 \\ 1.5866 & 0.1344 & 0.4095 & 1.1794 & -0.9925 & 0.8905 & 0.5602 & -1.1477 \\ 0.7641 & 0.6437 & -2.0927 & 1.5228 & 0.0533 & 0.8970 & 0.1428 & 0.5543 \\ -1.0982 & 1.4538 & -2.1948 & -1.4674 & -0.7619 & 0.1669 & 0.1910 & -1.4562 \\ 0.7249 & -1.8998 & -0.1476 & -0.7729 & 0.5174 & -2.3614 & -0.3332 & -0.3404 \\ 0.1476 & 0.8403 & -0.3028 & -0.4868 & 0.8683 & 0.4873 & -0.0583 & 1.8999 \\ 2.2642 & 1.8592 & 1.4312 & 0.6824 & 0.5707 & 1.9692 & 1.3696 & -0.6353 \\ -0.0637 & -0.4936 & 1.9980 & 1.9972 & -0.1334 & 0.8525 & -3.0381 & 0.5415 \end{pmatrix}.$$

We can show that Problem I is solvable. Then following the steps in the algorithm in §4, we obtain the required matrix $C^* \in \mathcal{L}_S$ as follows:

$$C^* = \begin{pmatrix} 1.0134 & -0.6262 & -0.6091 & 0.2024 & 0.8464 & 0.1507 & -1.2111 & -1.7916 \\ 0.3118 & 0.1653 & 1.1857 & 0.8940 & 0.0265 & -1.3515 & -1.3027 & -1.9541 \\ 0.1912 & 0.6515 & -0.9667 & 1.0504 & -0.5886 & -0.8704 & -0.6760 & -0.8166 \\ -0.7399 & 0.4515 & -0.6165 & -0.5674 & -0.9952 & -0.2698 & -0.8830 & -0.0169 \\ -0.0169 & -0.8830 & -0.2698 & -0.9952 & -0.5674 & -0.6165 & 0.4515 & -0.7399 \\ -0.8166 & -0.6760 & -0.8704 & -0.5886 & 1.0504 & -0.9667 & 0.6515 & 0.1912 \\ -1.9541 & -1.3027 & -1.3515 & 0.0265 & 0.8940 & 1.1857 & 0.1653 & 0.3118 \\ -1.7916 & -1.2111 & 0.1507 & 0.8464 & 0.2024 & -0.6091 & -0.6262 & 1.0134 \end{pmatrix}$$

which satisfies $\|C^* - \tilde{C}\| = \min_{C \in \mathcal{L}_S} \|C - \tilde{C}\|$. Finally, the following matrix $\Omega^* = TC^*G_d^{-1} +$

G_d^{-1} can be calculated:

$$\Omega^* = \begin{pmatrix} 6.0016 & -1.2368 & -1.2031 & 0.3997 & 1.6719 & 0.2976 & -2.3922 & -3.5388 \\ 0.6158 & 4.3265 & 2.3420 & 1.7659 & 0.0523 & -2.6695 & -2.5731 & -3.8598 \\ 0.3776 & 1.2868 & 2.0907 & 2.0748 & -1.1625 & -1.7192 & -1.3352 & -1.6129 \\ -1.4615 & 0.8918 & -1.2176 & 2.8793 & -1.9657 & -0.5329 & -1.7441 & -0.0333 \\ -0.0333 & -1.7441 & -0.5329 & -1.9657 & 2.8793 & -1.2176 & 0.8918 & -1.4615 \\ -1.6129 & -1.3352 & -1.7192 & -1.1625 & 2.0748 & 2.0907 & 1.2868 & 0.3776 \\ -3.8598 & -2.5731 & -2.6695 & 0.0523 & 1.7659 & 2.3420 & 4.3265 & 0.6158 \\ -3.5388 & -2.3922 & 0.2976 & 1.6719 & 0.3997 & -1.2031 & -1.2368 & 6.0016 \end{pmatrix}.$$

Example 2. In this example, we demonstrate our algorithm in another way. For simplicity , we consider $n = 10$, $m = 3$, $s = 6$. We first choose a random matrix $\hat{C} \in \mathcal{C}_{10}$:

$$\hat{C} = \begin{pmatrix} 1.6405 & -0.1078 & -0.8875 & 0.3703 & -0.2894 & -0.6384 & 0.7080 & 0.2080 & 0.3988 & 0.8062 \\ -0.4574 & -0.8891 & -0.1455 & -0.0858 & -0.2658 & -1.3510 & 0.7036 & -0.3054 & 0.4304 & 1.4557 \\ 0.1118 & -0.1969 & 0.1812 & -0.2555 & 1.1810 & 0.5378 & 0.4137 & 0.8233 & -1.2063 & 1.3373 \\ -0.7977 & -0.0109 & 0.3346 & -0.3387 & 0.3376 & 0.2088 & -0.0052 & 0.0533 & 0.8645 & -0.2588 \\ 0.1512 & -0.5887 & -0.3039 & -0.0137 & 0.4058 & 0.1813 & 0.5433 & -0.1110 & 0.4449 & -0.0643 \\ -0.0643 & 0.4449 & -0.1110 & 0.5433 & 0.1813 & 0.4058 & -0.0137 & -0.3039 & -0.5887 & 0.1512 \\ -0.2588 & 0.8645 & 0.0533 & -0.0052 & 0.2088 & 0.3376 & -0.3387 & 0.3346 & -0.0109 & -0.7977 \\ 1.3373 & -1.2063 & 0.8233 & 0.4137 & 0.5378 & 1.1810 & -0.2555 & 0.1812 & -0.1969 & 0.1118 \\ 1.4557 & 0.4304 & -0.3054 & 0.7036 & -1.3510 & -0.2658 & -0.0858 & -0.1455 & -0.8891 & -0.4574 \\ 0.8062 & 0.3988 & 0.2080 & 0.7080 & -0.6384 & -0.2894 & 0.3703 & -0.8875 & -0.1078 & 1.6405 \end{pmatrix}.$$

Then we compute its eigenpairs: Three of the eigenvalues of $\hat{C}$ are 2.1176, $1.0359 \pm 1.1570\sqrt{-1}$. Let $\mathbf{x}_1$, $\mathbf{x}_2 \pm \sqrt{-1}\mathbf{x}_3$ be the corresponding eigenvectors. We now take

$$X = [\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1] = \begin{pmatrix} -0.0659 & -0.2562 & -0.5799 \\ 0.0678 & 0.0191 & -0.2116 \\ -0.6079 & 0 & -0.2835 \\ 0.0422 & 0.0986 & 0.1571 \\ 0.0959 & -0.1867 & 0.1181 \\ 0.0959 & -0.1867 & 0.1181 \\ 0.0422 & 0.0986 & 0.1571 \\ -0.6079 & 0.0000 & -0.2835 \\ 0.0678 & 0.0191 & -0.2116 \\ -0.0659 & -0.2562 & -0.5799 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 1.0359 & 1.1570 & 0 \\ -1.1570 & 1.0359 & 0 \\ 0 & 0 & 2.1176 \end{pmatrix}.$$

Given such X, Λ, and

$$C_0 = \begin{pmatrix} 1.6405 & -0.1078 & -0.8875 & 0.3703 & -0.2894 & -0.6384 \\ -0.4574 & -0.8891 & -0.1455 & -0.0858 & -0.2658 & -1.3510 \\ 0.1118 & -0.1969 & 0.1812 & -0.2555 & 1.1810 & 0.5378 \\ -0.7977 & -0.0109 & 0.3346 & -0.3387 & 0.3376 & 0.2088 \\ 0.1512 & -0.5887 & -0.3039 & -0.0137 & 0.4058 & 0.1813 \\ -0.0643 & 0.4449 & -0.1110 & 0.5433 & 0.1813 & 0.4058 \end{pmatrix},$$

we can verify that Problem I is solvable. Hence $\mathcal{L}_S$ is nonempty. We now perturb $\hat{C}$ by a random matrix to obtain a matrix $\tilde{C} \notin \mathcal{C}_{10}$:

$$\tilde{C} = \begin{pmatrix} 1.6510 & -0.0907 & -0.8789 & 0.3653 & -0.2906 & -0.6402 & 0.7090 & 0.2027 & 0.4049 & 0.8028 \\ -0.4538 & -0.8976 & -0.1401 & -0.0693 & -0.2665 & -1.3369 & 0.6919 & -0.3034 & 0.4402 & 1.4466 \\ 0.1097 & -0.1948 & 0.1928 & -0.2422 & 1.1870 & 0.5320 & 0.4281 & 0.8313 & -1.2258 & 1.3349 \\ -0.7985 & -0.0205 & 0.3434 & -0.3477 & 0.3422 & 0.2029 & 0.0032 & 0.0747 & 0.8667 & -0.2572 \\ 0.1457 & -0.5941 & -0.2902 & -0.0137 & 0.4016 & 0.1786 & 0.5295 & -0.1168 & 0.4491 & -0.0602 \\ -0.0578 & 0.4532 & -0.1160 & 0.5324 & 0.1973 & 0.4021 & -0.0297 & -0.3002 & -0.5844 & 0.1455 \\ -0.2607 & 0.8667 & 0.0397 & -0.0128 & 0.1987 & 0.3351 & -0.3423 & 0.3373 & -0.0018 & -0.7974 \\ 1.3200 & -1.2080 & 0.8347 & 0.4064 & 0.5380 & 1.1847 & -0.2747 & 0.1803 & -0.1985 & 0.1151 \\ 1.4712 & 0.4299 & -0.2972 & 0.7098 & -1.3340 & -0.2750 & -0.0797 & -0.1331 & -0.9039 & -0.4427 \\ 0.8031 & 0.3804 & 0.2401 & 0.7207 & -0.6404 & -0.2924 & 0.3609 & -0.8630 & -0.1050 & 1.6367 \end{pmatrix}.$$

Using the proposed algorithm in §4, we can obtain $C^* \in \mathcal{L}_S$ such that $\|C^* - \tilde{C}\| = \min_{C \in \mathcal{L}_S} \|C - \tilde{C}\|$. Moreover, the solution C^* is given by:

$$C^* = \begin{pmatrix} 1.6405 & -0.1078 & -0.8875 & 0.3703 & -0.2894 & -0.6384 & 0.7141 & 0.2080 & 0.3972 & 0.8084 \\ -0.4574 & -0.8891 & -0.1455 & -0.0858 & -0.2658 & -1.3510 & 0.7013 & -0.3054 & 0.4310 & 1.4549 \\ 0.1118 & -0.1969 & 0.1812 & -0.2555 & 1.1810 & 0.5378 & 0.4160 & 0.8233 & -1.2068 & 1.3381 \\ -0.7977 & -0.0109 & 0.3346 & -0.3387 & 0.3376 & 0.2088 & -0.0054 & 0.0533 & 0.8646 & -0.2588 \\ 0.1512 & -0.5887 & -0.3039 & -0.0137 & 0.4058 & 0.1813 & 0.5433 & -0.1110 & 0.4449 & -0.0643 \\ -0.0643 & 0.4449 & -0.1110 & 0.5433 & 0.1813 & 0.4058 & -0.0137 & -0.3039 & -0.5887 & 0.1512 \\ -0.2588 & 0.8646 & 0.0533 & -0.0054 & 0.2088 & 0.3376 & -0.3387 & 0.3346 & -0.0109 & -0.7977 \\ 1.3381 & -1.2068 & 0.8233 & 0.4160 & 0.5378 & 1.1810 & -0.2555 & 0.1812 & -0.1969 & 0.1118 \\ 1.4549 & 0.4310 & -0.3054 & 0.7013 & -1.3510 & -0.2658 & -0.0858 & -0.1455 & -0.8891 & -0.4574 \\ 0.8084 & 0.3972 & 0.2080 & 0.7141 & -0.6384 & -0.2894 & 0.3703 & -0.8875 & -0.1078 & 1.6405 \end{pmatrix}.$$

In addition, we note that if in Problem I, we also assume that the required matrix C is symmetric, i.e. C is bisymmetric, then Problem I is reduced to the inverse problem for submatrix constrained bisymmetric matrices discussed in [18]. For the corresponding solvability conditions, the algorithm for finding the best approximation solution to the corresponding best approximation problem and the numerical examples, we can refer to [18].

These examples and many other numerical experiments with the algorithm proposed in §4 confirm our theoretical results in this paper.

6. Conclusions. In this paper, we discussed the inverse eigenproblem for the submatrix constrained centrosymmetric matrices. We also considered the best approximation solution in the corresponding solution set for the constrained inverse problem to a given matrix in Frobenius norm. The solvability conditions and the explicit formula for the solution are provided. We proposed an algorithm for finding the best approximation solution. Some tests are also given to illustrate our results.

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