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DETECTING NON LINEARITY

A REVIEW

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Résumé

Ces dernières années l'intérêt pour les modèles non linéaires a vu un grand essor. En conséquence la mise au point de tests performants permettant de détecter entre les différents modèles est fondamentale. On présente ici une revue d'une part des tests de linéarité, d'autre part des tests construits spécifiquement pour détecter une hypothèse particulière. On considère les tests pour lesquels la puissance a été étudiée.

Mots clés : tests de linéarité, modèles non linéaires, puissance des tests.

Classification AMS : 62 M 10 - 62 G 10.

Abstract

These last years the interest in modelling nonlinear time series has been growing steadily. Consequently, the knowledge of judicious tests to detect between linear models and nonlinear models is very important. We review the linear tests and the tests against specific alternatives. We explicit the tests for which the power has been investigated.

Keywords : linear tests, nonlinear models, power of the tests.

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I. Introduction.

During the last few years the interest in modelling nonlinear time series has been growing steadily, for a review one can see Guégan (1989). But selecting a proper nonlinear model is a real problem in applied time series analysis. At the beginning, tests have been defined to decide the nonadequacy of the linear hypothesis, but in case of rejection of the linear hypothesis from data generating process, it is not always clear how to identify a specific kind of nonlinearity. Concerning this problem, we review the recent works which have been proposed in the literature, specifying the nonlinear alternatives. The performance of these tests is evaluated on simulated data and many comparisons exist between the different tests, but in any case so few works investigate the theoretical power of the different tests that are proposed. The plan of this article is the following. In Section II we present "linear tests", this means tests which have been constructed without a specific nonlinear alternative in mind, in the frequency domain and in the time domain. In Section III we introduce linearity tests against specific alternatives. In Section IV we investigate the study of the power of the tests presented in the previous sections, and in Section V we precise the tests which have been used in real data.

II. Linear tests.

We present the linearity tests which have been constructed without a specific nonlinear alternative in mind. The idea has been to provide diagnostic tools for revealing the possible inadequacy of a linear model.

2.1. Frequency Domain

In the frequancy domain, tests for general nonlinearity have been outlined by Subba Rao and Gabr (1980), and Hinnich (1982). Subba Rao-Gabr's test is based on a general result using cumulant spectra, established by Brillinger (1965), namely the constancy over all k and over all $w_1, w_2, \ldots, w_k$ of the ratio

$$|h_k(w_1, \ldots, w_k)|^2 \, [h(w_1)h(w_2)\ldots h(w_k)h(w_1 + \ldots + w_k)]^{-1} ,$$

h_k being the cumulant spectral density function of order k and h being the spectral density function for a general stationary linear time series possessing moments of appropriate order. The Subba Rao-Gabr's test essentially looks for constancy of the ratio for $k = 2$. Hinnich (1982) has pointed out that Subba Rao-Gabr's test can be sensitive to "outliers" due to small values of the consistent estimate $\widehat{h}(w)$ of $h(w)$ at certain w. He has proposed an improved and robustified version of the test. Specially, Hinnich has replaced the usual sample second moment matrix adopted by Subba Rao and Gabr, by the known asymptotic covariance matrix of $\hat{h}_2(w_1,w_2)$ in the test statistic.

2.2 Time Domain

In time domain, different approaches have been considered.

a. Tests based on the squares of time series data

Several methods for detecting nonlinearity involves squares of time series data. This motivation is provided by a work of Granger and Newbold (1976). They showed that for a series X(t) which is normal (and therefore linear)

$$\rho_k\,(X(t)^2) = (\rho_k(X(t)))^2$$

where $\rho_k(\bullet)$ represents the lag k of the autocorrelation. Any departures from this result seems to indicate a degree of nonlinearity, as pointed out by Granger and Andersen (1978). In the same idea Maravall (1983) examines the ACF of $(X(t))^2$, or that one of $(\varepsilon(t))^2$, where $\varepsilon(t)$ is the sequence of white noise generating the process X(t).

b. Portmanteau test

A portmanteau test has been considered by McLeod and Li (1983) based on the autocorrelations of squared residual from a linear fit. It is analogous to the well known Box-Pierce statistic used to test the adequacy of an ARMA model. The McLeod-Li's test statistic is defined by

$$T_n = n(n+2)\sum_{i=1}^{M}\frac{\hat{\gamma}^2(i)}{n-i}$$

Where $$\hat{\gamma}(k) = \sum_{t=k+1}^{n}(\hat{\varepsilon}(t)-\hat{\sigma}^2)(\hat{\varepsilon}(t-k)-\hat{\sigma}^2)/\sum_{t=1}^{n}(\hat{\varepsilon}^2(t)-\hat{\sigma}(t))^2$$

with $$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n}\hat{\varepsilon}^2(t)$$

and $\hat{\varepsilon}(t) = X(t) - f(X(t),\hat{\theta}_n)$ if we define the process X(t) by the help of the function f , as

$$f(X)(t)),\theta) = \tilde{X}(t)'\cdot\theta \quad ,$$

where n is the sample size and M the number of the correlations we consider. Under the null hypothesis the distribution of the statistic T_n is asymptotically a $\chi^2(M)$. Note that these two types of tests (a) and (b) have no great discriminatory power.

c. CUSUM test (Portmanteau test)

A linearity test based to CUSUM (cumulative sums) is due to Petruccelli and Davies (1986). The proposed test is based on cumulative sums of standardized one step forecast errors from autogressive fits to the data. The idea which consists to use such a statistic in a piecewise linear regression context was developped by Artel and Fowlkes (1976) and Petruccelli and Davis followed the same idea. Let the time series data $(X_1,...,X_n)$ which are rearranged in ascending order, given $(X_{(1)},...,X_{(n)})$ such that $X_{(r)} \leq X_{(s)}$ for $r \leq s$. If r_{min} denotes a positive integer << n, for each $r << r_{min}$, an autoregressive model of the following form is fitted:

$$\begin{bmatrix} X_{(1)} \\ \vdots \\ X_{(r)} \end{bmatrix} = \begin{bmatrix} 1 & X_{(1)-1} & \cdots & X_{(1)-p} \\ 1 & X_{(2)-1} & \cdots & X_{(2)-p} \\ & \vdots & & \\ 1 & X_{(r)-1} & \cdots & X_{(r)-p} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_p \end{bmatrix} + \begin{bmatrix} \hat{e}_{(1)} \\ \vdots \\ \hat{e}_{(r)} \end{bmatrix}$$

Let z_{r+1} denotes the standartized innovation, i.e. :$z_{r+1} = \left\{X_{(r+1)} - \hat{X}_{(r+1)}\right\}/k$, where`

$$\hat{X}_{(r+1)} = \hat{a}_0 + \sum_{i=1}^{p} \hat{a}_i X_{(r+1)-i}$$

and

$$k^2 = \frac{1}{r}\sum_{i=1}^{r} \hat{e}^2_{(i)}$$

then the cumulative sums

$$Z_r = \sum_{i=r_{min}+1}^{r} z_i \,, \quad r = r_{min} + 1, \cdots, n-p$$

are formed and used as the basis for non linearity which "tracks" the cusum for systematic deviation. The statistic test used is

$$T_n = \max_{r_{min}+1 \leq r \leq n-p} |Z_r| \,.$$

And under the null hypothesis H_0 (linear fit), one uses the following result , which holds approximatively for moderate sample sizes :

$$(1) \qquad \Pr(T_n/(n-p-r_{min})^{1/2} \le t) \underset{n\to\infty}{\to} \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8t^2}\right)$$

Then for a given series, if $1-p^*$ denotes the value computed from the right hand side of (1), with t given by

$$t = \max_{r_{min}+1\le r\le n-p} |Z_r| / (n-p-r_{min})^{1/2}$$

thus the test rejects H_0 at the α level of significance if $p^* < \alpha$.

d. Test for additivity

Keenan's test (1985) has been obtained to test linearity against a second order Volterra expansion, namely

$$X_t = \mu + \sum_{u=-\infty}^{\infty} \theta_u a_{t-u} + \sum_{u,v=-\infty}^{+\infty} \theta_{uv} a_{t-u} a_{t-v} \ .$$

The test of linearity is equivalent to a test of no multiplicative terms in Volterra expansion. There is a striking ressemblance of this to the framework of Tukey's one degree of freedom test for non-additivity. Tukey's test in a regression setting is based upon the Fourier expansion of the residual. An analogous framework is used by Keenan. The test statistic T_n is based on the use of auxiliary regressions and is computed as follows :

Step 1 : For an user specified M regress X_t on $\{1, X_{t-1}, \ldots, X_{t-M}\}$. Let $\{\widehat{X}_t\}$ be the filted values, $\{\hat{a}_t\}$ the residual and SSE the residual sum of squares.

Step 2 : Regress $\{\widehat{X}_t^2\}$ on $\{1, X_{t-1}, \ldots, X_{t-M}\}$. Let $\{\widehat{\xi}_t\}$ be the residual.

Step 3 : Let $\widehat{\eta} = \sum_{t=M+1}^{n} \hat{a}_t \widehat{\xi}_t \,/\, (\sum_{t=M+1}^{n} \widehat{\xi}_t^2)$.

Step 4 : Let $T_n = \dfrac{(n-2M-2)\widehat{\eta}^2}{SSE - \widehat{\eta}^2}$.

Under the null hypothesis :

$$H_0 \ : \ \sum_{u,v=-\infty}^{+\infty} \theta_{uv} a_{t-u} a_{t-v} = 0 \ .$$

Keenan's test statistic T_n is asymptotically distributed as a F(1,n-2M-2) distribution.

Following Keenan, Tsay (1986) has formulated a test procedure by using auxiliary regressions, but in step 2, he regrèsses the products $\widehat{X}_{t-i}\widehat{X}_{t-j}$, in place of $\widehat{X}_t^2$, on $\{1, X_{t-1}, \ldots, X_{t-M}\}$, and thus performs Keenan's test.

e. Score Test

The Lagrange Multiplier or Score test used for a number of important testing procedures in linear series models has been derived by Saikonnen and Luukkonen (1988) to detect nonlinearity. This test is based on the derivative of the loglikelihood function. Consider a statistical model defined by a family of densities $f(x,\theta)$, θ being a vector parameter belonging to some open set $\Theta \subset \mathbb{R}^{m+k}$. One considers the null hypothesis $H_0 : \theta_2 = 0$, where θ_2 is formed by the k components of θ. Ones denotes by θ_1, the vector formed by the firsts m components of θ , so $\theta = (\theta_1, \theta_2)$. Let l denotes the loglikelihood function of the sample, then the score vector is defined by the first derivation of l :

$$\frac{\partial l}{\partial \theta} = \left[\frac{\partial l}{\partial \theta_1} ; \frac{\partial l}{\partial \theta_2}\right]' \quad \text{where ' represents the transposed .}$$

The Fisher information matrix J is the negative of the mathematical expectation of the second derivatives of the loglikelihood function l , and is denoted by:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = -E \begin{bmatrix} \dfrac{\partial^2 l}{\partial \theta_1^2} & \dfrac{\partial^2 l}{\partial \theta_1 \partial \theta_2} \\ \dfrac{\partial^2 l}{\partial \theta_2 \partial \theta_1} & \dfrac{\partial^2 l}{\partial \theta_2^2} \end{bmatrix}$$

Let $\widehat{\theta} = (\widehat{\theta}_1 , 0)$ be the maximun likelihood estimate of θ under H_0. Then the L.M. test or score test T_n is defined by

$$T_n = \frac{1}{n} \left(\frac{\partial l}{\partial \theta_2}\right)_{\widehat{\theta}}' \cdot \widehat{J}_{2,1}^{-1} \cdot \left(\frac{\partial l}{\partial \theta_2}\right)_{\widehat{\theta}} \qquad (2)$$

where $\widehat{J}_{2,1}$ is a consistent estimate of $J_{2,1} = J_{22} - J_{21} J_{11}^{-1} J_{12}$. Under adequate regularity assumptions, the test statistic T_n converges in distribution under the null hypothesis H_0 to the χ^2 distribution with k degrees of freedom.

f. Bootstrapped COX test.

Li (1989) apply the Cox's test for separate families of hypothesis to discrimate different nonlinear time series models. He uses bootstrap method to obtain the critical values. Let $(X_1, ..., X_n) = \mathbf{X}$ a random vector. Suppose that under H_0 the probability density function is $f(\mathbf{X},\alpha)$ where α is an unknown parameter, and under the alternative, the density function is $g(\mathbf{X},\beta)$ with β an unknown parameter. Let f and g belong to separate hypothesis, $\hat{\alpha}$ and $\hat{\beta}$ be the maximun likelihood estimates of α and β under H_0 and the alternative respectively, $L_f(\hat{\alpha})$ and $L_g(\hat{\beta})$ the corresponding maximized loglikelihood functions , then the Cox statistic T_n is:

$$T_n = L_f(\hat{\alpha}) - L_g(\hat{\beta}) - E_{\hat{\alpha}}(L_f(\hat{\alpha}) - L_g(\hat{\beta}))$$

where $E_{\hat{\alpha}}$ denotes the mathematical expectation under H_0. T_n under regularity conditions is asymptotically normally distributed under H_0. Li approximates the finite sample distribution of T_n using the bootstrap method.

Step 1: Given a realisation $(X_1, ..., X_n)$ of the time series, one fits the best models under the two separate hypothesis. Denote them M_0 and M_A.

Step 2: For a large enough positive integer B, B sets of artificial realisations $R_k = \{x^*_{1k}, \cdots, x^*_{nk}\}$ are generated under M_0. Maximum likelihood estimates $\hat{\alpha}^*_k$ and $\hat{\beta}^*_k$ are obtained for each of the realisation. An approximation of the distribution $C_n = L_f(\hat{\alpha}) - L_g(\hat{\beta})$ under H_0 is obtained from the empirical distribution $C^*_{nk} = L_f(\hat{\alpha}^*_k) - L_g(\hat{\beta}^*_k)$.

Step 3: The hypothesis H_0 is rejected at level α if C_n exceeds the $[B\alpha]$ order statistic of C^*_{nk}.

g. Neyman-Pearson test.

Chen (1989) consider a new test based on the Neyman-Pearson test, considering that this previous test presents certain disavantages as the acceptance of H_0 is not reliable, the power functions cannot often be obtained and the consistency can hardly discussed. His different sketch is the following: let the two hypothesis:

$$H_0 : \theta \in \Theta_0$$

and

$$H_1 : \theta \in \Theta_1.$$

If one can form statistics as

$$S_N = T + Q_N \text{ a.s.,}$$

where T is constant (known) and $Q_N \rightarrow 0$ a.s. if $\theta \in \Theta_0$, and Q_N keeps away from 0 a.s., if $\theta \in \Theta_1$. One may define R_n as acceptance region. R_n is usually defined by critical function C_n like:

$$R_N = \{S : S \leq T + C_N\}$$

or

$R_N = \{S : T - C_N \leq S \leq T + C_N\}$ so on.

Some examples for the choice of C_N are:

(1) $S_N = T + O(\log \log N/N)$ a.s. $\theta \in \Theta_0$,

$\liminf S_N = T' > T$, but T' may be unknown, $\theta \in \Theta_1$.

Take $C_N = \log N / N$.

(2) $\limsup S_N = T$, a.s. $\theta \in \Theta_0$,

$\lim S_N = \infty$, a.s. $\theta \in \Theta_1$,

Take $C_N = c$, a constant.

III. Linearity tests against specific alternatives.

Some works are more specified. They concern tests which have been obtained to identify specific kind of non-linearity. Sometimes they can be extended and used to detect other types of non-linearities.

3.1. Extension to the Bartlett-Quenouille's test.

An extension to the Bartlett-Quenouille's test has been derived by Guégan (1984) to detect bilinear models. The idea is the following: from the study of the moments of order fouth of the models, different statistics have been established in function of the particular bilinear models one wish to detect. Under the null hypothesis, using consistent estimates to the fourth moments, one shows that the considered statistic is Gaussian with adequate mean and finite variance. This procedure can be easily generalized for a wide class of bilinear models and for other kinds of non-linear models, but the computations become rapidly very tedious.

3.2. Log-likelihood ratio test.

Chan and Tong (1986) considered a likelihood ratio test for discrimating SETAR models from linear ones. Let X_t a SETAR process with two regimes:

$$\begin{cases} X_t - \sum_{i=1}^{p_1} \varphi_i^{(1)} X_{t-i} = \mu^{(1)} + a_t^{(1)} & \text{if } X_{t-d} \leq w \\ X_t - \sum_{i=1}^{p_2} \varphi_i^{(2)} X_{t-i} = \mu^{(2)} + a_t^{(2)} & \text{if } X_{t-d} > w \end{cases}$$

where d and w are referred to be the delay and threshold parameters respectively, and $\{ a_t^{(i)} \}$' s, i= 1, 2, are two independant Gaussian white noise processes with mean zero and variance $\sigma^2_{a,i} > 0$. The null hypothesis is $H_0 : p_1 = p$ and $\varphi_v^{(1)} = \varphi_v^{(2)}$ for v = 0, 1, ... , p. It is assumed that p and d are known. Under H_0 the threshold parameter w is absent. Let RSS_{AR} and $RSS_{TAR(w)}$ denote the residual sums of squares under H_0 and the alternative respectively after a least squares fit. Then the test statistic H_0 is:

$$T_n = \sup_{w \in R} (RSS_{AR} - RSS_{TAR(w)})/\hat{\sigma}^2$$

with $$\hat{\sigma}^2 = \inf_{w \in R} RSS_{TAR(w)} / n$$

where n = N -p + 1 is the effective number of observations. Chan and Tong (1988) show that the asymptotic null distribution of the likelihood ratio test statistic may be identified with the first passage probability of an m-dimensional Brownian bridge. They show too that, in some specific cases, the asymptotic null distribution of the test statistic depends of "degrees of freedom" and not of the exact null joint distribution if the time series.

3.3. Threshold test.

Tsay (1989a) from the idea of the CUSUM's test, derives a test for SETAR models. This test differs from the previous one in the way by which the predictive residuals are used. Basically, the predictive residuals of the arranged autoregression is orthogonal to the linear regressors if X_t is linear. Let the following TAR model:

$$X_t = \varphi_0^{(j)} \sum_{i=1}^{p} \varphi_i^{(j)} X_{t\ i} + a_t^{(j)} \qquad r_{j\ 1} \le X_{t\text{-}d} \le r_j$$

j = 1, ... , k and d a positive integer, the thresholds are $-\infty = r_0 < r_1 < \ldots < r_k = \infty$, and consider an arranged autoregression according to the threshold variable. In case of a TAR(2;p,d) model with n observations, the threshold variable $X_{t\ d}$ may assume values $\{X_h, \ldots, X_{n\ d}\}$ where h = max(1,p+1-d). Let π_i be the time index of the i^{th} smallest observation of $\{X_h, \ldots, X_{n\text{-}d}\}$. One may rewrite the model as:

$$X_{\pi_i+d} = \varphi_0^{(1)} + \sum_{v=1}^{p} \varphi_v^{(1)} X_{\pi_i+d\text{-}v} + a^{(1)}_{\pi_i+d} \qquad \text{if } i \le s$$

$$= \varphi_0^{(2)} + \sum_{v=1}^{p} \varphi_v^{(2)} X_{\pi_i+d\ v} + a^{(2)}_{\pi_i+d} \qquad \text{if } i > s$$

where s satisfies $X_{\pi_s} < r_1 < X_{\pi_{s+1}}$. This is an arranged autoregression with the first s cases in the first regime and the rest in the second regime. For n and d fixed, the effective number of observations is then n-d-h+1. One assumes that the recursive autoregressions begin with b observations so that there are n-d-b-h+1 predictive residuals available. Then one makes the least square regression

$$(1) \qquad \hat{e}_{\pi_i+d} = w_0 + \sum_{v=i}^{p} w_v X_{\pi_i+d-v} + \varepsilon_{\pi_i+d}$$

for i = b+1, ..., n-d-h+1, and one considers the test statistic

$$T_n = \frac{(\sum \hat{e}_t^2 - \sum \hat{\varepsilon}_t^2) / (p+1)}{\sum \hat{e}_t^2 / (n\text{-}d,b\text{-}p\text{-}h)}$$

whose the summations are over all the observations and where $\hat{\varepsilon}_t$ is the least squares residual of (1). Then if X_t is a linear AR process of order p, for n large, the distribution of the statistic T_n follows approximatively an F distribution with p+1 and n-d-b-p-h degrees of freedom. Furthermore the distribution of $(p+1)T_n$ is asymptotically a chi-squared random variable with (p+1) degrees of freedom.

3.4. Generalized threshold test.

Tsay (1989 b) has observed that:

- The idea of Lagrange Multiplier tests appear to be powerful in detecting the finite degree of non-linearity,

- The idea of arranged autoregression is useful in splotting threshold nonlinearity, so he derives a new test combining these two ways.

Let an autoregression of order m, denoted X_t, one knows that the associated residual $\{\hat{e}_t\}$ is asymptotically a white noise process if X_t is a linear AR(p) process. On the other hand, if X_t is bilinear then $\{\hat{e}_t\}$ is related to $X_{t-j}\varepsilon_{t-j}$ for some i and j. Consequently, to detect the possibility of bilinearity in X_t one may apply the technique of added variables such as $\{X_{t-j}\hat{e}_{t-i}\}$ and $\{\hat{e}_{t-i}\hat{e}_{t-i-1}\}$ for i = 1, ..., m. The same idea can be applied to EXPAR, with added variables $X_{t\ 1} \exp(- X_{t-1}^2 / \gamma)$ where γ is a normalized constant ($\gamma = \max (|X_{t-1}|)$). In case of SETAR models, one rearranges as in the previous test. So the schema of the test is the following:

Step 1: Fit recursively an arranged autoregression of order m to X_t, and calculate the normalized predictive residuals $\tilde{e}_t$, t = b+1, ..., n, where b is chosen such that there are sufficient cases to start the recursion.

Step 2: Regress $\tilde{e}_t$ on the regressors $\{1, X_{t-1}, ..., X_{t-m}\}$, $\{X_{t-i}\tilde{e}_{t-i}, \tilde{e}_{t-i}\tilde{e}_{t-i-1} / 1 \le i \le n\}$ and $X_{t-1} \exp(-X_{t-1}^2 / \gamma)$. ...

Compute the associate F statistic T_n as in the previous threshold test. If X_t is a linear AR(p) process of order $p \le m$, T_n follows approximatively an F-distribution with degrees of freedom 3(m+1) and n-b-3(m+1).

3.5. Correlation dimension test.

Brock, Dechert and Scheinkman (BDS) (1986) established a test based on the correlation dimension. They consider that the correlation dimension permits in principle to distinguish between data generated by independant and identically distributed random variables from data generated by a deterministic non-linear difference equation, even in situations where traditional linear techniques fall to distinguish between these two alternatives. Let $X_1, ..., X_N$ be independant with a common distribution F. For each $\varepsilon > 0$, let $I_\varepsilon : \mathbf{R}^2 \to \mathbf{R}$ be the indicator function of the set $B_\varepsilon = \{ (z,y) \in \mathbf{R}^2, |z-y| \le \varepsilon\}$ i.e. if $(z,y) \in B_\varepsilon$, $I_\varepsilon(z,y) = 1$ and if $(z,y) \notin B_\varepsilon$, $I_\varepsilon(z,y) = 0$ otherwise. For each $n \ge 1$, let

$$C(m,n,\varepsilon)\,(X_1, \cdots, X_n) = \frac{2}{n(n-1)} \sum_{\substack{1 \le i,j \le n \\ n+m \le N}} \prod_{k=0}^{m-1} I_\varepsilon(X_{i+k}, X_{j+l}), \text{ with } n+m \le N$$

If we consider the m-histories $Y_t = (X_1, ..., X_{t-m+1})$, then the correlation dimension of $(Y_t)_{t=0}^{\infty}$ is given by:

$$d = \lim_{\varepsilon \to 0} \{\frac{\log [\lim C(m,n,\varepsilon)(X_1, \cdots, X_{n+m})]}{\log \varepsilon}\}$$

BDS uses the previous notion to test non-linear dependence knowing that

$$\sqrt{n}\,(C(m,n,\varepsilon) - (C(1,n,\varepsilon))^m) \xrightarrow{D} N(0,\sigma)$$

where, the variance σ depends of $C(m,n,\varepsilon)$. Scheinman and Le Baron (1986) introduced the statistic $S(m,n,\varepsilon) = C(m,n,\varepsilon) / C(m-1,n,\varepsilon)$ in order to study non-linear dependence. And one have the following result. For any integer $m > 1$, as $n \to \infty$

$$\sqrt{n}\,(S(m,n,\varepsilon) - (C(1,n,\varepsilon))) \xrightarrow{D} N(0,\rho)$$

where the variance ρ depends also of the correlation $C(m,n,\varepsilon)$. Brock and Dechert (1988) apply the above method in case of GARCH model.

IV. Study of the power of the different tests.

More recently, most of the works try to discuss the power of these different tests. Thanks to the simulations, one can obtain an idea of the power of the tests under specific alternatives. Some works compare the empirical power of various tests under the same alternative, some of them point out that certain tests allow to have a better detection of certain linearities than others.

4.1. The frequency domain.

Ashley, Patterson and Hinnich (1986) present an artificial data analysis of the power of Hinnich's test for certain nonlinear models. They examine the power of the Hinnich's linearity test to detect the mispecification in case of a linear ARMA model fitted to data generated by a non-linear model as bilinear model, non-linear autoregressive model, threshold model and exponential model. They show that the Hinnich's test detect nonlinearity with good performance.

Brockett, Hinnich and Patterson (1988) apply these results in case of different kinds of series in various domains as economy, meteorology,....

4.2 The time domain.

Davies and Petruccelli (1986) compare two tests: McLeod-Li's test and Keenan's test with the help of a large collection of time series generated by SETAR models. They show that the Keenan's test statistic successfully detected naearly half the series as being nonlinear, while after fitting linear AR(1) models to all the SETAR series, the McLeod-Li's test statistic detect residual problem in just one series on six. These statistics did not agree on those series that each detected is being non-linear.

Chan and Tong (1986) construct a comparative study between Subba Rao-Gabr's test, Hinnich's test, the CUSUM test and the likelihood ratio statistic. They consider data generated by non-linear models as bilinear models, non-linear moving average model, threshold model and exponential model. They conclude that the comparative study suggests that the Hinnich's modification of Subba Rao-Gabr's frequency domain test exhauses the efficacity of the latter and may be used together with time domain tests such as Keenan's test, the CUSUM test and the likelihood ratio test.

Lukkonen, Saikonnen and Terasvirta (1988a) have investigated the power of different test statistics (Keenan's test, Tsay's test, McLeod-Li's test, LM test), when the true non-linear model is ARCH, bilinear EAR and SETAR models. They show that the power of the LM tests against "incorrect" alternatives varies widely depending on the parameters of the data generating process. One of the few regularities might be that the test of McLeod and Li responds to heteroscedasdicity in the errors but too little else, this test can be consistent against bilinear models and the LM tests do not perform very well if the true model is ARCH. Luukkonen, Saikkonen and Terasvirta (1988b) adapted Keenan's and Tsay's tests in case of SETAR models and investigate their empirical power. They conclude that the adapted test is a useful alternative to the CUSUM test in testing linearity of a univariate time series model against non-linear SETAR models. In the same way, Moenaddin and Tong (1988) make a comparison between CUSUM and likelihood ratio test for SETAR models. It seems clear that the CUSUM test is more sensitive to outliers and tends to regard a linear series with a outlier as nonlinear.

Petruccelli (1989) adapted the CUSUM test for SETAR models, and he compares his empirical power with other tests' power, as Tsay's test, likelihood ratio test and the LM test. He concludes that no one of the considered tests performs for all the models investigated. There are clear differences apparent in the performance of the various tests, but there is a uniform poor performance of the likelihood ratio test. The LM test seems to perform the others.

Brock, Dechert and Scheinkman (1986) compares the power of their test with the one of Hinnich, considering simulated non-linear moving average and threshold models. They show that the percentage of samples rejected is always higher than the one obtained with Hinnich's test, and they constate that increasing in the dimension, seems to lead to an increase on the power.

So few papers investigate the theoretical power of these different tests. Guégan-Pham (1989) investigate the local power of Lagrange Multipliers against diagonal bilinear alternatives which are contiguous to the null hypothesis. They show that the statistic T_n defined by (2) converge under the alternative to a noncentral χ^2 variate with P degrees of liberty, where P represents the order of the diagonal bilinear part. The theoretical computation of the local power has been compared with simulations, showing good agreement..

V. Uses of the tests with real datas.

Most of the previous above-mentioned works have used the different tests to compare the choice of different alternatives on real data. We particularly note:

- Luukkonen and Teravirta (1988) who have applied the previous results on econometric datas.

-Ray (1988) who has adjusted different non-linear models on indian econometric datas and uses different tests to decide the best adequate model.

-Tsay (1989a, 1989b) who has used the previous different tests to decide models for sunspot data.

-Li (1989) who has used his test to decide between bilinear or threshold models on sunspot datas.

VI Conclusion.

This paper is a review concerning the different tests existing to detect nonlinearity. An other approach which seems very interesting is the nonparametric one as it has been developed, in particular by Diebolt (1989). In his paper Diebolt presents two nonparametric goodness-of-fit tests for the nonlinear autoregressive process defined by

$$X_{n+1} = T(X_n) + U(X_n)\varepsilon_{n+1},$$

which contains in his formulation some of the nonlinear above-mentioned models. The comparison with this approach and the one developed in the previous sections can give interesting developments in the future.

Note that one can find also a review of the different tests presented here, in the chapter five of the recent and very interesting book of Tong (1990).

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