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BALANCE PROPERTIES OF THE FIXED POINT OF THE SUBSTITUTION ASSOCIATED TO QUADRATIC SIMPLE PISOT NUMBERS

Ondřej Turek[1]

Abstract. In this paper we will deal with the balance properties of the infinite binary words associated to β-integers when β is a quadratic simple Pisot number. Those words are the fixed points of the morphisms of the type $\varphi(A) = A^pB$, $\varphi(B) = A^q$ for $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p \geq q$, where $\beta = \frac{p+\sqrt{p^2+4q}}{2}$. We will prove that such word is t-balanced with $t = 1 + [(p-1)/(p+1-q)]$. Finally, in the case that $p < q$ it is known [B. Adamczewski, *Theoret. Comput. Sci.* **273** (2002) 197–224] that the fixed point of the substitution $\varphi(A) = A^pB$, $\varphi(B) = A^q$ is not m-balanced for any m. We exhibit an infinite sequence of pairs of words with the unbalance property.

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1. Introduction

A Pisot number β is a real algebraic integer greater than 1, all of whose conjugates are of modulus strictly less than 1. Since $\beta > 1$, we can define for every $x > 0$ the so-called β-expansion of x as a representation of the form

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots,$$

for x_i non-negative integers satisfying certain conditions. The β-expansion is a generalization of ordinary representations of real numbers in base 10 and can be defined for all $\beta > 1$. Analogically to the decimal expansions, coefficients of the β-expansion are found by the 'greedy algorithm', *i.e.* we find maximal $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$ and we set $x_k = [x/\beta^k]$ and $r_k = x/\beta^k - [x/\beta^k]$. For $i \in \mathbb{Z}$,

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[1] Department of Mathematics, FNSPE, Czech Technical University, Trojanova 13, 120 00 Praha 2, Czech Republic; oturek@centrum.cz

$i < k$ we put $x_i = [\beta r_{i+1}]$ and $r_i = \beta r_{i+1} - [\beta r_{i+1}]$. This algorithm implies that the coefficients x_i are integers in $\{0, 1, \ldots, [\beta]\}$. The term β-expansion of 1 is defined differently: we put $k = 1$ and then proceed analogically to the case $x \neq 0$. For more facts about β-expansions see Chapter 7 of [8].

The Pisot number β is said to be simple, if the β-expansion of 1 is finite, otherwise we call it non-simple. It can be shown that the simple quadratic Pisot numbers are exactly the positive roots of the polynomials $x^2 - px - q$ with $p \geq q$ (see [2, 5]). Consider the set $\mathbb{Z}_\beta$ of β-integers, *i.e.* those numbers $x \geq 0$ whose β-expansion is of the form $x = x_k\beta^k + \cdots + x_1\beta + x_0$.

From now on β is a simple quadratic Pisot number. Drawn on the real line, there are only two distances between neighbouring points of $\mathbb{Z}_\beta$. Conversely, there are exactly two types of distances between neighbouring points of $\mathbb{Z}_\beta$ for $\beta > 1$ only if β is a quadratic Pisot number. If we assign names A, B to the two types of distances and write down the order of distances in $\mathbb{Z}_\beta$ on the real line, we naturally obtain an infinite word; we will denote this word by u. It can be shown that the word u is a fixed point of a certain substitution φ (see *e.g.* [7]); in particular, for the simple quadratic Pisot number β (the root of $x^2 - px - q$ for $p \geq q \geq 1$), the generating substitution is

$$\varphi(A) = A^pB, \quad \varphi(B) = A^q,$$

$$A \mapsto A^pB \mapsto (A^pB)^p A^q \mapsto \cdots$$

A word v defined over the binary alphabet $\{A, B\}$ is said to be t-balanced ($t \in \mathbb{N}$) if for all pairs of factors w, $\hat{w}$ of v, which are of the same length, the difference between the number of letters A in w and $\hat{w}$ is less or equal to t. From Theorem 13 of [1] follows that there exists an integer t such that the word u is t-balanced for all simple Pisot numbers. However, the optimal bounds were known only for the following cases:

- $p = q = 1$: $t = 1$. The word u is Fibonacci word; see Chapter 2 of [8];
- $q = 1, p > 1$: $t = 1$. This case appeared in the paper [6], see Proposition 6.1;
- $p = q = 2$: $t = 2$. See Theorem 7.1 of the paper [11].

In this paper we compute the optimal bound for all values of p and q corresponding to minimal polynomials of quadratic simple Pisot numbers.

According to [1], if $p < q$, the word u is not m-balanced for all $m \in \mathbb{N}$. In the last section of this paper we exhibit a sequence of pairs of factors $v^{(n)}$, $w^{(n)}$ of u such that $v^{(n)}$ and $w^{(n)}$ are of the same length and the number of letters A in $v^{(n)}$ and $w^{(n)}$ differ at least by n.

2. Preliminaries

Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be a finite alphabet. A concatenation of letters in $\mathcal{A}$ is called a word. The set $\mathcal{A}^*$ of all finite words equipped with the empty word ϵ and the operation of concatenation is a free monoid. The length of a word

$w = w_0 w_1 \cdots w_{n-1}$ is denoted by $|w| = n$. One may consider also infinite words $v = v_0 v_1 v_2 \cdots$, the set of infinite words is denoted by $\mathcal{A}^{\mathbb{N}}$. A word w is called a factor of $v \in \mathcal{A}^*$, resp. $\mathcal{A}^{\mathbb{N}}$, if there exist words $w^{(1)} \in \mathcal{A}^*$, $w^{(2)} \in \mathcal{A}^*$, resp. $w^{(2)} \in \mathcal{A}^{\mathbb{N}}$ such that $v = w^{(1)} w w^{(2)}$. The word w is called a prefix of v, if $w^{(1)} = \epsilon$. It is a suffix of v, if $w^{(2)} = \epsilon$.

Denote by $|w|_{a_i}$ the number of letters a_i in the word w. The balance function B_v of the infinite word v is defined by:

$$B_v(n) = \max_{1 \le i \le k} \max_{w, \hat{w} \in \mathcal{L}_n(v)} \{ ||w|_{a_i} - |\hat{w}|_{a_i}| \} ,$$

where $\mathcal{L}_n(v)$ denotes the set of all factors of length n of the word v. We say that an infinite word v is t-balanced, if for every i, $1 \le i \le k$ and for every pair of factors w, $\hat{w}$ of v, $|w| = |\hat{w}|$ we have $||w|_{a_i} - |\hat{w}|_{a_i}| \le t$. The infinite word is thus t-balanced if and only if its balance function is bounded by t. Recall that Sturmian words are characterized by the property that they are 1-balanced (or simply balanced).

A morphism on the free monoid $\mathcal{A}^*$ is a map $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ satisfying $\varphi(w\hat{w}) = \varphi(w)\varphi(\hat{w})$ for all $w, \hat{w} \in \mathcal{A}^*$. Clearly, the morphism φ is determined if we define $\varphi(a_i)$ for all $a_i \in \mathcal{A}$.

A morphism φ is called a substitution, if $\varphi(a_i) \neq \epsilon$ for all $i = 1, 2, \ldots, k$ and if there exist $a_i \in \mathcal{A}$ such that $|\varphi(a_i)| > 1$. An infinite word u is said to be a fixed point of the substitution φ, or invariant under the substitution φ, if

$$\varphi(u_0)\varphi(u_1)\varphi(u_2) \cdots = u_0 u_1 u_2 \cdots \tag{1}$$

or $\varphi(u) = u$, after having naturally extended the action of φ to infinite words. Relation (1) implies that $\varphi(u_0) = u_0 \hat{u}$ and $\varphi^n(u) = u$ for every $n \in \mathbb{N}$. The length of the word $\varphi^n(A)$ grows to infinity with n, therefore for every $n \in \mathbb{N}$ the word $\varphi^n(u_0)$ is a prefix of the fixed point u, formally $u = \lim_{n\to\infty} \varphi^n(u_0)$.

With every substitution φ can be associated an incidence matrix M_φ, which is defined as

$$(M_\varphi)_{ij} = |\varphi(a_j)|_{a_i} .$$

From now on, we shall focus on the substitution φ on the alphabet $\{A, B\}$ given by

$$\begin{aligned} \varphi(A) &= A^p B \\ \varphi(B) &= A^q , \quad p \ge 1 , q \ge 1, \end{aligned} \tag{2}$$

and let us denote by

$$u = u_0 u_1 u_2 u_3 \cdots$$

the infinite word in the alphabet $\mathcal{A}$ invariant under φ. The substitution φ has a unique fixed point, namely

$$u = \lim_{n\to\infty} \varphi^n(A).$$

The incidence matrix associated with the substitution (2) is thus of the form

$$M_\varphi = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}. \tag{3}$$

This matrix has two real eigenvalues:

$$\theta_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \theta_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

Since $q > 0$, necessarily $\theta_1 > 1$. According to [1], Theorem 13, the balance properties of the substitution (2) are determined by the absolute value of the eigenvalue θ_2:

(i) if $|\theta_2| < 1$, then the balance function of the fixed point u is bounded;
(ii) if $|\theta_2| \geq 1$, then the balance function of the fixed point u is not bounded.

Obviously, the situation (i) corresponds to $p \geq q \geq 1$, the situation (ii) to $p < q$. We will find the uniform bound of the balance function for the case (i) (for fixed p and q). In the second case we will give an example of the sequence of the pairs of factors $v^{(n)}$, $w^{(n)}$ of u satisfying $\left|v^{(n)}\right| = \left|w^{(n)}\right|$ and $\left|v^{(n)}\right|_A - \left|w^{(n)}\right|_A \to +\infty$ for $n \to +\infty$.

3. Basic properties of u in relation to balances

In this section we state some properties of the infinite word u that follow from the form of the substitution (2). Results of this section will be used for investigation of balance properties of u.

Observation 3.1. *For every $n \in \mathbb{Z}$, $n \geq 2$ we have*

$$\varphi^n(A) = \left(\varphi^{n-1}(A)\right)^p \left(\varphi^{n-2}(A)\right)^q.$$

Proof. The statement can be proved easily by induction on n. □

Observation 3.2. *For every $n \in \mathbb{N}$,*

$$\begin{aligned} \varphi^{2n}(A) &= vBA^q \\ \varphi^{2n-1}(A) &= wA^pB \end{aligned}$$

for some words $v, w \in \mathcal{A}^*$.

Proof. The statement can be proved easily by induction on n. □

Observation 3.3. *Let BA^kB be a factor of u. Then $k = p$ or $k = p + q$. In particular, if A^k is a factor of u, then $k \leq p + q$.*

Proof. It suffices to show the statement for a finite word $\varphi^n(A)$ for every $n \in \mathbb{N}$. Since for $n \in \mathbb{N}$ the word $\varphi^n(A)$ begins with A^pB, we obtain the result using Observations 3.1, 3.2. □

Observation 3.4. *Let vB be a finite factor of u. Then there exists a unique finite factor w satisfying this condition: If vB is a suffix of $\varphi(\hat{w})$, then w is a suffix of $\hat{w}$. Moreover, there exists a unique nonnegative integer k such that $A^k vB = \varphi(w)$.*

Proof. Since vB is a factor of u, there exists $n \in \mathbb{N}$ such that vB is a factor of $\varphi^n(A)$. Therefore we can find a factor w in $\varphi^{n-1}(A)$ such that vB is a factor of $\varphi(w)$. Moreover, since B is a suffix of vB, there exists w such that vB is a suffix of $\varphi(w)$. We choose the factor w of $\varphi^{n-1}(A)$ so that it has minimal length. Assume that there exist two factors $w^{(1)}$, $w^{(2)}$, $\left|w^{(1)}\right| = \left|w^{(2)}\right| = |w|$ such that $w^{(1)} = z^{(1)}Az$, $w^{(2)} = z^{(2)}Bz$ for some factors $z^{(1)}$, $z^{(2)}$, z satisfying $|z| < |w|$. Hence $\varphi\left(w^{(1)}\right) = \varphi\left(z^{(1)}\right) A^p B\varphi(z)$, $\varphi\left(w^{(2)}\right) = \varphi\left(z^{(2)}\right) A^q\varphi(z)$. Since $|z| < |w|$, then $|\varphi(z)| < |vB|$, thus at the same time $B\varphi(z)$ is a suffix of vB and $A\varphi(z)$ is a suffix of vB. It is a contradiction, thus w is unique and with respect to its minimal length w satisfies the condition from this observation.

Assume that $\varphi(w) = \hat{v}BA^j v$; then $\varphi(w) = \hat{\hat{v}}\varphi(A)A^j vB$, thus $A^j vB = \varphi(z)$ for some factor z, which contradicts minimality of $|w|$. ☐

Observation 3.5. *For every finite word w we have*

$$|w|_A = |\varphi(w)|_B\,, \qquad |\varphi(w)|_A = p|w|_A + q|w|_B\,.$$

Observation 3.6. *Let v, w be factors of u, $|v| = |w|$. Then*

$$|v|_A - |w|_A = |w|_B - |v|_B\,.$$

4. Balances of binary infinite word in the Pisot case

In this section we will find a uniform bound of the balance function corresponding to the fixed point of the substitution φ given by $\varphi(A) = A^pB$, $\varphi(B) = A^q$, $p \geq q \geq 1$ and we will show that this bound is optimal.

Theorem 4.1. *The infinite word u invariant under the morphism $\varphi : \{A, B\} \to \{A, B\}$, given by $\varphi(A) = A^pB$, $\varphi(B) = A^q$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p \geq q$ is t-balanced, where $t = 1 + \left[\frac{p-1}{p+1-q}\right]$.*

Proof. We shall prove the theorem by contradiction. Assume that there exist an $n \in \mathbb{N}$ and two factors v, w of u, $|v| = |w| = n$, such that $|v|_A - |w|_A \geq t + 1$. We choose minimal n with this property. Therefore

$$|v|_A - |w|_A = t + 1. \tag{4}$$

Since $t + 1 \geq 2$, the words v and w are of the form $v = A\hat{v}A$, $w = B\hat{w}B$ for some factors $\hat{v}$, $\hat{w}$ of u.
Moreover, Observation 3.3 implies that one of the following situations occurs:

(i) $w = BA^pB$;
(ii) $w = BA^{p+q}B$;

(iii) $w = BA^pBA^pB$;
(iv) $w = BA^{p+q}BA^pB$;
(v) $w = BA^pBA^{p+q}B$;
(vi) $w = BA^{p+q}BA^{p+q}B$;
(vii) $w = BA^pB\hat{\hat{w}}BA^pB$ for some word $\hat{\hat{w}}$;
(viii) $w = BA^{p+q}B\hat{\hat{w}}BA^pB$ for some word $\hat{\hat{w}}$;
(ix) $w = BA^pB\hat{\hat{w}}BA^{p+q}B$ for some word $\hat{\hat{w}}$;
(x) $w = BA^{p+q}B\hat{\hat{w}}BA^{p+q}B$ for some word $\hat{\hat{w}}$.

In order to show that the only possible situation is (vii) let us at first prove the following statement:

Let v, w be the factors defined above, $n = |v| = |w|$. Then:

(a) if $w = BA^kB\bar{w}$, then there exists a factor $\bar{v}$ such that $v = A^{k+1}\bar{v}$;
(b) if $w = \bar{w}BA^kB$, then there exists a factor $\bar{v}$ such that $v = \bar{v}A^{k+1}$.

For the proof of (a) assume that $v = A^jB\bar{\bar{v}}$ and $j < k+1$. Then $|\bar{\bar{v}}| = \left|A^{k+1-j}B\bar{w}\right| < |v| = |w|$ and $|\bar{\bar{v}}|_A - \left|A^{k+1-j}B\bar{w}\right|_A = |v|_A - |w|_A$, which contradicts the minimality of n.

The proof of statement (b) is similar.

Statements (a) and (b) will be used for determination of the structure of the word v:

(i) $v = A^{p+2}$;
$t + 1 = |v|_A - |w|_A = 2 \Rightarrow t = 1 \Rightarrow q = 1$.
However, $v = A^{p+2} \Rightarrow q \geq 2$ according to Observation 3.3, which is a contradiction.

(ii) $v = A^{p+q+2}$. It is a contradiction with Observation 3.3.

(iii) $v = A^{2p+3}$ or $v = A^{p+1}BA^{p+1}$.
Since $v = A^{2p+3}$ contradicts Observation 3.3 ($2p + 3 > p + q$), then $v = A^{p+1}BA^{p+1}$. Thus $BABA = \varphi^{-1}(A^{q-1}vA^{q-1}B)$ is a factor of u according to Observations 3.3 and 3.4; occurence of the factor BAB in the word u together with Observation 3.3 imply that $p = 1$.
Since $p \geq q$, necessarily $q = 1$ and thus u is the Fibonacci word, which is known to be balanced. This is a contradiction with the assumption (4).

(iv) $v = A^{p+q+1}BA^{p+1}$ or $v = A^{2p+q+3}$. Both situations contradict Observation 3.3.

(v) $v = A^{p+1}BA^{p+q+1}$ or $v = A^{2p+q+3}$. It is a contradiction with Observation 3.3.

(vi) $v = A^{p+q+1}BA^{p+q+1}$ or $v = A^{2p+2q+3}$. It is a contradiction with Observation 3.3.

(vii) $v = A^{p+1}\check{v}A^{p+1}$ for some factor $\check{v}$.

(viii) $v = A^{p+q+1}\check{v}A^{p+1}$ for some factor $\check{v}$. It is a contradiction with Observation 3.3.

(ix) $v = A^{p+1}\check{v}A^{p+q+1}$ for some factor $\check{v}$. It is a contradiction with Observation 3.3.

(x) $v = A^{p+q+1}\check{v}A^{p+q+1}$ for some factor $\check{v}$. It is a contradiction with Observation 3.3.

Hence

$$w = BA^pB\hat{\hat{w}}BA^pB \quad \text{for some factor } \hat{\hat{w}} \tag{5}$$

and $v = A^{p+1}\check{v}A^{p+1}$ for some factor $\check{v}$. From the relation (5) follows that $|w| \geq 2p+4$, which implies that there exist $p+1 \leq i \leq p+q$, $p+1 \leq \ell \leq p+q$ such that $v = A^iB\hat{\hat{v}}BA^\ell$. Since $\ell \geq p+1 > q$, we can define $h = \ell - q \in \mathbb{N}$; then

$$v = A^iB\hat{\hat{v}}BA^{q+h}\,, \qquad p+1 \leq i \leq q+p, \quad p+1 \leq q+h \leq q+p. \tag{6}$$

Let us consider the word v' defined by the relation

$$v' = A^{q+p}B\hat{\hat{v}}BA^q\,.$$

Then

$$v'A^h = A^jv\,, \quad j+i = q+p\,. \tag{7}$$

Relations (6) together with Observation 3.3 imply that $v'A^pB$ is a factor of u, and from Observation 3.4 follows that it has uniquely determined preimage $\varphi^{-1}(v'A^pB) = xA$ for some factor x. Thus x is a factor of u and $\varphi(x) = v'$.

Similarly: $w' = A^pw = A^pB\hat{\hat{w}}BA^pB$ is a factor of u and has uniquely determined preimage $y = \varphi^{-1}(w')$.

The following relations for the unknown integers j and h could be obtained from (6) and (7):

$$0 \leq j \leq q-1\,, \qquad p+1-q \leq h \leq p\,. \tag{8}$$

Observation 3.5 implies

$$\begin{aligned}
|x|_A &= |v'|_B = |v|_B\,,\\
|y|_A &= |w'|_B = |w|_B\,,\\
|x|_B &= \frac{1}{q}\left(|v'|_A - p\cdot|v'|_B\right) = \frac{1}{q}\left(|v'| - (p+1)\cdot|v'|_B\right)\\
&= \frac{1}{q}\left(|v| + j - h - (p+1)\cdot|v|_B\right)\,,\\
|y|_B &= \frac{1}{q}\left(|w'|_A - p\cdot|w'|_B\right) = \frac{1}{q}\left(|w'| - (p+1)\cdot|w'|_B\right)\\
&= \frac{1}{q}\left(|w| + p - (p+1)\cdot|w|_B\right)\,.
\end{aligned}$$

Thus

$$|y|_A - |x|_A = |w|_B - |v|_B = |v|_A - |w|_A = t+1 \tag{9}$$

(according to Observation 3.6) and

$$\begin{aligned}
|x| &= |x|_A + |x|_B = |v|_B + \frac{1}{q}\left(|v| + j - h - (p+1)\cdot|v|_B\right)\,,\\
|y| &= |y|_A + |y|_B = |w|_B + \frac{1}{q}\left(|w| + p - (p+1)\cdot|w|_B\right)\,.
\end{aligned} \tag{10}$$

Using relation (4) and Observation 3.6 we obtain the difference between lengths of the words x and y:

$$\begin{aligned} |x|-|y| &= |v|_B-|w|_B+\frac{1}{q}\left(|v|-|w|-(p+1)(|v|_B-|w|_B)+j-h-p\right) \\ &= -(t+1)+\frac{1}{q}\left((p+1)(t+1)+j-h-p\right). \end{aligned} \tag{11}$$

Necessarily $|x| \in \mathbb{N}$, $|y| \in \mathbb{N}$, thus $|x|-|y| \in \mathbb{Z}$ and

$$q \mid (p+1)(t+1)+j-h-p\,.$$

Let us denote $V=(p+1)(t+1)+j-h-p$, then

$$V \equiv 0 \pmod q\,, \tag{12}$$

$$|x|-|y| = -(t+1)+\frac{V}{q}\cdot \tag{13}$$

From (8) follows

$$-p \le j-h \le 2q-2-p. \tag{14}$$

When we substitute (14) into the definition relation for V, we obtain

$$(p+1)(t+1)-2p \le V \le (p+1)(t+1)+2(q-1)-2p. \tag{15}$$

Let $V_{\min}=(p+1)(t+1)-2p$. We will show that

$$V_{\min} > tq: \tag{16}$$

$$t=1+\left[\frac{p-1}{p+1-q}\right] \Rightarrow t > \frac{p-1}{p+1-q} \Rightarrow t(p+1-q) > p-1 \Rightarrow (p+1)(t+1)-2p > tq.$$

From relations (15), (16) and (12) follows

$$V \ge (t+1)q\,. \tag{17}$$

Relation (17) together with relation (13) imply $|x|-|y| \ge -(t+1)+\frac{(t+1)q}{q}=0$, hence

$$|x| \ge |y|\,. \tag{18}$$

Relation (18) allows us to define the word $\hat{x}$ as a prefix of x of length $|y|$.

Thus $|\hat{x}|=|y|$ and from relation (9) we obtain

$$|y|_A-|\hat{x}|_A \ge |y|_A-|x|_A = t+1\,.$$

From relations (10) and (5) follows that $|y|<|w|$. Thus words $\hat{x}$, y are factors of the same length $|\hat{x}|=|y|<|w|=n$ and satisfy $|y|_A-|\hat{x}|_A \ge t+1$, which contradicts minimality of n. □

Theorem 4.2. *Let u be the infinite word invariant under the morphism φ : $\{A, B\} \to \{A, B\}$, given by $\varphi(A) = A^pB$, $\varphi(B) = A^q$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p \geq q$. Then there exist factors v, w of u such that $|v| = |w|$ and $|v|_A - |w|_A = t$, where $t = 1 + \left[\frac{p-1}{p+1-q}\right]$; i.e. the bound t is optimal.*

Proof. We will prove that for every n, $1 \leq n \leq t$ there exist factors $v^{(n)}$, $w^{(n)}$ of u, $\left|v^{(n)}\right| = \left|w^{(n)}\right|$, such that

$$\begin{aligned} &\left|v^{(n)}\right|_A - \left|w^{(n)}\right|_A = n\,, \\ &v^{(n)} = A\hat{v}^{(n)} \text{ for some factor } \hat{v}^{(n)}\,, \\ &w^{(n)} = \hat{w}^{(n)}B \text{ for some factor } \hat{w}^{(n)}\,. \end{aligned} \tag{19}$$

By induction on n:
(I) $\{n = 1\}$ Let us denote

$$v^{(1)} = A, \qquad w^{(1)} = B;$$

$v^{(1)}$, $w^{(1)}$ are factors of u of the same length and they obviously satisfy (19).
(II) $\{n - 1 \to n\}$ Let us define

$$d_n = 1 + (p + 1 - q)(n - 1), \quad v^{(n)} = \varphi\left(w^{(n-1)}\right)A^{d_n}, \quad A^pw^{(n)} = \varphi\left(v^{(n-1)}A\right).$$

Then

(a) $v^{(n)}$ is a factor of u,

Proof. $w^{(n-1)} = \hat{w}^{(n-1)}B \Rightarrow \hat{w}^{(n-1)}BA = w^{(n-1)}A$ is a factor of u due to Observation 3.3 $\Rightarrow \varphi\left(w^{(n-1)}A\right) = \varphi\left(w^{(n-1)}\right)A^pB$ is a factor of $u \Rightarrow \varphi\left(w^{(n-1)}\right)A^d$ is a factor of u for every $d \leq p$. Since $d_n = 1 + (p + 1 - q)(n-1) \leq 1 + (p+1-q)(t-1) = 1 + (p+1-q)\left[\frac{p-1}{p+1-q}\right] \leq 1 + (p-1) = p$, $v^{(n)}$ is a factor of u. □

(b) $w^{(n)}$ is a factor of u,

Proof.
$n = 2$: $A^pw^{(2)} = \varphi\left(v^{(1)}A\right) = \varphi(AA) = A^pBA^pB$ is a factor of u according to Observation 3.3.
$n > 2$: $v^{(n-1)} = \varphi\left(w^{(n-2)}\right)A^{d_{n-1}} \Rightarrow A^pw^{(n)} = \varphi\left(v^{(n-1)}A\right) = \varphi\left(\varphi\left(w^{(n-2)}\right)A^{d_{n-1}}A\right) = \varphi\left(\varphi\left(\hat{w}^{(n-2)}B\right)A^{d_{n-1}}A\right)$.
If $d_{n-1} < p$, then $\varphi\left(\varphi\left(\hat{w}^{(n-2)}B\right)A^{d_{n-1}}A\right)A^{p-d_{n-1}-1}B = \varphi\left(\varphi\left(\hat{w}^{(n-2)}BA\right)\right) = \varphi\left(\varphi\left(w^{(n-2)}A\right)\right)$ and $A^pw^{(n)}A^{p-d_{n-1}-1}B = \varphi\left(\varphi\left(w^{(n-2)}A\right)\right)$ is a factor of u. Thus $w^{(n)}$ is a factor of u if $d_{n-1} = 1 + (p + 1 - q)(n - 1) < p$. However, this inequality is valid for every $n \leq t$, because $d_{n-1} \leq d_{t-1} < d_t = 1 + (p + 1 - q)(t - 1) = 1 + (p + 1 - q)\left[\frac{p-1}{p+1-q}\right] \leq p$. □

(c) $\left|v^{(n)}\right| = \left|w^{(n)}\right|$,

Proof. From the construction of the words $v^{(n)}$, $w^{(n)}$ and Observation 3.5 follows:

$$\begin{aligned}
\left|v^{(n)}\right| &= (p+1)\left|w^{(n-1)}\right|_A + q\left|w^{(n-1)}\right|_B + d_n \\
&= (p+1)\left(\left|v^{(n-1)}\right|_A - (n-1)\right) + q\left(\left|v^{(n-1)}\right|_B + (n-1)\right) \\
&\quad +1 + (p+1-q)(n-1), \\
&= (p+1)\left|v^{(n-1)}\right|_A + q\left|v^{(n-1)}\right|_B + 1, \\
\left|w^{(n)}\right| &= (p+1)\left(\left|v^{(n-1)}\right|_A + 1\right) + q\left|v^{(n-1)}\right|_B - p = (p+1)\left|v^{(n-1)}\right|_A \\
&\quad +q\left|v^{(n-1)}\right|_B + 1,
\end{aligned}$$

hence $\left|v^{(n)}\right| = \left|w^{(n)}\right|$. □

(d) $v^{(n)}$, $w^{(n)}$ satisfy (19).

Proof. From the construction of the words $v^{(n)}$, $w^{(n)}$ follows:

$$\begin{aligned}
\left|v^{(n)}\right|_A &= p\left|w^{(n-1)}\right|_A + q\left|w^{(n-1)}\right|_B + d_n = p\left|w^{(n-1)}\right|_A + q\left|w^{(n-1)}\right|_B \\
&\quad +1 + (p+1-q)(n-1), \\
\left|w^{(n)}\right|_A &= p\left|v^{(n-1)}\right|_A + q\left|v^{(n-1)}\right|_B - p + p = p\left|v^{(n-1)}\right|_A + q\left|v^{(n-1)}\right|_B .
\end{aligned}$$

These relations together with Observation 3.6 imply

$$\begin{aligned}
\left|v^{(n)}\right|_A - \left|w^{(n)}\right|_A &= p\left(\left|w^{(n-1)}\right|_A - \left|v^{(n-1)}\right|_A\right) + q\left(\left|w^{(n-1)}\right|_B - \left|v^{(n-1)}\right|_B\right) \\
&\quad +1(p+1-q)(n-1) \\
&= -p(n-1) + q(n-1) + 1 + (p+1-q)(n-1) = n.
\end{aligned}$$

The fact that $v^{(n)} = A\hat{v}^{(n)}$ and $w^{(n)} = \hat{w}^{(n)}B$ for some factors $\hat{v}^{(n)}$ and $\hat{w}^{(n)}$ follows directly from the definition of φ. □

Now let us denote $v = v^{(t)}$, $w = w^{(t)}$. Then $|v| = |w|$ and $|v|_A - |w|_A = t$, which proves the theorem. □

Remark 4.3.

- If $p = q$, *i.e.* if $\varphi(A) = A^p B$, $\varphi(B) = A^p$, then u is p-balanced.
- For this type of substitution, the word u is Sturmian (*i.e.* $t = 1$), only if $q = 1$.

5. Balance properties of the fixed point u of the substitution $A \mapsto A^pB,\ B \mapsto A^q,\ p < q$

From [1], Theorem 13 follows, that if $p < q$, then u is not m-balanced for any $m \in \mathbb{N}$. In this section we will give a specific proof of this fact, in which we will find an explicit infinite sequence of pairs of factors with certain prescribed unbalance property.

Theorem 5.1. *The infinite word u invariant under the morphism $\varphi : \{A, B\} \to \{A, B\}$, given by $\varphi(A) = A^pB$, $\varphi(B) = A^q$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $p < q$ is not n-balanced for any $n \in \mathbb{N}$, i.e. for every $n \in \mathbb{N}$ there exist factors $v^{(n)}$, $w^{(n)}$ of u, $\left|v^{(n)}\right| = \left|w^{(n)}\right|$ such that $\left|v^{(n)}\right|_A - \left|w^{(n)}\right|_A \geq n$.*

Proof.

$p + 1 = q$: In this case $\theta_1 = p+1$ and $\theta_2 = -1$ is a root of unity, thus according to [1], Theorem 13, $B_u(n) = (O \cap \Omega)(\log N)$, where O and Ω are Landau symbols.
We define words $v^{(n)}$, $w^{(n)}$ recurrently, similarly as in the proof of Theorem 4.2, but for every $n \in \mathbb{N}$:

$$\begin{aligned} &v^{(1)} = A, \qquad w^{(1)} = B\,; \\ &n \in \mathbb{N},\ n \geq 2: \qquad v^{(n)} = \varphi\left(w^{(n-1)}\right) A, \qquad A^p w^{(n)} = \varphi\left(v^{(n-1)} A\right)\,. \end{aligned}$$

Then $v^{(n)}$, $w^{(n)}$ are factors of u and satisfy

$$\begin{aligned} &\left|v^{(n)}\right| = \left|w^{(n)}\right|\,; \\ &\left|v^{(n)}\right|_A - \left|w^{(n)}\right|_A = n\,; \\ &v^{(n)} = A\hat{v}^{(n)} \text{ for some factor } \hat{v}^{(n)}\,; \\ &w^{(n)} = \hat{w}^{(n)} B \text{ for some factor } \hat{w}^{(n)}\,, \end{aligned}$$

which can be proved similarly to the case $p \geq q$ (see proof of Th. 4.2).

$p \leq q - 2$: By induction. We will define a sequence of pairs of words $v^{(n)}$, $w^{(n)}$ satisfying

- $\left|v^{(n)}\right| = \left|w^{(n)}\right|$;
- $\left|w^{(n)}\right|_B - \left|v^{(n)}\right|_B \geq n$;
- $v^{(n)}A$ is a factor of u.

(I) $\{n = 1\}$ Let us define

$$v^{(1)} = A, \qquad w^{(1)} = B;$$

from Observation 3.3 follows that $v^{(1)}A = AA$ is a factor of u.

(II) $\{n - 1 \to n\}$ Let factors $v^{(k)}$, $w^{(k)}$ satisfy

- $\left|v^{(k)}\right| = \left|w^{(k)}\right|$;
- $\left|w^{(k)}\right|_B - \left|v^{(k)}\right|_B \geq k$;

- $v^{(k)}A$ is a factor of u

for all $k < n$, then we will define factors $v^{(n)}$, $w^{(n)}$ as follows.

At first, $v^{(n)}$ is given by

$$v^{(n)} = \varphi\left(w^{(n-1)}\right);$$

then

$$\left|v^{(n)}\right| = (p+1)\left|w^{(n-1)}\right|_A + q\left|w^{(n-1)}\right|_B$$

and $v^{(n)}A$ is a factor of u with respect to Observations 3.2 and 3.3.

Since $\left|\varphi\left(v^{(n-1)}A\right)\right| = (p+1)\left(\left|v^{(n-1)}\right|_A + 1\right) + q\left|v^{(n-1)}\right|_B$, we have

$$\begin{aligned}\left|v^{(n)}\right| - \left|\varphi\left(v^{(n-1)}A\right)\right| &= (p+1)\left(\left|w^{(n-1)}\right|_A - \left|v^{(n-1)}\right|_A - 1\right) \\ &\quad + q\left(\left|w^{(n-1)}\right|_B - \left|v^{(n-1)}\right|_B\right) \\ &= (q-p-1)\left(\left|v^{(n-1)}\right|_A - \left|w^{(n-1)}\right|_A\right) - p - 1 \geq \\ &\geq (q-p-1)(n-1) - p - 1 \geq -p - 1 - p - 1 \geq -p.\end{aligned} \tag{20}$$

Let $v^{(n-1)}Az^{(n-1)}$ be such factor of u, that $\left|\varphi\left(v^{(n-1)}Az^{(n-1)}\right)\right| \geq \left|v^{(n)}\right| + p$. Since A^p is a prefix of $\varphi(A)$ as well as of $\varphi(B)$ (because $p < q$), we can define a factor $\hat{w}^{(n)}$ by

$$A^p\hat{w}^{(n)} = \varphi\left(v^{(n-1)}Az^{(n-1)}\right);$$

it is obvious that $\left|\hat{w}^{(n)}\right| \geq \left|v^{(n)}\right|$. Factor $w^{(n)}$ will be now defined as a prefix of $\hat{w}^{(n)}$ of the length $\left|v^{(n)}\right|$, thus $\left|v^{(n)}\right| = \left|w^{(n)}\right|$. Relation (20) implies

$$\left|A^pw^{(n)}\right| - \left|\varphi\left(v^{(n-1)}A\right)\right| = p + \left|v^{(n)}\right| - \left|\varphi\left(v^{(n-1)}A\right)\right| \geq p + (-p) = 0\,. \tag{21}$$

From relation (21) follows that there exists a factor $\hat{\hat{w}}^{(n)}$, which satisfies

$$A^pw^{(n)} = \varphi\left(v^{(n-1)}A\right)\hat{\hat{w}}^{(n)}\,.$$

Then $v^{(n)}$, $w^{(n)}$ are factors of u, $\left|v^{(n)}\right| = \left|w^{(n)}\right|$,

$$\begin{aligned}\left|v^{(n)}\right|_B &= \left|w^{(n-1)}\right|_A\,, \\ \left|w^{(n)}\right|_B &\geq \left|\varphi\left(v^{(n-1)}A\right)\right|_B = \left|v^{(n-1)}\right|_A + 1\,,\end{aligned}$$

hence

$$\left|w^{(n)}\right|_B - \left|v^{(n)}\right|_B \geq \left|v^{(n-1)}\right|_A + 1 - \left|w^{(n-1)}\right|_A \geq n - 1 + 1 = n\,. \qquad (22)$$

Since there exists for every $n \in \mathbb{N}$ a pair of factors $v^{(n)}$, $w^{(n)}$ of the same length satisfying (22), the theorem is proved. □

6. Conclusions

We have described the balance properties of the infinite word u invariant under the substitution φ given by $\varphi(A) = A^p B$, $\varphi(B) = A^q$. The main result consists in the determination of the optimal bound of the balance function of u when u corresponds to some quadratic simple Pisot number.

When β is quadratic non-simple Pisot number, then the substitution associated to it is of the form $\varphi(A) = A^p B$, $\varphi(B) = A^q B$, $p > q$, and according to [1], the balance function of the corresponding word u is balanced in this case as well. Determination of its optimal bound remains to be an interesting open question.

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