Auctioning the Digital Dividend: a Model for Spectrum Auctions

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Abstract

This paper models a spectrum auction as a multi-unit auction where participants use the goods purchased to participate in a constrained, multi-good downstream market. We use dynamic programming techniques to solve for the optimal bidding strategy for firms in a clock auction. Firms often value constraining competitor market power highly, and inefficient firms will often bid aggressively to minimise competition. Regulators concerned with revenue maximisation have strong incentives to encourage this behaviour, capping more efficient firms or capping entrants to the market. In contrast, social welfare concerns suggest that allocating spectrum may be more efficient than using an auction.

Keywords: Clock Auction, Spectrum Auction, Telecommunications Market, Equilibrium Bidding, Capacity Constraints

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1 Introduction

For most modern economies, the "Digital Dividend" represents a substantial technological and financial windfall. Developments in television broadcasting allow the transmission of digital video and sound which require only one-sixth the bandwidth of inferior quality analogue transmissions. Replacing analogue transmissions with digital transmissions therefore frees up a substantial quantity of spectrum frequencies, which can be easily employed to transmit wireless data, as used by mobile phones, laptop computers and other devices.

This unallocated new spectrum presents a good opportunity for an economy in many ways. First, wireless telecommunications companies can expand their services. Firms who have access to the new spectrum can effectively provide superior data services relative to firms who do not. Second, consumers enjoy a wider range of services potentially delivered in a more timely manner. Third, the government benefits from the unallocated spectrum in two ways. On the one hand, the government, as an auctioneer, plans to maximize revenue from the auction. On the other hand, the government, as a regulator, can affect the degree of competition in the telecommunications market to increase total surplus. A common policy tool adopted by governments to achieve social optimum is to set caps on firms to limit their winnings. In this way, the regulator can prevent one firm from winning all units and becoming a monopolist.

Spectrum auctions are potentially an efficient way to allocate the new spectrum across firms. A spectrum auction is an example of a *multi-unit auction*. The auctioneer is selling a collection of relatively homogeneous goods to multiple firms.[1] Spectrum auctions do not represent an entirely new phenomenon in most countries. The United States began its auctions for spectrum licenses during the 1990s, and has since assessed their efficiency; see Cramton (1997), Cramton (1998), Kwerel and Rosston (2000), and Bush (2010). Meanwhile, the GSM (second generation mobile telecommunication) and UMTS (third generation) auctions in Europe from 1999 to 2001 attracted a lot of attention from the public for its interesting outcomes; see van Damme (2002), Klemperer (2005), and Grimm, Riedel, and Wolfstetter (2003). Economists have also been surprised by the huge revenues the British government realised from the sale of its 3G telecom licenses (Binmore and Klemperer (2002)).

Spectrum auctions can result in participants winning multiple units of spectrum, which in turn gives them market power. This is especially true if the number of firms is low. Therefore, our aim in this paper is to examine the equilibrium properties of a Digital Dividend auction, in which firms compete for market power in the telecommunications market. Our

[1] In reality, spectrum frequencies are not necessarily truly homogeneous, since there are technology-dependent synergies to having access to adjacent frequencies, while regional standards, technological limitations, and device manufacturers' decisions may make certain segments of a given band more or less desirable.

model consists of a downstream market, in which firms play a Cournot game with capacity constraints, similar to Laye and Laye (2008). In particular, firms produce two goods: low and high data use plans and are constrained in their ability to produce data plans by the amount of spectrum they have available.

Although multi-unit auctions can be set up in many ways, we consider a simultaneous clock auction to allocate the new spectrum; see Ausubel (2004). A clock auction consists of a sequence of rounds. In each round, auction participants report their demand for the units for sale. If the demand exceeds supply, the auctioneer increases the price, and the auction continues. Otherwise, the auction closes, and a subsequent proxy phase is used to allocate specific units of the good. An important feature of the clock auction is that auction participants must bid in a consistent fashion: no participant is allowed to raise demand as prices increase. By winning units in the clock auction, firms increase their production capacities for both products.

We solve the auction problem using a dynamic programming technique, which allows firms to bid strategically. We find many instances of firms following mixed strategies in their bidding, and thus our model frequently generates distributions of allocations, profits, levels of social welfare, and revenue for the auctioneer. Our technique also easily allows us to explore the way in which an auctioneer might impose caps on how much spectrum individual participants can win, a common tool used by regulators in spectrum auctions. Our findings show that regulators concerned solely with total surplus may find allocating spectrum by government policy to be more efficient than using a spectrum auction. In contrast, regulators concerned with revenue maximisation find that a capping structure which favours less efficient outcomes is an effective allocation mechanism.

We believe our work adds to the literature in several ways. First, in simultaneous multi-unit auctions, bidders are often assumed to have non-increasing marginal valuation (NIMV) for the goods in question. Hortaçsu and McAdams (2010) uses this feature for model tractability and nonparametric estimation. Unfortunately, this assumption is not necessarily valid in our case, since marginal value of a unit of spectrum may increase with a bidder's market power. In our model, units sold in the auction affect the production capacity. Therefore, winning more units not only increases a firm's production capacity, but also limits its competitors' capacities, which gives it market power. In such a case, marginal value of the spectrum increases for each additional unit of the spectrum. Since we cannot directly assume NIMV for firms participating the auction, we restrict our attention to a complete information case for model tractability. Second, potential market power in the downstream market may be concerning for a benevolent government who also happens to be the auctioneer. Das Varma (2003) investigates the interaction between market power in a downstream market and the equilibrium in an upstream auction. Under the independent private value paradigm, Das Varma considers a first-price sealed bid auction for a single unit which promotes process innovation. Jehiel and Moldovanu (2000) considers the effects of an auction of a single item, which can be seen as a patent, on the competition in the downstream market. In their model, Jehiel and Moldovanu assumes that profit is private information and represents a firm's type. They investigate the effects of the second-price sealed-bid auction (with a reserve price and an entry fee) on the downstream market. In our model, firms in the downstream market compete in a Cournot oligopoly market where they solve for a constrained profit maximization. The firms have capacity constraints which limit how much they can produce in each product market. This capacity-constraint Cournot model with linear product demand is previously analyzed by Laye and Laye (2008). However, in their case, no upstream activity takes place. Third, we consider a simultaneous multi-unit clock auction for our upstream market. In contrast, the previously mentioned papers with downstream markets only consider single-unit auctions. Finally, we model the government as both an auctioneer and a regulator, and examine how the government's objective function affects the optimal cap structure in the auction. There may be a trade-off between the total surplus and the auction revenue. In fact, even if an auction may be efficient, it may result in a downstream market which may be quite inefficient, provided that a firm wins most of the units. In this case, the government may act as a regulator and aim to maximize some combination of total surplus in the downstream market as well as the auction revenue. Dana and Spier (1994) also focuses on the government's problem to choose who produces in a downstream market. In their model, Dana and Spier assume a risk-neutral government determines market structure by giving the right to produce to a firm or to itself. We consider a more general objective function, and allow the government to be risk-neutral or risk-averse. Then, we compute the optimal caps as well as the mean and the covariance matrix of the total surplus and revenue.

The layout of the remainder of this paper is as follows. Section 2 outlines our model for the downstream market and the auction itself. This section also discusses our solution method. Section 3 presents our results for uncapped auctions, while Section 4 explores the optimal capping decision for a regulator organising a spectrum auction. Lastly, section 5 concludes.

2 Model

2.1 Downstream Market

We begin our theoretical model by describing the telecommunications market, which we will refer to as the downstream market throughout this paper. Suppose that there are M firms operating in the downstream market. The market is composed of two products: high and low data use plans (henceforth, products). Let q_{hi} and q_{li} denote firm i's output for the high and low products. Cost of production may differ across firms as well as products. Thus, let $C_{hi}(q)$ and $C_{li}(q)$ represent firm i's total cost of producing q units of the respective products.

Inverse demand for the two products are described as follows:

$$\text{Low Product: } P_l = P_l\left(\mathbf{q_l}; \boldsymbol{\beta}_l\right) \tag{1}$$
$$\text{where } \mathbf{q_l} = (q_{l1}, q_{l2}, \ldots, q_{lM})$$
$$\text{High Product: } P_h = P_h\left(\mathbf{q_h}; \boldsymbol{\beta}_h\right) \tag{2}$$
$$\text{where } \mathbf{q_h} = (q_{h1}, q_{h2}, \ldots, q_{hM}).$$

In equations (1) and (2), $\mathbf{q_l}$ and $\mathbf{q_h}$ are the vector of outputs for each product by all firms, while $\boldsymbol{\beta}_l$ and $\boldsymbol{\beta}_h$ are the corresponding demand parameters. Note that if all firms' outputs for a product are perfect substitutes, then one can simplify the inverse demand functions so that price depends only on the aggregate production.[2]

We assume that firms are endowed with an initial allocation of legacy spectrum, which we refer to as old spectrum, and denote it by $\mathbf{B_l}$. Meanwhile, depending on the current auction outcome, firms may win more of the new spectrum:

$$\text{Old Spectrum: } \mathbf{B_l} = (B_{l1}, B_{l2}, \ldots, B_{lM}) \tag{3}$$
$$\text{New Spectrum: } \mathbf{B_a} = (B_{a1}, B_{a2}, \ldots, B_{aM}). \tag{4}$$

[2] Alternatively, one can argue that the product prices may depend on both vectors of outputs, as the two products may be complements or substitutes. One can incorporate this relationship into the demand functions.

Using the old and new spectrum, firm i faces the following capacity constraints:

$$q_{hi} \leq \theta_{ai}\ B_{ai} \tag{5a}$$
$$q_{hi} + q_{li} \leq \theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} \tag{5b}$$
$$q_{hi} \geq 0 \tag{5c}$$
$$q_{li} \geq 0 \tag{5d}$$

where firm i can use the new spectrum to increase capacity for the high product, the low product, or both. In contrast, the legacy spectrum affects only the production of the low product. In equations (5a) and (5b), θ_{ai} and θ_{li} represent the marginal increase in capacity for the old and new spectrum, respectively. Finally, inequalities (5c) and (5d) are to ensure non-negative production for the high product.

We assume that the initial endowments by firms are already paid for or can be regarded as sunk costs. Given the spectrum allocations $\{\mathbf{B_l}, \mathbf{B_a}\}$, one can write down firm i's profit in the downstream market as follows:

$$\begin{aligned}\Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) =& P_h\left(\mathbf{q_h}; \boldsymbol{\beta}_h\right)\ q_{hi} - C_{hi}(q_{hi}) + \\ & P_l\left(\mathbf{q_l}; \boldsymbol{\beta}_l\right)\ q_{li} - C_{li}(q_{li})\end{aligned} \tag{6a}$$

$$\begin{aligned}\Pi_i(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}, P_a) =& \Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) - \\ & C_a(B_{ai}, P_a)\end{aligned} \tag{6b}$$

where $\{\mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}\}$ represents the vector of residual supply, and $\mathbf{B_{a,-i}}$ denotes the new spectrum allocated for firm i. The first equation in (6a) is the firm's profit in the downstream market, whereas the firm's profit net of the cost of the new spectrum is given in equation (6b). The right-hand side in equation (6a) is the sum of the profits from the high and low products, while the last term in equation (6b) is the cost of the new spectrum, which depends on the auction price P_a.

Finally, the assumption below describes the information structure in the model.

Assumption. *Assume that there is complete information in the market. In other words, all the information regarding the cost of production, the spectrum allocations, and the outputs by each firm, as well as the demand parameters are public information.*

Assuming complete information, firms can first solve for their profits in the downstream market, given the spectrum allocations. Then, they can enter the auction for the new spectrum and use the information on their marginal valuations to formulate their bidding strategies.

2.2 Clock Auction for the New Spectrum

We adopt a clock auction in this paper. Suppose that there are N items to be sold. The auctioneer's goal is to find a price such that there is no excess demand for the goods. We assume that the goods are homogeneous so firms' bids consist of a quantity of spectrum at the current auction price, rather than binary bids for individual units of spectrum. Let P_a represents the current price in the auction. A clock auction works in the following way:

1. The auctioneer starts P_a at zero.

2. At the current price P_a, firms submit their bids:

$$\mathbf{B_a}(P_a) = (B_{a1}(P_a), \ldots, B_{aM}(P_a)) \, .$$

   Firms may respond to each others' bids at the ongoing price.

3. If there is excess demand at the current price, then the auctioneer increases the price:

$$\sum_{i=1}^{M} B_{ai}(P_a) > N \ \Rightarrow P_a \text{ increases}$$

4. The previous step is repeated until the auctioneer increases the price to P_a^* such that:

$$P_a^* = \inf \left\{ P_a \ : \ \sum_{i=1}^{M} B_{ai}(P_a) \leq N \right\} .$$

When the auction ends, the equilibrium vector of spectrum allocation is $\mathbf{B_a^*}$. Firm i pays the auction price P_a^* for each unit it wins. Thus, the total cost of the spectrum to firm i is $P_a^* \ B_{ai}^*$. The total revenue collected by the auctioneer equals $P_a^* \ \sum_{i=1}^{M} B_{ai}^*$, which may be less than $P_a^* \ N$, if the auction ends with an excess supply.

2.3 Profit Maximization in Downstream Market

Given the spectrum allocations, we assume that the firms compete in a Cournot game by deciding how much to produce for both high and low products. Since firm outputs for the high and low products are constrained by the capacity constraints described in equations (5a), (5b), (5c) and (5d), the distribution of the new and old spectrum across firms can result in market power. Before we write down the profit maximization problem, we will first describe the feasible sets for firm i:

$$\mathcal{A}_{1i} = \{(q_{hi}, q_{li}) \ : \ (q_{hi}, q_{li}) \text{ satisfy capacity constraints in (5) given } \mathbf{B_a}\} \tag{7a}$$

$$\mathcal{A}_{2i} = \{(q_{hi}, q_{li}, B_{ai}) \ : \ (q_{hi}, q_{li}, B_{ai}) \text{ satisfy capacity constraints in (5)}\} \tag{7b}$$

The difference between the two feasible sets is that in equation (7a), the feasible set is for output levels, conditional on the new spectrum allocation. In equation (7b), the feasible set is defined for both output levels and the demand for the new spectrum. Using the feasible sets defined above, firm i's profit maximization problem is as follows:

$$\max_{<(q_{hi}, q_{li}) \in \mathcal{A}_{1i}>} \Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) \tag{$P1_i$}$$

where firm i chooses the outputs (q_{hi}, q_{li}) from the feasible set $\mathcal{A}_1$, conditional on the spectrum allocations. Therefore, we are interested in firms' production decisions assuming $\mathbf{B_a}$ is the result of the auction. In this way, we can derive the payoffs for firms and derive the marginal value of an additional unit of spectrum for each firm, which we then use to solve for the auction equilibrium. We first define the equilibrium in the downstream market.

Definition 2.1. A Nash Equilibrium in pure strategies in the Cournot game for the downstream market is such that given $(\mathbf{B_l}, \mathbf{B_a})$:

$$\{q_{hi}^*, q_{li}^*\}_{i=1}^M \text{ solves } (P1_i); \forall\ i = 1, \ldots, M.$$

In other words, given the equilibrium behavior of all other firms $\left(\mathbf{q^*_{h,-i}}, \mathbf{q^*_{l,-i}}\right)$, firm i does not find it profitable to change its production decision. Furthermore, if we assume that demand and cost functions are differentiable, then we can solve for the Nash equilibrium using the following complementarity problem from $(P1_i)$:

$$\begin{aligned}
\frac{\partial \Pi_i^D(q_{hi}, q_{li}, \cdot)}{\partial q_{hi}} + \eta_i - (\lambda_i + \mu_i) \geq 0 \perp q_{hi} \geq 0; \forall i = 1, \ldots, M \\
\frac{\partial \Pi_i^D(q_{hi}, q_{li}, \cdot)}{\partial q_{li}} - \lambda_i \geq 0 \perp q_{li} \geq 0; \forall i = 1, \ldots, M \\
\theta_{ai}\ B_{ai} - q_{hi} \geq 0 \perp \mu_i \geq 0; \forall i = 1, \ldots, M \\
\theta_{ai}\ B_{ai} + \theta_{li}\ B_{li} - (q_{hi} + q_{li}) \geq 0 \perp \lambda_i \geq 0; \forall i = 1, \ldots, M \\
q_{hi} \geq 0 \perp \eta_i \geq 0; \forall i = 1, \ldots, M \\
q_{li} \geq 0 \perp \xi_i \geq 0; \forall i = 1, \ldots, M
\end{aligned} \tag{CP1}$$

where $\{\mu_i, \lambda_i, \eta_i, \xi_i\}$ are the Lagrange multipliers on the capacity constraints given in equations (5). The complementarity problem (CP1) is a square problem with $6 \times M$ equations and $6 \times M$ unknowns $\{q_{hi}, q_{li}, \mu_i, \lambda_i, \eta_i, \xi_i\}_{i=1}^{M}$. However, the solution may not be unique; see Monderer and Shapley (1996) for a discussion of unique equilibrium in potential games. For the rest of the paper, we assume that the demand for both high and low products are linear, which ensures that the equilibrium for the capacity-constraint Cournot model in the downstream market is unique; see Laye and Laye (2008). We described the solution to the complementary problem in the appendix section A.

2.4 Solving the Auction: Non-Rival Bidding

Considerable intuition can be gained regarding the outcome of the spectrum auction by considering a simplified case. First, we relax the assumption of non-increasing bids, so there is no history dependence in the game. Second, we assume that there are N units available to *each firm*, so firms cannot be forced out of the market. These assumptions reduce the problem to a static framework so that we can solve the Cournot problem with either continuous or discrete bidding at any given auction price.

Considering first the case of continuous bidding, we modify the problem given in $(P1_i)$:

$$\max_{<(q_{hi}, q_{li}, B_{ai}) \in \mathcal{A}_{2i}>} \Pi_i^D(q_{hi}, q_{li}, B_{ai}, \mathbf{q_{h,-i}}, \mathbf{q_{l,-i}}, \mathbf{B_{a,-i}}, \mathbf{B_l}) \qquad (P2_i)$$

where firm i chooses for the capacity for the high product B_{ai} along with how much to produce for both products from the feasible set $\mathcal{A}_{2i}$. Assuming differentiability of the demand and cost functions, we can rewrite the complementarity problem in (CP1) as follows:

$$\text{(CP1) and } \theta_{ai} \ (\lambda_i + \mu_i) - P_a \geq 0 \perp B_{ai} \geq 0; \forall i = 1, \ldots, M \qquad \text{(CP3)}$$

where the extra equation in the complementarity problem (CP3) is because the new spectrum allocation is also a control variable. The complementarity problem (CP3) is a square problem with $6 \times M$ equations and $6 \times M$ unknowns $\{q_{hi}, q_{li}, B_{ai}, \mu_i, \lambda_i, \eta_i\}_{i=1}^{M}$. Assuming linear demand for both high- and low-product markets, the solution is unique for a given auction price. In this way, we can analyze the individual demand by firms as well as the aggregate demand for the new spectrum.

To consider discrete bidding, the firm's problem $(P2_i)$ is modified by constraining the firm's choice set to consist of only integer values for B_{ai}. In this case, Pure Strategy Nash Equilibria for the Cournot problem can be found by using a grid search over all possible demands for each firm. In this case, there can be multiple equilibria for some prices (see section 3.1).

2.5 Discussion

Before discussing rival bidding in section 2.6, we discuss three important features of a spectrum auction, which highlight the shortcomings of the non-rival analysis. Specifically, we discuss the effects of market power in the downstream market, spectrum as a scarce resource, and the non-increasing bid structure.

Degree of Competition in the Downstream Market Even though firms are endowed with some old spectrum, which they can use to produce the low-use product, only the new spectrum can be employed to provide production capacity for the new product, as modeled in equation 5. Therefore, unless a firm wins new spectrum at the auction, it cannot produce in the new-product market. Consequently, the auction outcome will affect the degree of competition in the downstream market. For example, suppose a single firm wins all the units in the auction, so the other firms are forced out of the new-product market. In this case, the firm with all the spectrum can operate as a monopolist in the market. This situation could occur if one of the firms is substantially more efficient in production, so that it faces much lower production costs. An alternative situation is one in which all firms are relatively similar in terms of production costs. In this case, all firms are likely to win at least some portion of the new spectrum, which makes the downstream market an oligopoly.

Potential market power affects marginal value of the new spectrum. When there is no market power, the standard assumption is that an additional unit of spectrum brings less and less profit to a firm. Therefore, it faces diminishing marginal valuation for spectrum. This can be justified by a concave profit as a function of the spectrum won. However, in our case, this is not necessarily true. In fact, the profit function may not be concave in certain regions. As a graphical illustration, we depict a firm's profit as a function of the spectrum units in a two firm case in figure 1. In this example, the firm has some old spectrum, so it can make positive profits despite losing all units at the auction. The different curves in figure 1 represents the firm's profits when the auction price increases. Keeping the auction winnings the same, a higher auction price will lower firms' profit. The circles on the curves represent the optima of the profit functions at given auction prices, while the maximum spectrum units available is 9. In figure 1, winning spectrum units has two distinct benefits to the firm. First, the more units won, the higher the capacity to produce the new product; therefore, profit goes up. However, this marginal increase in profit eventually declines to zero. Second, winning more units limits the competitor's capacity to produce the new product. In other words, the more units won by a firm, the greater the market power held by this firm. Due to these two effects, the profit function is not concave after a certain level. While the first factor dominates for the first few units won, the second effect kicks in for the later units won.

As expected, when the auction price is low, the firm aims to win all units to be a monopolist in the market. However, when the auction price goes up to a certain threshold, this is no longer profitable. Therefore, the firm agrees to share the market with other firms.

Spectrum as a Scarce Resource The number of available units in the spectrum auction is fixed at N units. This is important as it makes spectrum a scarce resource. In other words, the auctioneer (in this model, the government) has a resource constraint. This resource constraint introduces a dynamic structure to our model in the following way: at any auction price P_a, firms aim to maximize their own profits. If firms' bids lead to excess demand, the auction continues for at least another round, in which the auction price goes up. Since the firms submit their bids simultaneously, each bidding decision of the auction is essentially a Cournot game, where firms evaluate their own profits in both markets and decide if they would like to bid more aggressively to constrain the other firms'production in the new product market, or to bid what they need to produce optimally. As a result, they may engage in mixed strategies, where they may assign a positive probability to go another round and to finish the auction at the ongoing price, depending the auction price.

Non-Increasing Bids In a clock auction, bidders are often not allowed to increase their bid as the auction progresses. Given the dynamic setup for the auction, we should update the action set for each firm depending on what they bid at the end of the previous round. This introduces history dependence, so firms' latest bids from the last round are included in the set of state variables of the dynamic problem.

2.6 Solving the Auction: Rival Bidding

We now demonstrate how to solve for the equilibrium in both markets, incorporating the rival nature of the spectrum and the fact that firms cannot increase their bids, which makes the dynamics of the auction important. We assume that the firms can only submit their bids in discrete amounts. We list the important rules in discrete bidding below:

1. In each round, firms submit their bids in discrete units:

$$\mathbf{B_a} = (B_{a1}, \ldots, B_{aM}) \in \{0, 1, 2, 3, \ldots, N\}^M .$$

2. At the end of each round of activity, if demand for units of spectrum exceeds the supply of spectrum, the price of the units is increased by an increment ΔP.

3. Within each round, firms must decide whether to lower their reported demand or not. If multiple firms wish to lower their demand simultaneously, only one firm will succeed, and firms are considered equally likely to succeed. We also allow sequential dropping; i.e., if a firm drops demand by one unit, then the firm has priority to drop its demand further at the same price. Sequential dropping of demand becomes important since in the event of demand falling below supply, firms who have lowered demand may be required to purchase units of the good, if the terms of the auction stipulate that there can be no excess supply.

4. A round of bidding only ends when all firms have chosen not to act. A firm who does not initially succeed in lowering its demand may try again prior to the auction price increasing.

Given our solution to the downstream market and profit structure given in equations (6), we can calculate the payoffs that any firm receives at the end of the auction. Suppose that the vector $\mathbf{B_a}$ is the firms' winnings at the end of the auction. Then, firm i's profit net of spectrum costs equals:

$$\begin{aligned}\bar{V}_i(\mathbf{B_a}, P_a) &= \Pi_i(q^*_{hi}, q^*_{li}, B_{ai}, \mathbf{q}^*_{\mathbf{h},-\mathbf{i}}, \mathbf{q}^*_{\mathbf{l},-\mathbf{i}}, \mathbf{B}_{\mathbf{a},-\mathbf{i}}, \mathbf{B_l}, P_a) \\ &= \Pi^D_i(q^*_{hi}, q^*_{li}, B_{ai}, \mathbf{q}^*_{\mathbf{h},-\mathbf{i}}, \mathbf{q}^*_{\mathbf{l},-\mathbf{i}}, \mathbf{B}_{\mathbf{a},-\mathbf{i}}, \mathbf{B_l}) - P_a\ B_{ai}\end{aligned} \tag{8}$$

where the first term on the equation is the downstream profits, while the second term is the cost of the spectrum items to firm i. We will refer to the value function in equation (8) as the *terminal value* for firm i.

With this structure in mind, we solve the auction as a dynamic programming problem. We begin by considering a high level of price, and presume that at this stage, all firms would wish to exit the market. For the case of no requirement for all assets to be purchased, this would result in all firms purchasing zero units of the new spectrum (and earning payoffs based upon their profits due to producing using their existing spectrum holdings). For the case where all units must be held, we assume that all firms with outstanding demand attempting to drop their demand. The market will thus clear at this stage. In addition, if we consider the auction to allow for no excess supply, at each time step, we impose an additional boundary condition that the auction will immediately clear if $\sum_{i=1}^{M} B_{ai} = N$.

Solving the auction consists of calculating two sets of numbers. The first is the probability that each firm drops its demand given a particular combination of competitors' demands, its own demand, and the current price. The second is the expected payoff for a firm as a function of the current set of firms' bids, the probability of dropping demand, and the current price:

$$\{V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}), \pi_i(\mathbf{B_a}, P_a)\}_{i=1}^{M}.$$

To calculate V_i and π_i, we proceed by backward induction, working back from the terminal (high) price of the auction. For each price level, we work through the different combinations of $\mathbf{B_a}$ sequentially, beginning with cases where $\sum_{i=1}^{M} B_{ai} = 1$, before proceeding to cases where $\sum_{i=1}^{M} B_{ai} = 2$, etc. First we define the *continuation value*, which is attained if no firm reduces demand at the given price:

$$\widehat{V}_i(\mathbf{B_a}, P_a) = \begin{cases} V_i(\mathbf{B_a}, P_a + \Delta P) & \text{if } \sum_{i=1}^{M} B_{ai} > N \\ \bar{V}_i(\mathbf{B_a}, P_a) & \text{if } \sum_{i=1}^{M} B_{ai} \leq N. \end{cases} \tag{9}$$

In equation (9), the continuation value equals the terminal value, if there is no excess demand at the current price, in which case the auction ends. Otherwise, the auction continues with a higher price. Before we define the value function for firm i during the auction, we find it important to introduce one more notation: the change in firm i's expected payoff due to firm j dropping its demand by one unit $\Delta V_{ij}(\mathbf{B_a}, P_a)$ as:

$$\Delta V_{ij}(B_{aj}, \mathbf{B_{a,-j}}, P_a) = V_i(B_{aj} - 1, \mathbf{B_{a,-j}}, P_a) - \hat{V}_i(\mathbf{B_a}, P_a) \tag{10}$$

where $\Delta V_{ij}(\cdot)$ will only be well-defined for cases where $B_{aj} \geq 1$. Note that any reduction in demand results in the auction moving to a new level of demand, and that price remains at the same level, which allows further reductions in demand to take place.

Suppose that each firm chooses to drop its demand with probability π_i. Then firm i's expected payoff is given by:

$$V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = 0) = \widehat{V}_i(\mathbf{B_a}, P_a) + \sum_{\substack{\delta_1, \ldots, \delta_M \in \{0,1\}^M \\ \delta_1 + \ldots + \delta_M \neq 0}} \left\{ \left[\prod_{k=1}^{M} \pi_k^{\delta_k} (1 - \pi_k)^{1-\delta_k} \right] \left[\frac{\sum_{j=1}^{M} \Delta V_{ij}(\mathbf{B_a}, P_a)\ \delta_j}{\sum_{j'=1}^{M} \delta'_j} \right] \right\} \tag{11}$$

where ϕ indicates which firm has dropped the last time. Therefore, the case that δ_{-1} equals 0 implies that no firm has priority in lowering demand, so each firm which aims to drop demand is equally likely to do so successfully. In equation (11), the first term is the case where no firm drops demand, while the second term (starting with the summation) is the case where at least one firm drops demand. In particular, the first component in the summation is the probability of seeing a given combination of firms trying to drop their demand, while the final fraction weights the change in firm i's expected payoff from each demand reduction by the probability that each succeeds. If firm m dropped demand in the last period, then firm i's current valuation is:

$$\begin{aligned} V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = m) = & \hat{V}_i(\mathbf{B_a}, P_a) + \pi_m \; \Delta V_{im}(\mathbf{B_a}, P_a) + \\ & (1 - \pi_m) \; \left(V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \delta_{-1} = 0) - \widehat{V}_i(\mathbf{B_a}, P_a) \right) \end{aligned} \tag{12}$$

where equation (11) can be seen to be a special case of (12) where we define $\pi_0 = 0$. Given the current valuation, firm i follows a mixed strategy if it is indifferent between lowering its own demand and maintaining it at the current level. This condition will hold if:[3]

$$V_i(\mathbf{B_a}, P_a, \pi_i = 1, \boldsymbol{\pi}_{-i}) = V_i(\mathbf{B_a}, P_a, \pi_i = 0, \boldsymbol{\pi}_{-i}). \tag{13}$$

Next, we define the mixed-strategy Nash equilibrium using the system of equations (13).

Definition 2.2. Nash Equilibrium in mixed strategies for the downstream market and the auction with discrete bidding is such that given $(\mathbf{B_a}, \mathbf{B_l}, P_a)$, for each firm i, there exists a probability $\pi(\mathbf{B_a}, P_a)$ which maximizes firm i's expected payoff:

$$V_i(\mathbf{B_a}, P_a, \pi_i, \boldsymbol{\pi}_{-i}) \geq V_i(\mathbf{B_a}, P_a, \rho_i, \boldsymbol{\pi}_{-i}); \forall \rho_i \in [0, 1]; \forall i = 1, \ldots, M \tag{CP2}$$

Solving for the demand reduction probabilities consists of solving the system of equations defined by (13) for all firms, where all firms either follow mixed strategies and satisfy (13) or follow pure strategies and satisfy CP2.

3 Results

As a numerical analysis, we focus on a scenario whose parameters are given in table 1. Some comments regarding this choice of numbers are in order. We have chosen to consider an auction with $M = 3$ firms over $N = 9$ units. Three firms are sufficient to avoid the singular feature of the two firm case: that spectrum not won by a firm is necessarily either won by its one competitor or unused. With three firms, a firm who does not buy a particular

[3] We solve this system of equations using Newton's method. It is helpful to note that

$$\frac{\partial V_i(\mathbf{B_a}, P_a, \boldsymbol{\pi}, \phi = 0)}{\partial \pi_n} = \sum_{\substack{\delta_1, \ldots, \delta_M \in \{0,1\}^M \\ \delta_1 + \ldots + \delta_M \neq 0}} (2\delta_n - 1) \left[\prod_{k \neq n} \pi_k^{\delta_k} (1 - \pi_k)^{1 - \delta_k} \right] \left[\frac{\sum_{j=1}^M \Delta V_{ij}(\mathbf{B_a}, P_a) \; \delta_j}{\sum_{j'=1}^M \delta'_j} \right].$$

unit of spectrum may be uncertain which of its competitors will win the unit. The nine-unit auction is also interesting, because it avoids possible problems where the number of units is not divisible by the number of firms. With three symmetric firms, it is possible for each firm to win three units of spectrum.

With our set of base case parameters, we consider ten scenarios, which we believe provide some insight into the workings of a spectrum auction. Each scenario is a slight variation of the benchmark model, which assumes parameter values given in table 1. These scenarios are summarised in table 2. Scenario 0 is the benchmark model where all three firms are symmetric. We investigate the effect of marginal costs on the equilbirum behaviour in scenarios 1–6. In particular, scenarios 1 to 4 consider situations where one firm is more efficient than the others. We explore whether these result in the more efficient firm (firm 3) obtaining a greater market share. Scenario 5 explores the problems which develop when one firm is *less* efficient than its competitors. Scenario 6 experiments with a policy in which the market is forced to clear as soon as supply equals demand, in order to mitigate the problems of scenario 5. Scenarios 7 through 9 explore the extent to which existing constraints due to allocations of legacy spectrum influence the auction of the new spectrum. Scenario 7 reduces the firms' spectrum, so that all three firms are constrained in the existing low-product market. In scenario 8, we consider a market where two incumbents (firms 1 and 2) have ample existing spectrum, whereas firm 3 (a newer entrant) could use the new spectrum to not only supply high use customers, but also to expand its market share of the low use market. Scenario 9 reverses the situation to one where firm 3 has a large share of legacy spectrum, and firms 1 and 2 are constrained.

Note that in all cases where the firms are not symmetric, firm 3 is different to firms 1 and 2. In many of the tables which follow, equilibria which differ only because firms 1 and 2 have been reversed will be pooled. Hence when we report an equilibrium with allocation $(1, 2, 2)$ this represents the two equilibria $(1, 2, 2)$ and $(2, 1, 2)$. For the cases where all three firms are identical, we pool across all equilibria which achieve the same distribution across firms (so that $(1, 2, 2)$ includes $(2, 1, 2)$ and $(2, 2, 1)$).

In the following two subsections, we first present some preliminary analysis using the Non-Rival bidding approach of section 2.4, before proceeding to our main results, using the Rival approach outlined in section 2.6.

3.1 Non-Rival Bidding

We begin by considering Scenario 1, where firm 3 is slightly more efficient than firms 1 and 2. We present the aggregate demand in the continuous bidding case in figure 2. Overall, the aggregate demand is a continuous function of the price. When the auction price is low, the firms aim to win all the units to become a monopoly. As the price increases, they revert back to the triopoly case; recall the shape of the profit function displayed in figure 1.

Next, we present the results of ten scenarios in the non-rival case in table 3. In the benchmark model, all firms acquire the same amount of spectrum at the equilibrium, which is around 1.65 units for each firm. Since the firms initially bid to be a monopolist, the auction ends at the price \$1.4227, when each firm can no longer afford to be a monopolist. Meanwhile, the regulator faces excess supply, which is prevalent in the other scenarios as well.

We carry out comparative statics on the marginal cost of production and the initial endowment of the legacy spectrum. In scenario 1, firm 3 is more efficient in producing both products ($c_{h3} = c_{l3} = 0.9$). As a result, we expect firm 3 to bid more aggressively to win more items. Since there is still excess supply at the end of the auction in scenario 0, firm 3 can simply claim more spectrum. Meanwhile, the other firms' winnings go down since the other firms are relatively less efficient. The auction price increases in this scenario as firm 3 prefers to be a monopolist until the auction price reaches 1.4870. If firm 3 gets even more efficient ($c_{h3} = c_{l3} = 0.7$), then the difference between the auction winnings of firm 3 and the others increases, and the auction price goes further up. In scenario 4, where firm 3 has the lowest marginal cost in all scenarios ($c_{h3} = c_{l3} = 0.1$), the spectrum units won by firm 3 rises by almost 40 percent to 2.27 units compared to the benchmark model, and the auction price jumps to almost \$1.62. Nonetheless, in these scenarios 0–4, firms 1 and 2 still survive in the auction. This is because even though firm 3 is relatively more efficient in production, the marginal cost of production for firms 1 and 2 stayed reasonably lower than the choke price in both markets. Consequently, firm 3 could not force the other two firms out of the market. It is also worth noting that the regulator as an auctioneer enjoys the asymmetry between firms. While this seems a bit counterintuitive, we interpret this result in the following way: as one of the firms separates itself from the others in terms of production costs, the highest price that it is willing to pay to win all units increases. Therefore, the revenue collected by the regulator increases (which is largely driven by the increase in price, as opposed to due to a larger number of units being sold).

In scenario 5, firm 3 is made less efficient, which implies that it wins less units than in scenario 0. Meanwhile, the other two firms are now relatively more efficient, which leads to higher units won by these firms. The auction price also decreases to \$1.4227, which is the same price as in scenario 0, due to the discretization of the auction price. Furthermore, the revenue decreases below \$7. Scenario 6 is not valid in this framework, since there is no price which will clear the market exactly, due to the jumps in demand.

Given the parameter values, the three firms are endowed with enough legacy spectrum in scenarios 0–6 to produce their equilibrium output at the low-product market. In scenarios 7–9, we focus on the allocation of the legacy spectrum, denoted by $\mathbf{B}_l$, and investigate how firms react in the auction if they need to use some of the new spectrum in the low-product market. In scenario 7, all firms are again symmetric as in scenario 0, but they all lack enough spectrum to optimally produce the low product. Therefore, they all need more (1.72) units, and the resulting equilibrium is symmetric as expected. Higher demand for the new spectrum increases the auction price, which leads to an increase in revenue compared to scenario 0. In scenarios 8 and 9, some of the firms (firms 1 and 2 in scenario 8 and firm 3 in scenario 9) have enough of the legacy spectrum to produce in the low-product market. Nonetheless, we find out that the firms do not use any of the new spectrum to produce the low product. We interpret this result in the following way: if some firms have enough of the legacy spectrum, the low-product market turns into a Stackelberg game rather than a Cournot game, since the firms with enough spectrum can limit the others to produce at the same level as before. As a result, all the firms use their new spectrum in the high-product market, where they are symmetric. In contrast, when demand is forced to be discrete (see section 3.2) firms may find it optimal to use their marginal units to produce both products).

While we obtain a unique pure strategy Nash equilibrium (PSNE) in the continuous-bidding Non-Rival case, there may be multiple PSNE in the discrete-bidding Non-Rival case. At a given auction price and a given vector of bids, we check whether each vector of bids is a Nash equilibrium. We calculate the auction price at which the equilibrium bids change: either total demand reduces or the bid structure changes. For instance, it may be the case that for a low auction price, a firm would prefer the others to drop demand, whereas at a higher price, it may reduce demand first. In both of these cases, the total demand for spectrum may be the same. We display these bid structures and the threshold prices in figure 3, and the aggregate demand in figure 4. As mentioned above, when the auction price is low, the firms all aspire to be a monopolist. As the auction price increases, the bids go down. As depicted in figure 4, the aggregate demand for spectrum for the discrete bidding case is downward sloping as in the continuous-bidding case (see figure 2). However, it is now a step function, since the aggregate demand decreases only after the price reaches a certain level. Given these multiple equilibria, we should not be surprised to find that when we explore the Rival Bidding case, firms will follow mixed strategies in deciding when to drop and when not to drop.

3.2 Rival Bidding

We now investigate the Rival Bidding case, where firms are restricted to bid in discrete units. Two differences exist between the Continuous Non-Rival and the Discrete Rival bidding cases. First, we now allow firms to steal each other winnings by introducing the resource constraint. As a result, the problem becomes a dynamic problem. Second, as the results of the ten scenarios for the continuous-bidding non-rival case reveal, firms may not necessarily find it optimal to win in discrete units. Therefore, because of discrete bidding, they may prefer to win less or more items, depending on the auction price and the other firms' behavior. As a result, the firms may engage in playing mixed strategies, as discussed in section 2.4. While the discrete-bidding feature of the auction generates the multiple equilibria, the limited spectrum and non-increasing bid formation lead to a dynamic problem that needs to be solved computationally.

We begin by examining scenarios 1 to 4, since these exhibit fairly straightforward behaviour by auction participants, and return to scenario 0 subsequently. These four scenarios explore the auction outcomes where one firm has an efficiency advantage, manifested as lower marginal costs; see table 4. Scenario 1 posits that firm 3 has a marginal cost of 0.9, as compared with other firms' costs of 1.0. With this set of marginal costs, firm 1 (one of the less efficient firms) has a preference for an allocation $(2, 1, 2)$ over the allocation $(1, 1, 2)$ at any price below \$2 and prefers $(1, 1, 2)$ to $(1, 2, 2)$ at any price. If we ignore the problems of dynamic bidding strategies, the Nash equilibria at price \$ 1.20 (the price at which the auction clears) are $(1, 2, 2)$ or $(2, 1, 2)$; see figure 3. However, our dynamic game results in a single equilibrium of $(1, 1, 2)$.

The reason for this different equilibrium is the ability in a dynamic game for one firm to co-ordinate the others and the inability of firms to increase their demand. By preventing the auction from ending unless the other firms demands are sufficiently low, the dominant firm can circumvent the PSNE outcome. The specific mechanics of this co-ordination are as follows. Once the price in the auction reaches \$1.2, firm 3 lowers its demand to 7. Firms 1 and 2 are now presented with an opportunity to clear the market, if they are both willing to lower demand to 1. Once firms 1 and 2 have done this, firm 3 will drop its demand to 2, since this is firm 3's best response to both firms 1 and 2 having bid demand of 1. If either firm 1 or firm 2 refuses to drop demand to 1, the auction continues, which will result in higher prices for all firms. Since firm 3 is more efficient, this cost is mitigated by the prospect of obtaining even more market power at a higher spectrum price, and the threat to continue the auction is credible. Hence firm 3 helps to co-ordinate firms 1 and 2 past the Nash equilibria and on to the dynamic outcome which offers higher payoff for firm 3.

Note that as we decrease firm 3's marginal costs from 0.9 to 0.7 (in scenario 2), the same equilibrium occurs, but now with a higher auction clearing price. Firms 1 and 2 are keen for firm 3 not to obtain a third unit of spectrum. Given firm 3's lower marginal costs, it would make heavy use of the third unit of spectrum, resulting in considerably lower profits for firms 1 and 2. Firms 1 and 2 incur higher expenses for their own spectrum by staying in the auction longer, but this is more than offset by the increase in their downstream profit due to constraining firm 3.

In scenario 4 (firm 3's marginal cost is now 0.1) firms 1 and 2 cannot compete with firm 3, the market collapses, and firm 3 gets a third unit.

We see our first example of multiple equilibria in the dynamic auction in scenario 3. Here firms 1 and 2 realise that bidding up the market may not be the best strategy. An alternative strategy would be to co-exist with firm 3 at low prices. However, as in Scenario 1, each inefficient firm has a preference for holding 2 units of spectrum while the other inefficient firm holds one unit. At low prices ($0.3 - $0.5) the firms follow mixed strategies, where each drops demand with some probability, inducing its rival to entertain a similar probability of dropping demand. If neither firm drops demand, the price rises, and the process is repeated. The end result is a sequence of low price equilibria in which the efficient firm (firm 3) obtains two units of spectrum, while one inefficient firm obtains two units and the other one unit; see table 4.

Scenarios 1 through 4 suggest that from the firms' perspectives, a dynamic game may yield superior outcomes to those suggested by the static game: one firm may be able to help co-ordinate other firms and avoid a lengthy bidding process. As noted in section 2.4, firms will sometimes find themselves in a situation where each has a preference for the other to lower demand, but if this does not eventuate, it would prefer to lower demand itself rather than have the auction continue. This behaviour becomes more apparent in the dynamic game once we consider the case of symmetric firms.

Table 5 considers the case of three identical firms, each of whom is unconstrained in the low use market. Comparing table 5 to table 4 shows a profusion of additional equilibria. To provide a flavour for how these work, we explore the equilibria which occur at $0.3 and $1.1.

If we sort the equilibria with respect to the auction price, we observe the first equilibrium at the price $0.3, at which no firm demands a third unit. At this price, each firm has a 2.7 percent probability of dropping demand to one. As soon as one does so, the others bid two units and the auction clears. If firms do not end the auction at the price $0.3, then multiple equilibria occur at the price $1.1. Therefore, a firm who drops demand to one unit at the low price may end up with a considerably better outcome when the price increases.

Once the price reaches 1.1, a more complicated lottery is enacted among the firms. First one firm may drop its demand by three units (each has a 15.53 percent probability of doing this) from nine units to six. Conditional on this action, the same firm will drop its demand to 1 with a 3.64 percent probability (note that the other firms may also drop their demand at this stage: see next paragraph). Then, the remaining two firms, being in equal positions of strength, and seeing the possibility of clearing the market now, will drop their demands to 4. The market *could* now clear, but if it did, the two high demand firms would end up holding surplus units, paying for spectrum that they will not use in the downstream market. The two high-demand firms now randomise unit drops in their demand. If one firm drops its demand to 1, the other firm immediately drops its demand to 2 and the market clears. If both demand 2, demand reductions cease. Note that because the firms are following mixed strategies, there exists some probability that neither firm drops demand. At this stage, the market clears, and the firms may be left holding surplus units of spectrum.

An alternative scenario plays out if the first firm to drop demand is followed by one of the other firms dropping demand, which each does with probability 34.76 percent. In this case, one firm (who originally dropped demand) has demand 6, one firm (who has just dropped demand) demands 8 units and one firm still demands 9 units. The low demand firm and high demand firm now both have an interest in coordinating so that the middle firm will demand 1 unit. The small demand firm will be willing to lower demand to 3, but will not lower further until the middle firm lowers demand, nor will the high demand firm budge from bidding 9 units unless the middle firm drops demand. This creates a standoff in which the auction will continue unless the middle firm lowers demand. Since the next price at which a competitor would lower demand is \$2, this is sufficient disincentive to the middle demand firm to force it to capitulate and lower demand. The high demand firm can then maintain demand at 7 to ensure (as was the case in Scenario 1) that both other firms lower their demand to 1. Note that neither the low demand firm nor the high demand firm can individually credibly commit to not lowering demand, but collectively, their demands prevent the market from clearing, and ensure that the medium demand firm is forced to lower demand.

Subsequent prices represent the firms' continued attempts to clear the market as the auction price becomes very high, culminating in some very costly outcomes where firms randomising to see who is left as a monopolist in the market may hold surplus units of spectrum when the auction clears. These outcomes result in very high revenues for the regulator, and potentially high social welfare, but very low profits for firms.

Several of the subsequent scenarios are also characterised by large numbers of equilibria. For brevity, we report only those which occur with probability 10^{-4} or higher, but note their existence in the tables and text.

Table 6 outlines scenarios 5 and 6. Here we consider the converse of scenario 1, where firm 3 is less efficient than firms 1 and 2. Not surprisingly, given scenario 1 (see table 4) the inefficient firm stays in the market, attempting to keep its more efficient competitors from obtaining large shares of spectrum. While 50.62 percent of the time this results in the market clearing at price \$1.2, and with one of the efficient firms holding 2 units, the final outcome is sometimes ruinous for the firms. With 40.22 percent probability, the price rises to \$3, and each firm buys only one unit. In scenario 5, there are also a large number of other, low probability equilibria, which are not reported in table 6. The highest price observed is \$3.7 for this scenario, and while this does occur with low probability, it can be associated with all units of spectrum being sold, resulting in a collosal revenue for the regulator of \$33.3.

The inefficient firm's activities are clearly very costly for the other firms (compare scenario 5 to the results for scenario 0 in table 5). They also often result in low social welfare, since the more efficient firms (firms 1 and 2) are unable to obtain much spectrum. Firm 3 is effectively holding up the market. One solution to this problem would be to force the market to clear as soon as demand equals supply. This would mean that if the inefficient firm attempts to maintain high demand, it will potentially end up buying over-priced units which it does not utilize. Scenario 6 shows the outcomes in table 6. The policy has its desired effect: firm 3 is no longer able to hold up the market. Instead, one of the efficient firms wins most of the units, and the other two become weaker competitors. The net result is a far higher level of social welfare. Revenue, however, suffers because the holdup is cleared and all competition is between the two efficient firms, who are unwilling to over-bid, given that the winner must purchase 7 units of spectrum. Depending on a policy-maker's objective, this may or may not be a better outcome.

Table 7 contains our last set of results, exploring the role of old spectrum on the current spectrum auction. Scenario 7 extends scenario 0 by considering the case where firms are all equally constrained regarding the old spectrum. Not surprisingly, this results in considerably higher prices for the auction, and also higher demand. In comparison to scenario 0, there is a much higher probability of all units being sold about 2 percent of the time. Further, the price distribution has a long tail, with prices as high as \$6.9.

Scenarios 8 and 9 explore the cases where not all firms are equally constrained. Scenario 8 shows a case where firm 3 is constrained, meaning that firms 1 and 2 enjoy market power in the low use market. Since firms must buy discrete units, even if in a continuous bidding arrangement firms would not buy spectrum specifically to produce in the low-product market, a firm with spare spectrum may use its marginal unit to produce in both the low-product and the high-product markets. Consequently, firm 3 exhibits similar behaviour compared to scenario 1: having a higher value for the new spectrum, firm 3 bids aggressively, and coordinates its competitors into agreeing on a single unit each. Also, as in scenario 1, the incumbent (lower value) firms have an incentive to run the price up, since they stand to lose from firm 3 expanding its operations in the low use market.

Scenario 9 presents the opposite scenario, where firm 3 has a surplus of legacy spectrum, but firms 1 and 2 are both constrained. In this case, firm 3 has a strong incentive to try and win new spectrum to ensure that the other firms do not make inroads into its profitable low use market. Firm 3 is remarkably successful in this, with 99 percent probability, it limits the constrained firms from buying more than one unit of spectrum apiece. The asymmetry between scenarios 8 and 9 is quite striking. When two incumbents face a single entrant, the entrant is able to buy a larger share of the new spectrum. However, when a single incumbent faces two entrants, the incumbent prevents the entrants from enlarging their market shares.

4 Regulation through Caps

While an auctioneer aims to maximise revenue in a standard auction, a regulator tasked with running a spectrum auction may also take into account total surplus in the downstream market when considering the allocation of spectrum. In particular, the regulator may have a trade-off between the auction revenue and the total surplus. For example, suppose that one of the firms is substantially more efficient. Then, this firm may increase the price high enough to win all the units. While selling all units at a high price is quite desirable as an auctioneer, the regulator may find out that the firm becomes a monopoly in the downstream market, which decreases the total surplus. As a result, the regulator may decide to intervene in the auction by setting *caps* to regulate the maximum number of units each firm can win in the auction.

Given that our solution technique works recursively backwards from high prices and low demand to low prices and high demand, we model caps by starting the auction with participants bidding demands that do not equal the total number of units available in the market but rather the cap given them by the government. In fact, for each combination of starting bids, it is possible to evaluate the distribution of auction clearing bids. With this information available, a regulator optimally chooses a cap structure for the auction depending on its objective function, which we define as follows:

$$\max_{<\overline{\mathbf{B_a}} \geq 0>} \phi_1 \, \mathcal{E}(TS \mid \overline{\mathbf{B_a}}) + \phi_2 \, \mathcal{E}(R \mid \overline{\mathbf{B_a}}) + \phi_3 \, \mathcal{V}\left[\phi_1 \, TS + \phi_2 \, R \mid \overline{\mathbf{B_a}}\right] \tag{14}$$

where $\overline{\mathbf{B_a}}$ denotes the vector of caps set by the regulator, and $\mathcal{E}(\cdot)$ and $\mathcal{V}(\cdot)$ represent the mean and the variance of the relevant variables, respectively. These caps limit how much each firm can win in the auction. The first two terms in equation (14) are the weighted average of the expected total surplus and revenue, and the last term is the variance of this weighted sum. This objective function generalizes the one used in Dana and Spier (1994), and allows the regulator to be risk-neutral (when $\phi_3 = 0$) or risk-averse (when $\phi_3 < 0$). For our numerical illustration, we consider five different types of regulators, and explore what the optimal cap structure for each regulator is for each of our scenarios:[4]

1. **Regulator 1:** This risk-neutral regulator is solely concerned with expected total surplus: the auction exists as a mechanism to ensure optimal allocation of spectrum, and revenue is an unimportant side effect. In this case, the parameters of the social welfare function takes values $(\phi_1, \phi_2, \phi_3) = (1, 0, 0)$.

2. **Regulator 2:** This risk-neutral regulator is only concerned with maximising expected revenue. In this case, the parameters of the social welfare function takes values $(\phi_1, \phi_2, \phi_3) = (0, 1, 0)$.

3. **Regulator 3:** This risk-neutral regulator maximises the sum of the two.[5] In this case, the parameters of the social welfare function takes values $(\phi_1, \phi_2, \phi_3) = (1, 1, 0)$.

4. **Regulator 4:** This risk-averse regulator seeks to maximise expected revenue less the variance of revenue. In this case, the parameters of the social welfare function takes values $(\phi_1, \phi_2, \phi_3) = (0, 1, -1)$.

5. **Regulator 5:** This risk-averse regulator seeks to maximise the sum of expected revenue and total surplus less the variance of revenue and total surplus. In this case, the parameters of the social welfare function takes values $(\phi_1, \phi_2, \phi_3) = (1, 1, -1)$.

We display the results of our analysis in table 8. Examining first the baseline scenario (scenario 0), we see several types of behaviour. Regulator 1 finds it optimal to set each firm's cap to around one half of the total units. This will result in a more competitive outcome than the prevailing allocation from an uncapped auction: one firm with two units and the other firms with one unit each. The trade-off with this approach is that there will be almost no auction activity, and the revenue will be zero. Regulator 2 limits one of the inefficient firms to create a competition between the other two firms. This action raises the expected

[4] We find that regulators seeking to maximise expected total surplus choose cap structures which result in certain outcomes, so there would be no difference in the optimal cap structure chosen by a risk-averse total-surplus maximiser.

[5] A more libertarian view of the auction might note that government use of revenue earned might result in lower total surplus than would be achieved through this revenue being profits for the private sector. Our analysis could be extended to consider this by seeking to maximise total surplus with a penalty ascribed to revenue ($\phi_2 < 0$). However, for all our scenarios considered here, since the optimal strategy when maximising total surplus results in a collapse of the auction (see Table 8), a regulator with this viewpoint would choose the same cap structure as a regulator purely concerned with total surplus.

revenue to above \$11. This strategy is risky, however, and regulator 4 will instead opt for capping two firms at seven units, to minimise variance (the high cap firm will co-ordinate the other firms' bids to one each, which will take place at a price of \$ 1.1).

When we explore the market with one firm being higher efficiency (scenarios 1–4), we see that regulator 1 will choose to give a more generous cap to the more efficient firm. In contrast, regulators concerned with revenue (regulators 2, 3 and 5) give generous caps to the less efficient firms, enabling them to effectively hold up the efficient firm, and increasing revenue per unit, at the expense of a smaller number of units being sold. In scenarios 1 and 2, a regulator concerned with both revenue and total surplus will follow this strategy if risk is not an issue (regulator 3) but once risk is factored in (regulator 5) it is preferable to opt for allocating spectrum and receiving no revenue (the same strategy followed by regulator 1). In contrast, as firm 3 becomes more efficient (in scenarios 3 and 4), regulator 5 is willing to let the auction run by giving a more generous cap to the efficient firm and one of the less efficient firms. Regulator 3 opts to handicap the efficient firm in scenario 3, but not in scenario 4.

Scenarios 5 and 6 explore the holdup problem with firm 3 being less efficient than firms 1 and 2. In scenario 5 (conventional auction clearing), all regulators favour the efficient firms with more generous caps, except regulator 4, who finds that giving the highest cap to the inefficient firm leads to a lower risk revenue outcome. In contrast, in Scenario 6 (where the auction clears as soon as demand equals supply) the regulator can safely favour the inefficient firm without danger of an inefficient outcome. We thus see all regulators except regulator 1 choosing generous caps for the inefficient firm.

Turning our attention to scenarios 7–9, we see the different regulators' attitude to constrained firms. For scenario 7, where all firms are constrained, most regulators choose a balanced set of caps, except regulator 4, who chooses to cap one firm at 7 and regulator 2, who caps one firm at 3. Creating competition between two constrained firms leads to higher revenue outcomes than situations with three constrained firms competing.

Scenario 8 and 9 show differing attitudes to incumbent firms (those with large existing spectrum holdings) and entrants (firms with smaller existing holdings). In scenario 8, a regulator 1 chooses to favour the entrant over the two incumbents. In contrast, regulator 2 favours the incumbents, encouraging them to run up prices and shut out the entrant. This is a risky strategy, however, and if the regulator is risk-averse (regulator 4) a more balanced strategy which does not penalise the entrant so heavily dominates. Regulator 3 chooses higher caps, and allows the auction to generate revenue and for the entrant to secure market share (see Section 3).

In scenario 9, where firm 3 represents a single incumbent, facing off against two entrants, we observe different behaviour. Regulator 1 again favours entrants to the market. However, in this case *similar* behaviour is exhibited by regulators 2 and 4. Both risk-neutral and risk-averse revenue maximisers choose a lower cap for the incumbent firm. In contrast, a regulator whose objective is a balance between total surplus and revenue chooses either an identical set of caps (regulator 3) or a more generous cap for the incumbent (regulator 5). Examining the distribution of revenues and total surplus levels for these choices, we see that regulator 5 has chosen caps which lead to negative covariances between the two outcomes, effectively hedging his position by generating high revenue in situations where total surplus is low.

5 Conclusion

In this paper, we explore a multi-unit auction in which market participants use the auction good to provide services in a downstream market. Winning large quantities in the auction will allow a participant to wield considerable power in the downstream market. We show that market participants who make less efficient use of spectrum (either because of higher marginal costs, or having large existing positions) have strong incentives to hold up the market, running up prices with the objective of reducing other firms' positions in the market. Interestingly, this is exactly the behaviour which will maximise revenue for regulators, and a revenue focused regulator will structure an auction in such a way as to encourage hold ups. In contrast, in many cases, a social-welfare maximising regulator will often find better outcomes can be achieved by simply assigning spectrum to firms.

We note that our findings, regarding firm mixed strategies (and auction allocations becoming random variables) occur even in a case where firms face no uncertainty regarding rival characteristics. An interesting extension of our work would be to add uncertainty regarding rival characteristics.

While this paper has focused on the use of a clock auction in the context of an auction for radio spectrum frequencies, the techniques used here are far more general in their application. Multi-unit auctions are frequently used in internet auctions (Ockenfels and Roth (2006)). We believe that our dynamic programming technique could easily be used to explore this area further.

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A Numerical Example: Linear Demand

Suppose that the demand for both products are linear and firms have quadratic costs:

$$\begin{aligned} P_h =& a_h - b_h \ Q_h = a_h - b_h \ \sum_{i=1}^{M} q_{hi} \\ P_l =& a_l - b_l \ Q_l = a_l - b_l \ \sum_{i=1}^{M} q_{li} \\ MC_{hi} =& c_{hi} + 2d_{hi}q_{hi}; \forall i = 1, \ldots, M \\ MC_{li} =& c_{li} + 2d_{li}q_{li}; \forall i = 1, \ldots, M \\ FC_i =& 0; \forall i = 1, \ldots, M \end{aligned}$$

where FC_i denotes the fixed cost of production for firm i, which is assumed to be zero. We can write the Lagrangian function as follows:

$$\begin{aligned} L_i =& P_h q_{hi} + P_l q_{li} - F_i - L_i - C_{li}(q_{li}) - C_{hi}(q_{hi}) + \\ & \lambda_i(\theta_{hi}B_{hi} + \theta_{li}B_{li} - q_{hi} - q_{li}) + \mu_i(\theta_{hi}B_{hi} - q_{hi}) + \eta_i q_{hi} + \zeta_i q_{li}. \end{aligned}$$

Therefore, the Cournot-Nash equilibrium (CNE) in downstream market is the solution to the following problem where $\forall i = 1, \ldots, M$:

$$P_h - b_h \ q_{hi} - (c_{hi} + 2d_{hi}q_{hi}) + \eta_i - (\lambda_i + \mu_i) \geq 0 \perp q_{hi} \geq 0 \tag{15a}$$
$$P_l - b_l \ q_{li} - (c_{li} + 2d_{li}q_{li}) + \xi_i - \lambda_i \geq 0 \perp q_{li} \geq 0 \tag{15b}$$
$$\theta_{ai} \ B_{ai} - q_{hi} \geq 0 \perp \mu_i \geq 0 \tag{15c}$$
$$\theta_{ai} \ B_{ai} + \theta_{li} \ B_{li} - (q_{hi} + q_{li}) \geq 0 \perp \lambda_i \geq 0 \tag{15d}$$
$$q_{hi} \geq 0 \perp \eta_i \geq 0 \tag{15e}$$
$$q_{li} \geq 0 \perp \xi_i \geq 0. \tag{15f}$$

Given the functional forms for the demand and costs, we can solve the complementary problem as a linear programming problem. More specifically, we can categorize the solution set as follows:

1. $\zeta_i = 0$ ($q_{hi} > 0$) and $\xi_i = 0$ ($q_{li} > 0$). This cases consists 4 sub-cases: $\lambda_i > 0$ and $\mu_i > 0$; $\lambda_i > 0$ and $\mu_i = 0$; $\lambda_i = 0$ and $\mu_i > 0$; $\lambda_i = 0$ and $\mu_i = 0$.

2. $\zeta_i > 0$ ($q_{hi} = 0$) and $\xi_i = 0$ ($q_{li} > 0$): in this case, $q_{hi} = 0 \leq \theta_{hi} B_{hi}$, hence $\mu_i = 0$. We consider two sub-cases: $\lambda_i > 0$ or $\lambda_i = 0$

3. $\zeta_i = 0$ ($q_{hi} > 0$) and $\xi_i > 0$ ($q_{li} = 0$). We consider two sub-cases: $\mu_i = 0$ or $\mu_i > 0$ as $q_{hi} \leq \theta_{hi} B_{hi} \leq \theta_{hi} B_{hi} + \theta_{li} B_{li}$, we have $\lambda_i = 0$ in both sub-cases.

4. $\zeta_i > 0$ ($q_{hi} = 0$) and $\xi_i > 0$ ($q_{li} = 0$). As $q_{hi} = 0 \leq \theta_{hi} B_{hi}$ and $q_{li} = 0 \leq \theta_{hi} B_{hi} + \theta_{li} Bli$, we have $\lambda_i = 0$ and $\mu_i = 0$.

Therefore, we have 9 cases in total. As mentioned above, we can solve each case as a linear programming problem.[6]

B Tables

Symbol	Value(s)	Explanation
N	9	Number of units auctioned
M	3	Number of firms
a_h	9	Choke price in the new-product market
a_l	5	Choke price in the old-product market
b_h	1	slope of the demand curve for the new product
b_l	1	slope of the demand curve for the old product
$\mathbf{c}_h$	$(1,1,1)$	linear part of the cost of production of the new product
$\mathbf{c}_l$	$(1,1,1)$	linear part of the cost of production of the old product
$\mathbf{d}_h$	$(0,0,0)$	quadratic part of the cost of production of the new product
$\mathbf{d}_l$	$(0,0,0)$	quadratic part of the cost of production of the old product
$\boldsymbol{\theta}_h$	$(1,1,1)$	marginal increase in capacity for the new product
$\boldsymbol{\theta}_l$	$(1,1,1)$	marginal increase in capacity for the old product
$\mathbf{B}_o$	$(2,2,2)$	endowment of legacy spectrum

Table 1: Parameters for the benchmark model (see scenario 0 in table 5)

Note: All other scenarios are perturbations of these parameter values.

[6] We will not get into further details here, but the interested reader is welcome to read our supplementary document, which is available on request from the authors.

Scenario	Characteristics
0	Base case.
1	Firm 3 is more efficient: $c_{h3} = c_{l3} = 0.9$.
2	Firm 3 is more efficient: $c_{h3} = c_{l3} = 0.7$.
3	Firm 3 is more efficient: $c_{h3} = c_{l3} = 0.5$.
4	Firm 3 is more efficient: $c_{h3} = c_{l3} = 0.1$.
5	Firm 3 is less efficient: $c_{h3} = c_{l3} = 1.1$.
6	Firm 3 is less efficient: $c_{h3} = c_{l3} = 1.1$.
	The auction is ended as soon as supply equals demand (excess supply is not allowed).
7	All firms are currently constrained: $\mathbf{B}_l = (0.5, 0.5, 0.5)$.
8	Firm 3 is currently constrained: $\mathbf{B}_l = (3, 3, 0.5)$.
9	Firms 1 and 2 are currently constrained: $\mathbf{B}_l = (0.5, 0.5, 6)$.

Table 2: Scenarios

Note: Perturbations to the parameters outlined in table 1 to generate the scenarios explored in section 3. The parameter c_i is firm i's marginal cost for production of both products. The endowment for legacy spectrum is denoted by $\mathbf{B}_l$.

Scenario	B_{a1}	B_{a2}	B_{a3}	Price	Total Surplus	Revenue	Probability
0	1.6443	1.6443	1.6443	1.4227	34.7967	7.0181	1
1	1.6033	1.6033	1.7032	1.4870	35.0277	7.3008	1
2	1.5533	1.5533	1.8533	1.4870	35.8741	7.3751	1
3	1.4868	1.4868	1.9868	1.5527	36.6787	7.7021	1
4	1.3701	1.3701	2.2700	1.6198	38.7806	8.1154	1
5	1.6693	1.6693	1.5693	1.4227	34.4449	6.9826	1
6	-	-	-	-	-	-	-
7	1.7237	1.7237	1.7237	1.5527	32.6503	8.0289	1
8	1.6118	1.6118	1.6118	1.5527	34.1040	7.5080	1
9	1.5779	1.5779	1.5779	1.6884	33.5406	7.9924	1

Table 3: Continuous-Bidding Non-Rival equilibria

Note: Nash equilibria for the Continuous-Bidding Non-Rival auction, as outlined in section 2.4. Parameters for the different scenarios are given in table 2.

Scenario	B_{a1}	B_{a2}	B_{a3}	Price	Total Surplus	Revenue	Probability
1	1	1	2	1.2	31.832	4.8	1
2	1	1	2	1.4	32.54	5.6	1
3	1	2	2	0.3	36.805	1.5	0.29123
3	1	2	2	0.4	36.805	2	0.0017939
3	1	2	2	0.5	36.805	2.5	0.022393
3	1	1	2	1.6	33.305	6.4	0.68458
4	1	1	3	0.3	39.407	1.5	1

Table 4: Scenarios 1–4

Note: Parameters are as given in table 1, with the exception of firm 3's marginal cost of production in both markets. Firm 3 is more efficient than the other firms. In scenario 1, firm 1 has marginal cost 0.9, while scenarios 2, 3, and 4 feature firm 3 marginal costs of 0.7, 0.5 and 0.1, respectively.
Each row represents a possible equilibrium in the market where symmetric equilibria have been pooled (e.g. $(1, 2, 2)$ is identical to $(2, 1, 2)$). Columns represent (in order) allocations to the three firms, price for the equilibrium in question, social welfare, revenue and finally, probability of the equilibrium occurring.

Scenario	B_{a1}	B_{a2}	B_{a3}	Price	Total Surplus	Revenue	Probability
0	1	2	2	0.3	35	1.5	0.083828
0	1	1	2	1.1	31.5	4.4	0.77085
0	1	2	3	1.1	36.375	6.6	0.004455
0	1	3	3	1.1	36.778	7.7	0.018219
0	1	2	4	1.1	36.375	7.7	0.00051498
0	1	3	4	1.1	36.778	8.8	0.0049804
0	1	4	4	1.1	36.778	9.9	0.0030559
0	1	2	5	2.3	36.375	18.4	1.0723e-05
0	1	2	6	2.3	36.375	20.7	4.2253e-06
0	2	2	2	2.5	37.5	15	5.1467e-06
0	2	2	3	2.5	37.5	17.5	0.00010943
0	2	2	4	2.5	37.5	20	5.0477e-08
0	2	2	5	2.5	37.5	22.5	4.0801e-08
0	0	1	2	2.9	27	8.7	0.035497
0	0	2	2	2.9	31.5	11.6	0.065416
0	0	2	3	2.9	35	14.5	0.00088281
0	0	3	3	2.9	35.944	17.4	0.00027202
0	2	3	3	2.9	37.5	23.2	0.011835
0	3	3	3	3	37.5	27	6.3941e-05
0	0	0	2	3.6	21.5	7.2	4.9128e-08
0	0	1	4	3.6	33.375	18	6.2001e-09
0	0	2	4	3.6	35	21.6	7.233e-10
0	0	1	5	3.6	33.375	21.6	4.7194e-10
0	0	2	5	3.6	35	25.2	1.7651e-10
0	1	1	1	3.6	27	10.8	6.2574e-09
0	1	1	4	3.6	35	21.6	1.9809e-09
0	1	1	5	3.6	35	25.2	3.6015e-10

Table 5: Scenario 0

Note: All three firms are symmetric in this benchmark model. As a result, this case exhibits many equilibria, ranging from an early clearing of the market, where one competitor receives a small share, to a high revenue scenario where all spectrum is purchased.
Each row represents a possible equilibrium in the market where symmetric equilibria have been pooled (e.g. $(1,2,2)$ is identical to $(2,1,2)$). Columns represent (in order) allocations to the three firms, price for the equilibrium in question, social welfare, revenue and finally, probability of the equilibrium occurring.

Scenario	B_{a1}	B_{a2}	B_{a3}	Price	Total Surplus	Revenue	Probability
5	2	1	1	1.2	31.282	4.8	0.50618
5	2	2	1	1.2	34.782	6	0.0055026
5	2	3	1	1.2	36.157	7.2	0.0074612
5	2	4	1	1.2	36.157	8.4	0.0026519
5	2	5	1	1.2	36.157	9.6	0.0010285
5	2	6	1	1.2	36.157	10.8	0.00042005
5	2	1	1	1.3	31.282	5.2	0.0156356
5	2	2	1	1.3	34.782	6.5	0.00021919
5	2	3	1	1.3	36.157	7.8	0.00016769
5	2	1	1	1.4	31.282	5.6	0.0518
5	2	2	1	1.4	34.782	7	0.0013758
5	2	3	1	1.4	36.157	8.4	0.00065516
5	2	4	1	1.4	36.157	9.8	0.00022679
5	2	1	1	1.5	31.282	6	0.0038302
5	2	1	1	1.6	31.282	6.4	0.00021586
5	1	1	1	3	26.782	9	0.40221
6	1	7	1	0	34.782	0	0.013178
6	1	7	1	0.1	34.782	0.9	0.057638
6	1	7	1	0.4	34.782	3.6	0.20202
6	1	7	1	0.5	34.782	4.5	0.023359
6	1	7	1	0.6	34.782	5.4	0.0023613
6	1	7	1	0.7	34.782	6.3	0.033961
6	1	7	1	0.8	34.782	7.2	0.15185
6	1	7	1	0.9	34.782	8.1	0.15509
6	1	7	1	1	34.782	9	0.36054

Table 6: Scenarios 5–6

Note: Firm parameters are as given in table 1, with the exception of firm 3, who has marginal cost 1.1. In scenario 5, the market clears once demand is less than supply and no further reductions in demand are registered. This results in firm 3 running up the price of spectrum units to prevent the more efficient firms from obtaining significant market share. In scenario 6, the auction is cleared as soon as demand equals supply. Scenario 5 has additional equilibria (not reported) with probabilities lower than 10×-4. The maximum auction price for these equilibria is \$3.7.

Each row represents a possible equilibrium in the market where symmetric equilibria have been pooled (e.g. $(1, 2, 2)$ is identical to $(2, 1, 2)$). Columns represent (in order) allocations to the three firms, price for the equilibrium in question, social welfare, revenue and finally, probability of the equilibrium occurring.

Scenario	B_{a1}	B_{a2}	B_{a3}	Price	Total Surplus	Revenue	Probability
7	1	1	2	1.6	28.875	6.4	0.5541
7	1	2	2	1.6	32.097	8	0.19336
7	1	2	3	1.6	34.597	9.6	0.014864
7	1	2	4	1.6	35.66	11.2	0.0014136
7	1	2	5	1.6	35.66	12.8	0.0004944
7	1	2	6	1.6	35.66	14.4	0.00019009
7	0	2	2	2.1	28.875	8.4	0.1139
7	1	3	3	2.1	36.597	14.7	0.033015
7	2	2	2	2.1	34.688	12.6	0.039033
7	2	2	3	2.1	36.688	14.7	0.0036517
7	2	3	3	2.1	37.243	16.8	0.024659
7	3	3	3	2.1	37.5	18.9	0.021102
7	0	1	2	2.5	24.375	7.5	0.0001968
8	1	1	2	1.2	31.319	4.8	1
9	1	1	2	1.1	30.875	4.4	0.99117
9	1	2	2	1.1	34.375	5.5	0.0011586
9	1	2	3	1.1	35.66	6.6	0.0023052
9	1	2	4	1.1	35.66	7.7	0.00081626
9	1	2	5	1.1	35.66	8.8	0.00031593
9	1	2	6	1.1	35.66	9.9	0.00012886
9	1	1	2	1.2	30.875	4.8	0.0036759
9	1	1	2	1.3	30.875	5.2	0.00015677

Table 7: Scenarios 7–9

Note: These scenarios build upon scenario 0; see table 5. Parameters are as given in table 1 with the exception of initial spectrum allocations. For scenario 7, initial spectrum allocations are 0.5, meaning that firms will all want to use new spectrum purchased to service low use customers. Scenario 7 pools across all three firms, since they are identical. Scenario 8 has initial allocations $(3, 3, 0.5)$ meaning that firms 1 and 2 have sufficient old spectrum for the low use market, but firm 3 could use new spectrum to increase supply to the low use market. Scenario 9 has initial allocations $(0.5, 0.5, 6)$ so that roles are reversed, and firm 1 and 2 will want to use new spectrum to compete in the low market. Scenarios 7 and 9 have further equilibria (not tabulated) with probabilities lower than 10^{-4}. In scenario 7, the maximum auction clearing price is \$6.9, while in scenario 9 the maximum auction clearing price is \$3.6.
Each row represents a possible equilibrium in the market where symmetric equilibria have been pooled (e.g. $(1, 2, 2)$ is identical to $(2, 1, 2)$). Columns represent (in order) allocations to the three firms, price for the equilibrium in question, social welfare, revenue and finally, probability of the equilibrium occurring.

	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
					Regulator 1					
Caps	(4,4,5)	(3,3,5)	(3,3,5)	(2,4,6)	(4,5,6)	(3,3,3)	(4,4,2)	(5,5,5)	(3,3,5)	(4,5,3)
$\mathcal{E}(TS)$	37.5	37.889	38.754	39.734	42.039	37.139	37.139	37.5	37.5	37.5
$\mathcal{E}(R)$	0	0	0	0	0	0	0	0	0	0
σ^2_{TS}	0	0	0	0	0	0	0	0	0	0
σ^2_R	0	0	0	0	0	0	0	0	0	0
$\sigma_{TS,R}$	0	0	0	0	0	0	0	0	0	0
					Regulator 2					
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Caps	(3,9,9)	(6,6,3)	(6,6,4)	(8,8,3)	(9,9,6)	(9,9,3)	(2,2,6)	(3,9,9)	(9,9,3)	(8,8,3)
$\mathcal{E}(TS)$	29.318	29.484	27.629	31.537	32.745	28.118	36.678	27.019	29.137	29.146
$\mathcal{E}(R)$	11.283	11.64	11.227	11.754	10.22	11.789	15.312	10.608	11.283	9.5607
σ^2_{TS}	8.733	11.46	33.176	7.755	7.742	7.25	0.323	16.32	8.733	14.062
σ^2_R	18.565	20.496	13.426	9.295	5.72	18.854	0.013	13.548	18.565	11.236
$\sigma_{TS,R}$	11.257	13.956	14.729	6.373	3.753	10.849	-0.002	11.575	11.257	10.001
					Regulator 3					
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Caps	(7,8,8)	(6,6,2)	(6,6,3)	(9,9,3)	(7,8,8)	(6,6,3)	(2,2,6)	(8,8,8)	(5,5,5)	(4,4,4)
$\mathcal{E}(TS)$	31.378	30.469	29.717	31.541	38.301	29.437	36.678	30.71	31.158	33.831
$\mathcal{E}(R)$	10.938	10.806	10.949	11.751	6.63	10.866	15.312	9.581	10.866	5.659
σ^2_{TS}	3.741	8.327	23.778	7.736	3.727	7.624	0.323	7.216	4.946	7.31
σ^2_R	7.378	13.249	8.033	9.271	2.349	13.466	0.013	7.282	6.525	18.271
$\sigma_{TS,R}$	4.241	7.178	7.659	6.348	-0.146	5.465	-0.002	6.643	4.023	-6.651
					Regulator 4					
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Caps	(7,7,9)	(7,8,8)	(8,9,8)	(5,5,7)	(4,8,3)	(3,6,7)	(2,2,6)	(7,9,9)	(8,9,8)	(7,7,5)
$\mathcal{E}(TS)$	31.5	31.832	32.54	33.305	34.284	28.616	36.678	30.547	31.319	31.011
$\mathcal{E}(R)$	4.4	4.8	5.6	6.4	4.907	8.371	15.312	9.589	4.8	7.495
σ^2_{TS}	0	0	0	0	0.767	5.9	0.323	6.862	0	0.601
σ^2_R	0	0	0	0	0.287	1.138	0.013	6.2	0	1.464
$\sigma_{TS,R}$	0	0	0	0	0.435	-1.499	-0.002	6.313	0	0.579
					Regulator 5					
	S0	S1	S2	S3	S4	S5	S6	S7	S8	S9
Caps	(4,4,5)	(3,3,5)	(3,3,5)	(8,8,6)	(2,4,6)	(3,3,3)	(2,2,6)	(5,5,5)	(3,4,3)	(3,3,4)
$\mathcal{E}(TS)$	37.5	37.889	38.754	36.805	41.761	37.139	36.678	37.5	36.975	36.684
$\mathcal{E}(R)$	0	0	0	4	0.6	0	15.312	0	0.6	1.884
σ^2_{TS}	0	0	0	0	0	0	0.323	0	0	0.217
σ^2_R	0	0	0	0	0	0	0.013	0	0	1.61
$\sigma_{TS,R}$	0	0	0	0	0	0	-0.002	0	0	-0.517

Table 8: Optimal capping strategies for different auctions

Note: Scenarios (S0–S9) are as listed in table 2. For each regulator, we display the capping structure (Caps), expected total surplus $\mathcal{E}(TS)$, expected revenue $\mathcal{E}(R)$, and the covariances of the variables $\{\sigma^2_{TS}, \sigma^2_R, \sigma_{TS,R}\}$.

C Figures

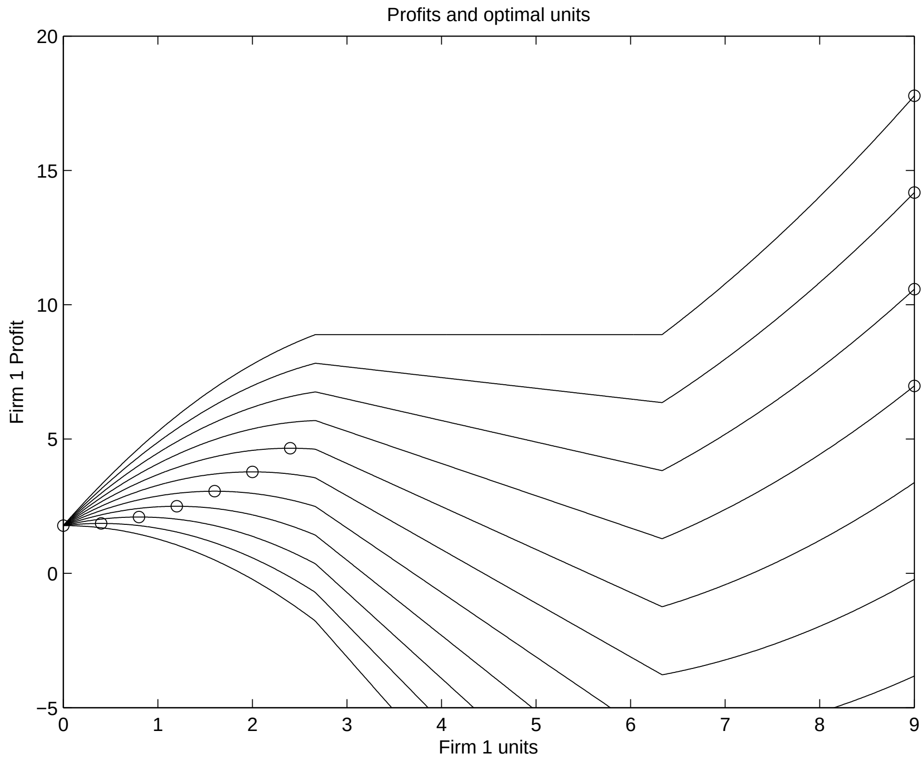

Figure 1: Duopoly Case: Profits vs. Auction Winnings

Note: These results assume market parameters as given in Table 1, with the exception of M (the number of firms) which is assumed to be two. The horizontal axis displays number of units firm 1 wins. Firm 2 is assumed to win the remaining units. Vertical axis displays firm 1's profit.

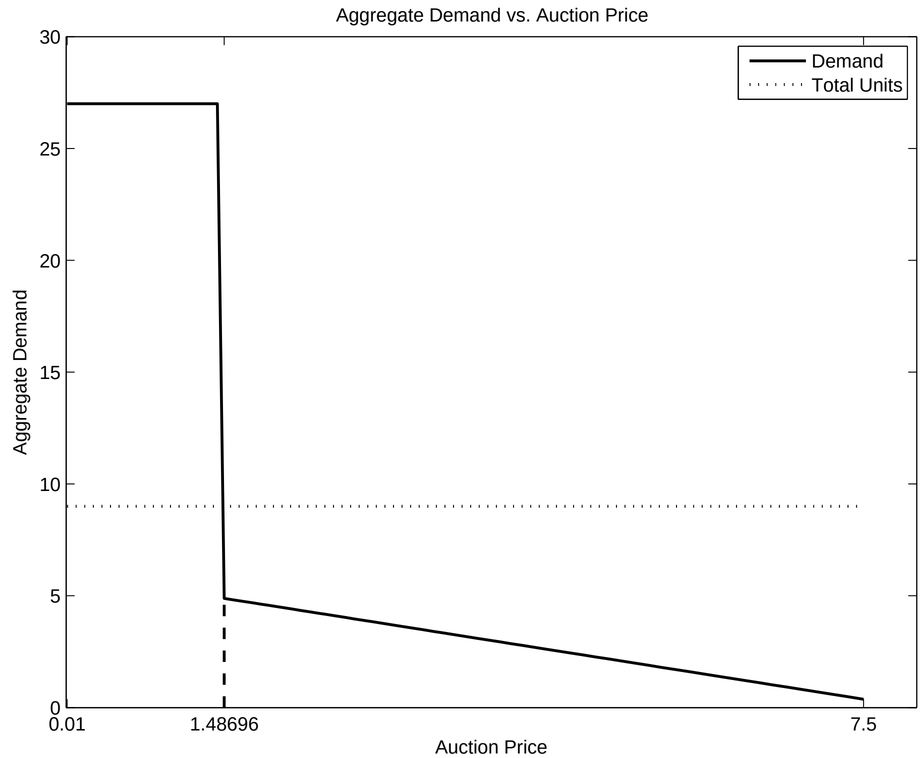

Figure 2: Non-rival bidding with continuous bids: Aggregate demand

Note: These results assume market parameters as given in Table 1 for scenario 1. The horizontal axis displays the auction price, while the vertical axis displays the aggregate demand for the three firms. These results assume Cournot behaviour by the firms: each firm bids their desired quantity given competitors' quantities.

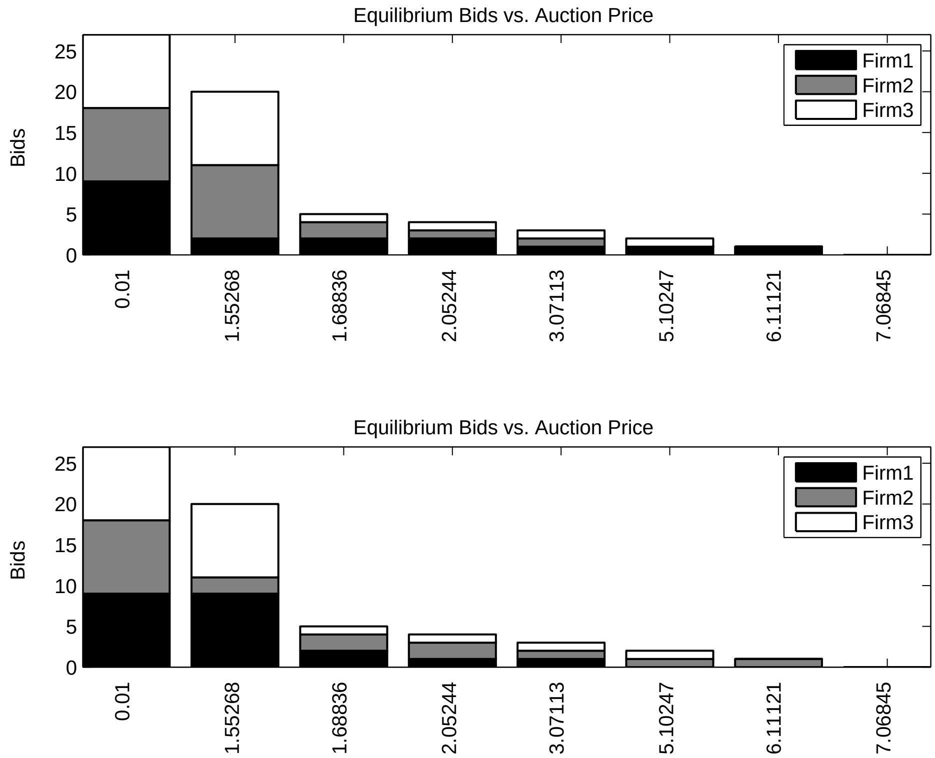

Figure 3: Non-rival bidding with discrete bids: Multiple equilibria

Note: These results assume market parameters as given in Table 1 for scenario 1. The horizontal axis shows the auction prices at which the aggregate demand changes (see figure 4). Blocks depict Pure Strategy Nash Equilibria for the Cournot game. The two cases depict alternative allocations which would be Nash equilibria at the given price.

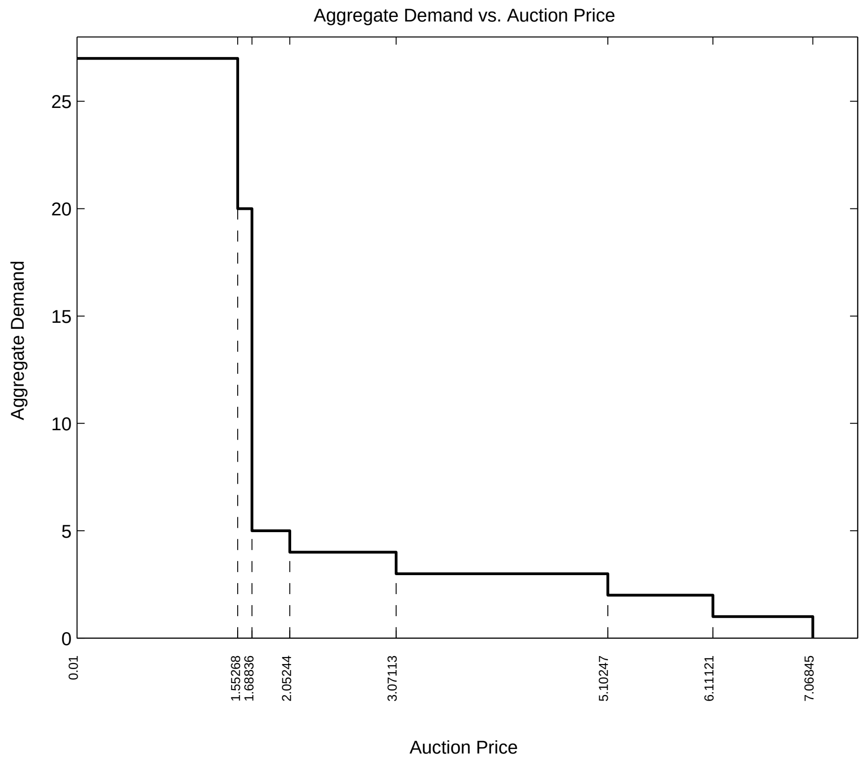

Figure 4: Non-rival bidding with discrete bids: Aggregate demand

Note: These results assume market parameters as given in Table 1 for scenario 1. The horizontal axis shows the auction price per unit of spectrum. The vertical axis shows the aggregate demand. In contrast to figure 2, firms are required to bid in discrete units.