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FROM CUBICAL TO GLOBULAR HIGHER CATEGORIES

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From cubical to globular higher categories (*)

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dedié à Andrée Ehresmann, en amitié et admiration

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Abstract. We show that a strict symmetric (infinite-dimensional) cubical category $\mathbb{A}$ has an associated ω-category $\mathrm{Glb}(\mathbb{A})$, consisting of its 'globular cubes'. The procedure of globularisation generally destroys important features of $\mathbb{A}$, like the existence of limits and colimits or the presence of symmetries. Then we examine the much more complex weak case, up to constructing the tricategory associated to a weak symmetric 3-cubical category.

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Introduction

This paper takes on a study of the theory of weak cubical categories, begun in [G1-G5] with the aim of extending to higher dimension the study of weak double categories developed in [GP1-GP4].

Here we prove that a strict symmetric (infinite-dimensional) cubical category $\mathbb{A}$ has an associated ω-category $\mathrm{Glb}(\mathbb{A})$, consisting of its 'globular cubes'. Then we examine in low dimension the much more complex weak case, up to constructing the tricategory associated to a weak symmetric 3-cubical category. As a general fact, the procedure of globularisation tends to destroy important features of cubical categories, like the existence of limits and colimits or the presence of symmetries; the main motivation of this analysis is that, presently, higher categories are mostly studied in the globular form - with some exceptions like those cited above.

Let us recall that a *weak cubical category* [G1-G5] has a cubical structure, with faces and degeneracies; moreover, there are weak compositions in countably many directions, which we call *cubical* (or *geometric*), and one strict composition, in the *transversal* (or *structural*) direction.

As a leading example, linked to higher dimensional cobordism (see [G1-G3]), one can think of the weak cubical category $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ of cubical cospans in a category with pushouts $\mathbf{X}$. An n-dimensional *object* is a functor $x\colon \wedge^n \to \mathbf{X}$, where $\wedge$ is the 'formal cospan' category

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$$(1) \qquad -1 \to 0 \leftarrow 1 \qquad \wedge \qquad\qquad \begin{array}{ccccc} (-1,-1) & \longrightarrow & (0,-1) & \longleftarrow & (1,-1) \\ \downarrow & & \downarrow & & \downarrow \\ (-1,0) & \longrightarrow & (0,0) & \longleftarrow & (1,0) \\ \uparrow & & \uparrow & & \uparrow \\ (-1,1) & \longrightarrow & (0,1) & \longleftarrow & (1,1) \end{array} \qquad \wedge^2.$$

An n-dimensional *transversal map*, or *structural map*, is a natural transformation $f\colon x \to y\colon \wedge^n \to \mathbf{X}$ of such functors. Their composition is also called *transversal*, or *structural*.

The ordinary categories $\mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{Cat}(\wedge^n, \mathbf{X})$ form a cubical object in $\mathbf{Cat}$, with obvious faces and degeneracies. Moreover, n-dimensional objects (and maps) have *cubical*, or *geometric*, composition laws $x +_i y$ in each direction $i = 1,..., n$, which are constructed with pushouts; these compositions are consistent with faces and degeneracies, *but only behave well up to suitable transversal maps,* which yield invertible comparisons for their associativity, unitarity and interchange.

Actually, as already stressed in the papers mentioned above, $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ is a *symmetric* weak cubical category, when equipped with the obvious action of the symmetric group S_n on $\mathbf{Cat}(\wedge^n, \mathbf{X})$; namely, the action of permuting the factors of $\wedge^n$, i.e. the directions of n-cubical cospans in $\mathbf{X}$. These symmetries allow one to only consider the faces, degeneracies and cubical compositions in a single direction (see 1.3), which greatly simplifies the coherence conditions. Notice also that cubical 1-truncation, keeping *one* weak direction *and* the strict transversal one, yields the weak *double* category $\mathbb{C}\mathrm{osp}(\mathbf{X})$ of ordinary cospans and their transversal maps, studied in [GP1]; here, symmetries 'disappear', since the groups S_0 and S_1 are trivial.

It should be noted that the importance of considering the transversal maps goes beyond the fact of containing the comparisons for the associativity, unitarity and interchange of the cubical compositions: this point only requires invertible transversal maps and disappears in the strict case. In fact, cubical limits (or colimits) have been dealt with in [G5], extending the theory of double limits developed in [GP1]: their projections (or injections) are transversal maps. Thus, if our previous category $\mathbf{X}$. is (co)complete, then $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ has all cubical (co)limits, in a way that is consistent with faces, degeneracies and transposition, and (co)lax functorial with respect to cubical composition. (The last fact is a general, straightforward consequence of the universal property; but here, where cubical compositions are based on pushouts, the cubical *co*limits are actually pseudo functorial.)

Outline. In the first two sections we begin by considering a (strict) symmetric cubical category $\mathbb{A}$ and its globularisation $\mathrm{Glb}\mathbb{A}$, a strict ω-category consisting of the 'globes', or globular cubes of $\mathbb{A}$. As an example, the symmetric cubical category $\omega\mathbb{R}\mathrm{el}$ of cubical relations of sets (1.8) yields an ω-category (2.7); the transversal maps of $\mathbf{A}$ do not intervene in this procedure of globularisation (but, again, are essential for the cubical (co)limits of $\omega\mathbb{R}\mathrm{el}$).

Then, in Sections 3 and 4, we recall the definition of a *weak* symmetric cubical category and some main examples, like cubical cospans on a category with pushouts. We also give a construction of the weak symmetric cubical category $\omega\mathbb{C}\mathrm{at}$ of cubical profunctors, modifying a similar construction given in [G4] with less adequate transversal maps.

Finally, in the last section, we examine globularisation in the weak case, up to constructing the tricategory (see Gordon, Power and Street [GPS]) associated to a weak symmetric 3-cubical category; here, the transversal maps contain the comparisons for associativity, unitarity and interchange, and intervene in the globularisation by their *companion* cubes (see 5.1). The problem of globularisation becomes very complicated in the (higher or) infinite-dimensional case, and should be based on one of the many - more or less equivalent - definitions of weak ω-category that have been proposed.

References to the rich literature on higher globular categories can be found in two recent books, by T. Leinster [Le] and E. Cheng - A. Lauda [CL]. Strict cubical categories with connections (and without transversal maps) have been studied in [ABS], and proved to be equivalent to (globular) ω-categories.

As a matter of notation, the indices α, β take the values $0, 1$, that are more often written as $-, +$.

1. Strict symmetric cubical categories

We begin by considering (strict) symmetric cubical categories; their globularisation will be defined in Section 2.

Some simpler notions will be used as preliminary steps. We use the term 'basic' when transversal maps have not yet been inserted in the structure, so that a cubical category can be defined as a category object in the category of basic cubical categories (cf. 1.5).

Moreover, as a preliminary step for the weak case dealt with in the sequel, we use the prefix 'pre' when we are not assuming the axioms of associativity, unitarity and interchange of the cubical compositions (and comparisons have not yet been introduced).

We end by constructing, as an example, the symmetric cubical category $\omega\mathbb{R}\mathrm{el}$ of cubical relations (1.8).

1.1. Cubical sets. A cubical set $X = ((X_n), (\partial^\alpha_i), (e_i))$, in the usual sense [K1, K2, BH1, BH2], has *faces* (∂^α_i) and *degeneracies* (e_i)

(1) $\partial^\alpha_i\colon X_n \rightleftarrows X_{n-1} :e_i \qquad (i = 1,..., n;\ \alpha = \pm),$

satisfying the cubical relations :

(2) $\partial^\alpha_i.\partial^\beta_j = \partial^\beta_j.\partial^\alpha_{i+1} \quad (j \leq i), \qquad e_j.e_i = e_{i+1}.e_j \quad (j \leq i),$

$\partial^\alpha_i.e_j = e_j.\partial^\alpha_{i-1} \quad (j < i), \qquad$ *or* $\qquad \mathrm{id} \quad (j = i), \qquad$ *or* $\qquad e_{j-1}.\partial^\alpha_i \quad (j > i).$

Elements of X_n are called n-*cubes*; *vertices* and *edges* for $n = 0$ or 1, respectively. Every n-cube $x \in X_n$ has 2^n vertices: $\partial^\alpha_1\partial^\beta_2\partial^\gamma_3(x) = \partial^\gamma_1\partial^\beta_1\partial^\alpha_1(x)$ for $n = 3$ and $\alpha, \beta, \gamma = \pm$.

A *morphism* $f = (f_n)\colon X \to Y$ is a sequence of mappings $f_n\colon X_n \to Y_n$ commuting with faces and degeneracies.

Small cubical sets and their morphisms form a category **Cub**, which has all limits and colimits and is cartesian closed. In fact, it is the presheaf category of functors $X\colon \mathbb{I}^{\mathrm{op}} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the *elementary cubes* $2^n = \{0, 1\}^n$, together with the maps $\{0, 1\}^m \to \{0, 1\}^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates [GM].

The terminal object $\top$ is freely generated by one vertex $*$ and will also be written $\{*\}$; but notice that *each* of its components is a singleton. The initial object is empty, i.e. all its components are; the other cubical sets have a non-empty component in each degree.

1.2. Symmetric cubical sets. As in [G1], a *symmetric cubical* set is a cubical set which is further equipped with mappings, called *transpositions*

(1) $s_i \colon X_n \to X_n$ $\qquad$ $(i = 1,..., n-1;\ n \geq 2)$.

These have to satisfy the Moore relations

(2) $s_i.s_i = 1,$ $\qquad$ $s_i.s_j.s_i = s_j.s_i.s_j \quad (i = j-1),$ $\qquad$ $s_i.s_j = s_j.s_i \quad (i < j-1),$

and the following equations of coherence with faces and degeneracies:

		$j < i$	$j = i$	$j = i+1$	$j > i+1$
(3) $\partial^\alpha_j.s_i$	$=$	$s_{i-1}.\partial^\alpha_j$	∂^α_{i+1}	∂^α_i	$s_i.\partial^\alpha_j,$
$s_i.e_j$	$=$	$e_j.s_{i-1}$	e_{i+1}	e_i	$e_j.s_i.$

Assigning the mappings (1) under conditions (2) amounts to letting the symmetric group S_n operate on X_n. Indeed, it is well known that S_n is generated, under the Moore relations, by the 'ordinary' transpositions $s_1,..., s_{n-1}$, where s_i, acting on the set $\{1,..., n\}$, exchanges i with $i+1$ (see Coxeter-Moser [CM], 6.2; or Johnson [Jo], Section 5, Thm. 3).

A *morphism* $f = (f_n) \colon X \to Y$ is a sequence of mappings $f_n \colon X_n \to Y_n$ commuting with faces, degeneracies and transpositions. The resulting category s**Cub** (of small symmetric cubical sets and their morphisms) is again a category of presheaves $X \colon \mathbb{I}_s{}^{op} \to \mathbf{Set}$, for the *symmetric cubical site* $\mathbb{I}_s$. The latter can be defined as the subcategory of **Set** consisting of the elementary cubes $2^n = \{0, 1\}^n$ together with the maps $2^m \to 2^n$ which delete some coordinates, permute the remaining ones and insert some 0's and 1's. It is a subcategory of the *extended cubical site* $\mathbb{K}$ of [GM], which also contains the 'connections' (higher degeneracies).

The truncated cases will also be of interest. A *symmetric n-cubical* set $X = ((X_k), (\partial^\alpha_i), (e_i), (s_i))$ has components indexed by $k = 0,..., n$. Of course, also its faces $\partial^\alpha_i \colon X_k \to X_{k-1}$, degeneracies $e_i \colon X_{k-1} \to X_k$ and transpositions $s_i \colon X_k \to X_k$ undergo the restriction $k \leq n$, and satisfy the symmetric cubical relations as far as appropriate.

A symmetric n-cubical set is a presheaf on the *truncated site* $\mathrm{Tr}_n\mathbb{I}_s$, namely the full subcategory of $\mathbb{I}_s$ with objects 2^k for $k \leq n$. We write as Tr_ns**Cub** the category of symmetric n-cubical set and their (obvious) morphisms.

1.3. A simpler presentation of symmetric cubical sets. In a symmetric cubical set, the presence of transpositions makes all faces and degeneracies determined by the 1-directed ones, $\partial^-_1, \partial^+_1$ and e_1. In fact, from $\partial^\alpha_{i+1} = \partial^\alpha_i.s_i$ and $e_{i+1} = s_i.e_i$, we deduce that:

(1) $\partial^\alpha_i = \partial^\alpha_1.\mathbf{s}'_i,$ $\qquad$ $e_i = \mathbf{s}_i.e_1$ $\qquad$ $(i = 2,..., n;\ \alpha = \pm),$

where we are using the inverse 'permutations' $\mathbf{s}_i$ and $\mathbf{s}'_i$

(2) $\mathbf{s}_i = s_{i-1}. \dots .s_1,$ $\qquad$ $\mathbf{s}'_i = s_1. \dots .s_{i-1}.$

This leads to a more economical presentation of our structure, as proved in [G3]. Namely, a symmetric cubical set can be equivalently defined as a system

(3) $\quad X \;=\; ((X_n),\, \partial_1^-,\, \partial_1^+,\, e_1,\, (s_i)),$

under the Moore relations for transpositions (1.2.2) and the axioms:

(4) $\quad \partial_1^\alpha.\partial_1^\beta \;=\; \partial_1^\beta.\partial_1^\alpha.s_1, \qquad e_1.e_1 \;=\; s_1.e_1.e_1, \qquad \partial_1^\alpha.e_1 \;=\; \mathrm{id},$

$\quad\;\; s_i.\partial_1^\alpha \;=\; \partial_1^\alpha.s_{i+1}, \qquad e_1.s_i \;=\; s_{i+1}.e_1.$

In other words, X can be presented as a system $((X_n), \partial_1^-, \partial_1^+, e_1)$ where each X_n is an S_n-set (equipped with an action of the symmetric group S_n) and the axioms (4) are satisfied.

1.4. Basic cubical categories. As a further step towards cubical categories, we define now the notion of a *basic cubical category* (called a 'reduced' cubical category in [G1, G4]), as a cubical set equipped with *cubical compositions* in all directions; these are assumed to be strictly categorical (i.e. strictly associative and unital, units being given by degeneracies) and to satisfy the interchange property.

More explicitly, our notion is defined as follows.

(cub.1) A basic cubical category $\mathbf{A}$ is, first of all, a cubical set (1.1):

(1) $\quad \mathbf{A} \;=\; ((A_n),\, (\partial_i^\alpha),\, (e_i)).$

(cub.2) Moreover, for $1 \leq i \leq n$, the i-*concatenation* $x +_i y$ (or i-composition) of two n-cubes x, y is defined when the latter are *i-consecutive*, i.e. $\partial_i^+(x) = \partial_i^-(y)$; the following 'geometric' interactions with faces and degeneracies are required:

(2) $\quad \partial_i^-(x +_i y) \;=\; \partial_i^-(x), \qquad \partial_i^+(x +_i y) \;=\; \partial_i^+(y),$

$$\partial_j^\alpha(x +_i y) \;=\; \begin{cases} \partial_j^\alpha(x) +_{i-1} \partial_j^\alpha(y), & \text{if } j < i, \\ \partial_j^\alpha(x) +_i \partial_j^\alpha(y), & \text{if } j > i, \end{cases}$$

(3) $$e_j(x +_i y) \;=\; \begin{cases} e_j(x) +_{i+1} e_j(y), & \text{if } j \leq i \leq n, \\ e_j(x) +_i e_j(y), & \text{if } i < j \leq n+1 \end{cases} \qquad (\textit{nullary interchange}).$$

(cub.3) For $1 \leq i \leq n$, we have a category $A_i^n = (A_{n-1}, A_n, \partial_i^-, \partial_i^+, e_i, +_i)$, where faces give domains and codomains, and degeneracy yields the identities. In other words, we have the following equations for i-consecutive n-cubes x, y, z:

(4) $\quad (x +_i y) +_i z \;=\; x +_i (y +_i z), \qquad e_i\partial_i^- x +_i x \;=\; x \;=\; x +_i e_i\partial_i^+ x.$

(cub.4) For $1 \leq i < j \leq n$, and n-cubes x, y, z, u, we have

(5) $\quad (x +_i y) +_j (z +_i u) \;=\; (x +_j z) +_i (y +_j u) \qquad$ (*middle-four interchange*),

whenever these compositions make sense:

(6) $\quad \partial_i^+(x) \;=\; \partial_i^-(y), \quad \partial_i^+(z) \;=\; \partial_i^-(u),$

$\quad\;\; \partial_j^+(x) \;=\; \partial_j^-(z), \quad \partial_j^+(y) \;=\; \partial_j^-(u),$

$$\begin{array}{|c|c|} \hline x & y \\ \hline z & u \\ \hline \end{array} \qquad\qquad \begin{array}{l} \bullet \longrightarrow i \\ \downarrow j \end{array}$$

A *cubical functor* $F: \mathbf{A} \to \mathbf{B}$ between basic cubical categories is a morphism of cubical sets which preserves all composition laws.

1.5. Cubical categories. Now, a *cubical category* $\mathbb{A}$ [G1] is a category object in the category of basic cubical categories (and their cubical functors)

$$(1) \qquad \mathbb{A}^{(0)} \underset{e_0}{\overset{\partial_0^\alpha}{\leftleftarrows\!\!\rightarrow}} \mathbb{A}^{(1)} \overset{c_0}{\longleftarrow} \mathbb{A}^{(2)} \qquad\qquad (\alpha = \pm),$$

or, equivalently, a basic cubical category in the category of categories

$$(2) \quad \mathbb{A} = ((\mathrm{tv}_n\mathbb{A}), (\partial_i^\alpha), (e_i), (+_i)), \qquad\qquad \mathrm{tv}_n\mathbb{A} = (A_n, M_n, (\partial_0^\alpha), e_0, c_0),$$

where A_n and M_n denote the set of objects and morphisms of the category $\mathrm{tv}_n\mathbb{A}$, respectively.

Explicitly, this statement means that $\mathbb{A}$ is a basic cubical category where each component $\mathrm{tv}_n\mathbb{A}$ is a category (namely, the category of n-cubes of $\mathbb{A}$ and their *transversal n-maps*, called the *transverse category of* $\mathbb{A}$ *of degree* n), while the cubical faces, degeneracies and concatenations are functors

$$(3) \quad \partial_i^\alpha: \mathrm{tv}_n\mathbb{A} \rightleftarrows \mathrm{tv}_{n-1}\mathbb{A} : e_i, \qquad\qquad +_i: \mathrm{tv}_n\mathbb{A} \times_i \mathrm{tv}_n\mathbb{A} \to \mathrm{tv}_n\mathbb{A}.$$

(The pullback $\mathrm{tv}_n\mathbb{A} \times_i \mathrm{tv}_n\mathbb{A}$ is the category of pairs of i-consecutive n-cubes.) We distinguish between the *cubical compositions* $x +_i y$ or $f +_i g$ (of i-consecutive n-cubes or n-maps), and the *transversal composition* $gf = c_0(f, g)$ of transversal maps $f: x \to y$, $g: y \to z$.

A basic cubical category amounts to a cubical category all of whose transversal maps are identities. A cubical category $\mathbb{A}$ has an associated basic cubical category $\mathbb{A}^{(0)}$, without transversal maps, that will be called the *basic form* of $\mathbb{A}$.

A *cubical functor* $F: \mathbb{A} \to \mathbb{B}$ between cubical categories strictly preserves the whole structure; in other words, F is an internal functor between category objects, as specified above.

A *transversal* (or *structural*) *transformation* $h: F \to G: \mathbf{A} \to \mathbf{B}$ between cubical functors is an internal transformation between internal functors. Concretely, it assigns, to every n-cube x of $\mathbf{A}$, a transversal map in $\mathbb{B}$

$$(4) \quad h(x): F(x) \to G(x),$$

consistently with faces, degeneracies, concatenations, and satisfying the naturality condition

$$(\mathrm{nat}) \quad hy.Ff = Gf.hx, \qquad\qquad \text{for every n-map } f: x \to y \text{ in } \mathbb{A}.$$

In a cubical category, as well as in all the weaker cases considered below, a transversal n-map $f: x \to x'$ is said to be *special* if its 2^n vertices are identities

$$(5) \quad \partial^{\boldsymbol{\alpha}}f: \partial^{\boldsymbol{\alpha}}x \to \partial^{\boldsymbol{\alpha}}x' \qquad\qquad \partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1}\, \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} \qquad (\alpha_i = \pm).$$

In degree 0, this just means an identity.

1.6. Remarks. (a) In a cubical category, a k-map between k-cubes should be viewed as a *(k+1)-dimensional cell*. Therefore an n-truncated cubical category will be called an *(n+1)-cubical category*.

Thus, a 1-cubical category is just a category, with objects and 0-maps. A 2-cubical category amounts to a (strict) double category: its double cells are given by 1-maps, and their boundary consists of 0-maps and 1-cubes. One level up, a 3-cubical category amounts to a (strict) triple category of a particular kind, with:

- objects (of one type);
- arrows in directions 0, 1 and 2, *where the last two types coincide*;
- 2-dimensional cells in directions 01, 02, 12, *where the first two types coincide*;
- and 3-dimensional cells (of one type).

(b) For fixed positive integers $i \leq n$, the cubical category $\mathbb{A}$ has an associated double category $\mathbb{A}_{ni}$, whose double cells are the transversal n-maps of $\mathbb{A}$; the two composition laws are the transversal composition and i-concatenation of $\mathbb{A}$. More explicitly: objects are the (n–1)-cubes, horizontal arrows are the transversal (n–1)-maps (with their composition) and vertical arrows are the n-cubes (with i-concatenation); the faces of a double cell are given by ∂_0^α and ∂_i^α.

(c) We speak of a *precubical category* when we do not want to assume the axioms of associativity, unitarity and interchange of the cubical compositions (cf. 1.4). This notion will be of interest later, for the weak case introduced in Section 3, where we will replace such axioms with 'comparisons', realised as transversal maps (invertible and special).

(d) It should be noted that transversal maps, apart from being essential in the weak case (as mentioned above) *are already important in the present strict case*. For instance, they allow us to define (and construct) cubical (co)products, and more generally cubical (co)limits, as shown in [G5] (or in [GP1] for the truncated case of double categories).

1.7. Symmetric cubical categories. A *symmetric cubical category*

(1) $\mathbb{A} = ((tv_n\mathbb{A}), (\partial_i^\alpha), (e_i), (+_i), (s_i)),$

is a cubical category (1.5) equipped with cubical functors $s_i\colon tv_n\mathbb{A} \to tv_n\mathbb{A}$ $(1 = 1,..., n–1)$ called *transpositions*, which make it a symmetric cubical set. Furthermore, concatenations and transpositions must be consistent, in the following sense

(2) $s_{i-1}(x +_i y) = s_{i-1}(x) +_{i-1} s_{i-1}(y),$ $\qquad s_i(x +_i y) = s_i(x) +_{i+1} s_i(y),$

$s_j(x +_i y) = s_j(x) +_i s_j(y)$ $\qquad (j \neq i–1, i),$

where the variables x, y can denote cubes or transversal maps.

As with symmetric cubical sets, all faces, degeneracies and concatenations are now determined by the 1-directed ones (i.e. ∂_1^α, $e_1, +_1$), together with transpositions (see 1.3).

The *involutive* case, further equipped with reversions under axioms which can be easily deduced from [GM], is also of interest - e.g. for higher relations, higher (co)spans and singular cubes of a space; however, we will not go here into such details.

A *symmetric cubical functor* is a cubical functor which also preserves transpositions. A *symmetric transversal transformation* (or *structural transformation*) $h\colon F \to G\colon \mathbf{A} \to \mathbf{B}$ between such functors is defined as above (1.5), by further requiring that the transversal maps $h(x)\colon F(x) \to G(x)$ commute with all transpositions.

1.8. Cubical relations. As a simple, non-trivial example we recall here the symmetric cubical category $\omega\mathbb{R}\mathrm{el}$ *of cubical relations* (of sets), introduced in [G4] (Sections 4.1, 4.2). We follow here the direct construction of cubical relations as subsets of suitable cartesian products, but that paper also gives another realisation of $\omega\mathbb{R}\mathrm{el}$, as a quotient of the *weak* symmetric cubical category of spans of sets; the latter can more easily be extended to other domains.

Items will be indexed by the three-element set $\{0, u, 1\}$ and its powers. A 1-cubical relation is an ordinary relation $a: a_0 \mathrel{\bullet\!\!\!\to} a_1$ of sets, viewed as a subset $a_u \subset a_0 \times a_1$, and will be written with a dot-marked arrow; their composition will be written in additive notation. The cubical structure so far is obvious:

$$\partial_1^\alpha(a: a_0 \mathrel{\bullet\!\!\!\to} a_1) = a_\alpha, \qquad\qquad e_1(x) = \Delta(x): x \mathrel{\bullet\!\!\!\to} x \qquad\qquad (\text{for } x \text{ a set}),$$

where $\Delta(x)$ is the diagonal of $x \times x$.

A *2-cubical relation* a consists of:

- four vertices $(a_{ij}): 2{\times}2 \to \mathbf{Set}$ (where $2{\times}2 = \{0, 1\}^2$ is a discrete category on four objects),
- four (binary) relations on the sides of a square, written a_{uj} and a_{iu} (see the diagram below, where no condition of commutativity is assumed),
- and one *quaternary relation* $a_{uu} \subset \prod a_{ij}$ whose projection on each side is contained in the corresponding binary relation

$$\text{(1)}\qquad \begin{tikzcd} a_{00} \arrow[r, "a_{u0}", "\bullet"'] \arrow[d, "a_{0u}"'] \arrow[dr, phantom, "a_{uu}"] & a_{10} \arrow[d, "a_{1u}"] \\ a_{01} \arrow[r, "a_{u1}"'] & a_{11} \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & 1 \\ 2 & \end{tikzcd} \qquad \begin{aligned} & a_{uu} \subset a_{00} \times a_{01} \times a_{10} \times a_{11}, \\ & (p_{0j}, p_{1j})(a_{uu}) \subset a_{uj}, \\ & (p_{i0}, p_{i1})(a_{uu}) \subset a_{iu}. \end{aligned}$$

(We write $p_{ij}: a_{00} \times a_{01} \times a_{10} \times a_{11} \to a_{ij}$ the four cartesian projections.)

The 1-concatenation $c = a +_1 b$ is defined when the 2-cubes a, b are consecutive in direction 1, i.e. $a_{1u} = b_{0u}$, and is shown below, at the right

$$\text{(2)}\qquad \begin{tikzcd} a_{00} \arrow[r, "a_{u0}"] \arrow[d, "a_{0u}"'] \arrow[dr, phantom, "a_{uu}"] & a_{10} \arrow[d, "a_{1u}"] \\ a_{01} \arrow[r, "a_{u1}"'] & a_{11} \end{tikzcd} \begin{matrix} = \\ = \\ = \end{matrix} \begin{tikzcd} b_{00} \arrow[r, "b_{u0}"] \arrow[d, "b_{0u}"'] \arrow[dr, phantom, "b_{uu}"] & b_{10} \arrow[d, "b_{1u}"] \\ b_{01} \arrow[r, "b_{u1}"'] & b_{11} \end{tikzcd} \qquad \begin{tikzcd} a_{00} \arrow[r, "a_{u0} + b_{u0}"] \arrow[d, "a_{0u}"'] \arrow[dr, phantom, "c_{uu}"] & b_{10} \arrow[d, "b_{1u}"] \\ a_{01} \arrow[r, "a_{u1} + b_{u1}"'] & b_{11} \end{tikzcd}$$

The subset

(3) $\quad c_{uu} = a_{uu} +_1 b_{uu} \subset a_{00} \times a_{01} \times b_{10} \times b_{11},$

is formed of those 4-tuples $(x_{00}, x_{01}, z_{10}, z_{11})$ for which there is some pair $(y, y') \in a_{10} \times a_{11} = b_{00} \times b_{01}$ such that $(x_{00}, x_{01}, y, y') \in a_{uu}$ and $(y, y', z_{10}, z_{11}) \in b_{uu}$. In other words, $a_{uu} +_1 b_{uu}$ *is an ordinary composition of relations*, provided we view a_{uu} and b_{uu} as binary relations, as follows:

(4) $\quad a_{uu}: a_{00} \times a_{01} \mathrel{\bullet\!\!\!\to} a_{10} \times a_{11}, \qquad\qquad b_{uu}: b_{00} \times b_{01} \mathrel{\bullet\!\!\!\to} b_{10} \times b_{11}.$

This proves that 1-concatenation is strictly categorical, i.e. strictly associative, with strict units provided by the following degeneracy $e_1(a)$ of an ordinary relation $a: a_0 \mathrel{\bullet\!\!\!\to} a_1$

(5)
$$\begin{tikzcd} a_0 \arrow[r, "e_1(a_0)"] \arrow[d, "a_u"'] \arrow[dr, phantom, "(e_1a)_{uu}"] & a_0 \arrow[d, "a_u"] \\ a_1 \arrow[r, "e_1(a_1)"'] & a_1 \end{tikzcd}$$
$$(e_1a)_{u0} \;=\; e_1(a_0), \qquad (e_1a)_{uu} \;=\; \Delta(a_u), \qquad (e_1a)_{u1} \;=\; e_1(a_1).$$

$$\Delta(a_u) \;=\; \{(x_0, x_1, x_0, x_1) \in a_0 \times a_1 \times a_0 \times a_1 \mid (x_0, x_1) \in a_u\},$$

The same holds for 2-concatenation, which can be defined in the symmetric way, or by transposition and the previous operation:

(6) $a +_2 a' \;=\; s_1(s_1(a) \;+_1 s_1(a'))$, $\qquad e_2a \;=\; s_1e_1(a)$.

We proceed analogously in higher dimension. An *n-cube*, or *n-cubical relation*, is a family $a = (a_{\mathbf{t}})$, indexed by the n-tuples $\mathbf{t} = (t_1,..., t_n) \in \{0, 1, u\}^n$ and satisfying the following conditions.

(a) If $\mathbf{t} \in 2^n$, then $a_{\mathbf{t}}$ is a set. Otherwise, let $w = |\mathbf{t}|$ be its *weight*, i.e. the number of u's in the n-tuple; then $a_{\mathbf{t}}$ is a 2^w-ary relation

(7) $a_{\mathbf{t}} \subset A_{\mathbf{t}} \;=\; \prod a_{i_1...i_n}$,

the cartesian product being indexed by those n-tuples $\mathbf{i} = (i_1,..., i_n) \in 2^n$ where i_j coincides with t_j when the latter is 0 or 1 (there are 2^w such n-tuples). For instance, $a_{0u} \subset a_{00} \times a_{01}$, as in diagram (1).

(b) If $\mathbf{t}'$ is a multi-index obtained by replacing *one* occurrence of u in $\mathbf{t}$ with 0 or 1 (with weight $w' = w - 1$), then the corresponding projection $p_{\mathbf{tt}'}$ must send the 2^w-ary relation $a_{\mathbf{t}}$ *into* the $2^{w'}$-ary relation $a_{\mathbf{t}'}$

(8) $p_{\mathbf{tt}'}\colon A_{\mathbf{t}} \to A_{\mathbf{t}'}$, $\qquad p_{\mathbf{tt}'}(a_{\mathbf{t}}) \subset a_{\mathbf{t}'}$.

We define now a transversal map $f\colon a \to b$, as a natural transformation on the discrete category 2^n which is 'coherent' with the 'multiple' relations inside a and b:

(9) $f \;=\; (f_{\mathbf{i}})\colon a \to b\colon 2^n \to \mathbf{Set}$, $\qquad f_{\mathbf{i}}\colon a_{\mathbf{i}} \to b_{\mathbf{i}}$ $\qquad (\mathbf{i} = (i_1,..., i_n) \in 2^n)$,

(*coherence condition*) for every multi-index $\mathbf{t} \in \{0, 1, u\}^n$, the mapping $f_{\mathbf{t}}\colon A_{\mathbf{t}} \to B_{\mathbf{t}}$ defined by the cartesian product of the components $f_{\mathbf{i}}$ singled out in (7), carries the subset $a_{\mathbf{t}}$ into $b_{\mathbf{t}}$.

Faces are easily defined, using the maps

(10) $\partial_i^\alpha\colon \{0, 1, u\}^{n-1} \to \{0, 1, u\}^n$, $\qquad \partial_i^\alpha(t_1,..., t_{n-1}) \;=\; (t_1,..., \alpha,..., t_{n-1})$ $\qquad (\alpha = 0, 1)$,

Transpositions come from permuting the factors of $\{0, 1, u\}^n$.

Degeneracies are defined inductively, extending (5) (here $a_{\mathbf{t}}$ is written as $a(\mathbf{t})$ and $\alpha \in 2$)

(11) $(e_1a)(\alpha, t_2,..., t_n) \;=\; a(t_2,..., t_n)$,

$(e_1a)(u, \alpha,..., t_n) \;=\; (e_1\partial_1^\alpha a)(\alpha,..., t_n)$,

$(e_1a)(u, u,..., t_n) \;=\; \Delta(a(u,..., t_n))$.

Here, a *special* transversal map (see 1.5.5) amounts to an inclusion of subsets of cartesian products of the vertices.

The importance of transversal maps (in the present strict case, where comparisons are not needed) is the same as for double categories: the category (or 2-category) Rel(**Ab**) of relations of abelian groups even lacks finite products and coproducts, while the *double category* $\mathbb{R}$el(**Ab**) of abelian groups, their homomorphisms and relations has all double limits and colimits (cf. [GP1]). Similarly, the cubical categories of relations (on sets or abelian groups) have all cubical limits and colimits (cf. [G5]).

2. From symmetric cubical categories to globular categories (the strict case)

$A = ((A_n), (\partial_i^\alpha), (e_i), (s_i))$ is always a symmetric cubical set. In such a structure, we single out the *globular cubes* (in direction 1), which form a globular set $\mathrm{Glb}(A)$. Note that, even though transpositions do not appear in the definition of a globular cube, without them we would have non-equivalent notions of globularity in the various directions.

Similarly, a basic symmetric cubical category **A** gives a globular ω-category Glb(**A**), while a symmetric cubical category $\mathbb{A}$ has a sort of 'cylindrical' ω-category $\mathrm{Cyl}(\mathbb{A})$.

2.1. Globular cubes. An n-cube x of the symmetric cubical set A is said to be *globular*, or an *n-globe* (in direction 1), if it satisfies the following condition:

(a) for each $i = 1,..., n$ and $\alpha = \pm$, the i-directed face $\partial_i^\alpha(x)$ belongs to the image of the iterated degeneracy $(e_1)^{i-1}\colon A_{n-i} \to A_{n-1}$.

Here $(e_1)^{i-1}$ is an abuse of notation, for the composite

$$e_1 \dots e_1 = e_{i-1} \dots e_1\colon A_{n-i} \to \dots \to A_{n-2} \to A_{n-1}.$$

Thus, all 0-cubes and 1-cubes are globular; the following diagrams show the cases $n = 2, 3$

(1)
$$\begin{tikzcd} x_2^- \arrow[r, equal] \arrow[d, "x_1^-"'] \arrow[dr, phantom, "x"] & x_2^- \arrow[d, "x_1^+"] \\ x_2^+ \arrow[r, equal] & x_2^+ \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "2"'] & 1 \\ {} & \end{tikzcd} \qquad \partial_2^\alpha(x) = e_1(x_2^\alpha),$$

(2)
$$\begin{tikzcd}[column sep=small] x_3^- \arrow[rr, equal] \arrow[dr, equal] \arrow[dd, "x_2^-"'] & & x_3^- \arrow[dr, equal] & \\ & x_3^- \arrow[rr, equal, "e_1e_1(x_3^-)"] \arrow[dd, "x_2^+"'] & & x_3^- \arrow[dd, "x_2^+"] \\ x_3^+ \arrow[dr, equal] & x_1^- & e_1(x_2^+) & \\ & x_3^+ \arrow[rr, equal] & & x_3^+ \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "3"'] \arrow[dr] & 1 \\ {} & 2 \end{tikzcd} \qquad \begin{aligned} \partial_2^\alpha(x) &= e_1(x_2^\alpha), \\ \partial_3^\alpha(x) &= e_1e_1(x_3^\alpha). \end{aligned}$$

In the second, the 1-directed faces x_1^α are arbitrary 2-cubes; the 2-indexed faces $e_1(x_2^\alpha)$ are degenerate in direction 1; the 3-indexed faces $e_1e_1(x_3^\alpha)$ are totally degenerate.

One can notice the following points, made precise in the lemma below.

- There is no condition on the 1-directed faces $x_1^\alpha = \partial_1^\alpha(x) \in A_{n-1}$ (since $(e_1)^0 = \mathrm{id}(A_{n-1})$).

- Provided that $n \geq 2$, we have $\partial_2^\alpha(x) = e_1(x_2^\alpha)$, for some $x_2^\alpha \in A_{n-2}$; but both x_2^α are determined by x, as the 1-faces of its 1-faces x_1^β (independently of $\beta = \pm$):

$$x_2^\alpha \;=\; \partial_1^\beta e_1(x_2^\alpha) \;=\; \partial_1^\beta\partial_2^\alpha(x) \;=\; \partial_1^\alpha\partial_1^\beta(x) \;=\; \partial_1^\alpha(x_1^\beta).$$

- And so on. Finally, the highest-directed faces $\partial_n^\alpha(x)$ are totally degenerate, produced by two vertices

$$\partial_n^\alpha(x) \;=\; (e_1)^{n-1}(p^\alpha), \qquad\qquad p^\alpha \;=\; x_n^\alpha \;=\; \partial_1^\alpha\,\partial_1^{\alpha_2} \ldots \partial_1^{\alpha_n}(x) \;=\; \partial_1^{\alpha_n} \ldots \partial_{n-1}^{\alpha_2}\,\partial_n^\alpha(x),$$

(where the indices $\alpha_2,\ldots,\alpha_n$ are arbitrary).

In the symmetric cubical category $\mathbb{A}$, a *globular cube* is defined in the same way. A transversal map $f\colon x \to y$ is said to be *cylindrical* if it satisfies the same condition (a); then the cubes x, y are globular, while f itself has a sort of 'cylindrical' geometry, analysed in 2.7. We say that f is *special cylindrical* if, moreover, it is special as defined in 1.5, i.e. if all its 2^n vertices are identities.

2.2. Lemma and Definition (Globular faces). Let x be an n-globe of the symmetric cubical set A.

Its i-indexed faces $\partial_i^\alpha(x)$ are degenerate in the first $i-1$ directions, and can be obtained as follows (independently of $\alpha_2,\ldots,\alpha_i = \pm$):

(1) $\partial_i^\alpha(x) \;=\; (e_1)^{i-1}(d_i^\alpha(x))$,

$$d_i^\alpha(x) \;=\; x_i^\alpha \;=\; \partial_1^\alpha\,\partial_1^{\alpha_2} \ldots \partial_1^{\alpha_i}(x) \;=\; \partial_1^{\alpha_i} \ldots \partial_{i-1}^{\alpha_2}\,\partial_i^\alpha(x).$$

The (n–i)-dimensional cube $d_i^\alpha(x)$ is globular. It will be called a *globular face of* x *of dimension* n–i. In particular, the 1-directed faces

(2) $\partial_1^\alpha(x) \;=\; d_1^\alpha(x) \;=\; x_1^\alpha$,

will be called the *main faces* of x. They have the same faces

(3) $\partial_j^\beta(x_1^\alpha) \;=\; (e_1)^{j-1}(x_j^\beta)$,

independently of α. (We will see in the next lemma that these faces determine the faces of x.)

Proof. By hypothesis, for each $i = 1,\ldots n$, there exist two cubes x_i^α such that $\partial_i^\alpha(x) = (e_1)^{i-1}(x_i^\alpha)$. But then x_i^α is determined as in formula (1):

$$x_i^\alpha \;=\; \partial_1^{\alpha_2} \ldots \partial_1^{\alpha_i}\, e_1 \ldots e_1(x_i^\alpha) \;=\; \partial_1^{\alpha_2} \ldots \partial_1^{\alpha_i}\,\partial_i^\alpha(x) \;=\; \partial_1^\alpha\,\partial_1^{\alpha_2} \ldots \partial_1^{\alpha_i}(x).$$

Moreover the (n–i)-cube x_i^α is globular, because its j-directed faces

$$\partial_j^\beta(x_i^\alpha) \;=\; \partial_j^\beta\,\partial_1^\alpha\,\partial_1^{\alpha_2} \ldots \partial_1^{\alpha_i}(x) \;=\; \partial_1^\alpha\,\partial_1^{\alpha_2} \ldots\; \partial_1^{\alpha_i}(\partial_{j+i}^\beta x),$$

belong to $\partial_1^\alpha\,\partial_1^{\alpha_2} \ldots \partial_1^{\alpha_i}(\mathrm{Im}(e_1)^{j+i-1}) \;=\; \mathrm{Im}(e_1)^{j-1}$.

Taking $i = 1$ in the last computation we get the formula (3)

$$\partial_j^\beta(\partial_1^\alpha(x)) \;=\; \partial_1^\alpha\partial_{j+1}^\beta(x) \;=\; \partial_1^\alpha(e_1)^j(x_j^\beta) \;=\; (e_1)^{j-1}(x_j^\beta). \qquad\square$$

2.3. Lemma (Inductive form). Let x be an n-cube of the symmetric cubical set A, with $n \geq 1$. Then x is globular if and only if:

(i) the main faces $\partial_1^\alpha(x)$ are (n–1)-globes,

(ii) the other faces are degenerate in direction 1, i.e. $\partial^\beta_{i+1}(x) \in \mathrm{Im}(e_1)$, for $i < n$ and $\beta = \pm$.

Moreover, when x is globular, its main faces $\partial^\alpha_1(x)$ have the same boundary.

The second condition above can be equivalently replaced with:

(ii') $\partial^\beta_{i+1}(x) \;=\; e_1\, \partial^\beta_i\, \partial^\alpha_1(x)$ (for all i, α, β).

Proof. First, condition (ii') obviously implies (ii). Conversely, if $\partial^\beta_{i+1}(x) = e_1(u)$, then:

$$u \;=\; \partial^\alpha_1\, e_1(u) \;=\; \partial^\alpha_1\, \partial^\beta_{i+1}(x) \;=\; \partial^\beta_i\, \partial^\alpha_1(x).$$

Now, let x be an n-globe, so that $\partial^\alpha_i(x) = (e_1)^{i-1}(x^\alpha_i)$. We already know from the previous lemma that the main faces $\partial^\alpha_1(x) = d^\alpha_1(x) = x^\alpha_1$ are (n–1)-globes and have the same boundary. Property (ii) also holds: $\partial^\beta_{i+1}(x) \in \mathrm{Im}(e_1)^i \subset \mathrm{Im}(e_1)$.

Conversely, let x satisfy (i) and (ii'). Then $\partial^\beta_{i+1}(x) = e_1\, \partial^\beta_i\, \partial^\alpha_1(x)$. But $\partial^\alpha_1(x)$ is an (n-1)-globe, whence $\partial^\beta_{i+1}(x)$ belongs to $e_1(\mathrm{Im}(e_1)^{i-1}) = \mathrm{Im}(e_1)^i$, so that x is an n-globe. □

2.4. Theorem (Closure properties). In a symmetric cubical set, globular cubes are closed under e_1 and all faces ∂^α_i. In a (possibly basic) symmetric cubical category they are also closed under all i-concatenations. In a symmetric cubical category, globular transversal maps are closed under transversal composition and all i-concatenations; the same holds for the special ones.

Proof. First, it is obvious that e_1 preserves globular cubes (because $\partial^\alpha_i e_1 = e_1 \partial^\alpha_{i+1}$).

We already know from the previous lemma that the (n–i)-dimensional cube $d^\alpha_i(x)$ is a globular cube; it follows that $\partial^\alpha_i(x) = (e_1)^{i-1}(d^\alpha_i(x))$ is also.

Now, in a basic symmetric cubical category, let us start from noting that all 1-cubes are globular and obviously closed under 1-concatenation. Suppose that globular cubes of dimension $n-1$ are closed under concatenation in all directions (i.e. $1,..., n-1$). Now, consider the i-concatenation $x +_i y$ of two globular n-cubes, with $i \leq n$, and *let us prove that it is globular, using the previous lemma.*

Let us recall the formulas

$$\partial^\alpha_j(x +_i y) \;=\; \begin{cases} \partial^\alpha_j(x) +_{i-1} \partial^\alpha_j(y), & \text{if } j < i \leq n, \\ \partial^\alpha_j(x) +_i \partial^\alpha_j(y), & \text{if } i < j \leq n. \end{cases}$$

$$e_j(x +_i y) \;=\; \begin{cases} e_j(x) +_{i+1} e_j(y), & \text{if } j \leq i \leq n, \\ e_j(x) +_i e_j(y), & \text{if } i < j \leq n+1. \end{cases}$$

As to property (i), we must check that the main faces $\partial^\alpha_1(x +_i y)$ are (n–1)-globes. Indeed, for $i = 1$, $\partial^\alpha_1(x +_1 y)$ is x or y, and is globular; for $i > 1$, $\partial^\alpha_1(x +_i y)$ is a concatenation of lower globular cubes, hence is globular by the inductive hypothesis.

As to property (ii), suppose that $\partial^\beta_j(x) = e_1(u)$ and $\partial^\beta_j(y) = e_1(v)$, with $j > 1$. We must check various cases for $\partial^\beta_j(x +_i y)$.

(a) Case $j < i$ (with $j > 1$ and $i > 2$):

$$\partial^\beta_j(x +_i y) \;=\; \partial^\beta_j(x) +_{i-1} \partial^\beta_j(y) \;=\; e_1(u) +_{i-1} e_1(v) \;=\; e_1(u +_{i-2} v).$$

(b) Case $1 < i < j$:

$$\partial^\beta_j(x +_i y) \;=\; \partial^\beta_j(x) +_i \partial^\beta_j(y) \;=\; e_1(u) +_i e_1(v) \;=\; e_1(u +_{i-1} v).$$

(c) Case $1 = i < j$:

$$\partial^\beta_j(x +_1 y) \;=\; \partial^\beta_j(x) +_1 \partial^\beta_j(y) \;=\; e_1(u) +_1 e_1(v) \;=\; e_1(u),$$

where the existence of $e_1(u) +_1 e_1(v)$ means that: $u = \partial^+_1 e_1(u) = \partial^-_1 e_1(v) = v$.

(d) Case $i = j$. This is obvious, because $\partial^\beta_i(x +_i y)$ is x or y.

The last assertion, about transversal maps, is proved in the same way (for i-concatenations), or is obvious (for transversal composition). □

2.5. The globular set associated to a symmetric cubical one. Starting from the symmetric cubical set A, we begin by constructing the *associated globular set* $\mathrm{Glb}A$ of its globular cubes.

An *n-globe* of $\mathrm{Glb}A$ is a globular n-cube of A. Faces and degeneracies come from the 1-directed faces of cubes (because of Theorem 2.4):

(1) $d^\alpha\colon \mathrm{Glb}_n A \rightleftarrows \mathrm{Glb}_{n-1}A :e, \qquad\qquad d^\alpha \;=\; \partial^\alpha_1, \quad e \;=\; e_1.$

We have thus a *globular set* $((\mathrm{Glb}_n A), (d^\alpha), (e))$

$$(2) \qquad \dots\ \mathrm{Glb}_n A \;\overset{d^\alpha}{\underset{e}{\rightleftarrows}}\; \mathrm{Glb}_{n-1}A \;\overset{d^\alpha}{\underset{e}{\rightleftarrows}}\; \dots \;\overset{d^\alpha}{\underset{e}{\rightleftarrows}}\; \mathrm{Glb}_1 A \;\overset{d^\alpha}{\underset{e}{\rightleftarrows}}\; \mathrm{Glb}_0 A \;=\; A_0.$$

This means that faces and degeneracies satisfy the *globular identities*:

(3) $d^\alpha d^- \;=\; d^\alpha d^+, \qquad\qquad d^\alpha e \;=\; \mathrm{id},$

and indeed we already know that $\partial^\alpha_1\partial^\beta_1(x) = d^\alpha_2(x)$ does not depend on β (cf. 2.2.1).

In any globular set $G = ((G_n), (d^\alpha), (e))$ one can define the *higher faces* (or *iterated faces*) d^α_i, for $i = 1,\dots, n$ (and independently of $\alpha_2,\dots, \alpha_n$)

(4) $d^\alpha_i\colon G_n \to G_{n-i}, \qquad\qquad d^\alpha_i \;=\; d^\alpha_1\, d^{\alpha_2}_1 \dots d^{\alpha_i}_1.$

For $G = \mathrm{Glb}A$, this definition plainly agrees with the globular faces defined above (in 2.2). An n-globe $x \in G_n$ can be written in the following form, showing all its globular faces:

(5) $x\colon d^-_1 x \;\to_1\; d^+_1 x\colon d^-_2 x \;\to_2\; d^+_2 x\colon\; \dots\; d^-_n x \;\to_n\; d^+_n x.$

We write **Glb** (resp. Tr_n**Glb**) the category of globular sets (resp. n-truncated globular sets) and their (obvious) morphisms.

2.6. Compositions. If **A** is a *basic* symmetric cubical category, we make the globular set $\mathrm{Glb}A$ (on the underlying symmetric cubical set A) into a globular category $\mathrm{Glb}\mathbf{A}$, by defining its compositions.

Let x, y be two n-globes of **A**, *consecutive in direction* $i = 1,\dots, n$. This is equivalently expressed by the following conditions, globular or cubical respectively

(1) $d^+_i(x) \;=\; d^-_i(y), \qquad\qquad \partial^+_i(x) \;=\; \partial^-_i(y),$

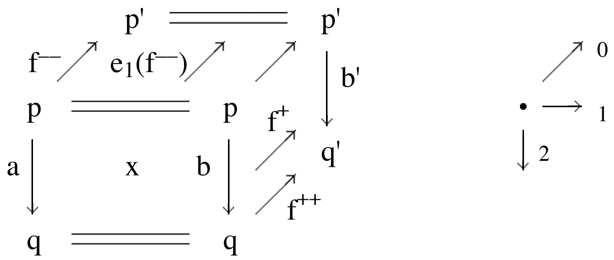
because $\partial_i^\alpha = (e_1)^{i-1} d_i^\alpha$ (by 2.5.2). Therefore, we define their i-composition $x +_i y$ as in **A**: the result is globular, because of Theorem 2.4.

Since our compositions are strictly associative and unitary, Glb**A** is indeed a globular category.

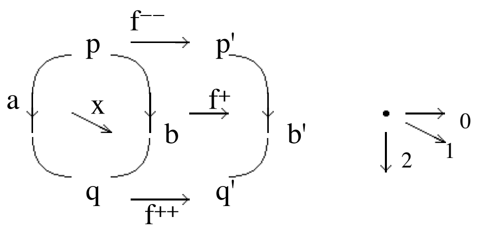
If $\mathbb{A}$ is an (extended) symmetric cubical category, the same arguments apply to cylindrical transversal maps (between globular cubes), and we obtain a category object Cyl$\mathbb{A}$ in the category of globular categories. This will be called a *cylindrical category*.

By a further restriction to *special* cylindrical transversal maps (between globular cubes) we obtain a 'special' category object SCyl$\mathbb{A}$ in the category of globular categories, where the 'transversal' morphisms between vertices are identities. This will be called a *special cylindrical category*.

Cylindrical transversal maps have indeed a cylindrical shape: for instance, a cylindrical 2-map $f: x \to x'$ (where the faces $\partial_2^\alpha(f)$ are 1-degenerate)



can be pictured as a solid cylinder on a 2-globe



$x: a \to_1 b: p \to_2 q,$
$x': a' \to_1 b': p' \to_2 q',$

$f^-: a \to a', \quad f^{--}: p \to p',$
$f^+: b \to b', \quad f^{++}: q \to q'.$

After its transversal 0-faces x, x', the map f has the following globular faces

(3) $f^\alpha = d_1^\alpha(f): \partial_1^\alpha(x) \to \partial_1^\alpha(x'), \qquad f^{\alpha\alpha} = d_2^\alpha(f): d_2^\alpha(x) \to d_2^\alpha(x') \qquad (\alpha = \pm).$

It is special cylindrical if its vertices $f^{\alpha\alpha} = d_2^\alpha(f)$ are identities.

2.7. Globular relations. The *basic* cubical category $\omega\mathbb{R}\mathrm{el}^{(0)}$ of cubical relations yields a globular category Glb($\omega\mathbb{R}\mathrm{el}^{(0)}$), where a 2-globe a consists of a 2-cubical relation (as in 1.8.1) whose 2-faces are 1-degenerate. Writing $a: a' \to_1 a'': x \to_2 y$, we have

- two sets x, y,

- two (binary) relations $a' \subset x \times y$, $a'' \subset x \times y$,

- and one *quaternary relation* a satisfying the following conditions (where $\Delta(x)$ is the diagonal of $x \times x$)

$$(1) \qquad \begin{array}{ccc} & x & \\ a' \downarrow & \xrightarrow{\;a\;} & \downarrow a'' \\ & y & \end{array} \qquad\qquad \begin{array}{l} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array} \qquad\qquad \begin{array}{ll} a \subset x \times y \times x \times y, & \\ (p_1, p_3)(a) \subset \Delta(x), & (p_2, p_4)(a) \subset \Delta(y), \\ (p_1, p_2)(a) \subset a', & (p_2, p_4)(a) \subset a''. \end{array}$$

On the other hand, the cubical category $\omega\mathbb{R}\mathrm{el}$ (*including transversal maps*) yields a cylindrical category $\mathrm{Cyl}(\omega\mathbb{R}\mathrm{el})$. Recalling the description of a *special* transversal map of $\omega\mathbb{R}\mathrm{el}$ as an inclusion of subsets of cartesian products of the vertices (at the end of 1.8), the *special* cylindrical category $\mathrm{SCyl}(\omega\mathbb{R}\mathrm{el})$ can be called an *ordered* ω*-category*.

As to low-dimensional truncated cases, the double category of relations $2\mathbb{R}\mathrm{el} = \mathrm{tr}_1(\omega\mathbb{R}\mathrm{el})$ stays unchanged under Cyl (since all 1-cubes are globular), while $\mathrm{SCyl}(2\mathbb{R}\mathrm{el})$ is the ordinary 2-category of relations (an ordered category). One level up, the triple category of relations $3\mathbb{R}\mathrm{el} = \mathrm{tr}_2(\omega\mathbb{R}\mathrm{el})$ yields:

- a structure $\mathrm{Cyl}(3\mathbb{R}\mathrm{el})$, which is a category object in the category of 2-categories,
- an *ordered 2-category* $\mathrm{SCyl}(3\mathbb{R}\mathrm{el})$ (having consistent orderings of 1-cells and 2-cells with the same 0-dimensional faces).

3. Weak symmetric cubical categories

We now recall the definition of a *weak* symmetric cubical category, introduced in [G1, G4].

3.1. Symmetric precubical categories. First, a *basic symmetric precubical category*

(1) $\mathbf{A} = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i)),$

is a symmetric cubical set with compositions, satisfying the consistency axioms (cub.1-2) of 1.4, where compositions are consistent with transpositions (in the sense of 1.7.2). Notice that we are not assuming that the i-compositions behave in a categorical way or satisfy interchange, in any sense, even weak. The morphisms of this structure are obvious.

Second, a *symmetric precubical category* is a category object $\mathbb{A}$ in the category of *basic* symmetric precubical categories (as defined above)

$$(2) \qquad \mathbb{A}^{(0)} \underset{e_0}{\overset{\partial_0^\alpha}{\leftleftarrows\!\!\rightarrow}} \mathbb{A}^{(1)} \xleftarrow{\;c_0\;} \mathbb{A}^{(2)} \qquad\qquad (\alpha = \pm).$$

Explicitly, this means the following data and axioms.

(wcub.1) A basic symmetric precubical category $\mathbb{A}^{(0)} = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-cubes*, or *n-dimensional objects* of $\mathbb{A}$.

(wcub.2) A basic symmetric precubical category $\mathbb{A}^{(1)} = ((M_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-maps* of $\mathbb{A}$, or also *(n+1)-cells*.

(wcub.3) Symmetric cubical functors ∂_0^α and e_0, called 0-*faces* and 0-*degeneracy*, with $\partial_0^\alpha.e_0 = \mathrm{id}$.

Also here, an n-map is written as $f\colon x \to x'$, where $\partial_0^- f = x$, $\partial_0^+ f = x'$ are n-cubes. Every n-dimensional object x has an *identity* $e_0(x)\colon x \to x$. Note that ∂_0^α and e_0 preserve cubical faces (∂_i^α, with $i > 0$), cubical degeneracies (e_i), transpositions (s_i) and cubical concatenations ($+_i$). In particular, given two i-consecutive n-maps f, g, their 0-faces are also i-consecutive and we have:

(3) $f +_i g\colon\ x +_i y \to x' +_i y'$ (for $f\colon x \to x'$, $g\colon y \to y'$; $\partial_i^+ f = \partial_i^- g$).

(wcub.4) A composition law c_0 which assigns to 0-consecutive n-maps $f\colon x \to x'$, $h\colon x' \to x''$ (of the same dimension), an n-map $hf\colon x \to x''$ (also written $h.f$). This composition law is (strictly) categorical, and forms a category $\mathbb{A}_n = (A_n, M_n, \partial_0^\alpha, e_0, c_0)$. It is also consistent with the basic symmetric precubical structure, in the following sense

(4) $\partial_i^\alpha(hf) = (\partial_i^\alpha h).(\partial_i^\alpha f)$, $\quad e_i(hf) = (e_i h)(e_i f)$, $\quad s_i(hf) = (s_i h)(s_i f)$,

$(h +_i k).(f +_i g) = hf +_i kg$,

$$
\begin{tikzcd}
\bullet \arrow[r, "\partial_i^- f"] \arrow[d, "x"'] \arrow[dr, phantom, "- f \to"] & \bullet \arrow[r, "\partial_i^- h"] \arrow[d] \arrow[dr, phantom, "- h \to"] & \bullet \arrow[d, "x''"] \\
\bullet \arrow[r] \arrow[d, "y"'] \arrow[dr, phantom, "- g \to"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "- k \to"] & \bullet \arrow[d, "y''"] \\
\bullet \arrow[r, "\partial_i^+ g"'] & \bullet \arrow[r, "\partial_i^+ k"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "0"] \arrow[d, "i"] & {} \\
{} &
\end{tikzcd}
$$

The last condition is the (strict) middle-four interchange *between the strict composition* c_0 *and any weak one*.

An n-map $f\colon x \to x'$ is said to be *special* if its 2^n vertices $\partial^{\boldsymbol{\alpha}} f$ are identities, where:

(5) $\partial^{\boldsymbol{\alpha}} f\colon\ \partial^{\boldsymbol{\alpha}} x \to \partial^{\boldsymbol{\alpha}} x'$ $\qquad \partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$ $\qquad (\alpha_i = \pm)$.

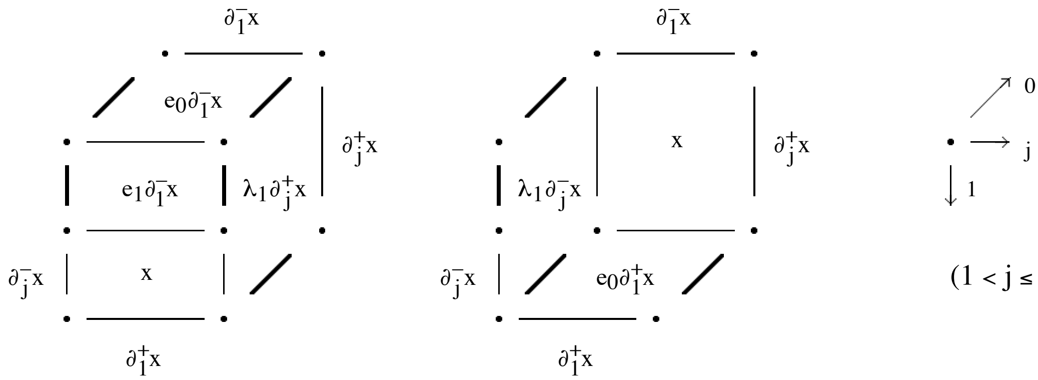
In degree 0, this just means an identity.

3.2. Comparisons. We now define a *weak symmetric cubical category* $\mathbb{A}$ as a symmetric precubical category (3.1), which is further equipped with invertible special transversal maps, playing the role of comparisons for units, associativity and cubical interchange, as follows. (We only assign the comparisons in direction 1; all the others can be obtained with transpositions.)

(wcub.5) For every n-cube x, we have a special invertible n-map $\lambda_1 x$, which is natural on n-maps and has the following faces (for $n > 0$)

(1) $\lambda_1 x\colon (e_1 \partial_1^- x) +_1 x \to x$ (*left-unit 1-comparison*),

$\partial_1^\alpha \lambda_1 x = e_0 \partial_1^\alpha x$, $\qquad \partial_j^\alpha \lambda_1 x = \lambda_1 \partial_j^\alpha x \quad (1 < j \leq n)$



(1 < j ≤ n).

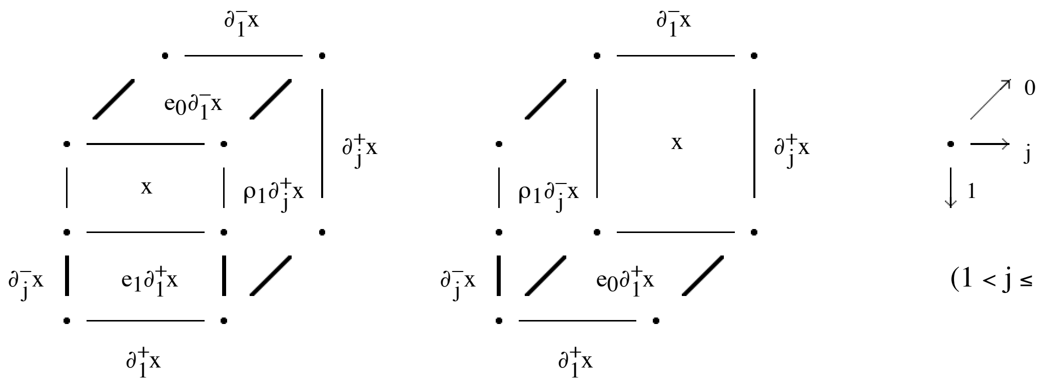
The naturality condition means that, for every n-map $f\colon x \to x'$, the following square of n-maps commutes

$$
(2) \qquad \begin{array}{ccc}
(e_1\partial_1^- x) +_1 x & \xrightarrow{\lambda_1 x} & x \\
{\scriptstyle (e_1\partial_1^- f) +_1 f}\downarrow & & \downarrow{\scriptstyle f} \\
(e_1\partial_1^- x') +_1 x' & \xrightarrow[\lambda_1 x']{} & x'
\end{array}
$$

(wcub.6) For every n-cube x, we have an invertible special n-map $\rho_1 x$, which is natural on n-maps and has the following faces (the naturality diagram, similar to diagram (2), is not written down)

(3) $\rho_1 x\colon x +_1 (e_1\partial_1^+ x) \to x$, (*right-unit 1-comparison*),

$\partial_1^\alpha \rho_1 x = e_0\partial_1^\alpha x$, $\qquad \partial_j^\alpha \rho_1 x = \rho_1\partial_j^\alpha x \quad (1 < j \leq n)$,



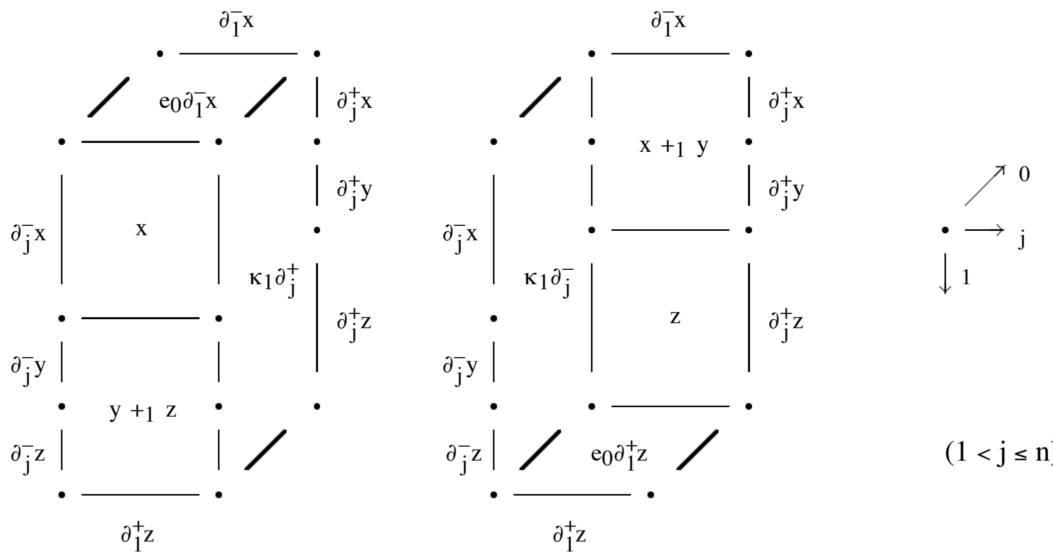
(1 < j ≤ n).

(wcub.7) For three 1-consecutive n-cubes x, y, z, we have an invertible special n-map $\kappa_1(x, y, z)$, which is natural on n-maps and has the following faces

(4) $\kappa_1(x, y, z)\colon x +_1 (y +_1 z) \to (x +_1 y) +_1 z$ (*associativity 1-comparison*),

$\partial_1^- \kappa_1(x, y, z) = e_0\partial_1^- x$, $\qquad \partial_1^+ \kappa_1(x, y, z) = e_0\partial_1^+ z$,

$\partial_j^\alpha \kappa_1(x, y, z) = \kappa_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z) \quad (1 < j \leq n)$,



$(1 < j \leq n)$.

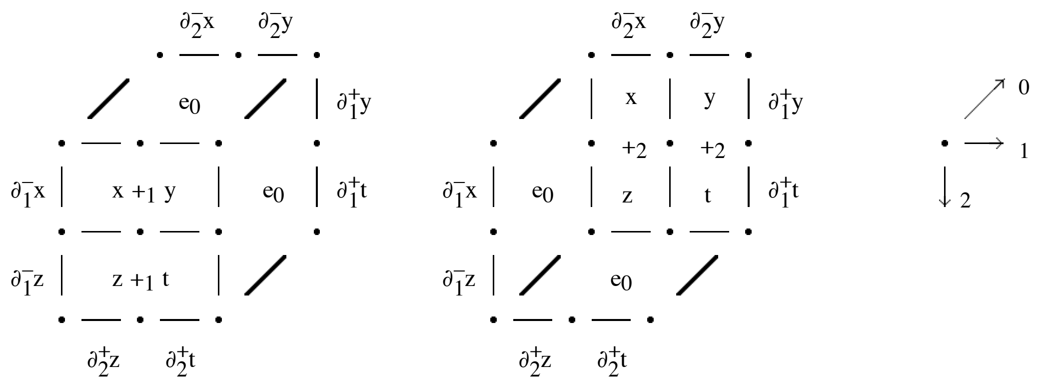
(wcub.8) Given four n-cubes x, y, z, u which satisfy the boundary conditions making the following concatenations possible, we have an invertible n-map $\chi_1(x, y, z, u)$, which is natural on n-maps and has the following faces (partially displayed below)

(5) $\chi_1(x, y, z, t)\colon (x +_1 y) +_2 (z +_1 t) \;\to\; (x +_2 z) +_1 (y +_2 t)$ (*interchange 1-comparison*),

$\partial_1^-\chi_1(x, y, z, t) \;=\; e_0(\partial_1^- x +_2 \partial_1^- z)$, $\quad \partial_1^+\chi_1(x, y, z, t) \;=\; e_0(\partial_1^+ y +_2 \partial_1^+ t)$,

$\partial_2^-\chi_1(x, y, z, t) \;=\; e_0(\partial_2^- x +_1 \partial_2^- y)$, $\quad \partial_2^+\chi_2(x, y, z, t) \;=\; e_0(\partial_2^+ z +_1 \partial_2^+ t)$,

$\partial_j^\alpha\chi_1(x, y, z, t) \;=\; \chi_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z, \partial_j^\alpha t)$ $\quad (2 < j \leq n)$,



(wcub.9) Finally, these comparisons must satisfy some conditions of coherence, listed in 3.3 below.

There is a more general 'u-lax' case, dealt with in [G6], which will not be used here. For a *u-lax symmetric cubical category* we do not assume the comparisons for left and right unitarity to be invertible (writing them as *directed towards simpler expressions*, as above); but we still require that the comparisons for associativity and interchange be invertible. The main example is the cubical structure $\mathbb{S}\mathrm{ng}(X)$ of singular cubes of a topological space, with transversal maps defined by reparametrisations. In that version, the axioms above are denoted as (ucub.5-9).

3.3. Coherence. The coherence axiom (wcub.9) means that the following diagrams of transversal maps commute (assuming that all the cubical compositions make sense).

(i) *Coherence pentagon for* $\kappa = \kappa_1$:

(1)
$$
\begin{tikzcd}[column sep=small]
 & & (x +_1 y) +_1 (z +_1 u) \arrow[drr, "\kappa"] & & \\
x +_1 (y +_1 (z +_1 u)) \arrow[urr, "\kappa"] \arrow[dr, "1+\kappa"'] & & & & ((x +_1 y) +_1 z) +_1 u \\
 & x +_1 ((y +_1 z) +_1 u) \arrow[rr, "\kappa"] & & (x +_1 (y +_1 z)) +_1 u \arrow[ur, "\kappa+1"'] &
\end{tikzcd}
$$

(ii) *Coherence conditions for the unit comparisons*:

(2)
$$
\begin{tikzcd}
e_1\partial_1^- x +_1 (x +_1 y) \arrow[rr, "\kappa"] \arrow[dr, "\lambda"'] & & (e_1\partial_1^- x +_1 x) +_1 y \arrow[dl, "\lambda+1"] \\
 & x +_1 y &
\end{tikzcd}
$$

(3)
$$
\begin{tikzcd}
x +_1 (e_1\partial_1^- y +_1 y) \arrow[rr, "\kappa"] \arrow[dr, "1+\lambda"'] & & (x +_1 e_1\partial_1^+ x) +_1 y \arrow[dl, "\rho+1"] \\
 & x +_1 y &
\end{tikzcd}
$$

(4)
$$
\begin{tikzcd}
x +_1 (y +_1 e_1\partial_1^+ y) \arrow[rr, "\kappa"] \arrow[dr, "1+\rho"'] & & (x +_1 y) +_1 e_1\partial_1^+ y \arrow[dl, "\rho"] \\
 & x +_1 y &
\end{tikzcd}
$$

(iii) *Coherence hexagon for* $\chi = \chi_1$ *and* $\kappa = \kappa_1$

(5)
$$
\begin{tikzcd}
(x +_1 (y +_1 z)) +_2 (x' +_1 (y' +_1 z')) \arrow[r, "\kappa+\kappa"] \arrow[d, "\chi"'] & ((x +_1 y) +_1 z) +_2 ((x' +_1 y') +_1 z') \arrow[d, "\chi"] \\
(x +_2 x') +_1 ((y +_1 z) +_2 (y' +_1 z')) \arrow[d, "1+\chi"'] & ((x +_1 y) +_2 (x' +_1 y')) +_1 (z +_2 z') \arrow[d, "\chi+1"] \\
(x +_2 x') +_1 ((y +_2 y') +_1 (z +_2 z')) \arrow[r, "\kappa"'] & ((x +_2 x') +_1 (y +_2 y')) +_1 (z +_2 z')
\end{tikzcd}
$$

(iv) *Coherence conditions for* $\chi = \chi_1$, $\lambda = \lambda_1$ *and* $\rho = \rho_1$

(6)
$$
\begin{tikzcd}
(e_1\partial_1^- x +_1 x) +_2 (e_1\partial_1^- y +_1 y) \arrow[r, "\lambda+\lambda"] \arrow[d, "\chi"'] & x +_2 y \arrow[dd, equal] & (x +_1 e_1\partial_1^+ x) +_2 (y +_1 e_1\partial_1^+ y) \arrow[l, "\rho+\rho"'] \arrow[d, "\chi"] \\
(e_1\partial_1^- x +_2 e_1\partial_1^- y) +_1 (x +_2 y) \arrow[d, equal] & & (x +_2 y) +_1 (e_1\partial_1^+ x +_2 e_1\partial_1^+ y) \arrow[d, equal] \\
e_1\partial_1^- (x +_2 y) +_1 (x +_2 y) \arrow[r, "\lambda"'] & x +_2 y & (x +_2 y) +_1 e_1\partial_1^+ (x +_2 y) \arrow[l, "\rho"]
\end{tikzcd}
$$

(The equality in the left and right column of this diagram follows from the 'geometric interactions' of axiom (cub.2), in 1.4.)

Truncation works as described in1.8. Again, a 1-truncated symmetric weak cubical category has no transpositions and is the same as a pseudo double category.

3.4. Remarks. As in 1.6(b), given two positive integers $i \leq n$, the weak symmetric cubical category $\mathbb{A}$ has an associated *weak* double category $\mathbb{A}_{ni}$, whose double cells are the transversal n-maps of $\mathbb{A}$; again, the two composition laws are the transversal composition and i-concatenation of $\mathbb{A}$.

The axioms (wcub.5-7) and (wcub.9)(i)(ii) just express this fact, for $i = 1$ (which is sufficient, because of symmetry).

3.5. Unitarity. The weak symmetric cubical category $\mathbb{A}$ is said to be *unitary* if, for every cube x, its comparisons $\lambda_1 x$ and $\rho_1 x$ are transversal identities, namely $e_0(x)$. (By symmetry, the same holds for every cubical i-composition.)

Then, for a transversal map $f \colon x \to y$, the naturality of λ_1 and ρ_1 proves that:

(i) $e_1\partial_1^- f +_1 f \;=\; f \;=\; f + e_1\partial_1^+ f.$

$$\begin{array}{ccccc}
e_1\partial_1^- x +_1 x & = & x & = & x +_1 e_1\partial_1^+ x \\
{\scriptstyle e_1\partial_1^- f +_1 f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f +_1 e_1\partial_1^+ f} \\
e_1\partial_1^- y +_1 y & = & y & = & y +_1 e_1\partial_1^+ y
\end{array}$$

But there is a more general notion, that is sufficient to ensure that globular cubes be closed under concatenation, and can often be 'forced' on weak symmetric cubical categories without heavy modifications. We say that $\mathbb{A}$ is *preunitary* if, for every 1-*degenerate* cube $e_1(x)$, its comparisons $\lambda_1(e_1(x))$ and $\rho_1(e_1(x))$ are transversal identities, actually the same: $e_0 e_1(x) \colon e_1(x) +_1 e_1(x) \to e_1(x)$.

4. Examples: cubical cospans and cubical profunctors

We recall, from [G1], the construction of the weak symmetric cubical category $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ of cubical cospans on a category with pushouts $\mathbf{X}$. Then, starting from the structure $\omega\mathbb{C}\mathrm{osp}(\mathbf{Emb})$ of cubical cospans of full embeddings of categories, we give a construction of the weak symmetric cubical category $\omega\mathbb{C}\mathrm{at}$ of cubical profunctors. (This modifies a similar construction given in [G4]: besides using more general transversal maps, we also correct an error of that paper with respect to cubes.)

Here, an embedding of categories always means a faithful functor, injective on objects.

4.1. Cubical cospans. We follow a construction given in [G1] for *cubical cospans*. Modifications of this structure for higher dimensional cobordism can be found in [G2, G3].

Let $\mathbf{X}$ be a category with pushouts, and equipped with a choice of them, called *distinguished pushouts*. A cospan in $\mathbf{X}$ is a diagram of shape

(1) $\quad u \;=\; (u^- \colon X^- \to X^0 \leftarrow X^+ \colon u^+),$

viewed as a morphism $u: X^- \to\!\!\!\!\!\to X^+$; they are composed with distinguished pushouts, forming a bicategory; or, also, the weak arrows of a larger structure, the pseudo double category $\mathbb{C}\mathrm{osp}(\mathbf{X})$, as in [GP1].

The model of the construction is the *formal cospan* $\wedge$, together with its cartesian powers $\wedge^n$

$$
(2) \qquad -1 \to 0 \leftarrow 1 \quad \wedge, \qquad
\begin{tikzcd}
(-1,-1) \arrow[r] \arrow[d] & (0,-1) \arrow[d] & (1,-1) \arrow[l] \arrow[d] \\
(-1,0) \arrow[r] & (0,0) & (1,0) \arrow[l] \\
(-1,1) \arrow[r] \arrow[u] & (0,1) \arrow[u] & (1,1) \arrow[l] \arrow[u]
\end{tikzcd} \quad \wedge^2. \qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "2"] & 1 \\
{}
\end{tikzcd}
$$

(In these diagrams, identities and composed arrows are understood.)

An *n-cubical cospan* in $\mathbf{X}$ is thus defined as a functor $x: \wedge^n \to \mathbf{X}$.

Faces and degeneracies are obvious, and these diagrams form a cubical set. Moreover, for $i = 1,..., n$, there are compositions $x +_i y$ of i-consecutive n-cubes, computed by distinguished pushouts. These operations behave 'categorically' and satisfy interchange in a weak sense, up to suitable comparisons.

To make room for the latter, the n-th component of $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$

$$(3) \quad \mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{Cat}(\wedge^n, \mathbf{X}),$$

is *not* reduced to the set of functors $x: \wedge^n \to \mathbf{X}$ (the *n-cubes*, or *n-dimensional objects*, of the structure), but is the *category* of such functors and their natural transformations $f: x \to x': \wedge^n \to \mathbf{X}$. The latter are the *n-maps* of the structure, or *(n+1)-cells*.

The comparisons are *invertible* special n-maps (between different realisations of the same colimit); but *general* n-maps are also important, e.g. to define limits and colimits (cf. [G1], Section 4.6, or [G5] for a general theory of cubical limits).

4.2. Remarks. Let us make precise that our choice of pushouts in $\mathbf{X}$ assigns, to each span (f, g) *one distinguished* pushout (f', g') (in a symmetric way, i.e. consistently with permutation of pairs)

$$
(1) \qquad
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "g"'] & \bullet \arrow[d, "f'"] \\
\bullet \arrow[r, "g'"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, "1"] \arrow[d, "1"'] & x \arrow[d, "1"] \\
x \arrow[r, "1"'] & x
\end{tikzcd}
$$

We assume a first *unitarity constraint*:

(i) each square of identities is a distinguished pushout,

so that $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ is preunitary (3.5).

If one assumes a stronger unitarity constraint (as in [G1]):

(ii) the distinguished pushout of the span $(f, 1)$ is $(1, f)$, and symmetrically, the weak symmetrical cubical category $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ becomes *unitary*, which would simplify even more our procedures of globularisation, in the next section. But we prefer not to oversimplify this example.

4.3. Profunctors as cospans. Now, to deal with cubical profunctors, we start from cubical cospans of full embeddings of (small) categories.

The crucial point is the fact that an ordinary profunctor $x\colon x_{-1} \nrightarrow x_1$ has a *collage* x_0, which consists of the sum of the categories x_{-1} and x_1, supplemented with new homs $x_0(a, b) = x(a, b)$, for a in the domain and b in the codomain. (Formally, the collage of a profunctor is a double colimit, the *cotabulator*, in the weak double category of categories, functors and profunctors, see [GP1].)

Thus, the profunctor x can be described as a cospan

(1) $\quad x_{-1} \xrightarrow{x^-} x_0 \xleftarrow{x^+} x_1$

satisfying the following conditions (which imply that x^-, x^+ have disjoint images)

(i) x^-, x^+ are full embeddings,

(ii) the embeddings x^-, x^+ cover all the objects of x_0,

(iii) there are no arrows in x_0 going from an object coming from x_1 to one coming from x_{-1}.

We have already seen how a profunctor yields such a cospan. Conversely, given the cospan (1), the profunctor is reconstructed as:

(2) $\quad x\colon (x_{-1})^{\mathrm{op}} \times x_1 \to \mathbf{Set}, \qquad (a, b) \mapsto x_0(x^-(a), x^+(b)).$

Note that condition (ii) is not closed under concatenation; we shall modify this operation (in 4.5).

In particular a functor $f\colon x \to y$ is identified with the profunctor

$$f\colon x^{\mathrm{op}} \times y \to \mathbf{Set}, \qquad f(a, b) = y(f(a), b),$$

and can be described as a cospan $x \to z \leftarrow y$ where z is the coproduct $x + y$, supplemented with new arrows $z(a, b) = y(f(a), b)$, for each object a of x and b of y.

4.4. Cospans of categories. We begin from considering condition 4.3(i). Let $\mathbb{C} = \omega\mathbb{C}\mathrm{osp}(\mathbf{Cat})$ be the weak symmetric cubical category of cubical cospans of functors between small categories. Let **Emb** be the category of small categories and their full embeddings, and $\mathbb{E}$ the *transversally full* substructure of $\mathbb{C}$ whose cubes belong to $\omega\mathbb{C}\mathrm{osp}(\mathbf{Emb})$.

Thus, an n-cube of $\mathbb{E}$ is a functor $x\colon \wedge^n \to \mathbf{Emb}$, but an n-map is a natural transformation $f\colon x \to y\colon \wedge^n \to \mathbf{Cat}$.

To show that this is legitimate, we prove that **Emb** has pushouts, which are also pushouts in **Cat**.

Given a span of full embeddings $A \leftarrow X \to B$, let us rename the items of A and B so that these functors are full *inclusions* and $X = A \cap B$. (The letters a, a', α, α' will denote objects and arrows of A, while b, b', β, β' belong to B and x, x' are objects of X.)

Now, the pushout W contains the obvious set-theoretical union $A \cup B$, supplemented with:

(a) new arrows $[\beta\alpha]\colon a \to x \to b$, for $\alpha\colon a \to x$ in A and $\beta\colon x \to b$ in B, modulo the equivalence relation generated by identifying $\beta\alpha = \beta'\alpha'$ if there exists some $\xi\colon x \to x'$ in X such that $\alpha' = \xi\alpha$ in A and $\beta = \beta'\xi$ in B,

(b) and, symmetrically, new arrows $[\alpha\beta]\colon b \to x \to a$.

The composition in W is easily defined, as in the following examples:

(1) $[\beta\alpha].\alpha' = [\beta(\alpha\alpha')]$, for $\alpha'\colon a' \to a$, $\alpha\colon a \to x$, $\beta\colon x \to b$,

$[\alpha'\beta'].[\beta\alpha] = \alpha'.(\beta'\beta).\alpha$, for $\alpha\colon a \to x$, $\beta\colon x \to b$, $\beta'\colon b \to x'$, $\alpha'\colon x' \to a'$,

where the last composition is in A (since $\beta'\beta\colon x \to x'$ belongs to the full subcategory $X = A \cap B$).

4.5. Essential cubical cospans. We now take condition 4.3(ii) into account, and construct a weak symmetric cubical category $\mathbb{E}'$ of *essential cubical cospans of full embeddings*; it is contained in $\mathbb{E}$, but has different cubical compositions. (It might be viewed as a quotient of $\mathbb{E}$, but using such a construction would be longer.)

We say that an n-cospan (of full embeddings) $x\colon \mathbf{\wedge}^n \to \mathbf{Emb}$ is *essential* if, in each category $x(\mathbf{t})$ ($\mathbf{t} \in \mathrm{Ob}\mathbf{\wedge}^n$), every object is the image of some object in a category occupying a vertex of the cube. (Plainly, there is nothing to check about such categories.)

Every n-cospan $x\colon \mathbf{\wedge}^n \to \mathbf{Emb}$ has an *associated essential cospan* $\hat{x}$, obtained as follows: the vertices of the cube (marked with bigger bullets, in the 2-dimensional case below) are left unchanged, but we replace each other category $x(\mathbf{t})$ ($\mathbf{t} \in \mathrm{Ob}\mathbf{\wedge}^n$) with the full subcategory $\hat{x}(\mathbf{t})$ determined by the objects which are the image of some object of a vertex

$$(1) \qquad \begin{array}{ccccc} \bullet & \longrightarrow & \cdot & \longleftarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longleftarrow & \cdot \\ \uparrow & & \uparrow & & \uparrow \\ \bullet & \longrightarrow & \cdot & \longleftarrow & \bullet \end{array}$$

Essential cubical cospans form a structure $\mathbb{E}'$ that inherits from $\mathbb{E}$ faces, degeneracies, transpositions and transversal maps. But we redefine the i-composition $x \mathbin{\hat{+}_i} y$ of i-consecutive essential n-cubical cospans as the essential n-cubical cospan associated to their i-composition in $\mathbb{E}$

(2) $x \mathbin{\hat{+}_i} y = (x +_i y)\hat{}$.

It is easy to see that the associativity comparisons restrict to (invertible) comparisons

(3) $\hat{\kappa}_i(x, y, z)\colon x \mathbin{\hat{+}_i} (y \mathbin{\hat{+}_i} z) \to (x \mathbin{\hat{+}_i} y) \mathbin{\hat{+}_i} z$,

and similarly for unitarity and interchange.

Finally, $\mathbb{E}'$ is again a weak symmetric cubical category.

4.6. Cubical profunctors. Finally, we take care of condition 4.3(iii), and construct a weak symmetric cubical category $\omega\mathbb{C}\mathrm{at}$ of *cubical profunctors*; it is contained in $\mathbb{E}'$, but has different degeneracies.

We say that an n-cube x of $\mathbb{E}'$ is an *n-profunctor* if, for each ordinary cospan which appears in x in a given direction i

(1) $x(\mathbf{t}') \to x(\mathbf{t}) \leftarrow x(\mathbf{t}'')$ $\qquad (t'_i = -1,\ t_i = 0,\ t''_i = 1;\ \ t'_j = t_j = t''_j\ \text{ for } j \neq i),$

condition (iii) of 4.3 is satisfied: there are no arrows in $x(\mathbf{t})$ going from an object coming from $x(\mathbf{t}'')$ to one coming from $x(\mathbf{t}')$.

Cubical profunctors inherit from $\mathbb{E}'$ faces, transpositions, transversal maps, compositions and comparisons for associativity and interchange. Degeneracies make some problems (as it is also the case for cospans in the domain of cobordism, see [G3]). Indeed, already in degree 1, the degenerate cospan of a (non-empty) category

$$e_1(x) = (x = x = x),$$

is essential but does not satisfy condition (iii). However (as in [G3]) we can replace degeneracies with *cylindrical degeneracies*: the 1-cube $E_1(x)$ on the category x is the following essential cospan of *disjoint* embeddings (which is also reduced):

(2) $$x \xrightarrow{x^-} x_0 \xleftarrow{x^+} x$$

where the category $x_0 = x \times \mathbf{2}$ is the *collage* of the identity profunctor of x (and x^-, x^+ are the obvious embeddings). It is easy to see that $E_1(x)$ is a weak identity for concatenation *with* 1-profunctors (but not with general 1-cubes of $\mathbb{E}'$). Similarly one defines the cylindrical degeneracy $E_1(x)$ of any n-cube.

We restrain the construction of composition so that $E_1(x) +_1 E_1(x) = E_1(x)$.

Finally, we have obtained a preunitary weak symmetric cubical category $\omega\mathbb{C}\mathrm{at}$ of cubical profunctors, contained in $\mathbb{E}'$; the embedding preserves all the structure, except degeneracies (and their comparisons), and is transversally full.

Notice that the bicategory of ordinary profunctors can be realised as a *strict* 2-category, as recalled in [GP1]: a profunctor $u: A \to\!\!\!\!\!\!\bullet\;\; B$ can be defined as a colimit-preserving functor

(3) $u\hat{}: \mathbf{Set}^A \to \mathbf{Set}^B,$ $\qquad u\hat{}(F)(b) = \int^a u(a, b) \times F(a).$

This realisation might perhaps be used for another construction of $\omega\mathbb{C}\mathrm{at}$ as a strict symmetric cubical category.

5. Globularisation in the weak case

For a weak symmetric cubical category $\mathbb{A}$, a globularisation procedure has to make use of the transversal maps, that contain the comparisons. This can be done associating to any transversal n-map its *companion*, an (n+1)-cube.

Companions in double categories have been introduced and studied in [GP2-4]. They are related with connections in cubical sets, in the sense of Brown-Higgins [BH1, BH2, ABS].

In this section we suppose, for the sake of simplicity, that $\mathbb{A}$ is *preunitary*, as defined in 3.5. This ensures that the globes of $\mathbb{A}$ are closed under cubical compositions.

5.1. Companions. A *companion* of a transversal n-map $f\colon x^- \to x^+$ is an (n+1)-cube x such that $\partial^\alpha_{n+1}(x) = x^\alpha$, equipped with (n+1)-maps $\eta\colon e_1(x^-) \to_0 x$ and $\varepsilon\colon x \to_0 e_1(x^+)$

(1)
$$
\begin{tikzcd}
x^- \arrow[r, "e_0(x^-)"] \arrow[d, "e_{n+1}(x^-)"'] \arrow[dr, phantom, "\eta"] & x^- \arrow[d, "x"] \\
x^- \arrow[r, "f"'] & x^+
\end{tikzcd}
\qquad
\begin{tikzcd}
x^- \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\varepsilon"] & x^+ \arrow[d, "e_{n+1}(x^+)"] \\
x^+ \arrow[r, "e_0(x^+)"'] & x^+
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "n+1"] & 0 \\
{} &
\end{tikzcd}
$$

such that $\varepsilon.\eta = e_{n+1}(f)$ and $\eta +_{n+1} \varepsilon$ coincides with $e_0(x)$ *up to unitarity comparisons*, as in the following diagram:

(2)
$$
\begin{tikzcd}
x^- \arrow[r, "e_0(x^-)"] \arrow[dd, "x"'] & x^- \arrow[r, "e_0(x^-)"] \arrow[d, "e_{n+1}(x^-)"'] \arrow[dr, phantom, "\eta"] & x^- \arrow[r, "e_0(x^-)"] \arrow[d, "x"'] \arrow[dr, phantom, "\rho_{n+1}(x)"] & x^- \arrow[dd, "x"] & {} & x^- \arrow[r] \arrow[dd, "x"'] \arrow[ddr, phantom, "e_0(x)"] & x^- \arrow[dd, "x"] \\
{} \arrow[r, phantom, "\lambda^{-1}_{n+1}(x)"] & x^- \arrow[r] \arrow[d, "x"] \arrow[dr, phantom, "\varepsilon"] & x^+ \arrow[d, "e_{n+1}(x^+)"] & {} \arrow[r, phantom, "="] & {} & & \\
x^+ \arrow[r, "e_0(x^+)"'] & x^+ \arrow[r, "e_0(x^+)"'] & x^+ \arrow[r, "e_0(x^+)"'] & x^+ & {} & x^+ \arrow[r] & x^+
\end{tikzcd}
$$

The companion is determined up to invertible (n+1)-maps.

We say that $\mathbb{A}$ *has structural companions* if every transversal n-map $f\colon x^- \to x^+$ is equipped with a companion f_*, and the global assignment is consistent with faces and degeneracies, in the sense that:

(3) $(\partial^\alpha_i(f))_* = \partial^\alpha_i(f_*)$, $\qquad (e_i(f))_* = e_i(f_*)$ $\qquad (i = 1,..., n)$.

Below, we only work with low-dimensional cases, up to 2-truncation (and dimension 3). As a consequence, *we only need companions for transversal 1-maps*.

5.2 Low-dimensional cases. Let $\mathbb{A}$ be a weak symmetric cubical category.

(i) Truncating $\mathbb{A}$ in degree 1 we get a weak 2-cubical category $2\mathbb{A} = tr_1(\mathbb{A})$, with one strict direction and one weak direction; in other words, a weak double category. Its 1-cubes and special transversal maps form a bicategory.

(ii) Truncating $\mathbb{A}$ in degree 2 we get a weak symmetric 3-cubical category $3\mathbb{A} = tr_2(\mathbb{A})$, with one strict direction and two weak directions.

We want now to show that, if $\mathbb{A}$ has structural companions, we can use the 2-globes and certain cylindrical 2-maps to construct a tricategory T, as defined in Gordon, Power and Street [GPS].

(TD1) the objects of T are the 0-globes (or 0-cubes) $p, q,...$ of $\mathbb{A}$.

(TD2) For objects p^-, p^+ we have a bicategory $T(p^-, p^+)$ where:

(a) a 0-cell u is a 1-globe $u: p^- \to_1 p^+$ (i.e. a 1-cube with $\partial_1^\alpha u = p^\alpha$), called a 1-cell of T,

(b) a 1-cell $x: u^- \to_1 u^+$ is a 2-globe $x: u^- \to_1 u^+: p^- \to_2 p^+$ (i.e. a 2-cube with $\partial_1^\alpha x = u^\alpha$, $\partial_2^\alpha x = e_1(p^\alpha)$), called a *2-cell of* T,

(c) a 2-cell $f: x^- \to_0 x^+: u^- \to_1 u^+$ is a *globular* 2-map $f: x^- \to_0 x^+: u^- \to_1 u^+: p^- \to_2 p^+$ (i.e. a 2-map $f: x^- \to_0 x^+$ whose cubical faces $\partial_i^\alpha f$ are transversal identities, for $i = 1, 2$), called a *3-cell of* T. (Such globular 2-maps are necessarily cylindrical, because

$$\partial_2^\alpha(f) = e_0(\partial_0^\beta \partial_2^\alpha(f)) = e_0(\partial_2^\alpha \partial_0^\beta(f)) = e_0(\partial_2^\alpha x^\beta) = e_0(e_1(p^\alpha)) = e_1(e_0(p^\alpha)).\ )$$

(d) 2-cells of T have a composition law, defined by the 1-composition of 2-cubes (*taking into account the preunitarity property, which ensures that 2-globes are closed under 1-composition*):

(1) $x +_1 y$ (for $\partial_1^+ x = \partial_1^- y$).

(e) 3-cells of T have two composition laws, defined by transversal composition and 1-composition in $\mathbb{A}$:

(2) $f +_0 g = gf$ (for $\partial_0^+ f = \partial_0^- g$),

(3) $f +_1 g$ (for $\partial_1^+ f = \partial_1^- g$).

(f) The 0-composition of 3-cells is categorical, i.e. strictly associative and unitary.

(g) The 1-composition of 2-cells is associative up to its 0-invertible comparisons in $\mathbb{A}$:

(4) $\kappa_1(x, y, z): x +_1 (y +_1 z) \to_0 (x +_1 y) +_1 z$,

$$\begin{tikzcd} p^- \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "x"] & p^- \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "y"] & p^- \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "z"] & p^- \arrow[d] \\ p^+ \arrow[r, equal] & p^+ \arrow[r, equal] & p^+ \arrow[r, equal] & p^+ \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "2"'] & 1 \\ {} \end{tikzcd}$$

We verify now that $\kappa_1(x, y, z)$ is a globular 2-map (i.e. 2-cells of $T(p^-, p^+)$):

$$\partial_1^- \kappa_1(x, y, z) = e_0 \partial_1^- x, \qquad \partial_1^+ \kappa_1(x, y, z) = e_0 \partial_1^+ z,$$

$$\partial_2^\alpha \kappa_1(x, y, z) = \kappa_1(\partial_2^\alpha x, \partial_2^\alpha y, \partial_2^\alpha z) = \kappa_1(e_1(p^\alpha), e_1(p^\alpha), e_1(p^\alpha)) = e_0(e_1(p^\alpha)).$$

where the penultimate equality comes from a partial unitarity property (cf. 3.3(v)).

Their naturality with respect to transversal maps $f: x^- \to_0 x^+$, $g: y^- \to_0 y^+$, $h: z^- \to_0 z^+$ holds in $\mathbb{A}$.

(h) The composition of 2-cells is unitary up to its comparisons in $\mathbb{A}$, which again are globular (also because of preunitarity).

(TD3) For objects p, q, r, the last composition is defined as a weak functor of bicategories:

(5) $C: T(p, q) \times T(q, r) \to T(p, r)$,

$C(u: p \to_1 q,\ v: q \to_1 r) = (u +_1 v: p \to_1 r)$,

$C(x: u^- \to_1 u^+: p \to_2 q,\ y: v^- \to_1 u^+: q \to_2 r) = (x +_2 y: u^- +_1 v^- \to_1 u^+ +_1 v^+: p \to_2 r)$,

$C(f: x^- \to_0 x^+: u^- \to_1 u^+: p \to_2 q,\ g: y^- \to_0 y^+: v^- \to_1 u^+: q \to_2 r)$

$= (f +_2 g: x^- +_2 y^- \to_0 x^+ +_2 y^+: u^- +_1 v^- \to_1 u^+ +_1 v^+: p \to_2 r)$.

To define the comparison of C with respect to the 1-composition of 2-cells (2-globes of $\mathbb{A}$), take four 2-cells x, y, z, t disposed as below. Then the interchange comparison of $\mathbb{A}$

$$\chi_1(x, y, z, t)\colon (x +_1 y) +_2 (z +_1 t) \to_0 (x +_2 z) +_1 (y +_2 t),$$

$$\begin{tikzcd}
p \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "x"] & p \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "y"] & p \arrow[d] \\
q \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "z"] & q \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "t"] & q \arrow[d] \\
r \arrow[r, equal] & r \arrow[r, equal] & r
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & 1 \\
2 &
\end{tikzcd}$$

yields the comparison we want:

(6) $\chi_1(x, y, z, t)\colon C(x +_1 y, z +_1 t) \to_0 C(x, z) +_1 C(y, t)$.

This is indeed a globular cell (also because of preunitarity):

$$\partial_1^-\chi_1(x, y, z, t) = e_0(\partial_1^- x +_2 \partial_1^- z), \qquad \partial_1^+\chi_1(x, y, z, t) = e_0(\partial_1^+ y +_2 \partial_1^+ t),$$
$$\partial_2^-\chi_1(x, y, z, t) = e_0(e_1(p)) = e_1(e_0(p)), \qquad \partial_2^+\chi_2(x, y, z, t) = e_0(e_1(r)) = e_1(e_0(r)).$$

(TD4) The unit of an object p is a strict functor of bicategories, defined on the singleton 2-category **1**:

(7) $U\colon \mathbf{1} \to T(p, p)$,

$U(*) = (e_1(p)\colon p \to_1 p)$,

$U(1\colon 1 \to_1 1\colon * \to_2 *) = (e_1e_1(p)\colon e_1(p) \to_1 e_1(p)\colon p \to_2 p)$,

$U(1\colon 1 \to_0 1\colon 1 \to_1 1\colon * \to_2 *) = (e_0e_1e_1(p)\colon e_1e_1(p) \to_0 e_1e_1(p)\colon e_1(p) \to_1 e_1(p)\colon p \to_2 p)$.

(TD5) For objects p, q, r, s we define a pseudo natural equivalence of pseudo functors:

(8) $k\colon C(1 \times C) \to C(C \times 1)\colon T(p, q) \times T(q, r) \times T(r, s) \to T(p, s)$.

First, for a triple of 1-cells (u, v, w) forming an object of $T(p, q) \times T(q, r) \times T(r, s)$, we have a comparison 2-cell:

(9) $k(u, v, w)\colon u +_1 (v +_1 w) \to_1 (u +_1 v) +_1 w\colon p \to_2 s$,

$k(u, v, w) = (\kappa(u, v, w))_*$,

given by the companion of the 1-map

$$\kappa(u, v, w)\colon u +_1 (v +_1 w) \to_0 (u +_1 v) +_1 w.$$

Second, for a triple of 2-cells (x, y, z) forming a 1-cell of $T(p, q) \times T(q, r) \times T(r, s)$

(10) $x\colon u^- \to_1 u^+\colon p \to_2 q, \qquad y\colon v^- \to_1 v^+\colon q \to_2 r, \qquad z\colon w^- \to_1 w^+\colon r \to_2 s$,

we construct a comparison 3-cell:

(11) $k(x, y, z)\colon x +_2 (y +_2 z) +_1 k(u^+, v^+, w^+) \to_0 k(u^-, v^-, w^-) +_1 (x +_2 y) +_2 z\colon$

$u^- +_1 (v^- +_1 w^-) \to_1 (u^+ +_1 v^+) +_1 w^+\colon p \to_2 s$.

As a first step we consider the cell $\zeta = \eta +_1 (\kappa(x, y, z) +_1 \varepsilon)$, obtained by 1-composition of the comparison 3-cell $\kappa(x, y, z)$ with a unit η and counit ε of companionship for $\kappa(u^\alpha, v^\alpha, w^\alpha)$

$$
\begin{tikzcd}[column sep=large]
u^- +_1 (v^- +_1 w^-) \arrow[r, "e_0(-)"] \arrow[d, "e_1(-)"'] \arrow[dr, phantom, "\eta"] & u^- +_1 (v^- +_1 w^-) \arrow[d, "k^-"] \\
u^- +_1 (v^- +_1 w^-) \arrow[r, dashed, "\kappa(u^-, v^-, w^-)"] \arrow[d, "X"'] \arrow[dr, phantom, "\kappa(x, y, z)"] & (u^- +_1 v^-) +_1 w^- \arrow[d, "Y"] \\
u^+ +_1 (v^+ +_1 w^+) \arrow[r, dashed, "\kappa(u^+, v^+, w^+)"] \arrow[d, "k^+"'] \arrow[dr, phantom, "\varepsilon"] & (u^+ +_1 v^+) +_1 w^+ \arrow[d, "e_1(-)"] \\
(u^+ +_1 v^+) +_1 w^+ \arrow[r, "e_0(-)"'] & (u^+ +_1 v^+) +_1 w^+
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & 0 \\
1 &
\end{tikzcd}
$$

$$
\begin{aligned}
k^\alpha &= k(u^\alpha, v^\alpha, w^\alpha), \\
X &= x +_2 (y +_2 z), \\
Y &= (x +_2 y) +_2 z.
\end{aligned}
$$

Then we obtain $k(x, y, z)$ by correcting the previous 3-cell ζ with the appropriate comparisons for units, as follows:

$$
\begin{tikzcd}[column sep=small]
u^-{+_1}(v^-{+_1}w^-) \arrow[r, equal] \arrow[dd, "X +_1 k^+"'] \arrow[ddr, phantom, "\lambda_1^{-1}(X +_1 k^+)"] & u^-{+_1}(v^-{+_1}w^-) \arrow[r, equal] \arrow[d, "e_1(-)"] \arrow[ddr, phantom, "\zeta"] & u^-{+_1}(v^-{+_1}w^-) \arrow[r, equal] \arrow[d, "k^-"'] \arrow[ddr, phantom, "e_1k^- +_1 \rho_1 Y"] & u^-{+_1}(v^-{+_1}w^-) \arrow[dd, "k^- +_1 Y"] \\
& u^-{+_1}(v^-{+_1}w^-) \arrow[d, "X"] & (u^-{+_1}v^-){+_1}w^- \arrow[d, "Y{+_1}e_1(-)"'] & \\
(u^+{+_1}v^+){+_1}w^+ \arrow[r, equal] & (u^+{+_1}v^+){+_1}w^+ \arrow[r, equal] & (u^+{+_1}v^+){+_1}w^+ \arrow[r, equal] & (u^+{+_1}v^+){+_1}w^+
\end{tikzcd}
$$

5.3. Lemma (Cospans and companions). The weak symmetric cubical categories $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ and $\omega\mathbb{C}$ at have structural companions.

Proof. (a) We first address cubical cospans. The natural transformation $f\colon x^- \to x^+\colon \wedge^n \to \mathbf{X}$ can be viewed as a functor $f\colon \mathbf{2} \times \wedge^n \to \mathbf{X}$. The functors p, η, ε defined as follows (on the objects)

$$
\begin{array}{llll}
(1) & p\colon \wedge \to \mathbf{2}, & p(-1) = 0, & p(0) = p(1) = 1\,, \\
& \eta\colon \mathbf{2} \times \wedge \to \mathbf{2}, & \eta(0, i) = 0, & \eta(1, i) = p(i), \\
& \varepsilon\colon \mathbf{2} \times \wedge \to \mathbf{2}, & \varepsilon(0, i) = p(i), & \varepsilon(1, i) = 1,
\end{array}
$$

yield an (n+1)-cube and (n+1)-maps that satisfy our conditions:

$$
\begin{array}{lll}
(2) & f_* = f.(p \times \wedge^n)\colon \wedge^{n+1} \to \mathbf{X}, & \\
& \eta_f = f.(\eta \times \wedge^n)\colon \mathbf{2} \times \wedge^{n+1} \to \mathbf{X}, & \varepsilon_f = f.(\varepsilon \times \wedge^n)\colon \mathbf{2} \times \wedge^{n+1} \to \mathbf{X}.
\end{array}
$$

(b) A natural transformation $f\colon x^- \to x^+\colon \wedge^n \to \mathbf{Cat}$ between cubical profunctors can be transformed into an (n+1)-cubical profunctor f_*, following the procedure sketched at the end of 4.3 for the 0-dimensional case (when f is a functor between small categories. □

5.4. Globularisation of cubical cospans. Truncating the cubical structure $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ in degree 1, we get a weak *2-cubical* category $2\mathbb{C}\mathrm{osp}(\mathbf{X}) = \mathrm{tr}_1(\omega\mathbb{C}\mathrm{osp}(\mathbf{X}))$, with one strict direction and one weak direction. This coincides with the pseudo (or weak) double category defined and studied in [GP1, GP2]. Its associated special cylindrical structure $\mathrm{SCyl}(2\mathbb{C}\mathrm{osp}(\mathbf{X}))$ is the ordinary bicategory of cospans.

Truncating $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ in degree 2 we get a weak symmetric *3-cubical* category $3\mathbb{C}\mathrm{osp}(\mathbf{X}) = \mathrm{tr}_2(\omega\mathbb{C}\mathrm{osp}(\mathbf{X}))$, with one strict direction and two weak directions. Then the procedure expounded in 5.2 gives a tricategory,

The same can be done with $\omega\mathbb{C}\mathrm{at}$.

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