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三角范畴定义中八面体公理的等价命题及其应用

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摘要:三角范畴是一个带有自同构的加法范畴,并且满足 4 条公理,其中的 1 条重要公理是八面体公理. 由 Grothendick-Verdier 在上个世纪 60 年代提出的八面体公理相对于其它 3 条公理形式比较复杂,应用起来比较不方便. 因此研究八面体公理的其它等价命题引起了人们的兴趣. 本文在王济荣工作的基础上给出八面体公理的第 1 个等价命题,再利用对偶的思想导出八面体公理的第 2 个等价命题。最后利用 homotopy cartesian 得到八面体公理的第 3 个等价命题,并利用第 3 个等价命题简化 Peng 和 Tan 的证明.

关键词:三角范畴;八面体公理;homotopy cartesian

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1　预备知识

设 C 是加法范畴,C 上的两个态射 $f:X\to Y$,$g:Y\to Z$ 的合成记为 gf,态射 $(a,b):X\oplus Y\to Z$ 的转置记为 $(a,b)^T$.

定义 1[1]　设 C 是加法范畴,$T:C\to C$ 是加法范畴自同构,C 中一个六元组是 $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}TX$ 使得 $vu=0$,$wv=0$,$(Tu)w=0$.

定义 2[1]　六元组 $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}TX$ 到六元组 $X'\xrightarrow{u'}Y'\xrightarrow{v'}Z'\xrightarrow{w'}TX'$ 的一个态射是指一个态射组 (f,g,h) 使得

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & TX \arrow[d, "Tf"'] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX'
\end{tikzcd}$$

可交换.

定义 3[1]　设 C 是带有自同构 T 的加法范畴,ε 是 C 中的六元组的一个集合. 称 (C,T,ε) 为一个预三角范畴,如果 ε 满足如下公理(TR1),(TR2),(TR3):

(TR1) 与 ε 中六元组同构的六元组仍在 ε 中;对每个态射 $X\xrightarrow{u}Y$,在 ε 中存在一个六元组 $X\xrightarrow{u}Y\to Z\to TX$;且 $X\xrightarrow{1}X\to 0\to TX\in\varepsilon$;

(TR2) 若 $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}TX\in\varepsilon$,则 $Y\xrightarrow{v}Z\xrightarrow{w}TX\xrightarrow{-Tu}TY\in\varepsilon$;

(TR3) 若 $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}TX$,$X'\xrightarrow{u'}Y'\xrightarrow{v'}Z'\xrightarrow{w'}TX'\in\varepsilon$,且 $f:X\to X'$,$g:Y\to Y'$ 使得 $u'f=gu$,则存在 $h:Z\to Z'$ 使得 (f,g,h) 是六元组的一个态射.

定义 4[1]　C 是预三角范畴时,ε 中的六元组称为三角.

定义 5[2]　设 C 是预三角范畴,称 C 是三角范畴,若满足(TR4):

(TR4)(八面体公理)设 $X\xrightarrow{f}Y\xrightarrow{a}Z'\to TX$ 和 $L\xrightarrow{g}Y\xrightarrow{b}Z\to TL$ 均是三角,则存在 W 和下列交换图,使得第 2 行,第 3 行,第 2 列,第 3 列均是三角.

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d, "g"'] & L \arrow[d] & \\
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "b"'] & Z' \arrow[r] \arrow[d, "c"'] & TX \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r, "d"] \arrow[d] & W \arrow[r, "e"] \arrow[d, "i"'] & TX \arrow[d, "Tf"'] \\
 & TL \arrow[r, equal] & TL \arrow[r] & TY
\end{tikzcd}$$

定义 6[2]　C 是三角范畴时,C 中的三角称为好三角.

性质 1①　若 $X\xrightarrow{u}Y\xrightarrow{v}Z$ 是三角,则 $T^{-1}X\xrightarrow{-T^{-1}w}X\xrightarrow{u}Y\xrightarrow{v}Z$ 也是三角.

性质 2①　在(TR3)中,(f,g,h) 中任意两个是同构,第 3 个也是同构.

2　主要定理及其证明

定义 7[3]　称下列交换图

$$\begin{tikzcd}
X \arrow[r, "\alpha'"] \arrow[d, "\alpha''"'] & Y' \arrow[d, "\beta'"] \\
Y'' \arrow[r, "\beta''"] & Z
\end{tikzcd}$$

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① 章璞. 导出范畴七讲. 2005:1 - 15.

为 homotopy cartesian,如果存在三角 $X \xrightarrow{(\alpha',\alpha'')^T} Y' \oplus Y'' \xrightarrow{(\beta',-\beta'')} Z \xrightarrow{\gamma} TX$,其中 γ 称为微分.

定理 1 设 (C,T,ε) 是预三角范畴,则(TR4)等价于(TR4-1):

(TR4-1) 对任意的态射对 $Z' \xleftarrow{a} Y \xrightarrow{b} Z$,则有下列交换图,使得第 2 行,第 3 行,第 2 列,第 3 列均是三角.

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d, "g"] & L \arrow[d] & \\
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "b"] & Z' \arrow[r] \arrow[d, "c"] & TX \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r, "d"] \arrow[d] & W \arrow[r, "e"] \arrow[d, "i"] & TX \arrow[d, "Tf"] \\
 & TL \arrow[r, equal] & TL \arrow[r, "-Tg"] & TY
\end{tikzcd}$$

证明 (必要性) 设 $Y \xrightarrow{a} Z'$ 可嵌入三角 $X \xrightarrow{f} Y \xrightarrow{a} Z' \to TX$, $Y \xrightarrow{b} Z$ 可嵌入三角 $L \xrightarrow{g} Y \xrightarrow{b} Z \to TL$,则由(TR4)知(TR4-1)成立.

(充分性) 设 $X \xrightarrow{f} Y \xrightarrow{a} Z' \xrightarrow{k_1} TX$ 和 $L \xrightarrow{g} Y \xrightarrow{b} Z \xrightarrow{k_2} TL$ 是任意三角,对 $Z' \xleftarrow{a} Y \xrightarrow{b} Z$,由(TR4-1)知有下列交换图,

$$\begin{tikzcd}
 & Q' \arrow[r, equal] \arrow[d, "g'"] & Q' \arrow[d] & \\
Q \arrow[r, "f'"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "b"] & Z' \arrow[r, "k'_1"] \arrow[d, "c"] & TQ \arrow[d, equal] \\
Q \arrow[r] & Z \arrow[r, "d"] \arrow[d, "k'_2"] & W \arrow[r, "e"] \arrow[d, "i"] & TQ \arrow[d, "Tf'"] \\
 & TQ' \arrow[r, equal] & TQ' \arrow[r, "-Tg'"] & TY
\end{tikzcd}$$

使得第 2 行,第 3 行,第 2 列,第 3 列均是三角. 因为 $X \xrightarrow{f} Y \xrightarrow{a} Z' \xrightarrow{k_1} TX$ 和 $Q \xrightarrow{f'} Y \xrightarrow{a} Z' \xrightarrow{k'_1} TX'$ 都是三角,所以由(TR3)知存在 $\sigma: X \to Q$ 使得

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "a"] \arrow[d, equal] & Z' \arrow[r, "k_1"] \arrow[d, equal] & TX \\
Q \arrow[r, "f'"] & Y \arrow[r, "a"] & Z' \arrow[r, "k'_1"] & TQ
\end{tikzcd}$$

可交换. 同理存在 $\tau: L \to Q'$ 使得

$$\begin{tikzcd}
L \arrow[r, "g"] & Y \arrow[r, "b"] \arrow[d, equal] & Z \arrow[r, "k_2"] \arrow[d, equal] & TL \\
Q' \arrow[r, "g'"] & Y \arrow[r, "b"] & Z \arrow[r, "k'_2"] & TQ'
\end{tikzcd}$$

可交换. 所以 $f'\sigma = f, k'_1 = (T\sigma)k_1, g'\tau = g, k'_2 = (T\tau)k_2$,且 σ, τ 是同构. 则

$$\begin{tikzcd}
X \arrow[r, "bf"] \arrow[d, "\sigma"] & Z \arrow[r, "d"] \arrow[d, equal] & W \arrow[r, "(T\sigma^{-1})e"] \arrow[d, equal] & TX \arrow[d, "T\sigma"] \\
Q \arrow[r, "bf'"] & Z \arrow[r, "d"] & W \arrow[r, "e"] & TQ
\end{tikzcd}$$

和

$$\begin{tikzcd}
L \arrow[r, "ag"] \arrow[d, "\tau"] & Z' \arrow[r, "c"] \arrow[d, equal] & W \arrow[r, "(T\tau^{-1})i"] \arrow[d, equal] & TL \arrow[d, "T\tau"] \\
Q' \arrow[r, "ag'"] & Z' \arrow[r, "c"] & W \arrow[r, "i"] & TQ'
\end{tikzcd}$$

可交换. 因为 σ, τ 是同构,所以 $X \xrightarrow{bf} Z \xrightarrow{d} W \xrightarrow{(T\sigma^{-1})e} TX$ 和 $L \xrightarrow{ag} Z' \xrightarrow{c} Z \xrightarrow{(T\tau^{-1})} TX$ 是三角. 又 $(T\sigma^{-1})ec = (T\sigma^{-1})k_1' = k_1, Tf(T\sigma^{-1})e = (Tf')e = -(Tg')i = -(Tg)(T\tau^{-1})i, (T\tau^{-1})id = (T\tau^{-1})k_2' = (T\tau^{-1})(T\tau)k_2 = k_2$,所以有下列交换图,且第 2,第 3 行,第 2,第 3 列是三角,即(TR4)成立.

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d, "g"] & L \arrow[d] & \\
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "a"] \arrow[d, "b"] & Z' \arrow[r, "k_1"] \arrow[d, "c"] & TX \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r, "d"] \arrow[d, "k_2"] & W \arrow[r, "(T\sigma^{-1})e"] \arrow[d, "(T\tau^{-1})i"] & TX \arrow[d, "Tf"] \\
 & TL \arrow[r, equal] & TL \arrow[r, "-Tg"] & TY
\end{tikzcd}$$

由于 $Z' \xleftarrow{a} Y \xrightarrow{b} Z$ 的对偶是 $Z' \xrightarrow{a} Y \xleftarrow{b} Z$,所以我们可以类似地证得定理 2.

定理 2 设 (C,T,ε) 是预三角范畴,则(TR4)等价于(TR4-2):

(TR4-2) 对任意的态射对 $Z' \xrightarrow{a} Y \xleftarrow{b} Z$,则存在下列交换图

$$\begin{tikzcd}
 & L \arrow[r, equal] \arrow[d, "h"] & L \arrow[d] & \\
X \arrow[r, "e"] \arrow[d, equal] & W \arrow[r, "d"] \arrow[d, "c"] & Z \arrow[r] \arrow[d, "b"] & TX \arrow[d, equal] \\
X \arrow[r] & Z' \arrow[r, "a"] \arrow[d] & Y \arrow[r, "f"] \arrow[d] & TX \arrow[d] \\
 & TL \arrow[r, equal] & TL \arrow[r] & TW
\end{tikzcd}$$

使得第 2,第 3 行,第 2,第 3 列是三角.

引理 1[3] 设 (C,T,ε) 是预三角范畴,则(TR4)等价于(TR4-3):

(TR4-3) 如下图

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & TX \\
X' \arrow[r] & Y' & &
\end{tikzcd}$$

若第 1 行为三角,方框为 homotopy cartesian,微分为 δ,则有下列交换图

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r, "\beta"] \arrow[d, equal] & TX \arrow[d] \\
X' \arrow[r] & Y' \arrow[r, "\alpha"] & Z \arrow[r] & TX'
\end{tikzcd}$$

使得第 2 行为三角,且 $\delta = \beta\alpha$.

定理 3 设 (C,T,ε) 是预三角范畴,则(TR4-3)

等价于(TR4-4):

(TR4-4) 对任意的下列交换图且行为三角

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[r, "w"] & TX \arrow[d, equal] \\
X \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX
\end{tikzcd}$$

则存在 $g:Z \to Z'$ 使得上图可交换,且

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y' \arrow[r, "v'"] & Z'
\end{tikzcd}$$

为 homotopy cartesian,微分 $\delta = -(Tu)w'$.

证明　(必要性) 设 $Y \xrightarrow{(v,f)^T} Z \oplus Y'$ 可嵌入三角 $Y \xrightarrow{(v,f)^T} Z \oplus Y' \xrightarrow{(\varphi,-\psi)} Q \xrightarrow{\delta} TY$. 对

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[r, "w"] \arrow[d, "\phi"'] & TX \arrow[r, "-Tu"] & TY \\
Y' \arrow[r, "\psi"] & Z'
\end{tikzcd}$$

由(TR4-3) 知有下列交换图,

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[r, "w"] \arrow[d, "\phi"'] & TX \arrow[r, "-Tu"] \arrow[d, equal] & TY \arrow[d, "Tf"'] \\
Y' \arrow[r, "\psi"] & Q \arrow[r, "\psi'"] & TX \arrow[r, "\psi''"] & TY'
\end{tikzcd}$$

且 $\delta = -(Tu)\psi'$. 所以 $\psi'' = (Tf)(-Tu) = -Tu'$, $\varphi v = \psi f$, $\psi'\varphi = w$. 因此 $Y' \xrightarrow{\psi} Q \xrightarrow{\psi'} TX \xrightarrow{-Tu'} TY'$ 是三角,于是,由(TR3) 知存在 $\sigma:\theta \to Z'$ 使得下图可交换,

$$\begin{tikzcd}
Y' \arrow[r, "\psi"] \arrow[d, equal] & Q \arrow[r, "\psi'"] \arrow[d, "\sigma"'] & TX \arrow[r, "-Tu'"] \arrow[d, equal] & TY' \arrow[d, equal] \\
Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX \arrow[r, "-Tu'"] & TY'
\end{tikzcd}$$

所以 σ 是同构,且 $\sigma\psi = v'$, $\psi' = w'\sigma$. 故 $\psi = \sigma^{-1}v'$, $\delta = -(Tu)\psi' = -(Tu)w'\sigma$. 所以 $Y \xrightarrow{(v,f)^T} Z \oplus Y' \xrightarrow{(\varphi,-\sigma^{-1}v')} Q \xrightarrow{(-Tu)w'\sigma} TY$ 是三角. 又有下列交换图

$$\begin{tikzcd}
Y \arrow[r, "{(v,f)^T}"] \arrow[d, equal] & Z \oplus Y' \arrow[r, "{(\sigma\phi,-v')}"] \arrow[d, equal] & Z' \arrow[r, "-(Tu)w'"] \arrow[d, "\sigma^{-1}"'] & TY \arrow[d, equal] \\
Y \arrow[r, "{(v,f)^T}"] & Z \oplus Y' \arrow[r, "{(\phi,-\sigma^{-1}v')}"] & Q \arrow[r, "-(Tu)w'\sigma"] & TY
\end{tikzcd}$$

且 σ^{-1} 是同构,所以

$$Y \xrightarrow{(v,f)^T} Z \oplus Y' \xrightarrow{(\sigma\varphi,-v')} Z' \xrightarrow{(-Tu)w'} TY \tag{1}$$

是三角. 令 $g = \sigma\varphi$,则 $gv = \sigma\varphi v = \sigma\psi f = v'f$, $w'g = w'\sigma\varphi = \psi'\varphi = w$,于是有下列交换图

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[r, "w"] \arrow[d, "g"'] & TX \arrow[r, "-Tu"] \arrow[d, equal] & TY \arrow[d, "Tf"'] \\
Y' \arrow[r, "v'"] & Q \arrow[r, "w'"] & TX \arrow[r, "-Tu'"] & TY'
\end{tikzcd}$$

故有下列交换图

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[r, "w"] \arrow[d, "g"'] & TX \arrow[d, equal] \\
X \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & TX
\end{tikzcd}$$

且由式(1) 知

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y' \arrow[r, "v'"] & Z'
\end{tikzcd}$$

为 homotopy cartesian,微分 $\delta = -(Tu)w'$.

(充分性) 设有

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & TX \\
X' \arrow[r, "u'"] & Y'
\end{tikzcd}$$

第 1 行为三角,且

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "u'"] & Y'
\end{tikzcd}$$

为 homotopy cartesian, 微分 $\delta:Y' \to TX$. 设 $f(-T^{-1}w):T^{-1}Z \to X'$ 可嵌入三角 $T^{-1}Z \xrightarrow{f(-T^{-1}w)} X' \xrightarrow{\alpha} Q \xrightarrow{\beta} Z$,对

$$\begin{tikzcd}
T^{-1}Z \arrow[r, "-T^{-1}w"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] & Z \arrow[d, equal] \\
T^{-1}Z \arrow[r, "f(-T^{-1}w)"] & X' \arrow[r, "\alpha"] & Q \arrow[r, "\beta"] & Z
\end{tikzcd}$$

由(TR4-4) 知存在 $g':Y \to Q$ 使得上图可交换,且

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g'"] \\
X' \arrow[r, "\alpha"] & Q
\end{tikzcd}$$

为 homotopy cartesian, 微分 $\delta = w\beta$. 所以存在 $\sigma:Q \to Y'$ 使得下图可交换

$$\begin{tikzcd}
X \arrow[r, "{(u,f)^T}"] \arrow[d, equal] & Y \oplus X' \arrow[r, "{(g',-\alpha)}"] \arrow[d, equal] & Q \arrow[r, "w\beta"] & TX \arrow[d, equal] \\
X \arrow[r, "{(u,f)^T}"] & Y \oplus X' \arrow[r, "{(g,-u')}"] & Y' \arrow[r, "\delta"] & TX
\end{tikzcd}$$

所以 σ 是同构,且 $\delta\sigma = w\beta$, $\sigma(g',-\alpha) = (g,-u')$. 从而 $\sigma g' = g$, $\sigma\alpha = u'$. 所以有下列交换图

$$\begin{tikzcd}
T^{-1}Z \arrow[r, "f(-T^{-1}w)"] \arrow[d, equal] & X' \arrow[r, "\alpha"] \arrow[d, equal] & Q \arrow[r, "\beta"] \arrow[d, "\sigma"'] & Z \arrow[d, equal] \\
T^{-1}Z \arrow[r, "f(-T^{-1}w)"] & X' \arrow[r, "u'"] & Y' \arrow[r, "\beta\sigma^{-1}"] & Z
\end{tikzcd}$$

所以 $T^{-1}Z \xrightarrow{f(-T^{-1}w)} X' \xrightarrow{u'} Y' \xrightarrow{\beta\sigma^{-1}} Z$ 是三角. 因为 $\beta\sigma^{-1}g = \beta g' = v$,所以有下列交换图

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, "Tf"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "\beta\sigma^{-1}"] & Z \arrow[r, "(Tf)w"] & TX' \end{tikzcd}$$

又 $w\beta\sigma^{-1} = \delta\sigma\sigma^{-1} = \delta$,所以(TR4-3)成立.

3 定理的应用

八面体公理的等价命题有时应用起来更方便. 在文献[4]中有如下引理,利用(TR4-3)和(TR4-4)其证明可以简化.

引理 2[4] 设 C 是三角范畴,T 为 C 上自同构,如果有好三角 $X \xrightarrow{(u,f)^T} Y \oplus X' \xrightarrow{(g,-u_1)} Y' \xrightarrow{\delta} TX$,则在同构意义下存在唯一的 $Z \in C$ 使得下图可交换,且行为好三角,$\delta = -(Tu)w_1$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, "Tf"'] \\ X' \arrow[r, "u_1"] & Y' \arrow[r, "v_1"] & Z \arrow[r, "w_1"] & TX' \end{tikzcd} \quad (*)$$

反之若有如上交换图(*),则存在 $g_1: Y \to Y'$ 使得 $X \xrightarrow{(u,f)^T} Y \oplus X' \xrightarrow{(g,-u_1)} Y' \xrightarrow{-(Tu)w_1} TX$ 是好三角.

证明 设 $X \xrightarrow{u} Y$ 可嵌入好三角 $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$,(在同构意义下唯一). 对

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & TX \\ X' \arrow[r, "u_1"] & Y' & & \end{tikzcd}$$

因为

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

是 homotopy cartesian,由(TR4-3)知有如下三角交换图,且 $\delta = -(Tu)w_1$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, "Tf"'] \\ X' \arrow[r, "u_1"] & Y' \arrow[r, "v_1"] & Z \arrow[r, "w_1"] & TX' \end{tikzcd}$$

反之,若有交换图(*),对下图

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] & Z \arrow[r, "w"] \arrow[d, equal] & TX \arrow[d, "Tf"'] \\ X' \arrow[r, "u_1"] & Y' \arrow[r, "v_1"] & Z \arrow[r, "w_1"] & TX' \end{tikzcd}$$

由(TR4-4)知存在 $g_1: Y \to Y'$ 使得 $X \xrightarrow{(u,f)^T} Y \oplus X' \xrightarrow{(g,-u_1)} Z \xrightarrow{-(Tu)w_1} TX$ 是好三角.

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The Equivalent Propositions of the Octahedral Axiom of Triangulated Categories and Applications

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Abstract: A triangulated categoty is an addition category with an automorphism. It satisfies four axioms. One of which is the octahedral axiom. It plays an important role. The notion of the octahedral axiom was first introduced by Grothendieck-Verdier,inearly sixties of the last century. Its form is quite complex. And it is not convenient to use. Then studying the other equivalent propositions arouses people's interest. In this paper,the homotopy Cartesian is equivalent with the octahedral axiom if theother three axioms are satisfied are proved. Moreover,two general equivalent propositions of the octahedral axiom are given. As an application,the proof of Peng and Tan is simplified by the third equivalent proposition.

Key words: triangulated categories;the octahedral axiom;homotopy cartesian