Symmetric Tridiagonal Inverse Quadratic Eigenvalue Problems with Partial Eigendata

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Abstract

In this paper we concern the inverse problem of constructing the n-by-n real symmetric tridiagonal matrices C and K so that the monic quadratic pencil $Q(\lambda) := \lambda^2 I + \lambda C + K$ (where I is the identity matrix) possesses the given partial eigendata. We first provide the sufficient and necessary conditions for the existence of an exact solution to the inverse problem from the self-conjugate set of prescribed four eigenpairs. To find a physical solution for the inverse problem where the matrices C and K are weakly diagonally dominant and have positive diagonal elements and negative off-diagonal elements, we consider the inverse problem from the partial measured noisy eigendata. We propose a regularized smoothing Newton method for solving the inverse problem. The global and quadratic convergence of our approach is established under some mild assumptions. Some numerical examples and a practical engineering application in vibrations show the efficiency of our method.

Keywords. Quadratic pencil, symmetric tridiagonal matrix, inverse quadratic eigenvalue problem, Newton's method.

AMS subject classifications. 15A22, 15A18, 65F18, 65K10, 90C33

1 Introduction

In the vibration analysis of many structural engineering problems, we often need to solve a second-order differential equation

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t), \tag{1}$$

where the n-by-n matrices M, C, and K are known as the mass, damping and stiffness matrices, respectively, $u(t)$ is an n-vector, and $f(t)$ is a time-dependent external force vector. By the separation of variables $u(t) = x\, e^{\lambda t}$, where x is a constant vector, we can get the general solution to the homogeneous equation of (1) and this solution is given in terms of the solution of the following quadratic eigenvalue problem (QEP):

$$Q(\lambda)x := (\lambda^2 M + \lambda C + K)\, x = 0. \tag{2}$$

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The scalar λ and the associated nonzero vector x are, respectively, called the eigenvalue and the eigenvector of the quadratic pencil $Q(\lambda)$. QEPs arise in a remarkable variety of applications, including, for example, the vibrating analysis of structural mechanical and acoustic system, the electrical circuit simulation, fluid mechanics, the modeling microelectronic mechanical systems, and signal processing. A good survey on the applications, mathematical properties and numerical methods of QEPs is included in [47] by Tisseur and Meerbergen.

In this paper, we consider the inverse problem of constructing the matrices M, C, and K such that the quadratic pencil $Q(\lambda)$ (defined in (2)) has the prescribed partial eigendata. In particular, we focus on the symmetric tridiagonal inverse quadratic eigenvalue problem (IQEP). The problem is stated as follows:

TriIQEP. Construct a nontrivial quadratic pencil

$$Q(\lambda) = \lambda^2 I + \lambda C + K$$

from a set of prescribed eigendata $\{(\lambda_i, x^i)\}_{i=1}^p$, where C and K are both n-by-n symmetric and tridiagonal matrices defined by

$$C = \begin{bmatrix} a_1 & -b_2 & & & \\ -b_2 & a_2 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & a_{n-1} & -b_n \\ & & & -b_n & a_n \end{bmatrix} \tag{3}$$

and

$$K = \begin{bmatrix} c_1 & -d_2 & & & \\ -d_2 & c_2 & -d_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -d_{n-1} & c_{n-1} & -d_n \\ & & & -d_n & c_n \end{bmatrix}, \tag{4}$$

where the real numbers $\{a_i\}_1^n$, $\{b_i\}_2^n$, $\{c_i\}_1^n$, and $\{d_i\}_2^n$ are unknown parameters.

Inverse eigenvalue problems have been of great value for many applications, see for instance [10, 23]. Recent developments include the finite model updating problems in structural dynamics (e.g., [11, 21]) and the partial eigenstructure assignment problems in control theory (e.g., [14, 15, 40, 49]).

In general, an IQEP is very difficult to solve because of the additional physical structure constraints, which are the inherent properties of the original model. In particular, the solution matrices should preserve the exploitable structure properties such as symmetry, definiteness, bandedness, and sparsity etc.. For the computation purpose, many numerical methods were developed to solve various simplified versions of the IQEP, see for instance [13, 14, 20, 21, 34, 39, 49]. However, these methods may fail to generate a physically realizable solution, which is of great importance in applications. Recently, Chu, Kuo, and Lin [11] constructed a physical quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$, where the matrices M, C and K are real and symmetric with the matrices M and K being positive definite and positive semidefinite, respectively. Given complete information on eigenvalues and eigenvectors with all eigenvalues being simple and complex, Lancaster and Prells [31] got a quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$, where M, C and K are real symmetric matrices with both M and K being positive definite and C being positive semi-definite. Bai, Chu, and Sun [3] proposed an optimization method for constructing a quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ such that the updated matrices M, C, and K are all real and symmetric with the matrices M and K being positive definite and positive semidefinite, respectively. However, all these methods may not preserve the inherent structure connectivity.

The tridiagonal inverse problem arises in the vibrations, see for instance [10, 23, 35, 39]. We should point out that the given eigendata is often measured from a physically realizable structure and the number of available eigendata is much smaller than the problem dimension (i.e., $p \ll n$) [21]. Moreover, the matrices C and K denote the physical damping and stiffness matrices, respectively, which should be weakly diagonally dominant and have positive diagonal elements and negative off-diagonal elements [39, 47].

In this paper, we consider the TriIQEP with the matrices C and K defined as in (3) and (4), respectively. We first discuss the solvability of the TriIQEP from the self-conjugate set of specific four eigenvalues and the self-conjugate set of associated four eigenvectors and provide the sufficient and necessary conditions for the existence of an exact solution to the TriIQEP.

Because the solution matrices C and K should be weakly diagonally dominant and the corresponding parameters $\{a_i\}_1^n$, $\{b_i\}_2^n$, $\{c_i\}_1^n$, and $\{d_i\}_2^n$ should be real and positive, we discuss the TriIQEP in a new way. In this paper, we will reformulate the TriIQEP with the noisy eigendata $\{(\lambda_i, x^i)\}_{i=1}^p$ as a box constrained variational inequality (BVI) (i.e., a well-known nonlinear complementary problem (NCP)). Then, a regularized smoothing Newton algorithm is proposed for solving the BVI. Our method is motivated by the recent development of the numerical computation for structured IQEPs and the BVI/NCP. Burak and Ram [4] constructed the structured pencil $Q(\lambda) = \lambda^2 M + K$ with

$$M = \mathrm{diag}(m_1, \ldots, m_n), \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_n & k_n \end{bmatrix} \tag{5}$$

from a single natural frequency, two mode shapes and a static deflection due to a unit load for the undamped case (i.e., $C = 0$) and expressed the solution parameters in terms of a certain generalized eigenvalue problem. However, the positiveness of the parameters determined by (5) is not guaranteed. Bai [2] determined the structured quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ with K defined in (5) and

$$\left\{ \begin{aligned} M &= \begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & m_n & 2m_n \end{bmatrix}, \\ C &= \begin{bmatrix} c_1 + c_2 & -c_2 & & & \\ -c_2 & c_2 + c_3 & -c_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -c_n & c_n \end{bmatrix} \end{aligned} \right. \tag{6}$$

from the following two situations: (i) two real eigenvalues and three real eigenvectors or (ii) a real eigenvector and a self-conjugate set of two complex eigenpairs. The solvability conditions were provided. Chu, Del Buono, and Yu [9] discussed the IQEP for the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ with M and K defined in (5) and C defined in (6). Given two real eigenpairs or a self-conjugate set of two complex eigenpairs, they showed the solvability of the IQEP is equivalent to the consistency of a certain system of inequalities. The BVI/NCP is a fundamental problem in mathematical programming. Recently, the regularized/smoothing and nonsmoothing Newton methods for the BVI/NCP have been discussed in a large literature, see for instance [6, 8, 16, 27, 28, 29, 36, 37, 45, 46]. Given the estimate of the analytic model and the measured noisy eigendata, we reformulate the TriIQEP as a quadratically constrained quadratic programming problem, and then convert the optimization problem into a nonsmooth BVI. We show that the BVI is monotone. To prevent the singularity of the BVI, by using the well-known *Tikhonov* regularization techniques as in [36], we present a regularized smoothing approach for solving the BVI. Under some mild conditions, the global and quadratic convergence is established. Numerical examples and an engineering application in vibrations demonstrate the efficiency of our method.

Throughout the paper, we use the following notations. Let A^T denote the transpose of a matrix $A \in \mathbb{R}^{m\times n}$. Let $\mathbb{R}^n$ be a real vector space of dimension n with the Euclidean inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$. If $x \in \mathbb{R}^n$, $\mathrm{diag}(x)$ denotes the diagonal matrix whose ith diagonal element is x_i. Let $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ stand for the nonnegative orthant of $\mathbb{R}^n$ and the strictly positive orthant of $\mathbb{R}^n$, respectively. Let $\mathcal{N} := \{1,\ldots,n\}$. If $\mathcal{I}$ and $\mathcal{J}$ are index sets such that $\mathcal{I},\mathcal{J} \subseteq \mathcal{N}$, we use $A_{\mathcal{I}\mathcal{J}}$ to denote the $|\mathcal{I}| \times |\mathcal{J}|$ submatrix of an $n\times n$ matrix A consisting of entries A_{ij}, $i \in \mathcal{I}$, $j \in \mathcal{J}$. If $A_{\mathcal{I}\mathcal{I}}$ is nonsingular, we denote by $A/A_{\mathcal{I}\mathcal{I}}$ the Schur complement of $A_{\mathcal{I}\mathcal{I}}$ in A, i.e.,

$$A/A_{\mathcal{I}\mathcal{I}} = A_{\mathcal{J}\mathcal{J}} - A_{\mathcal{J}\mathcal{I}}A_{\mathcal{I}\mathcal{I}}^{-1}A_{\mathcal{I}\mathcal{J}}$$

where $\mathcal{J} = \mathcal{N}\backslash\mathcal{I}$. We denote by $x_{\mathcal{I}}$ the subvector of an n-vector with entries $x_i, i \in \mathcal{I}$. Finally, a set is described as *self-conjugate* if the complex conjugate of each of its members is contained in the set.

The paper is organized as follows. In the next section, we discuss the existence of a solution to the TriIQEP from a self-conjugate set of prescribed four eigenpairs. In Section 3 we reformulate the TriIQEP as a BVI. In Section 4 we provide our regularized smoothing Newton algorithm for solving the TriIQEP. In Section 5 we establish the global and quadratic convergence of our method. In Section 6, numerical examples and a practical application in vibrations are presented to illustrate the efficiency of our proposed method. Finally, the concluding remarks are given in Section 7.

2 Solvability of TriIQEP from Four Eigenpairs

In this section, we consider the TriIQEP such that the corresponding quadratic pencil has the given four eigenpairs $\{(\lambda_i, x^i)\}_1^4$ exactly. For real symmetric matrices M, C, K, it follows from (2) that if λ and x are the eigenvalue and the associated eigenvector of $Q(\lambda)$, then their complex conjugates are also one eigenpair of $Q(\lambda)$. Therefore, the TriIQEP can be described as the following three subproblems:

Problem A1. Construct the parameters $\{a_j\}_1^n$, $\{b_j\}_2^n$, $\{c_j\}_1^n$, $\{d_j\}_2^n$ from the prescribed four real eigenpairs.

Problem A2. Construct the parameters $\{a_j\}_1^n$, $\{b_j\}_2^n$, $\{c_j\}_1^n$, $\{d_j\}_2^n$ from the given two real eigenpairs and the self-conjugate set of specific two complex eigenpairs.

Problem A3. Construct the parameters $\{a_j\}_1^n$, $\{b_j\}_2^n$, $\{c_j\}_1^n$, $\{d_j\}_2^n$ from the self-conjugate set of prescribed four complex eigenpairs.

We first investigate the solvability of Problem A2. Let $\{(\lambda_i, x^i)\}_1^4$ be the given eigenpairs of the quadratic pencil $Q(\lambda)$, i.e.,

$$(\lambda_i^2 I + \lambda_i C + K)x^i = 0, \quad i = 1, 2, 3, 4. \tag{7}$$

For the sake of simplicity, we assume the two eigenpairs $\{(\lambda_i, x^i)\}_1^2$ are real and the two complex eigenpairs $\{(\lambda_i, x^i)\}_3^4$ are complex conjugate to each other, i.e.,

$$\begin{cases} \lambda_3 = \alpha + \imath\beta, & x^3 = x_R + \imath x_I \\ \lambda_4 = \alpha - \imath\beta, & x^4 = x_R - \imath x_I, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ with $\imath := \sqrt{-1}$, x_R and x_I are real n-vectors.

Let x_{iR} denote the ith component of the vector x_R. Then, (7) can be expressed in a system of $4n$ real equations:

$$\begin{cases} \lambda_1 x_1^1 a_1 + x_1^1 c_1 - \lambda_1 x_2^1 b_2 - x_2^1 d_2 &= -\lambda_1^2 x_1^1, \\ \lambda_2 x_1^2 a_1 + x_1^2 c_1 - \lambda_2 x_2^2 b_2 - x_2^2 d_2 &= -\lambda_2^2 x_1^2, \\ (\alpha x_{1R} - \beta x_{1I}) a_1 + x_{1R} c_1 - (\alpha x_{2R} - \beta x_{2I}) b_2 - x_{2R} d_2 &= -[(\alpha^2 - \beta^2) x_{1R} - 2\alpha\beta x_{1I}], \\ (\beta x_{1R} + \alpha x_{1I}) a_1 + x_{1I} c_1 - (\beta x_{1R} + \alpha x_{1I}) b_2 - x_{2I} d_2 &= -[2\alpha\beta x_{1R} + (\alpha^2 - \beta^2) x_{1I}], \end{cases} \tag{8}$$

$$\begin{cases} \lambda_1 x_i^1 a_i - \lambda_1 x_{i-1}^1 b_i + x_i^1 c_i - x_{i-1}^1 d_i - \lambda_1 x_{i+1}^1 b_{i+1} - x_{i+1}^1 d_{i+1} = -\lambda_1^2 x_i^1, \\ \lambda_2 x_i^2 a_i - \lambda_2 x_{i-1}^2 b_i + x_i^2 c_i - x_{i-1}^{(2)} d_i - \lambda_2 x_{i+1}^2 b_{i+1} - x_{i+1}^2 d_{i+1} = -\lambda_2^2 x_i^2, \\ (\alpha x_{iR} - \beta x_{iI}) a_i - (\alpha x_{i-1,R} - \beta x_{i-1,I}) b_i + x_{iR} c_i - x_{i-1,R} d_i \\ \qquad -(\alpha x_{i+1,R} - \beta x_{i+1,I}) b_{i+1} - x_{i+1,R} d_{i+1} = -[(\alpha^2 - \beta^2) x_{iR} - 2\alpha\beta x_{iI}], \\ (\beta x_{iR} + \alpha x_{iI}) a_i - (\beta x_{i-1,R} + \alpha x_{i-1,I}) b_i + x_{iI} c_i - x_{i-1,I} d_i \\ \qquad -(\beta x_{i+1,R} + \alpha x_{i+1,I}) b_{i+1} - x_{i+1,I} d_{i+1} = -[2\alpha\beta x_{iR} + (\alpha^2 - \beta^2) x_{iI}] \end{cases} \tag{9}$$

for $i = 2, 3, \ldots, n-1$, and

$$\begin{cases} \lambda_1 x_n^1 a_n - \lambda_1 x_{n-1}^1 b_n + x_n^1 c_n - x_{n-1}^1 d_n &= -\lambda_1^2 x_n^1, \\ \lambda_2 x_n^2 a_n - \lambda_2 x_{n-1}^2 b_n + x_n^2 c_n - x_{n-1}^2 d_n &= -\lambda_2^2 x_n^2, \\ (\alpha x_{nR} - \beta x_{nI}) a_n - (\alpha x_{n-1,R} - \beta x_{n-1,I}) b_n + x_{nR} c_n - x_{n-1,R} d_n &= -[(\alpha^2 - \beta^2) x_{nR} - 2\alpha\beta x_{nI}], \\ (\beta x_{nR} + \alpha x_{nI}) a_n - (\beta x_{n-1,R} + \alpha x_{n-1,I}) b_n + x_{nI} c_n - x_{n-1,I} d_n &= -[2\alpha\beta x_{nR} + (\alpha^2 - \beta^2) x_{nI}]. \end{cases} \tag{10}$$

To show our main results, we let

$$y = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix} \in \mathbb{R}^{4n-2} \text{ with } y^1 = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \in \mathbb{R}^2 \text{ and } y^i = \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} \in \mathbb{R}^4 \text{ for } 2 \leq i \leq n, \tag{11}$$

$$g = \begin{pmatrix} g^1 \\ g^2 \\ \vdots \\ g^n \end{pmatrix} \in \mathbb{R}^{4n} \text{ with } g^i = -\begin{pmatrix} \lambda_1^2 x_i^1 \\ \lambda_2^2 x_i^2 \\ (\alpha^2-\beta^2)x_{iR} - 2\alpha\beta x_{iI} \\ 2\alpha\beta x_{iR} + (\alpha^2-\beta^2)x_{iI} \end{pmatrix} \in \mathbb{R}^4 \text{ for } 1 \leq i \leq n,$$

$$A_{ii} = \begin{bmatrix} \lambda_1 x_i^1 & -\lambda_1 x_{i-1}^1 & x_i^1 & -x_{i-1}^1 \\ \lambda_2 x_i^2 & -\lambda_2 x_{i-1}^2 & x_i^2 & -x_{i-1}^2 \\ \alpha x_{iR} - \beta x_{iI} & -(\alpha x_{i-1,R} - \beta x_{i-1,I}) & x_{iR} & -x_{i-1,R} \\ \beta x_{iR} + \alpha x_{iI} & -(\beta x_{i-1,R} + \alpha x_{i-1,I}) & x_{iI} & -x_{i-1,I} \end{bmatrix} \text{ for } 2 \leq i \leq n,$$

$$B_{ii} = -\begin{bmatrix} 0 & \lambda_1 x_i^1 & 0 & x_i^1 \\ 0 & \lambda_2 x_i^2 & 0 & x_i^2 \\ 0 & \alpha x_{i,R} - \beta x_{i,I} & 0 & x_{i,R} \\ 0 & \beta x_{i,R} + \alpha x_{i,I} & 0 & x_{i,I} \end{bmatrix} \text{ for } 2 \leq i \leq n,$$

$$A = \begin{bmatrix} A_{11} & B_{22} & & & \\ & A_{22} & B_{33} & & \\ & & \ddots & \ddots & \\ & & & A_{n-1,n-1} & B_{nn} \\ & & & & A_{nn} \end{bmatrix} \text{ with } A_{11} = \begin{bmatrix} \lambda_1 x_1^1 & x_1^1 \\ \lambda_2 x_1^2 & x_1^2 \\ \alpha x_{1R} - \beta x_{1I} & x_{1R} \\ \beta x_{1R} + \alpha x_{1I} & x_{1I} \end{bmatrix}.$$

Expressions (8), (9), and (10) can be rewritten as the following linear system:

$$Ay = g. \tag{12}$$

Therefore, the solvability of Problem A2 is equivalent to that of equation (12). On the existence and uniqueness of the solution to Problem A2, we have the following results.

Theorem 2.1 *Problem A2 has a solution if and only if the following conditions are satisfied:*

(1) $\text{rank}(A_{nn}) = \text{rank}([A_{nn}, g^n])$*;*

(2) $\text{rank}(A_{ii}) = \text{rank}([A_{ii}, g^i - B_{i+1,i+1}y^{i+1}])$ *for some vector* $y^{i+1} \in \mathbb{R}^4$, $i = n-1, \ldots, 1$.

Proof: Problem A2 has a solution if and only if equation (12) has a solution. Equation (12) has a solution if and only if conditions *(1)* and *(2)* of Theorem 2.1 are satisfied. $\square$

Theorem 2.2 *Problem A2 has a unique solution if and only if the following conditions are satisfied:*

(1) $\det(A_{ii}) \neq 0, (i = 2, \dots, n)$*;*

(2) $\mathrm{rank}(A_{11}) = \mathrm{rank}([A_{11}, g^1 - B_{22}y^2]) = 2$ *for some vector* $y^2 \in \mathbb{R}^4$.

Proof: Problem A2 has a unique solution if and only if equation (12) has a unique solution. Equation (12) has a unique solution if and only if conditions *(1)* and *(2)* of Theorem 2.2 are satisfied. □

Remark 2.3 *As Theorems 2.1 and 2.2, we can establish the sufficient and necessary conditions for the solvability of Problems A1 and A3.*

Next, under the conditions of Theorem 2.1 or Theorem 2.2, we can find a solution to Problem A2. If the given eigendata satisfies the conditions *(1)* and *(2)* in Theorem 2.1, the general solution to Problem A2 is given as follows:

Suppose the singular value decomposition (SVD) [24] of A_{ii} is given by

$$A_{ii} = U_{ii} \begin{bmatrix} \Sigma_{ii} & 0 \\ 0 & 0 \end{bmatrix} V_{ii}^T$$

for $i = 1, \dots, n$, where $\Sigma_{ii} = \mathrm{diag}(\sigma_1^i, \dots, \sigma_{r_i}^i)$, $\sigma_1^i \geq \sigma_2 \geq \dots \geq \sigma_{r_i}^i > 0$, $r_i = \mathrm{rank}(A_{ii})$, and U_{ii} and V_{ii} are both orthogonal matrices with appropriate dimensions. Then, the Moore-Penrose generalized inverse A_{ii}^+ of A_{ii} has the form

$$A_{ii}^+ = V_{ii} \begin{bmatrix} \Sigma_{ii}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U_{ii}^T.$$

Therefore, by (12) (or (8), (9) and (10)), we can obtain the solution as follows:

1. $y^n = A_{nn}^+ g^n + \sum_{k=r_n+1}^4 s_k v_k$, where s_k is an arbitrary real scalar and v_k denotes the kth column of the matrix V_{nn}.

2. $y^i = A_{ii}^+(g^i - B_{i+1,i+1}y^{i+1}) + \sum_{k=r_j+1}^4 s_k v_k$, $(i = n-1, \dots, 2)$, where s_k is an arbitrary real scalar and v_k denotes the kth column of the matrix V_{ii}.

3. $y^1 = A_{11}^+(g^1 - B_{22}y^2) + \sum_{k=r_1+1}^2 s_k v_k$, where s_k is an arbitrary real scalar and v_k denotes the kth column of the matrix V_{11}.

On the other hand, if the given eigendata satisfies the conditions *(1)* and *(2)* in Theorem 2.2, we can find the unique solution to Problem A2 as follows:

Expression (10) gives

$$A_{nn} \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} = g^n,$$

which uniquely determines the parameters a_n, b_n, c_n, and d_n since $\det(A_{nn}) \neq 0$.

Next, expression (9) yields

$$A_{ii}\begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{bmatrix} = g^i - B_{i+1,i+1}\begin{bmatrix} a_{i+1} \\ b_{i+1} \\ c_{i+1} \\ d_{i+1} \end{bmatrix}, \quad i = n-1, \ldots, 2. \tag{13}$$

Therefore, we can successively find the unique a_i, b_i, c_i, and d_i by solving equation (13) for $i = n-1, \ldots, 2$ under the condition $\det(A_{ii}) \neq 0$.

Finally, expression (8) gives rise to

$$A_{11}\begin{bmatrix} a_1 \\ c_1 \end{bmatrix} = g^1 - B_{22}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix},$$

the unique a_1 and b_1 are determined by the above equation under the condition *(2)* of Theorem 2.2.

Remark 2.4 *We point out that, in practice, we may not find a physical solution by the above procedure. That is, some of the parameters $\{a_j\}_1^n$, $\{b_j\}_2^n$, $\{c_j\}_1^n$, $\{d_j\}_2^n$ may not be positive and so the corresponding symmetric tridiagonal matrices C and K may not be diagonally dominant. Therefore, it needs further study to determine the necessary and sufficient condition on the eigendata so that the constructed solution is physically feasible.*

For the purpose of demonstration we present the following example.

Let $n = 5$ and we randomly generate two real eigenpairs $\{(\lambda_i, x^i)\}_1^2$ and the self-conjugate set of two complex eigenpairs $\{(\lambda_3 = \alpha + \imath\beta, x^3 = x_R + \imath x_I), (\lambda_4 = \alpha - \imath\beta, x^4 = x_R - \imath x_I)\})$ as follows:

$$\begin{cases} \lambda_1 = -1.9254, \lambda_2 = -3.8372, \alpha = -2.0531, \beta = 2.6361, \\ x^1 = (-0.4558, -0.0601, -0.0272, -0.0187, -0.0033)^T, \\ x^2 = (0.1611, 0.1822, -0.0607, 0.0122, -0.0250)^T, \\ x_R = (-0.0946, -0.1039, -0.0945, -0.0748, -0.0335)^T, \\ x_I = (-0.0249, -0.0725, -0.1213, -0.1098, -0.1191)^T. \end{cases}$$

Then we can easily check that these data satisfies the conditions of Theorem 2.2. The constructed damping and stiffness matrices are as follows:

$$C = \begin{bmatrix} 13.3506 & -7.4981 & 0 & 0 & 0 \\ -7.4981 & 19.7065 & -7.8325 & 0 & 0 \\ 0 & -7.8325 & 13.4431 & -4.2948 & 0 \\ 0 & 0 & -4.2948 & 15.7152 & -7.6594 \\ 0 & 0 & 0 & -7.6594 & 11.6351 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 22.2242 & -16.1481 & 0 & 0 & 0 \\ -16.1481 & 51.5355 & -24.6453 & 0 & 0 \\ 0 & -24.6453 & 52.1586 & -21.1986 & 0 \\ 0 & 0 & -21.1986 & 45.9344 & -17.7634 \\ 0 & 0 & 0 & -17.7634 & 35.6038 \end{bmatrix}.$$

From this example we observe that, by our procedure, we find a physical realizable solution for Problem A2 from the prescribed eigendata.

3 Problem Reformulation

In this section, we shall first rewrite the TriIQEP as a constrained optimization problem and then consider the corresponding BVI. As in [3, 11], we assume that the set of prescribed eigendata $\{(\lambda_i, x^i)\}_{i=1}^p$ is self-conjugate. Without loss of generality, for $i = 1, 2, \ldots, s$ $(2s \leq p)$, let

$$\begin{cases} \lambda_{2i-1} & = & \alpha_i + \imath\beta_i, & x^{2i-1} & = & x_R^i + \imath x_I^i, \\ \lambda_{2i} & = & \alpha_i - \imath\beta_i, & x^{2i} & = & x_R^i - \imath x_I^i. \end{cases},$$

where $\alpha_i, \beta_i \in \mathbb{R}$ with $\beta_i \neq 0$ and $x_R^i, x_I^i \in \mathbb{R}^n$ and let $\lambda_{2s+1}, \ldots, \lambda_p \in \mathbb{R}$ and $x^{2s+1}, \ldots, x^p \in \mathbb{R}^n$. To simplify the discussion, we will describe the given eigendata in the real matrix form $(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ with

$$\Lambda = \mathrm{diag}(\lambda_1^{[2]}, \ldots, \lambda_s^{[2]}, \lambda_{2s+1}, \ldots, \lambda_p)$$

and

$$X = [x_R^1, x_I^1, \ldots, x_R^s, x_I^s, x^{2s+1}, \ldots, x^p],$$

where

$$\lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_i \neq 0, \quad \text{for} \quad i = 1, \ldots, s.$$

Let the matrices C_o and K_o be some a-priori estimates for the unknown matrices C and K and have the structure as in (3) and (4) with the corresponding parameters $\{a_i^o\}_1^n$, $\{b_i^o\}_2^n$, $\{c_i^o\}_1^n$, and $\{d_i^o\}_2^n$. We point out that such a-priori estimates C_o and K_o are, respectively, called the estimated analytic damping and stiffness matrices in the finite element model updating [21]. As in [3], the TriIQEP is to find the $n \times n$ real symmetric matrices C and K such that

$$\begin{cases} \inf & \frac{1}{2}(\|C - C_o\|^2 + \|K - K_o\|^2) \\ \text{s.t.} & X\Lambda^2 + CX\Lambda + KX = 0, \\ & C, K \in \mathcal{WDD}, \end{cases} \tag{14}$$

where $\mathcal{WDD}$ is the set of all n-by-n symmetric tridiagonal and weakly diagonally dominant matrices with positive diagonal and negative off-diagonal.

We remark that it is not easy to find the positive parameters $\{a_i\}_1^n$, $\{b_i\}_2^n$, $\{c_i\}_1^n$, and $\{d_i\}_2^n$ such that the corresponding $n \times n$ matrices $C, K \in \mathcal{WDD}$ satisfy the system

$$X\Lambda^2 + CX\Lambda + KX = 0,$$

see [2, 4, 9, 39] for the exploration of the physical solvability of the structured quadratic pencils. Moreover, we note that the eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$ is experimentally obtained and it is inevitably corrupted by noise [1, 21]. Thus the reconstructed matrices C and K need not satisfy exactly the equality constrains in (14). To reduce the sensitivity, instead of solving (14), we consider the following quadratically constrained quadratic programming problem (QCQP):

$$\begin{cases} \inf & \frac{1}{2}(\|C - C_o\|^2 + \|K - K_o\|^2) \\ \text{s.t.} & \|X\Lambda^2 + CX\Lambda + KX\| \leq \delta_n, \\ & C, K \in \mathcal{WDD}, \end{cases} \tag{15}$$

where δ_n is a small positive number which depends on the noise level of the measured eigendata [1]. We note that Problem (14) is recovered as $\delta_n \to 0$. Without causing any confusion, we refer to Problem (15) as our TriIQEP.

Since the matrices C and K are defined by (3)-(4), respectively, we can rewrite the quadratical constraint in (15) in terms of the parameters $\{a_i\}_1^n$, $\{b_i\}_2^n$, $\{c_i\}_1^n$, and $\{d_i\}_2^n$. To achieve this, let the vector y be defined as in (11). To simplify the notations, in what follows, we let

$$g = \begin{pmatrix} g^1 \\ g^2 \\ \vdots \\ g^n \end{pmatrix} \in \mathbb{R}^{4n} \text{ with } g^i = -\begin{pmatrix} (\alpha_1^2 - \beta_1^2)x_{iR}^1 - 2\alpha_1\beta_1 x_{iI}^1 \\ 2\alpha_1\beta_1 x_{iR}^1 + (\alpha_1^2 - \beta_1^2)x_{iI}^1 \\ \vdots \\ (\alpha_s^2 - \beta_s^2)x_{iR}^s - 2\alpha_s\beta_s x_{iI}^s \\ 2\alpha_s\beta_s x_{iR}^s + (\alpha_s^2 - \beta_s^2)x_{iI}^s \\ \lambda_{s+1}^2 x_i^{s+1} \\ \vdots \\ \lambda_p^2 x_i^p \end{pmatrix} \in \mathbb{R}^4 \text{ for } 1 \leq i \leq n,$$

$$y_o = \begin{pmatrix} y_o^1 \\ y_o^2 \\ \vdots \\ y_o^n \end{pmatrix} \in \mathbb{R}^{4n-2} \text{ with } y_o^1 = \begin{pmatrix} a_1^o \\ c_1^o \end{pmatrix} \in \mathbb{R}^2 \text{ and } y_0^i = \begin{pmatrix} a_i^o \\ b_i^o \\ c_i^o \\ d_i^o \end{pmatrix} \in \mathbb{R}^4 \text{ for } 2 \leq i \leq n,$$

$$A_{ii} = \begin{bmatrix} \alpha_1 x_{iR}^1 - \beta_1 x_{iI}^1 & -(\alpha_1 x_{i-1,R}^1 - \beta_1 x_{i-1,I}^1) & x_{iR}^1 & -x_{i-1,R}^1 \\ \beta_1 x_{iR}^1 + \alpha_1 x_{iI}^1 & -(\beta_1 x_{i-1,R}^1 + \alpha_1 x_{i-1,I}^1) & x_{iI}^1 & -x_{i-1,I}^1 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_s x_{iR}^s - \beta_s x_{iI}^s & -(\alpha_s x_{i-1,R}^s - \beta_s x_{i-1,I}^s) & x_{iR}^s & -x_{i-1,R}^s \\ \beta_s x_{iR}^s + \alpha_s x_{iI}^s & -(\beta_s x_{i-1,R}^s + \alpha_s x_{i-1,I}^s) & x_{iI}^s & -x_{i-1,I}^s \\ \lambda_{s+1} x_i^{s+1} & -\lambda_{s+1} x_{i-1}^{s+1} & x_i^{s+1} & -x_{i-1}^{s+1} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_p x_i^p & -\lambda_p x_{i-1}^p & x_i^p & -x_{i-1}^p \end{bmatrix} \text{ for } 2 \leq i \leq n,$$

$$
B_{ii} = -\begin{bmatrix} 0 & \alpha_1 x_{i,R}^1 - \beta_1 x_{i,I}^1 & 0 & x_{i,R}^1 \\ 0 & \beta_1 x_{i,R}^1 + \alpha_1 x_{i,I}^1 & 0 & x_{i,I}^1 \\ \vdots & \vdots & \vdots & \\ 0 & \alpha_s x_{i,R}^s - \beta_s x_{i,I}^s & 0 & x_{i,R}^s \\ 0 & \beta_s x_{i,R}^s + \alpha_s x_{i,I}^s & 0 & x_{i,I}^s \\ 0 & \lambda_{s+1} x_i^{s+1} & 0 & x_i^{s+1} \\ \vdots & \vdots & \vdots & \\ 0 & \lambda_p x_i^p & 0 & x_i^p \end{bmatrix} \quad \text{for } 2 \le i \le n,
$$

and

$$
\mathcal{A} = \begin{bmatrix} A_{11} & B_{22} & & & 0 \\ & A_{22} & B_{33} & & \\ & & \ddots & \ddots & \\ & & & A_{n-1,n-1} & B_{nn} \\ 0 & & & & A_{nn} \end{bmatrix} \text{ with } A_{11} = \begin{bmatrix} \alpha_1 x_{1R}^1 - \beta_1 x_{1I}^1 & x_{1R}^1 \\ \beta_1 x_{1R}^1 + \alpha_1 x_{1I}^1 & x_{1I}^1 \\ \vdots & \vdots \\ \alpha_s x_{1R}^s - \beta_s x_{1I}^s & x_{1R}^s \\ \beta_s x_{1R}^s + \alpha_s x_{1I}^s & x_{1I}^s \\ \lambda_{s+1} x_1^{s+1} & x_1^{s+1} \\ \vdots & \vdots \\ \lambda_p x_1^p & x_1^p \end{bmatrix}.
$$

Then the quadratic constraint in (15) becomes

$$
\|\mathcal{A}y - g\| \le \delta_n.
$$

Therefore, the QCQP (15) is reduced to the following form:

$$
\begin{cases} \inf & \frac{1}{2}\|y - y^o\|^2 \\ \text{s.t.} & \|\mathcal{A}y - g\| \le \delta_n, \\ & \mathcal{B}y \ge 0, \\ & y \in \mathbb{R}_{++}^{4n-2}, \end{cases} \tag{16}
$$

where $\mathcal{B} \in \mathbb{R}^{2n \times (4n-2)}$, whose entries are all zeros except that

$$
\begin{cases} \mathcal{B}_{1,1} = 1, & \mathcal{B}_{1,4} = -1, \\ \mathcal{B}_{2,2} = 1, & \mathcal{B}_{2,6} = -1, \end{cases}
$$

$$
\begin{cases} \mathcal{B}_{2i-1,4(i-1)-2+1} = 1, & \mathcal{B}_{2i-1,4(i-1)-2+2} = \mathcal{B}_{2i-1,4(i-1)-2+6} = -1, \\ \mathcal{B}_{2i,4(i-1)-2+3} = 1, & \mathcal{B}_{2i,4(i-1)-2+4} = \mathcal{B}_{2i,4(i-1)-2+8} = -1 \end{cases}
$$

for $i = 2, 3, \ldots, n-1$, and

$$
\begin{cases} \mathcal{B}_{2n-1,4(n-1)-2+1} = 1, & \mathcal{B}_{2n-1,4(n-1)-2+2} = -1, \\ \mathcal{B}_{2n,4(n-1)-2+3} = 1, & \mathcal{B}_{2n,4(n-1)-2+4} = -1. \end{cases}
$$

We note that the inequality constraint $\mathcal{B}y \ge 0$ corresponds to the constraint that the solution matrices C and K defined in (3) and (4) should be weakly diagonally dominant.

For the convenience of numerical computation, we shall consider the following relaxed form:

$$\begin{cases} \inf & f_0(y) := \frac{1}{2}\|y - y^o\|^2 \\ \text{s.t.} & f_1(y) := \|\mathcal{A}y - g\|^2 - \delta_n^2 \le 0, \\ & f_2(y) := \mathcal{B}y \ge 0, \\ & y \in \mathbb{R}_+^{4n-2}. \end{cases} \tag{17}$$

We observe that Problem (17) admits a strictly feasible solution, i.e., there exists a point $y^0 \in \mathbb{R}_{++}^{4n-2}$ such that $f_1(y^0) < 0$ and $f_2(y^0) > 0$. Thus the strong Slater constraint qualification [26] holds for the problem. Thus solving Problem (17) is equivalent to finding $y \in \mathbb{R}_+^{4n-2}$, $\xi \in \mathbb{R}_+$, and $\zeta \in \mathbb{R}_+^{2n}$ such that

$$\begin{cases} \nabla f_0(y) + \xi \nabla f_1(y) - \nabla f_2(y)^T \zeta = 0, \\ \xi \ge 0, \quad \zeta \ge 0, \quad -f_1(y) \ge 0, \quad f_2(y) \ge 0, \\ \xi f_1(y) = 0, \quad f_2(y)^T \zeta = 0. \end{cases} \tag{18}$$

where $\nabla f_i(y)$ is the gradient of $f_i(y)$ at $y \in \mathbb{R}^{4n-2}$, $i = 0, 1, 2$. We point out that a solution of (18) is a KKT point of Problem (17).

Let

$$\mathcal{G} := \mathbb{R}_+^{4n-2} \times \mathbb{R}_+ \times \mathbb{R}_+^{2n} = \mathbb{R}_+^m \quad \text{with} \quad m := 6n - 1$$

and

$$z := (y, \xi, \zeta), \quad F(z) := \begin{pmatrix} \nabla f_0(y) + \xi \nabla f_1(y) - \nabla f_2(y)^T \zeta \\ -f_1(y) \\ f_2(y) \end{pmatrix}. \tag{19}$$

It is well-known that solving Problem (18) is equivalent to the solution of the BVI defined as follows: Find $z^* \in \mathcal{G}$ such that

$$F(z)^T (z - z^*) \ge 0, \quad \text{for all } z \in \mathcal{G}. \tag{20}$$

Let $\Pi_{\mathcal{G}}(\cdot)$ denote the Euclidean projection onto $\mathcal{G}$. Then, solving (20) is reduced to solving the following Robinson's normal equation:

$$E(z) := F(\Pi_{\mathcal{G}}(z)) + z - \Pi_{\mathcal{G}}(z) = 0 \tag{21}$$

in the sense that if $\hat{z}^*$ is a solution of (21), then

$$z^* := \Pi_{\mathcal{G}}(\hat{z}^*)$$

is a solution of (20). Conversely if z^* is a solution of (20), then

$$\hat{z}^* := z^* - F(z^*)$$

is a solution of (21) [41]. We note that (21) is a nonsmooth equation since $\Pi_{\mathcal{G}}(\cdot)$ is not differentiable everywhere.

By using the Chen-Harker-Kanzow-Smale (CHKS) smoothing function [5, 30, 43] for $\Pi_{\mathcal{G}}(\cdot)$, we can approximate $E(\cdot)$ by

$$\widetilde{G}(\varepsilon, z) := F(\phi(\varepsilon, z)) + z - \phi(\varepsilon, z), \quad (\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m,$$

where $\phi(\varepsilon, z)$ is defined by

$$\phi_i(\varepsilon, z) := \varphi(\varepsilon, z_i), \quad i \in \mathcal{M} := \{1, 2, \ldots, m\}. \tag{22}$$

Here the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is the CHKS smoothing function defined by

$$\varphi(a, b) := \frac{1}{2}\left(b + \sqrt{b^2 + 4a^2}\right), \quad \forall\, (a, b) \in \mathbb{R} \times \mathbb{R} \tag{23}$$

We point out that the function φ is continuously differentiable everywhere but the origin and

$$\varphi'(a, b) = \left(\frac{2a}{\sqrt{b^2 + 4a^2}}, \frac{1}{2}\big(1 + \frac{b}{\sqrt{b^2 + 4a^2}}\big)\right). \tag{24}$$

For all $b \in \mathbb{R}$ and $a \neq 0$, $\varphi'_b(a, b) \in [0, 1]$, see also [6, 7].

Let $w := (\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m$. Then, it is easy to see that $\widetilde{G}$ is continuously differentiable for any $w \in \mathbb{R}_{++} \times \mathbb{R}^m$.

To prevent the singularity of the derivative of the mapping F, we can use the regularization technique. The simplest regularization technique is the well-known Tikhonov regularization, i.e., the function F is replaced by F_ε, where

$$F_\varepsilon(z) := F(z) + \varepsilon z, \quad \varepsilon \in \mathbb{R}_{++},$$

see for instance [16, 45] for regularization techniques for NCP. For the BVI (20), we define a regularized function $H : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ by

$$H(w) := \begin{pmatrix} \varepsilon \\ G(w) \end{pmatrix}, \quad w := (\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m, \tag{25}$$

where

$$G(w) := \widetilde{G}(w) + \varepsilon z.$$

It follows that the function H is continuously differentiable at any point $w = (\varepsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$ (see the next section for detail). Moreover, $z^* = (y^*, \xi^*, \zeta^*) \in \mathbb{R}^{4n-2} \times \mathbb{R} \times \mathbb{R}^{2n}$ solves (21) if and only if $w^* := (0, z^*) \in \mathbb{R} \times \mathbb{R}^m$ solves $H(w) = 0$. In this case, the projection $\Pi_{\mathcal{G}}(z^*)$ of z^* on the set $\mathcal{G}$ is a solution of the BVI (20). We point out that these results hold only under the assumption that the solution set of the BVI (20) is nonempty (See Section 5 for detail).

The following definitions related to F will be used in this paper.

Definition 3.1 [6] *Let Ω be a nonempty subset of $\mathbb{R}^n$. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a*

- *P_0-function over the set Ω if there exists a index i such that*

$$x_i \neq y_i, \quad \text{and} \quad (x_i - y_i)[F_i(x) - F_i(y)] \geq 0 \quad \text{for all } x, y \in \Omega \text{ and } x \neq y;$$

- *uniform P-function over the set Ω if, for some $\mu > 0$,*

$$\max_{1\le i\le n}(x_i - y_i)[F_i(x) - F_i(y)] \ge \mu\|x-y\|^2 \quad \textit{for all } x, y \in \Omega;$$

- *monotone function over the set Ω if*

$$(x-y)^T[F(x) - F(y)] \ge 0 \quad \textit{for all } x, y \in \Omega;$$

- *strongly monotone function over the set Ω if, for some $\mu > 0$,*

$$(x-y)^T[F(x) - F(y)] \ge \mu\|x-y\|^2 \quad \textit{for all } x, y \in \Omega.$$

It is obvious that every monotone function is a P_0-function, every strongly monotone function is a uniform P-function.

We now show the monotonicity of the function F defined in (19) over the set $\mathcal{G}$.

Proposition 3.2 *The function $F : \mathbb{R}^m \to \mathbb{R}^m$ defined in (19) is a monotone function over the set $\mathcal{G}$, i.e.,*

$$\big(F(z^1) - F(z^2)\big)^T (z^1 - z^2) \ge 0, \quad \textit{for all } z^1, z^2 \in \mathcal{G}.$$

Proof: From (19), we have for any $z^1, z^2 \in \mathcal{G}$,

$$\begin{aligned} & F(z^1) - F(z^2) \\ = & \begin{bmatrix} (y^1 - y^2) + 2(\xi^1 \mathcal{A}^T\mathcal{A}y^1 - \xi^2\mathcal{A}^T\mathcal{A}y^2) - 2(\xi^1 - \xi^2)\mathcal{A}^T g - \mathcal{B}^T(\zeta^1 - \zeta^2) \\ -\big((y^1)^T\mathcal{A}^T\mathcal{A}y^1 - (y^2)^T\mathcal{A}^T\mathcal{A}y^2\big) + 2g^T\mathcal{A}(y^1 - y^2) \\ \mathcal{B}(y^1 - y^2) \end{bmatrix} \\ = & \begin{bmatrix} (y^1 - y^2) + 2(\xi^1 - \xi^2)\mathcal{A}^T\mathcal{A}y^1 + 2\xi^2\mathcal{A}^T\mathcal{A}(y^1 - y^2) - 2(\xi^1 - \xi^2)\mathcal{A}^T g - \mathcal{B}^T(\zeta^1 - \zeta^2) \\ -(y^1 - y^2)^T\mathcal{A}^T\mathcal{A}y^1 - (y^2)^T\mathcal{A}^T\mathcal{A}(y^1 - y^2) + 2g^T\mathcal{A}(y^1 - y^2) \\ \mathcal{B}(y^1 - y^2) \end{bmatrix}. \end{aligned}$$

Thus we have that, for any $z^1, z^2 \in \mathcal{G}$,

$$\begin{aligned} & (F(z^1) - F(z^2))^T(z^1 - z^2) \\ = \ & \|y^1 - y^2\|^2 + 2(\xi^1 - \xi^2)(y^1)^T\mathcal{A}^T\mathcal{A}(y^1 - y^2) + 2\xi^2(y^1 - y^2)^T\mathcal{A}^T\mathcal{A}(y^1 - y^2) \\ & \quad -(\xi^1 - \xi^2)(y^1)^T\mathcal{A}^T\mathcal{A}(y^1 - y^2) - (\xi^1 - \xi^2)(y^1 - y^2)^T\mathcal{A}^T\mathcal{A}y^2 \\ & = \|y^1 - y^2\|^2 + (\xi^1 + \xi^2)(y^1 - y^2)^T\mathcal{A}^T\mathcal{A}(y^1 - y^2) \ge 0. \end{aligned}$$

The proof is completed. □

We note that F is a continuously differentiable function. This, together with the monotonicity of F over the set $\mathcal{G}$, shows that $F'(x)$ is positive semidefinite for all $x \in \mathcal{G}$, see for instance [22].

4 A Regularized Smoothing Newton Method

In this section, we propose a regularized smoothing Newton-type algorithm for solving the equation $H(w) = 0$, where H is defined by (25). This is motivated by the super numerical performance of the regularized/smoothing Newton's methods, see for instance [8, 28, 36, 37, 45]. For example, Chen, Qi, and Sun [8] designed the first globally and superlinearly convergent smoothing Newton-type method by exploiting the Jacobian consistency and applying the infinite sequence of smoothing approximation functions.

Noticing that F is a monotone function over the set $\mathcal{G}$, we have the following result on the nonsingularity of the Jacobian of H.

Proposition 4.1 *a) The function H defined in (25) is continuously differentiable for all $w = (\varepsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$ and*

$$H'(w) = \left[\begin{array}{cc} 1 & 0 \\ G'_\varepsilon(w) & G'_z(w) \end{array} \right], \tag{26}$$

where

$$\begin{array}{rcl} G'_\varepsilon(w) & := & (F'(\phi(w)) - I)\, q(\varepsilon) + z, \\ G'_z(w) & := & F'(\phi(w))D(z) + (1+\varepsilon)I - D(z) \end{array}$$

with $D(z) = \mathrm{diag}(d_i(z), i \in \mathcal{M})$, $d_i(z) = \partial\phi_i(\varepsilon, z)/\partial z_i$, $q_i(\varepsilon) = \partial\phi_i(\varepsilon, z)/\partial\varepsilon$, and $d_i(z) \in [0,1]$, $i \in \mathcal{M}$.

b) For any $w \in \mathbb{R}_{++} \times \mathbb{R}^m$, the matrix $H'(w)$ is nonsingular.

Proof: a) Since the CHKS function φ defined in (23) is continuously differentiable for all $(a,b) \in \mathbb{R}^2$ but $(a,b) = (0,0)$, it is obvious that the function $H(\cdot)$ is continuously differentiable for any $w = (\varepsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$. By direct computation, we have (26). It follows from (24) that $d_i(z) \in [0,1]$.

b) Since F is a monotone function over $\mathcal{G}$, we obtain that $F'(z)$ is positive semidefinite for all $z \in \mathcal{G}$. Let $w := (\varepsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$. Then we know that $\phi(w) \in \mathcal{G}$ and $F'(\phi(w))$ is positive semidefinite. Now, suppose that there exists $h \in \mathbb{R}^m$ such that

$$G'_z(w)h = \big(F'(\phi(w))D(z) + (1+\varepsilon)I - D(z)\big)\, h = 0. \tag{27}$$

We claim that $D(z)h = 0$. Otherwise, if $D(z)h \neq 0$, then there exists a index $j \in \mathcal{M}$ such that $[D(z)h]_j = d_j h_j \neq 0$. By noting that $d_i(z) \in [0,1]$ for all $i \in \mathcal{M}$, we have

$$\begin{array}{rl} & (D(z)h)^T F'(u(w))(D(z)h) = -(D(z)h)^T[((1+\varepsilon)I - D(z))h] \\ = & -h^T(1+\varepsilon)D(z)h + h^T D^2(z)h = \sum\limits_{i=1}^m (-1-\varepsilon+d_i)d_i h_i^2 \leq (-1-\varepsilon+d_j)d_j h_j^2 < 0, \end{array}$$

which contradicts the fact that $F'(\phi(w)$ is positive semidefinite. Hence $D(z)h = 0$. Also, it follows from (27) that

$$(1+\varepsilon)h = 0,$$

which implies that $h = 0$ since $\varepsilon \in \mathbb{R}_{++}$. This shows that $G'_z(w)$ is nonsingular. Therefore, $H'(w)$ is nonsingular for any $w = (\varepsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$. $\square$

Now, we propose a regularized smoothing Newton method for solving $H(w) = 0$. Given $\bar{\varepsilon} \in \mathbb{R}_{++}$ and $\tau \in (0, 1)$ such that $\tau\,\bar{\varepsilon} < 1$. Let $\bar{w} := (\bar{\varepsilon}, 0) \in \mathbb{R} \times \mathbb{R}^m$. Define the merit function $\psi : \mathbb{R}^{m+1} \to \mathbb{R}_+$ by

$$\psi(w) := \|H(w)\|^2$$

and define $\gamma : \mathbb{R}^{m+1} \to \mathbb{R}_+$ by

$$\gamma(w) := \tau \min\{1, \psi(w)\}.$$

Let

$$\mathcal{V} := \{w = (\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m |\, \varepsilon > \gamma(w)\,\bar{\varepsilon}\}.$$

Then, for any $w \in \mathbb{R}^{m+1}$, $\gamma(w) \leq \tau < 1$. Thus for any $z \in \mathbb{R}^m$, we have

$$(\bar{\varepsilon}, z) \in \mathcal{V}.$$

Next, we state our regularized smoothing Newton's method as follows.

Algorithm 4.2 *(A regularized smoothing Newton's method)*

Step 0. *Give $\delta, \tau \in (0, 1)$, $\sigma \in (0, 1/2)$, and $\bar{\varepsilon} \in \mathbb{R}_{++}$ such that $\tau\,\bar{\varepsilon} < 1$. Let $w^0 := (\varepsilon^0, z^0)$ with $\varepsilon^0 := \bar{\varepsilon}$ and $z^0 \in \mathbb{R}^m$ being arbitrary. Let $\bar{w} := (\bar{\varepsilon}, 0)$ and $k := 0$.*

Step 1. *If $\|H(w^k)\| = 0$, then stop. Otherwise, let $\gamma_k := \gamma(w^k)$.*

Step 2. *Compute*

$$\Delta w^k := (\Delta\varepsilon^k, \Delta z^k) \in \mathbb{R} \times \mathbb{R}^m$$

by

$$H(w^k) + H'(w^k)\Delta w^k = \gamma_k(w^k)\bar{w}. \tag{28}$$

Step 3. *Let l_k be the smallest nonnegative integer l such that*

$$\psi(w^k + \delta^l \Delta w^k) \leq [1 - 2\sigma(1 - \tau\bar{\varepsilon})\delta^l]\psi(w^k)$$

Step 4. *Define*

$$w^{k+1} := w^k + \delta^{l_k}\Delta w^k.$$

Then replace k by $k + 1$ and go to Step 1.

This algorithm is based on regularized/smoothing Newton's method in [36, 37] for the BVI/NCP. By Proposition 4.1, $H(\cdot)$ is continuously differentiable for any $w^k \in \mathbb{R}_{++} \times \mathbb{R}^m$, and $H'(w^k)$ is nonsingular for any $w^k \in \mathbb{R}_{++} \times \mathbb{R}^m$. By following the similar proof of Lemma 5, Propositions 5 and 6 in [37], we can obtain the following results for Algorithm 4.2.

Proposition 4.3 *The followings are the properties of Algorithm 4.2.*

(a) *Algorithm 4.2 is well-defined.*

(b) *Algorithm 4.2 generates an infinite sequence $\{w^k = (\varepsilon^k, z^k)\}$.*

(c) *$\varepsilon^k \in \mathbb{R}_{++}$ and $w^k \in \mathcal{V}$ for all $k \geq 0$.*

5 Convergence Analysis

In this section, we shall establish the global and quadratic convergence of Algorithm 4.2. In particular, we shall prove that Algorithm 4.2 generates an infinite sequence $\{w^k\}$ such that the sequence $\{\psi(w^k)\}$ converges to zero, and the projection of any accumulation point on $\mathcal{G}$ is a solution to the BVI (20) (i.e., Problem (17)). For any given $\varepsilon \in \mathbb{R}_{++}$, we define the merit function $\theta_\varepsilon(z) : \mathbb{R}^m \to \mathbb{R}_+$ by

$$\theta_\varepsilon(z) := \|G(\varepsilon, z)\|^2. \tag{29}$$

It is obvious that for any $\varepsilon \in \mathbb{R}_{++}$, $\theta_\varepsilon(z)$ is continuously differentiable and

$$\nabla\theta_\varepsilon(z) = 2G^T(\varepsilon, z)G'_z(\varepsilon, z),$$

where $G'_z(\varepsilon, z) = F'(\phi(w))D(z) + (1+\varepsilon)I - D(z)$ and $D(z)$ is given as in Proposition 4.1. From the proof of a) of Proposition 4.1 we see that for any $(\varepsilon, z) \in \mathbb{R}_{++} \times \mathbb{R}^m$, $G'_z(\varepsilon, z)$ is nonsingular. We note that

$$\theta_0(z) = \|E(z)\|^2,$$

where $E(z)$ is defined in (21) and for any $w = (\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m$,

$$\psi(w) = \varepsilon^2 + \theta_\varepsilon(z).$$

We note from Proposition 4.1 and Proposition 4.3 that, for every $k \geq 0$, if $\varepsilon^k \in \mathbb{R}_{++}$ and $w^k \in \mathcal{V}$, then $H'(w^k)$ is nonsingular. Also, for any accumulation point $w^* = (\varepsilon^*, z^*)$ of $\{w^k\}$, if $\varepsilon^* \in \mathbb{R}_{++}$ and $w^* \in \mathcal{V}$, then $H'(w^*)$ is nonsingular. Therefore, following the similar proof of [37, Theorem 4], we can show the following result on Algorithm 4.2.

Lemma 5.1 *Algorithm 4.2 generates an infinite sequence $\{w^k\}$ with $\lim_{k\to\infty} \psi(w^k) = 0$ and any accumulation point w^* of $\{w^k\}$ is a solution of $H(w) = 0$.*

We note that Lemma 5.1 only shows that any accumulation point of the sequence $\{w^k\}$, if exists, is a solution of $H(w) = 0$. The following result ensures the existence of such an accumulation point. Our result can be seen as a generalization of [22, Theorem 3.8] or [36, Lemma 4.2].

Theorem 5.2 *Suppose that $\tilde{\varepsilon}^1$ and $\tilde{\varepsilon}^2$ are given two positive numbers such that $\tilde{\varepsilon}^1 < \tilde{\varepsilon}^2$. Let the level set $\mathcal{L}(w^0)$ be defined by*

$$\mathcal{L}(w^0) := \{w = (\varepsilon, z) \in [\tilde{\varepsilon}^1, \tilde{\varepsilon}^2] \times \mathbb{R}^m : \psi(w) \leq \psi(w^0)\}.$$

Then $\mathcal{L}(w^0)$ is bounded.

Proof: For the sake of contradiction, suppose that there exists a sequence $\{w^k = (\varepsilon^k, z^k) \in \mathbb{R} \times \mathbb{R}^m\}$ such that

$$\varepsilon^k \in [\tilde{\varepsilon}^1, \tilde{\varepsilon}^2], \quad \psi(w^k) \leq \psi(w^0) \quad \text{and} \quad \|w^k\| \to \infty. \tag{30}$$

Obviously, $\|z^k\| \to \infty$. Then, it is easy to show that

$$\max\{0, z_i^k\} \to \infty \Longrightarrow z_i^k \to +\infty \quad \text{and} \quad |z_i^k - \max\{0, z_i^k\}| \to 0, \quad i \in \mathcal{M}. \tag{31}$$

Notice that for any $i \in \mathcal{M}$, $\phi_i(w^k)$ defined in (22) is Lipschitz continuous with the Lipschitz constant 1 [6] and satisfies

$$0 \leq \phi_i(w^k) - \max\{0, z_i^k\} \leq \varepsilon^k, \quad i \in \mathcal{M}. \tag{32}$$

By (31) and (32), it follows that for all sufficiently large k,

$$\max\{0, z_i^k\} \to \infty \Longrightarrow |\phi_i(w^k) - z_i^k| \leq 2\varepsilon^k, \quad i \in \mathcal{M}.$$

Define the index set $\mathcal{J}$ by $\mathcal{J} := \{i | \, \phi_i(w^k) \text{ is unbounded}, i \in \mathcal{M}\}$. Then the set $\mathcal{J}$ is nonempty because otherwise

$$\|G(w^k)\| = \|F(\phi(w^k)) + (1+\varepsilon^k)z^k - \phi(w^k)\| \to \infty.$$

Let $\widehat{w}^k = (\widehat{\varepsilon}^k, \widehat{z}^k) \in \mathbb{R}_{++} \times \mathbb{R}^m$ be defined by

$$\widehat{\varepsilon}^k = \begin{cases} \varepsilon^k & \text{if } i \notin \mathcal{J}, \\ 0 & \text{if } i \in \mathcal{J}, \end{cases} \quad i \in \mathcal{M}$$

and

$$\widehat{z}^k = \begin{cases} z^k & \text{if } i \notin \mathcal{J}, \\ 0 & \text{if } i \in \mathcal{J}, \end{cases} \quad i \in \mathcal{M}.$$

Then

$$\phi_i(\widehat{w}^k) = \begin{cases} \phi_i(w^k) & \text{if } i \notin \mathcal{J}, \\ 0 & \text{if } i \in \mathcal{J}, \end{cases} \quad i \in \mathcal{M}.$$

Hence $\{\|\phi(\widehat{w}^k)\|\}$ is bounded. By using the monotonicity of F over the set $\mathcal{G}$, we get

$$\begin{aligned} 0 &\leq (\phi(w^k) - \phi(\widehat{w}^k))^T[F(\phi(w^k)) - F(\phi(\widehat{w}^k))] \\ &= \sum_{i=1}^{m}(\phi_i(w^k) - \phi_i(\widehat{w}^k))[F_i(\phi(w^k)) - F_i(\phi(\widehat{w}^k))] \\ &= \textstyle\sum_{i\in\mathcal{J}} \phi_i(w^k)[F_i(\phi(w^k)) - F_i(\phi(\widehat{w}^k))] \end{aligned} \tag{33}$$

Since $\{\|\phi(\widehat{w}^k)\|\}$ is bounded and F is continuous, $\{\|F(\phi(\widehat{w}^k))\|\}$ remains bounded. Since for any $i \in \mathcal{J}$, $\phi_i(w^k) \to +\infty$, (33) implies that, for any $i \in \mathcal{J}$, $F_i(\phi(w^k))$ does not tend to $-\infty$. This, together with the boundedness of $|\phi_i(w^k) - z_i^k|$, $i \in \mathcal{M}$, in turn implies that, for any $i \in \mathcal{J}$,

$$\|G_i(w^k)\| = \|F_i(\phi(w^k)) + (1+\varepsilon^k)z_i^k - \phi_i(w^k)\| \to \infty$$

since $\varepsilon^k z_i^k \to +\infty$. This contradicts (30) because $\|H(w^k)\| \geq \|G(w^k)\|$. This completes our proof. $\square$

Corollary 5.3 *For any $\varepsilon \in \mathbb{R}_{++}$, the function $\theta_\varepsilon(z)$ defined in (29) is coercive, i.e.,*

$$\lim_{\|z\|\to\infty} \theta_\varepsilon(z) = \infty.$$

Assumption 5.4 *The solution set $\mathcal{S}$ of (21) is nonempty*[1].

Remark 5.5 *By [36, Lemma 4.1], Assumption 5.4 is equivalent to that the solution set of (20) is nonempty.*

To establish the global convergence, we need the following lemma.

Lemma 5.6 *Let $\mathcal{C} \subset \mathbb{R}^m$ be a compact set. Then, for any $\eta > 0$, there exists a scalar $\widetilde{\varepsilon} \in \mathbb{R}_{++}$ such that*

$$|\theta_\varepsilon(z) - \theta_0(z)| \le \eta$$

for all $z \in \mathcal{C}$ and $\varepsilon \in [0, \widetilde{\varepsilon}]$.

Proof: It easily follows from the continuity of $\theta_\varepsilon(z)$ on the compact set $\mathcal{C} \times [0, \widetilde{\varepsilon}]$ and the uniform continuity of $\theta_\varepsilon(z)$ there. □

Now, we state the global convergence result for our algorithm.

Theorem 5.7 *Suppose that Assumption 5.4 is satisfied. Then the infinite sequence $\{w^k\}$ generated by Algorithm 4.2 is bounded and any accumulation point w^* of $\{w^k\}$ is a solution of $H(w) = 0$.*

Proof: From Lemma 5.1, it follows that there exists an infinite sequence $\{w^k\}$ generated by Algorithm 4.2 such that

$$\lim_{k\to\infty} \psi(w^k) = 0 \tag{34}$$

and any accumulation point of $\{w^k\}$ is a solution of $H(w) = 0$. It remains to show the boundedness of $\{w^k\}$. For the sake of contradiction, suppose that the sequence $\{w^k\}$ is unbounded. Then, by taking the subsequence if necessary, we get $\|w^k\| \to \infty$. Since $\{\varepsilon^k\}$ is bounded, there exists a compact set $\mathcal{C} \subset \mathbb{R}^m$ such that $\mathcal{S} \subset \text{int}\mathcal{C}$, where "int" denotes the topological interior of a given set, and for all k large enough,

$$z^k \notin \mathcal{C}. \tag{35}$$

Let $\widetilde{z} \in \mathcal{S}$ be an arbitrary solution of (21). Then

$$\theta_0(\widetilde{z}) = 0 \quad \text{and} \quad \psi(\widetilde{w}) = 0, \quad \widetilde{w} := (0, \widetilde{z}).$$

Notice that

$$\widetilde{c} := \min_{z\in\partial\mathcal{C}} \theta_0(z) > 0.$$

By Lemma 5.6, for $\eta = \widetilde{c}/3$, we have that, for sufficiently large $k \ge 0$,

$$\theta_{\varepsilon^k}(\widetilde{z}) \le \frac{1}{3}\widetilde{c} \quad \text{and} \quad \theta_{\varepsilon^k}(z) \ge \frac{2}{3}\widetilde{c}. \tag{36}$$

Also, (34) implies that

$$\varepsilon^k \to 0 \quad \text{and} \quad \theta_{\varepsilon^k}(z^k) \to 0 \quad \text{as } k \to \infty$$

[1]This implies that the solution set $\mathcal{S}$ of (21) is nonempty and bounded since the function F is monotone over $\mathcal{G}$, see for instance [32].

and so, for sufficiently large $k \geq 0$,

$$\theta_{\varepsilon^k}(z^k) \leq \frac{1}{3}\widetilde{c}. \tag{37}$$

Now, let an index k be fixed such that (35)-(37) hold. By using the well-known Mountain Pass Theorem (see for instance [16, Theorem 5.3]), there exists a point $\widehat{z} \in \mathbb{R}^m$ such that

$$\nabla\theta_{\varepsilon^k}(\widehat{z}) = 0 \quad \text{and} \quad \theta_{\varepsilon^k}(\widehat{z}) \geq \frac{2}{3}\widetilde{c} > 0.$$

This means that the stationary point $\widehat{z}$ of θ_{ε^k} is not a global minimizer of θ_{ε^k}. This contradiction shows that $\{w^k\}$ is bounded. □

In the following, we establish the quadratic convergence of Algorithm 4.2. For this purpose, we need the definition of semismoothness. Semismoothness was originally introduced by Mifflin [33] for functionals and was extend to vector valued functions by Qi and Sun [38].

Definition 5.8 *Suppose that $\Psi : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is a locally Lipschitzian function and has a generalized Jacobian $\partial\Psi$ in the sense of Clarke [12]. Then*

1) Ψ is said to be semismooth at $x \in \mathbb{R}^{n_1}$ if

$$\lim_{\substack{V \in \partial\Psi(x+th') \\ h' \to h, t \downarrow 0}} \{Vh'\}$$

exists for any $h \in \mathbb{R}^{n_1}$.

2) Ψ is said to be strongly semismooth at x if Ψ is semismooth at x and for any $V \in \partial\Psi(x + th), h \to 0$, it follows that

$$\Psi(x+h) - \Psi(x) - Vh = O(\|h\|^2).$$

We note that the function $\varphi(\cdot)$ defined in (23) is strongly semismooth for any $(a, b) \in \mathbb{R}^2$. Then the function H defined by (25) is strongly semismooth everywhere [18]. By the strong semismoothness of H, we have the following theorem on the quadratic convergence for Algorithm 4.2. Since the proof is similar as in [37, Theorem 8], we omit it here.

Theorem 5.9 *Suppose that w^* is an accumulation point of the sequence $\{w^k\}$ generated by Algorithm 4.2. If all $V \in \partial H(w^*)$ are nonsingular, then the sequence $\{w^k\}$ converges to w^* with*

$$\|w^{k+1} - w^*\| = O(\|w^k - w^*\|^2) \quad \text{and} \quad \varepsilon^{k+1} = O\big((\varepsilon^k)^2\big).$$

Theorem 5.9 shows that Algorithm 4.2 is quadratically convergent under the nonsingularity assumption of $\partial H(w^*)$. We now discuss the nonsingularity of $\partial H(w^*)$. For convenience, we define three index associated with the solution $w^* = (\varepsilon^*, z^*)$ as follows:

$$\mathcal{I} = \{i : z_i^* > 0\}, \quad \mathcal{J} = \{i : z_i^* = 0 = [F(\Pi_{\mathcal{G}}(z^*))]_i\}, \quad \mathcal{K} = \{i : [F(\Pi_{\mathcal{G}}(z^*))]_i > 0\}.$$

The BVI (20) is said to be R-regular at z^* if $\nabla F_{\mathcal{I}\mathcal{I}}(\Pi_{\mathcal{G}}(z^*))$ is nonsingular and the Schur complement of $\nabla F_{\mathcal{I}\mathcal{I}}(\Pi_{\mathcal{G}}(z^*))$ in

$$\begin{bmatrix} \nabla F_{\mathcal{I}\mathcal{I}}(\Pi_{\mathcal{G}}(z^*)) & \nabla F_{\mathcal{I}\mathcal{J}}(\Pi_{\mathcal{G}}(z^*)) \\ \nabla F_{\mathcal{J}\mathcal{I}}(\Pi_{\mathcal{G}}(z^*)) & \nabla F_{\mathcal{J}\mathcal{J}}(\Pi_{\mathcal{G}}(z^*)) \end{bmatrix}$$

is a P-matrix, i.e., all its principal minors are positive, see for instance [17].

Before discussing the nosningularity of any element in $\partial H(w^*)$, we provide the estimate on $\partial H(\cdot)$ at the solution $w^* = (\varepsilon^*, z^*)$.

Proposition 5.10

$$\partial H(w^*) \subseteq \begin{bmatrix} 1 & 0 \\ U(\varepsilon^*) & U(z^*) \end{bmatrix} \quad \text{and} \quad U := (U(\varepsilon^*), U(z^*)) \in \partial G(w^*).$$

Here

$$U(\varepsilon^*) \in \mathbb{R}^m \quad \text{and} \quad U(z^*) \in \mathbb{R}^{m \times m}$$

with

$$U(z^*) \subseteq \nabla F(\phi(\varepsilon^*, z^*))\, D(z^*) + (1+\varepsilon^*)I - D(z^*),$$

where $D(z^*) := \mathrm{diag}(d_i(z^*), i \in \mathcal{M})$ *with*

$$\begin{cases} d_i(z^*) = 1 & \text{if } i \in \mathcal{I} \\ d_i(z^*) \in [0,1] & \text{if } i \in \mathcal{J} \\ d_i(z^*) = 0 & \text{if } i \in \mathcal{K} \end{cases}.$$

Proof: The proof is similar to that of [17, Proposition 3.1]. □

Based on Proposition 5.10 and the R-regularity assumption of the solution z^*, we can show the nonsingularity of all the elements in $\partial H(w^*)$ in a way similar to that in [37, Proposition 9] or [17, Proposition 3.2] .

Theorem 5.11 *Suppose that*

$$w^* := (\varepsilon^*, z^*) \in \mathbb{R} \times \mathbb{R}^m$$

is a solution of $H(w) = 0$. If the BVI (20) is R-regular at z^*, then all the matrices $V \in \partial H(w^*)$ are nonsingular.

6 Numerical Results

In this section, we report the numerical performance of Algorithm 4.2 for solving the TriIQEP (16). All the numerical tests were done using `MATLAB 7.0`. As in [1], we set the measurement noise level to be $r = 0.08$. An upper bound estimate for the noise parameter δ_n is given in terms of the measured model data

$$\delta_n = r(\|M_o X \Lambda^2\| + \|C_o X \Lambda\| + \|K_o X\|)$$

Throughout the numerical experiments, we set $\bar{\varepsilon} = 0.1$ and choose the starting point (`SP.`) as

i) $\varepsilon^0 = \bar{\varepsilon}, \quad y^0 = (0, \ldots, 0)^T \in \mathbb{R}^{4n-2}, \quad \xi^0 = 0 \in \mathbb{R}, \quad \zeta^0 = (0, \ldots, 0)^T \in \mathbb{R}^{2n};$

ii) $\varepsilon^0 = \bar{\varepsilon}, \quad y^0 = (1, \ldots, 1)^T \in \mathbb{R}^{4n-2}, \quad \xi^0 = 1 \in \mathbb{R}, \quad \zeta^0 = (1, \ldots, 1)^T \in \mathbb{R}^{2n}.$

The other parameters used in the algorithm are as follows:

$$\delta = 0.5, \quad \sigma = 0.5 \times 10^{-4}, \quad \tau = 0.2 \times \min(1, 1/\bar{\varepsilon}).$$

The stopping criterion is set to be

$$\|H(w^k)\| \leq 10^{-6}$$

where the function H is defined in (25). To demonstrate the numerical performance of Algorithm 4.2, we solve the linear system (28) iteratively[2]. From Proposition (4.1), we observe that, in general, the matrix $H'(w^k)$ in Algorithm 4.2 is square and nonsingular but not necessarily symmetric. Thus we can solve (28) by the iterative methods such as the QMR [19], the GMRES [42], the BICG [48], and the CGS [44] methods. For simplicity of demonstration, we solve (28) by the QMR method using the `MATLAB`-provided `QMR` function with the default tolerance.

Example 6.1 *Suppose that the parameters $\{a_i^o\}_1^n$, $\{b_i^o\}_2^n$, $\{c_i^o\}_1^n$, and $\{d_i^o\}_2^n$ are generated randomly for different values of n and that the measured noisy eigendata $(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ is also generated randomly for different values of p.*

Our numerical results are given in Tables 1 and 2, where `IT.`, `NF.`, and `VAL.` stand for the number of iterations, the number of function evaluations, and the value of $\|H(\cdot)\|$ at the final iterate of our algorithm (the largest number of iterations in QMR is set to be $\max(2000, 6n)$), respectively. The numerical results in Tables 1 and 2 show that our proposed algorithm is very efficient for solving the TriIQEP.

$p = 15$, $s = 3$				
SP.	n	IT.	NF.	VAL.
i)	50	13	19	9.0×10^{-13}
	100	13	19	6.5×10^{-11}
	200	13	19	5.9×10^{-10}
	300	13	18	5.2×10^{-9}
	400	13	19	1.4×10^{-8}
	500	13	20	4.6×10^{-8}
ii)	50	12	16	9.9×10^{-9}
	100	13	19	7.4×10^{-11}
	200	13	18	7.4×10^{-10}
	300	13	19	5.6×10^{-9}
	400	13	18	2.2×10^{-8}
	500	13	18	6.1×10^{-8}

Table 1: Numerical results for Example 6.1

[2]As an anonymous referee pointed out, unless the dimension of the problem is very large (say, far beyond one thousand), the linear system (28) should be solved by the direct methods since it is quite sparse.

$n = 100$					
SP.	p	s	IT.	NF.	VAL.
i)	10	3	13	20	3.0×10^{-11}
	20	6	13	18	1.4×10^{-11}
	30	9	13	20	6.0×10^{-11}
	40	12	14	22	2.5×10^{-12}
	50	15	14	23	1.8×10^{-11}
ii)	10	3	12	15	4.1×10^{-11}
	20	6	13	19	2.1×10^{-11}
	30	9	13	19	1.8×10^{-11}
	40	12	13	19	2.5×10^{-11}
	50	15	13	19	1.3×10^{-11}

Table 2: Numerical results for Example 6.1

Example 6.2 *In this example, we consider an engineering application as in [39]. The vibrations of a simple connected, damped mass-spring system with masses of unit weight are governed by*

$$I\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0$$

with

$$\begin{cases} C &= P\mathrm{diag}(0, e_1, e_2, \ldots, e_{n-1})P^T + \mathrm{diag}(\pi_1, \pi_2, \ldots, \pi_n), \\ K &= P\mathrm{diag}(0, f_1, f_2, \ldots, f_{n-1})P^T + \mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_n), \end{cases}$$

where $P = [\delta_{ij} - \delta_{i+1,j}]$ *with* δ_{ij} *the Kronecker delta. In consideration of physical realizability, all the damping and stiffness constants* $\{e_i\}_1^{n-1}$, $\{\pi_i\}_1^n$, $\{f_i\}_1^{n-1}$, *and* $\{\kappa_i\}_1^n$ *should be positive. This requires that the damping and stiffness matrices (i.e.,* C *and* K*) are weakly diagonally dominant and have positive diagonal entries and negative off-diagonal entries.*

We first randomly generate the parameters $\{e_i^o\}_1^{n-1}$, $\{\pi_i^o\}_1^n$, $\{f_i^o\}_1^{n-1}$, *and* $\{\kappa_i^o\}_1^n$ *with* $n = 5$ *by*

$$\begin{cases} \{e_i^o\}_1^4 &= \{1.9010, 1.7347, 1.8652, 2.7087\}, \\ \{\pi_i^o\}_1^5 &= \{2.3015, 2.5923, 2.2725, 3.2452, 3.3226\}, \\ \{f_i^o\}_1^4 &= \{4.6148, 7.8653, 7.1597, 3.8038\}, \\ \{\kappa_i^o\}_1^5 &= \{9.5716, 7.9270, 4.6954, 5.5770, 9.3244\}. \end{cases}$$

The corresponding quadratic pencil $Q(\lambda) := \lambda^2 I + \lambda C_o + K_o$ *has* 2 *real eigenvalues and* 4 *pairs of complex conjugate eigenvalues:*

$$\begin{cases} -7.3094 \\ -2.5927 \\ -4.1460 \pm 3.2582\imath, \\ -2.7996 \pm 2.8901\imath, \\ -1.6421 \pm 2.8652\imath, \\ -1.5378 \pm 2.3032\imath, \end{cases}$$

and the associated eigenvectors are given by

$$\begin{bmatrix} 0.0011 & 0.0006 & -0.0424 \mp 0.0541\imath & -0.1076 \mp 0.0401\imath & -0.1038 \mp 0.1811\imath & 0.0418 \mp 0.0949\imath \\ -0.0044 & -0.0197 & 0.1061 \pm 0.0834\imath & 0.0725 \pm 0.0575\imath & -0.0878 \mp 0.0778\imath & -0.0344 \mp 0.1152\imath \\ 0.0240 & -0.0641 & -0.0882 \mp 0.0476\imath & 0.1029 \pm 0.1062\imath & -0.0855 \pm 0.0314\imath & -0.1003 \mp 0.1502\imath \\ -0.1086 & -0.2809 & -0.0063 \pm 0.0249\imath & -0.0105 \mp 0.0540\imath & -0.0119 \pm 0.0975\imath & -0.1357 \mp 0.1143\imath \\ 0.0773 & 0.2147 & 0.0326 \pm 0.0001\imath & -0.0921 \mp 0.0554\imath & 0.0073 \mp 0.0930\imath & -0.1014 \mp 0.1256\imath \end{bmatrix}.$$

We now perturb randomly the eigenvectors corresponding the real eigenvalue -2.5927 *and the pair of complex conjugate eigenvalues* $-1.5378 \pm 2.3032\imath$ *and use the perturbed eigendata as the measured noisy data* $\{(\lambda_i, x_i)\}_{i=1}^3$ *, where*

$$\lambda_1 = -2.5927, \quad x^1 = \begin{bmatrix} -0.0519 \\ -0.0106 \\ -0.0346 \\ -0.2307 \\ 0.2715 \end{bmatrix}, \quad \lambda_{2,3} = -1.5378 \pm 2.3032\imath, \quad x^{2,3} = \begin{bmatrix} -0.0157 \mp 0.0713\imath \\ -0.0467 \mp 0.1616\imath \\ -0.1312 \mp 0.1142\imath \\ -0.0950 \mp 0.0701\imath \\ -0.1448 \mp 0.1823\imath \end{bmatrix}.$$

We then use Algorithm 4.2 with any one of the prescribed starting points to reconstruct the physical model. We find a physically realizable solution for the TriIQEP, which is determined by the parameters as follows:

$$\begin{cases} \{e_i\}_1^4 & = \{1.7823, 1.8607, 1.6389, 2.6919\}, \\ \{\pi_i\}_1^5 & = \{2.5823, 2.3289, 2.5823, 3.5158, 3.2211\}, \\ \{f_i\}_1^4 & = \{4.6364, 7.8503, 7.0908, 3.6905\}, \\ \{\kappa_i\}_1^5 & = \{9.5229, 7.8932, 4.7126, 5.8491, 9.2809\}. \end{cases}$$

In the following, let the estimated analytic parameters $\{e_i^o\}_1^{n-1}$, $\{\pi_i^o\}_1^n$, $\{f_i^o\}_1^{n-1}$, and $\{\kappa_i^o\}_1^n$ and the noisy eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$ be generated randomly. Tables 3 and 4 list the numerical results for various values of n and p. It is seen that our method still performs efficiently as expected.

7 Concluding Remarks

In this paper we focus on the inverse eigenvalue problem for the symmetric tridiagonal monic quadratic pencil $Q(\lambda) := \lambda^2 I + \lambda C + K$. We first discuss the solvability of the inverse problem from the self-conjugate set of prescribed four eigenpairs. The solvability conditions are presented. In many practical applications, both the matrices C and K should be weakly diagonally dominant and have positive diagonal elements and negative off-diagonal elements. Thus, it is difficult to find the condition on the eigendata to ensure the existence of an exact physical solution [39]. However, it is well-known that the eigendata is, in general, measured from the physical structure and inevitably corrupted by noise [1, 21]. To preserve the structural connectivity and overcome the erroneous estimate of C and K, Our problem is reformulated as a constrained optimization problem which is then transformed into a new BVI. An important advantage of the BVI is its monotonicity as shown. We propose a regularized smoothing Newton method for solving the monotone BVI. The global and quadratic convergence is established under some mild

$p = 15$, $s = 3$				
SP.	n	IT.	NF.	VAL.
i)	50	10	16	7.8×10^{-13}
	100	9	10	2.1×10^{-12}
	200	10	16	2.0×10^{-11}
	300	10	15	5.3×10^{-10}
	400	8	9	1.4×10^{-9}
	500	9	11	4.0×10^{-9}
ii)	50	11	17	1.6×10^{-9}
	100	12	21	3.7×10^{-12}
	200	12	21	1.2×10^{-10}
	300	11	18	5.2×10^{-7}
	400	11	18	3.0×10^{-9}
	500	11	17	6.2×10^{-7}

Table 3: Numerical results for Example 6.2

$n = 100$					
SP.	p	s	IT.	NF.	VAL.
i)	10	3	8	9	5.3×10^{-12}
	20	6	9	11	3.5×10^{-12}
	30	9	9	11	1.2×10^{-11}
	40	12	9	12	6.6×10^{-12}
	50	15	9	12	7.1×10^{-12}
ii)	10	3	9	10	4.9×10^{-12}
	20	6	11	19	1.5×10^{-8}
	30	9	10	13	1.2×10^{-11}
	40	12	12	22	7.0×10^{-12}
	50	15	12	21	7.3×10^{-12}

Table 4: Numerical results for Example 6.2

assumptions, which essentially require the existence of a solution of the BVI, see Assumption 5.4. We demonstrate the efficiency of our algorithm by some numerical tests and a practical engineering application in vibrations.

An interesting problem is whether our method can be extended to other structured IQEPs, which arise in many applications. This needs further study.

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