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二元概率型算子族的收敛定理①

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摘要　利用随机向量构造了一类二元概率型算子族, 并利用量化的 Poisson 极限定律和概率分布的性质给出这类算子族的收敛阶估计.

关键词　二元概率型算子, 随机向量, 收敛阶

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线性算子序列(或算子族) 用来逼近另一类线性算子, 这是80年代以来出现在算子逼近论领域中的新课题, 概率论的方法结合逼近论的方法已经成功地用来处理此类问题[1, 3, 6], 本文的目的是对如下二元概率型算子进行这类问题的研究.

1　二元概率型算子

设 $\Omega=\{(x,y)|x,y\geq 0\}$ 为平面上的区域, $(X_{r,x},Y_{t,u})(W_{r,x},Z_{t,y})$ 和 $(R_{r,x},T_{t,y})$ 是具有参数 (r,t,x,y) 的随机向量, 其联合概率分布如下:

$$P(Z_{r,x}=k,Y_{t,y}=j)=\binom{r+k-1}{k}\binom{t+j-1}{j}\frac{x^k}{(1+x)^{r+k}}\frac{y^j}{(1+y)^{t+j}} \tag{1}$$

$$P(W_{r,x}=k,Z_{t,y}=j)=e^{-(rx+ty)}\frac{(rx)^k}{k!}\frac{(ty)^j}{j!} \tag{2}$$

随机向量 $(R_{r,x},T_{t,y})$ 的分布密度函数如下:

$$g_{rt}^{x,y}(u,v)=\begin{cases}\dfrac{r^rt^t}{x^ry^t\Gamma(r)\Gamma(t)}u^{r-1}v^{t-1}e^{-(\frac{ru}{x}+\frac{rv}{y})}, & (u,v)\in\Omega\text{ 且 }x,y\neq 0\\ 0, & \text{其余的}(u,v)\text{ 或 }x,y\text{ 之一为零}\end{cases} \tag{3}$$

这里 $r,t>0$; $x,y\geq 0$; k,j 为非负整数, 而 $\binom{r+k-1}{k}=\dfrac{\Gamma(r+k)}{k!\Gamma(r)}$

分别记

$$p_{rk}(x)=\binom{r+k-1}{k}\frac{x^k}{(1+x)^{r+k}}\text{ 和 }q_{rk}(x)=e^{-rx}\frac{(rx)^k}{k!}\qquad k=0,1,2,\cdots\cdots \tag{4}$$

对在 Ω 上定义的实函数 $f(x,y)$ 满足 $E\left|f\left(\frac{X_{rx}}{r},\frac{Y_{t,y}}{t}\right)\right|<+\infty$, 即 $f\left(\frac{X_{rx}}{r},\frac{Y_{t,y}}{t}\right)$ 的数学期望存在时, 定义如下二元概率型算子:

$$V_{r,t}(f;x,y)=Ef\left(\frac{Z_{r,x}}{r},\frac{Y_{t,y}}{t}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f\left(\frac{k}{r},\frac{j}{t}\right)P_{rk}(x)P_{tj}(y) \tag{5}$$

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同样地, 对 Ω 上的实函数 $f(x,y)$ 满足 $E\left|f\left(\frac{W_{r,x}}{r},\frac{Z_{r,y}}{t}\right)\right|<+\infty$ 和 $E\left|f\left(\frac{R_{r,x}}{r},\frac{T_{r,y}}{t}\right)\right|<+\infty$ 时, 分别定义如下二元概率型算子。

$$S_{r,t}(f;x,y)=Ef\left(\frac{X_{r,x}}{r},\frac{Y_{t,y}}{t}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f\left(\frac{k}{r},\frac{i}{t}\right)q_{rk}(x)q_{tj}(y) \tag{6}$$

和

$$G_{r,t}(f;x,y)=Ef\left(\frac{Z_{r,x}}{r}\frac{Y_{t,y}}{t}\right)=\iint_{\Omega}f(u,v)^{g}x,y_{r,t},(u,v)\,\mathrm{d}u\mathrm{d}v \tag{7}$$

正线性算子族 $V_{r,t}(f;x,y)$ $S_{r,t}(f;x,y)$ 和 $G_{r,t}(f;x,y)$ 分别称为连续参数型的二元 Lupas-Baskakov 算子, Szász-Mirakjan 算子和 Gamma 算子, 由算子族 $V_{r,t}(f;x,y)$ 出发, 进一步构造如下一类算子: 设对 Ω 上的可测的实函数 $f(x,y)$ 和 $\alpha>0$, 有

$$V_{\alpha,\alpha}\left(\left|f(\alpha u,\alpha v)\right|;\frac{x}{\alpha},\frac{y}{\alpha}\right)<+\infty \tag{8}$$

则定义

$$V_{r,t}^{(\alpha)}(f;x,y)=V_{\alpha r,\alpha t}\left(f(\alpha u,\alpha v);\frac{x}{\alpha},\frac{y}{\alpha}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f\left(\frac{k}{r},\frac{i}{t}\right)p_{\alpha r,k}\left(\frac{x}{\alpha}\right)p_{\alpha t,j}\left(\frac{y}{\alpha}\right) \tag{9}$$

2　主要定理及其证明

由直接计算得到. 对任意的 $\alpha>0$, 有

$$G_{\alpha,\alpha}(1;x,y)=1;\quad G_{\alpha r,\alpha t}(u;x,y)=x;\quad G_{\alpha,\alpha}(v;x,y)=y;$$

$$G_{\alpha,\alpha}(uv;x,y)=xy;\quad G_{\alpha r,\alpha t}((u-x)^2;x,y)=\frac{x^2}{\alpha r};G_{\alpha r,\alpha t}((v-y)^2;x,y)=\frac{y^2}{\alpha t} \tag{10}$$

引理1　对任意的 $\alpha>0$, 有

$$V_{r,t}^{(\alpha)}f=G_{\alpha r,\alpha t}(S_{r,t}f) \tag{11}$$

证　由定义注意到条件(8), 有

$$G_{\alpha,\alpha}(S_{r,t}f)=\iint_{\Omega}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f\left(\frac{k}{r},\frac{i}{t}\right)q_{rk}(u)q_{tj}(v)g^{x,y},_{\alpha,\alpha},(u,v)\,\mathrm{d}u\mathrm{d}v$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f\left(\frac{k}{r},\frac{i}{t}\right)\iint_{\Omega}q_{rk}(u)q_{tj}(v)g^{x,y},_{\alpha,\alpha},(u,v)\,\mathrm{d}u\mathrm{d}v$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f\left(\frac{k}{r},\frac{i}{t}\right)\cdot\binom{\alpha r+k-1}{k}\binom{\alpha t+j-1}{j}\frac{\left(\frac{x}{\alpha}\right)^k}{\left(1+\frac{x}{\alpha}\right)^{\alpha r+k}}\frac{\left(\frac{y}{\alpha}\right)^j}{\left(1+\frac{y}{\alpha}\right)^{\alpha+j}}$$

$$=V_{r,t}^{(\alpha)}f$$

由量化的 Poisson 极限定律[3, 4] 得到

引理2　设 $x>0$, $\tau>0$ 则有

$$\sum_{k=0}^{\infty}\left|\left(\binom{\tau+k-1}{k}\right)\left(\frac{\tau}{\tau+x}\right)^{\tau}\left(\frac{x}{\tau+x}\right)^{k}-e^{-x}\frac{x^k}{k!}\right|\le\min\left\{2\frac{x^2}{\tau},2\frac{x}{\tau}\right\} \tag{12}$$

现在给出如下估计

定理1　设 $\alpha>0$, $f(x,y)$ 在 Ω 上有界, 则有

$$\left|V_{r,t}^{(\alpha)}(f,x,y)-S_{r,t}(f,x,y)\right|\le\min\left\{\frac{2}{\alpha}(x+y),\ \frac{2}{\alpha}(rx^2+\tau y^2)\right\}\|f\|_{\Omega} \tag{13}$$

证　注意到 $p_{rk}(x)q_{\alpha t,j}\left(\frac{y}{\alpha}\right)$ 为边缘概率分布. 并由引理2得

$$
\begin{aligned}
&\left| V_{r,t}^{(\alpha)}(f;x,y) - S_{r,t}(f;x,y)\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f\left(\frac{k}{r},\frac{i}{t}\right) q_{\alpha r,k}\left(\frac{x}{\alpha}\right) q_{\alpha t,j}\left(\frac{y}{\alpha}\right) - \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f\left(\frac{k}{r},\frac{i}{t}\right) p_{rk}(x) p_{tj}(y)\right| \\
&\le \|f\|_{\Omega} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left\{ q_{\alpha t,j}\left(\frac{y}{\alpha}\right)\left| q_{\alpha r,k}\left(\frac{x}{\alpha}\right) - p_{rk}(x)\right| + p_{rk}(x)\left| p^{\alpha t,j}\left(\frac{y}{\alpha}\right) - p_{tj}(y)\right|\right\} \\
&\le \|f\|_{\Omega}\left(\sum_{k=0}^{\infty}\left| q_{\alpha r,k}\left(\frac{x}{\alpha}\right) - p_{rk}(x)\right| + \sum_{k=0}^{\infty}\left|\left| q_{\alpha t,j}\left(\frac{y}{\alpha}\right) - p_{tj}(y)\right|\right. \\
&\le \|f\|_{\Omega} \cdot \min\left\{ \frac{2}{\alpha}(x+y), \frac{2}{\alpha}(rx^2+ty^2)\right\}
\end{aligned}
$$

推论　若 f 在 Ω 上有界, 则对任意的 $(x,y)\in\Omega$, 有

$$\lim_{\alpha\to+\infty} V_{r,t}^{(\alpha)}(f;x,y) = S_{rt}(f;x,y) \qquad (14)$$

并且在 Ω 的任一有界子区域上, 收敛是一致的。

对于 $f\in C(\Omega)$, 陈文忠等人在[5]中定义差分和光滑模如下:

$$\Delta_{he_1} f(x,y) = f(x+h,y) - f(x,y) \qquad (15)$$

$$\Delta_{he_1}^2 f(x,y) = f(x+h,y) - 2f(x,y) + f(x-h,y) \qquad (16)$$

这里　$x\pm h\in(0,+\infty)$, $y\in(0,\infty)$

$$\Delta_{he_1}\Delta_{he_2} f(x,y) = f\left(x+\frac{h}{2},y+\frac{l}{2}\right) - f\left(x-\frac{h}{2},y+\frac{l}{2}\right) + f\left(x-\frac{h}{2},y-\frac{l}{2}\right) \qquad (17)$$

这里　$x\pm\frac{h}{2}\in(0,+\infty)$, $y\pm\frac{l}{2}\in(0,+\infty)$

$$\omega_1(f,h) = \sup\{|\Delta_{\delta e_1} f(x,y)| \mid (x+\delta,y)\in(0,+\infty)\times(0,+\infty), 0<\delta\le h\} \qquad (18)$$

$$\omega_{2e_1}(f,h) = \sup\{|\Delta_{\delta e_1}^2 f(x,y)| \mid (x\pm\delta,y)\in(0,+\infty)\times(0,+\infty), 0<\delta\le h\} \qquad (19)$$

$$\omega_{e_1e_2}(f,h,l) = \sup\left\{|\Delta_{\delta_1 e_1}\Delta_{\delta_2 e_2} f(x,y)| \,\Big|\, \left(x\pm\frac{\delta_1}{2}, y\pm\frac{\delta_1}{2}\right)\in(0,+\infty)\times(0,+\infty), 0<\delta_1\le h, 0<\delta_2\le l\right\} \qquad (20)$$

这里 $e_1=(1,0)$, $e_2=(0,1)$

$\omega_2(f,h)$ 和 $\omega_{2e_2}(f,h)$ 可类似定义

利用上面的光滑模, 有如下估计式:

定理2　对 $f\in C(\Omega)$, 有

① $$\left|V_{r,t}^{(\alpha)}(f;x,y) - S_{r,t}(f;x,y)\right| \le \frac{1}{\sqrt{\alpha}}\left[\sqrt{r}\,x\,\omega_1\left(f,\frac{1}{r}\right) + \sqrt{t}\,y\,\omega_2\left(f,\frac{1}{t}\right)\right] \qquad (21)$$

② $$\left|V_{r,t}^{(\alpha)}(f;x,y) - S_{r,t}(f;x,y)\right|$$

$$\le \frac{1}{2\alpha} r x^2 \omega_{2e_1}\left(f,\frac{1}{r}\right) + \frac{1}{\alpha}\sqrt{rt}\,xy\,\omega_{e_1e_2}\left(f,\frac{1}{r},\frac{1}{t}\right) + \frac{1}{2\alpha} t y^2 \omega_{2e_1}\left(f,\frac{1}{t}\right) \qquad (22)$$

证　先证②记 $S_{r,t}f = f_{r,t}$, 由 Taylor 展开式得

$$
\begin{aligned}
f_{r,t}(u,v) &= f_{r,t}(x,y) + (u-x)\frac{\partial f_{r,t}}{\partial x}(x,y) + (v-y)\frac{\partial f_{r,t}}{\partial y}(x,y) \\
&+ \frac{1}{2}\left[(u-x)^2\frac{\partial^2 f_{r,t}}{\partial x^2}(\theta_1,\theta_2) + 2(u-x)(v-y)\frac{\partial^2 f_{r,t}}{\partial x\partial y}(\theta_1,\theta_2) + (v-y)^2\frac{\partial^2 f_{r,t}}{\partial y^2}(\theta_1,\theta_2)\right]
\end{aligned}
$$

这里 θ_1 是介于 u 与 x 之间的一个数, θ_2 是介于 v 与 y 之间的一个数. 但是

$$\left|\frac{\partial^2 f_{r,t}}{\partial x^2}(\theta_1,\theta_2)\right|$$

$$= \left| r^2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left[f\left(\frac{k+2}{r}, \frac{j}{t}\right) - 2f\left(\frac{k+1}{r}, \frac{j}{t}\right) + f\left(\frac{k}{r}, \frac{j}{t}\right) \right] e^{-(r\theta_1 + t\theta_2)} \cdot \frac{(r\theta_1)^k}{k!} \frac{(t\theta_2)^j}{j!} \right| \le r^2 \omega_{2e_1}\left(f, \frac{1}{r}\right)$$

同理可得:

$$\left| \frac{\partial^2 f_{r,t}}{\partial x \partial y}(\theta_1, \theta_2) \right| \le rt\omega_{e_1 e_2}\left(f, \frac{1}{r}, \frac{1}{t}\right); \quad \left| \frac{\partial^2 f_{r,t}}{\partial y^2}(\theta_1, \theta_2) \right| \le t^2 \omega_{2e_2}\left(f, \frac{1}{t}\right)$$

于是由(10)和引理1, 并利用 Schwarz 不等式得

$$\left| V_{r,t}^{(\alpha)}(f; x, y) - S_{r,t}(f; x, y) \right| \le \frac{1}{2} r^2 \omega_{2e_1}\left(f, \frac{1}{r}\right) G_{\alpha,\alpha}((u-x)^2; x, y) +$$

$$T rt\omega_{2e_1e_2}\left(f, \frac{1}{r}, \frac{1}{t}\right) G_{\alpha,\alpha}(|u-x||v-y|; x, y) + \frac{1}{2} t^2 \omega_{2e_2}\left(f, \frac{1}{t}\right) G_{\alpha,\alpha}((v-y)^2; x, y)$$

$$\le \frac{1}{2\alpha} rx^2 \omega_{2e_1}\left(f, \frac{1}{r}\right) + \frac{1}{\alpha} \overline{rt} xy \omega_{2e_1e_2}\left(f, \frac{1}{r}, \frac{1}{t}\right) + \frac{1}{2\alpha} ty^2 \omega_{2e_2}\left(f, \frac{1}{t}\right)$$

①可类似证明

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Convergence Theorems of Twodimensional Probability Type Operators

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Abstract A class of twodimensional probability type operators was constructed by means of twodimensional probability distributions. Furthermore, by using continuous type Poisson limit law and properties of probability distributions, the convergence rates of these probability type operators were given.

Key words Twodimensional probability type operators, Random vector, Rate of Comvergence