RAIRO-Inf. Theor. Appl. **39** (2005) 1-29
DOI: 10.1051/ita:2005001

THE GLOBALS OF PSEUDOVARIETIES OF ORDERED SEMIGROUPS CONTAINING B_2 AND AN APPLICATION TO A PROBLEM PROPOSED BY PIN *

Jorge Almeida[1] and Ana P. Escada[2]

Abstract. Given a basis of pseudoidentities for a pseudovariety of ordered semigroups containing the 5-element aperiodic Brandt semigroup B_2, under the natural order, it is shown that the same basis, over the most general graph over which it can be read, defines the global. This is used to show that the global of the pseudovariety of level 3/2 of Straubing-Thérien's concatenation hierarchy has infinite vertex rank.

Mathematics Subject Classification. 20M05, 20M07, 20M35.

1. Introduction

Three concatenation hierarchies of rational languages have been considered since the 1970's: the *dot-depth hierarchy* introduced by Brzozowski [10], the *Straubing-Thérien hierarchy* [32, 33], and the *group hierarchy* presented in [19]. Pin (see [22]) has observed that all these hierarchies may be regarded as being indexed by half integers, that is numbers of the form n or $n + 1/2$ where n is a non-negative integer, and where each level other than level zero is obtained in the following way: the languages of level $n + 1/2$ are finite unions of products of the

Keywords and phrases. Semigroup, pseudovariety, semigroupoid, category, pseudoidentity, dot-depth, concatenation hierarchies.

* *Both authors acknowledge partial support by the project POCTI/32817/MAT/2000, which is co-financed by the European Community Fund FEDER, and by the INTAS grant #99-1224. Partial support by FCT, through the* Centro de Matemática da Universidade do Porto *and through the* Centro de Matemática da Universidade de Coimbra*, respectively, is also gratefully acknowledged.*

[1] Centro de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal; jalmeida@fc.up.pt

[2] Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal.

form $L_0a_1L_1 \dots a_rL_r$, where $L_0, \dots, L_r$ are languages of level n and $a_0, \dots, a_r$ are letters[1], and the languages of level $n+1$ are members of the Boolean closure of the class of languages of level $n + 1/2$. Thus each hierarchy is completely determined by its level zero. The dot-depth hierarchy $\mathcal{B}_n$ has the finite or cofinite languages of A^+ as a basis, the Straubing-Thérien hierarchy $\mathcal{V}_n$ is based on the languages $\emptyset$ and A^*, and the level zero of the group hierarchy $\mathcal{G}_n$ is obtained by taking the group languages. These three hierarchies are infinite and strict (see [11, 22]). The main problem, which is open in almost all cases, is whether each level is decidable. In fact, for the Straubing-Thérien hierarchy, this problem has been solved (positively) for $n \leq 3/2$ [8, 9, 30] and some partial results are known for the level two [24, 34, 35, 38]. For the other two hierarchies, the membership problem is only known to be decidable for $n \leq 1$ [14, 16, 17, 19]. See [31] for related problems concerning the product of rational languages.

Some of the early results concerning the dot-depth and Straubing-Thérien hierarchies follow from deep theorems of Simon [30]. The problem of the effective characterization of the *locally testable languages* was solved independently by McNaughton [20] and by Brzozowski and Simon [12]. This characterization implies that locally testable languages are of dot-depth one. The graph-theoretic arguments which were implicit in these works have had a strong impact on language and semigroup theories, namely in Knast's and Tilson's results. Knast [17] obtained a complex theorem on graphs and achieved an effective characterization of the whole class of languages of dot-depth one.

Tilson [37] (inspired by work of Simon [30], Knast [16,17], Thérien and Weiss [36], and Straubing [33]) developed a theory of (small) categories as partial algebras over graphs and showed its importance to study semigroups. In particular, his Derived Category Theorem turned out to be a powerful tool to deal with semidirect products of pseudovarieties of semigroups [7, 27]. In the profinite context, Weil and the first author [7] have used the derived category theorem to describe a basis of pseudoidentities of the semidirect product $\mathsf{V} * \mathsf{W}$ of pseudovarieties of semigroups from a basis of semigroupoid pseudoidentities of the pseudovariety $g\mathsf{V}$ of semigroupoids generated by the semigroups of V viewed as one-vertex semigroupoids. The proof of this result, that has come to be known as the *basis theorem*, has a gap, as has been observed by Rhodes and Steinberg, but the theorem stands in case W is locally finite or $g\mathsf{V}$ has finite vertex-rank.

Since each level of the considered hierarchies is a positive variety of languages, the variety theorem [13, 21] guarantees a one-to-one correspondence between the level n of each hierarchy and a pseudovariety of ordered monoids (or semigroups in case of the dot-depth hierarchy). The problem of decidability for a level n of the hierarchies is now reduced to determine whether the pseudovarieties B_n, V_n and G_n are decidable, where B_n, V_n, and G_n denote respectively the pseudovarieties of ordered semigroups (monoids) associated to $\mathcal{B}_n$, $\mathcal{V}_n$, and $\mathcal{G}_n$. There are interesting

[1] This is the definition adopted when the empty word is considered, that is for languages of A^*. Otherwise, for languages of A^+, one takes instead finite unions of languages of the form $u_0L_1u_1 \dots L_ru_r$, where $L_1, \dots, L_r$ are languages of level n and $u_0, \dots, u_r \in A^*$ are words with u_0 nonempty in case $r = 0$.

connections between V_n, B_n and G_n: $\mathsf{B}_n = \mathsf{V}_n * \mathcal{L}\mathsf{I}$ $(n > 0)$ [32], where $\mathcal{L}\mathsf{I}$ is the pseudovariety of *locally trivial semigroups*, and $\mathsf{G}_n = \mathsf{V}_n * \mathsf{G}$ $(n \geq 0)$ [23], where G is the pseudovariety of all finite groups. It is not immediately clear if these results reduce the study of one hierarchy to another. However, it is known that, for each integer n, B_n is decidable if and only if V_n is decidable [33].

For a pseudovariety V of finite ordered semigroups, $\ell\mathsf{V}$ denotes the pseudovariety of all finite ordered semigroupoids whose finite one-vertex subsemigroupoids with at least one edge lie in V, and $g\mathsf{V}$ is the pseudovariety of ordered semigroupoids generated by the ordered semigroups of V viewed as ordered one-vertex semigroupoids. A basis of pseudoidentities of ordered semigroupoids which defines $\ell\mathsf{V}$ can be easily obtained from a given basis for V. However it is not so easy to compute a basis of semigroupoid pseudoidentities for $g\mathsf{V}$ given such a basis for V. Based on results of Reilly [28], Azevedo, Teixeira and the first author [5] proved that the problem may be systematically treated when the pseudovariety V of semigroups contains the 5-element aperiodic Brandt semigroup B_2.

In Section 3, we show that a similar approach works in case V is a pseudovariety of ordered semigroups containing the ordered inverse semigroup B_2. Indeed, given a basis Σ of pseudoidentities of V, then the set consisting of the pseudoidentities of Σ over the most general graphs over which they can be read is a basis for the global of V.

In Section 4, we use our result as well as techniques introduced in [6] to show that the pseudovariety $\mathsf{V}_{\frac{3}{2}}$ of ordered semigroups, which corresponds to the level $3/2$ of the Straubing-Thérien hierarchy, has infinite vertex rank. Initially, as suggested by J.-E. Pin, our goal was to investigate, using the "basis theorem" for semidirect products, the decidability of the pseudovariety of finite semigroups $\mathsf{G}_{\frac{3}{2}} = \mathsf{V}_{\frac{3}{2}} * \mathsf{G}$, which corresponds to the level $3/2$ of the group hierarchy. The infinity of the vertex-rank of $\mathsf{V}_{\frac{3}{2}}$ renders this problem out of reach for the currently known range of validity of the basis theorem.

2. Preliminaries

We recall in Sections 2.1 and 2.2 some definitions and results concerning ordered semigroups and semigroupoids, free profinite semigroups and semigroupoids, and pseudovarieties of ordered semigroups and semigroupoids. For more details the reader is referred to [3, 4, 15, 22, 27]. Section 2.3 introduces basic facts about inverse semigroups and a version in the "ordered semigroups context" of a result of Reilly [28].

2.1. Ordered semigroups

An *ordered semigroup* S is a semigroup equipped with a partial order relation $\leq$ such that, for all $x, y, z \in S$, $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. A *homomorphism of ordered semigroups* $\varphi : (S, \leq) \to (T, \leq)$ is a semigroup homomorphism such that, for all $x, y \in S$, $x \leq y$ implies $\varphi(x) \leq \varphi(y)$. Semigroups are viewed as ordered semigroups under the equality order relation. An ordered semigroup $(S, \leq)$ is an *ordered subsemigoup* of $(T, \leq)$ if S is a subsemigroup of T and the order on S is the restriction to S of the order on T. If there is a surjective homomorphism of ordered semigroups $\varphi : (S, \leq) \to (T, \leq)$, then we say that $(T, \leq)$ is an *ordered quotient* of $(S, \leq)$. The *product* of a family $(S_i, \leq_i)_{i \in I}$ of ordered semigroups is the ordered semigroup $(\prod_{i \in I} S_i, \leq)$ where the multiplication is defined by $(s_i)_{i \in I}(t_i)_{i \in I} = (s_i t_i)_{i \in I}$ and the order is given by

$$(s_i)_{i \in I} \leq (t_i)_{i \in I} \text{ if and only if, for all } i \in I,\ s_i \leq_i t_i. \tag{1}$$

Throughout this paper X represents a finite non-empty set. Let X^+ be the free semigroup on X. Then $(X^+, =)$ is an ordered semigroup and it is the free ordered semigroup on X.

A *topological semigroup* is a semigroup endowed with a topology such that the multiplication is continuous. Finite semigroups are viewed as discrete topological semigroups. For a finite set X, we say that a topological semigroup S is X-*generated* if a mapping $\eta : X \to S$ is given such that $\eta(X)$ generates a dense subsemigroup of S.

For a finite set X, the *projective limit* of X-generated finite semigroups is denoted by $\widehat{X^+}$, and it is called the *free profinite semigroup* [4]. This semigroup is compact, totally disconnected and X-generated *via* the natural mapping $\iota : X \to \widehat{X^+}$. One may show that $\widehat{X^+}$ has the following universal property: every mapping $\varphi : X \to S$ into a finite semigroup S can be extended to a unique continuous homomorphism $\hat{\varphi} : \widehat{X^+} \to S$ such that $\hat{\varphi}\iota = \varphi$. Moreover, the topological structure of $\widehat{X^+}$ is metrizable, that is $\widehat{X^+}$ may be endowed with a metric d such that d induces the given topology of $\widehat{X^+}$ [4, Prop. 7.4].

Note that the subsemigroup generated by ιX is (isomorphic to) the free semigroup X^+ so we may assume that X^+ is a subsemigroup of $\widehat{X^+}$.

Let $u, v \in \widehat{X^+}$. We say that a finite ordered semigroup $(S, \leq)$ *satisfies the (pseudo)identity* $u \leq v$ and we write $S \models u \leq v$ if, for every continuous homomorphism $\varphi : \widehat{X^+} \to S$, we have $\varphi(u) \leq \varphi(v)$.

A *pseudovariety of ordered semigroups* is a class of finite ordered semigroups which is closed under taking finite ordered subsemigroups, finitary products and ordered quotients.

Pin and Weil have extended Reiterman's theorem [29] to pseudovarieties of ordered semigroups [25]. For a set Σ of pseudoidentities, we denote by $[\![\Sigma]\!]$ the class of all finite ordered semigroups which satisfy all pseudoidentities of Σ.

Theorem 2.1. *Let* V *be a class of finite ordered semigroups. Then* V *is a pseudovariety of ordered semigroups if and only if there exists a set* Σ *of pseudoidentities over finite sets such that* $\mathsf{V} = [\![\Sigma]\!]$.

2.2. Ordered semigroupoids

A *graph* is a set $G = V(G) \mathring{\cup} E(G)$ consisting of two sorts of elements, *vertices* and *edges*, endowed with two operations $\alpha, \omega : E(G) \to V(G)$ which give respectively the *beginning* and *end* vertices of each edge. For $a, b \in E(G)$, we say that a and b are *coterminal* if $\alpha(a) = \alpha(b)$ and $\omega(a) = \omega(b)$, and they are *consecutive* if $\omega(a) = \alpha(b)$. A *path* in a graph Γ is a finite sequence $a_1 \dots a_n$ of edges of Γ such that a_i and a_{i+1} are consecutive, for $i = 1, \dots, n-1$. Graphs with one vertex may be identified with their sets of edges. A *graph homomorphism* is a mapping $\varphi : G \to H$ between two graphs respecting sorts and operations. A *subgraph* H of G is a graph contained in G such that the inclusion $H \hookrightarrow G$ is a graph homomorphism.

By a *semigroupoid* we mean a graph S endowed with a partial associative multiplication on $E(S)$ given by: for $s, t \in E$, st is defined if and only if $\omega(s) = \alpha(t)$ and, then, $\alpha(st) = \alpha(s)$ and $\omega(st) = \omega(t)$. If S admits a local identity at each vertex, then we say that S is a *category*. For vertices $c, d \in V(S)$, the *hom-set* $S(c, d)$ is the set of edges $s \in E(S)$ such that $\alpha(s) = c$ and $\omega(s) = d$; in case $c = d$ we put $S(c) = S(c, d)$.

For a semigroupoid S, a binary relation τ on $E(S)$ is said to be *compatible* if, for all $x, y \in E(S)$,

(i) if $(x, y) \in \tau$ then x and y are coterminal;
(ii) if $(x, y) \in \tau$ and x, z are consecutive edges, then $(xz, yz) \in \tau$;
(iii) if $(x, y) \in \tau$ and z, x are consecutive edges, then $(zx, zy) \in \tau$.

A *congruence* τ on a semigroupoid S is an equivalence relation on $E(S)$ which is compatible. The *quotient semigroupoid* S/τ has $V(S/\tau) = V(S)$ and $E(S/\tau) = E(S)/\tau$, with the induced maps α and ω and composition of edges.

We say that the semigroupoid S is an *ordered semigroupoid* if it is equipped with a compatible partial order relation $\leq$ on $E(S)$. It is important to notice that two edges must be coterminal to be comparable under $\leq$.

For a finite graph Γ, the *free semigroupoid* Γ^+ on Γ has as vertex-set $V(\Gamma)$ and as edges the non-empty paths of Γ.

A *homomorphism of ordered semigroupoids* is a map φ between $(S, \leq)$ and $(T, \leq)$ which respects sorts, operations and the partial order, that is, $\varphi(x) \leq \varphi(y)$ whenever $x \leq y$. A homomorphism of ordered semigroupoids $\varphi : S \to T$ is *full* if the restriction of φ to each hom-set $S(v_1, v_2)$ is surjective, φ is *order-faithful* if for every two coterminal edges x, y, $\varphi(x) \leq \varphi(y)$ implies $x \leq y$, and it is said to be a *quotient* if φ is full and $\varphi|_{V(S)}$ is bijective. Note that, if S is a semigroupoid and τ is a congruence on S, then the *canonical homomorphism* $\eta : S \to S/\tau$ is a quotient homomorphism.

An ordered semigroupoid S is said to *divide* an ordered semigroupoid T if there are an ordered semigroupoid U, a quotient homomorphism $U \to S$ and an order-faithful homomorphism $U \to T$. The *product* of a family $(S_i)_{i \in I}$ of semigroupoids has set of vertices $\prod_{i \in I} V(S_i)$, set of edges $\prod_{i \in I} E(S_i)$, and the partial operations α, ω, and edge composition are defined component-wise. If the factors S_i are ordered semigroupoids, then so is the product under the order given by (1). The *coproduct* of $(S_i)_{i\in I}$ is obtained by taking the disjoint union of the S_i, both for the set of vertices, the set of edges, and the partial operations α, ω, and edge composition. If the cofactors S_i are ordered semigroupoids then so is the coproduct under the (disjoint) union of the order relations of the cofactors.

A *pseudovariety of ordered semigroupoids* is a class of finite ordered semigroupoids containing the one-element semigroup, which is closed under taking divisors of ordered semigroupoids, and finitary direct products and coproducts. The pseudovariety of all finite ordered semigroupoids is denoted by OSd.

An ordered semigroup S is viewed as an ordered semigroupoid by taking the set of edges S with both ends at an added vertex. Conversely, for an ordered semigroupoid S and a vertex v of S, the set $S(v)$ of all *loops* at vertex v constitutes an ordered semigroup called the *local semigroup* of S at v.

The pseudovariety of ordered semigroupoids generated by a given pseudovariety V of ordered semigroups is called the *global* of V and is denoted $g\mathsf{V}$. Note that $g\mathsf{V}$ is the smallest pseudovariety of ordered semigroupoids whose ordered semigroups are precisely those of V. The largest such pseudovariety is called the *local* of V and is denoted $\ell\mathsf{V}$; it consists of all finite ordered semigroupoids such that all local semigroups are members of V. A pseudovariety V of ordered semigroups is said to be *local* if $g\mathsf{V} = \ell\mathsf{V}$.

A *topological semigroupoid* S is a semigroupoid endowed with a topology with respect to which the partial operations α, ω, and edge multiplication are continuous. Finite semigroupoids equipped with the discrete topology become topological semigroupoids.

A topological semigroupoid is *profinite* if it is a projective limit of finite semigroupoids; its topology is said to be the *profinite topology.* For a finite graph Γ, a topological semigroupoid S is said to be Γ-*generated* if there exists a graph homomorphism $\varphi : \Gamma \to S$ such that $\varphi_{|V(\Gamma)}$ is injective and the subgraph of S generated by $\varphi(\Gamma)$ is dense.

We denote by $\widehat{\Gamma^+}$ the projective limit of all Γ-generated finite semigroupoids. Note that $\widehat{\Gamma^+}$ is a Γ-generated semigroupoid *via* the natural mapping $\phi : \Gamma \to \widehat{\Gamma^+}$. The semigroupoid $\widehat{\Gamma^+}$ has the usual universal property: for every homomorphism $\varphi : \Gamma \to S$ into a profinite semigroupoid there exists a unique continuous homomorphism $\hat{\varphi} : \widehat{\Gamma^+} \to S$ such that $\hat{\varphi}\phi = \varphi$. Moreover, we may endow $\widehat{\Gamma^+}$ with a metric d such that the topology induced by d is the profinite topology [15].

By a V-*pseudoidentity over a graph* Γ we mean an ordered pair of the form $(x \le y, \Gamma)$ where x and y are coterminal edges of $\widehat{\Gamma^+}$. A finite ordered semigroupoid S *satisfies the pseudoidentity* $(x \le y, \Gamma)$ and we write $S \models (x \le y, \Gamma)$ if, for each continuous homomorphism of semigroupoids $\varphi : \widehat{\Gamma^+} \to S$, we have $\varphi(x) \le \varphi(y)$.

Given a set Σ of pseudoidentities over finite graphs it is readily checked that the class $\mathsf{V} = [\![\Sigma]\!]$, consisting of all finite ordered semigroupoids S which satisfy all pseudoidentities from Σ, is a pseudovariety of ordered semigroupoids. Then we also say that the pseudovariety V *is defined by* Σ or that Σ is a *basis of pseudoidentities* for V. The analog of Reiterman's Theorem in this context states that every pseudovariety of ordered semigroupoids has a basis of pseudoidentities. The proof of the ordered case can be readily obtained from the unordered case [7, 15] using the necessary adaptations to accommodate the order as in [25].

Following [1], we say that a pseudovariety V of semigroupoids has *vertex-rank* or *v-rank* n if n is the smallest non-negative integer such that V admits a basis of pseudoidentities over graphs with at most n vertices. If no such integer n exists, then we say that V has *infinite v-rank*.

2.3. Inverse semigroups

In this subsection, we recall and extend some results on inverse semigroups. The reader unfamiliar with this topic may wish to consult [18] for a much more thorough introduction.

A semigroup S is *inverse* if, for every $s \in S$ there is a unique *inverse* $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. The *natural order* on a inverse monoid S is defined as follows:

$$x \leq y \text{ if and only if } x = ey \text{ for some } e = e^2 \in S.$$

An inverse semigroup endowed with the natural order is an ordered semigroup and it is said to be an *ordered inverse semigroup*.

Inverse semigroups considered as algebras with the binary operation of multiplication and the unary operation of inversion form the variety $\mathcal{I}$ defined by the equations

$$xx^{-1}x = x,\ (x^{-1})^{-1} = x,\ xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Note that homomorphisms of (inverse) semigroups respect the natural order on inverse semigroups.

Denote by X^{-1} a set disjoint from X and in one-to-one correspondence $x \mapsto x^{-1}$ with X. We call X^{-1} the set of *formal inverses* of elements of X. For $x \in X$, we let $(x^{-1})^{-1} = x$. There is a free inverse semigroup over X denoted by FI_X. It is well known that

$$FI_X \simeq (X \cup X^{-1})^+/\tau$$

where τ is the congruence on $(X \cup X^{-1})^+$ generated by the set

$$\left\{ \left(uu^{-1}u, u\right),\ \left(uu^{-1}zz^{-1}, zz^{-1}uu^{-1}\right) :\ u, z \in \left(X \cup X^{-1}\right)^+ \right\}.$$

Let $w \in (X \cup X^{-1})^*$. The *content* of w, denoted $c(w)$, is the set of $x \in X$ such that either x or x^{-1} occur in w. A word w is said to be *reduced* if w does not contain factors xx^{-1} or $x^{-1}x$ for any $x \in X$. If $w = y_1 \dots y_n$ ($y_i \in X \cup X^{-1}$) is reduced then 1 and every word $y_i \dots y_j$ ($1 \leq i \leq j \leq n$) are *segments* of w, and 1 and every segment of w which begins at y_1 are *initial segments* of w. Let K_X be the set of all words of $(X \cup X^{-1})^+$ of the form

$$w = a_1a_1^{-1} \dots a_na_n^{-1}g$$

where $a_1, \dots, a_n, g$ are reduced words, no a_i is an initial segment of any a_j $(j \neq i)$ and g is an initial segment of a_1.

Theorem 2.2.

1. *For every $w \in FI_X$, there is $u \in K_X$ such that $w = u\tau$.*
2. *Let $u = a_1a_1^{-1} \dots a_ma_m^{-1}g$ and $v = b_1b_1^{-1} \dots b_nb_n^{-1}h$ be words of K_X. Then $u\tau = v\tau$ if and only if $\{a_1, \dots, a_m\} = \{b_1, \dots, b_n\}$ and $g = h$.*

We say that $u \in K_X$ is a *canonical form* of $u\tau \in FI_X$. By abuse of notation, we may also use u for $u\tau$.

Let I be a finite set. The set $I \times I \cup \{0\}$ with multiplication given by

$$(i,j)(i',j') = \begin{cases} (i,j') & \text{if } i' = j \\ 0 & \text{otherwise} \end{cases}$$

and $0a = a0 = 0$, for any $a \in I \times I \cup \{0\}$ is the $I \times I$ *aperiodic Brandt semigroup* and is denoted by B_I or $B_{|I|}$. All aperiodic Brandt semigroups are inverse. Let $\mathcal{B}$ be the subvariety of the variety $\mathcal{I}$ of all inverse semigroups generated by B_2.

The following is well known.

Lemma 2.3. *Let I be an arbitrary set. Then B_I belongs to $\mathcal{B}$.*

Let $a \in FI_X$ have canonical form

$$a = a_1a_1^{-1} \dots a_na_n^{-1}g$$

and define

$$S[a] = \bigcup_{i=1}^{n} \{b \neq 1 : b \text{ is an initial segment of } a_i\}.$$

Note that, if $S[a] = \{b_1, \dots, b_k\}$ then $a = (b_1b_1^{-1} \dots b_kb_k^{-1}g)\tau$.

Reilly [28] defines two relations γ_a and δ_a on $Y = X \cup X^{-1}$ as follows:

- $(x, y) \in \gamma_a$ if and only if $x, y \in S[a]$ or there exists $u \in FI_X \cup \{1\}$ such that $ux^{-1}y \in S[a]$ or $uy^{-1}x \in S[a]$.
- δ_a is the reflexive and transitive closure of γ_a.

Since γ_a is a symmetric relation it follows that δ_a is an equivalence relation.

If $a_i = a_{i1} \dots a_{in_i}$ for $i = 1, \dots, n$ and $g = g_1 \dots g_r$ with $a_{ij}, g_k \in Y$, then we put $s(a) = a_{11}\delta_a$ and

$$e(a) = \begin{cases} s(a) & \text{if } g = 1, \\ g_r^{-1}\delta_a & \text{otherwise.} \end{cases}$$

Note that, for all $i, j = 1, \dots, n$, $a_{i1}, a_{j1} \in S[a]$ so $(a_{i1}, a_{j1}) \in \delta_a$.

We say that a variety $\mathcal{V}$ of inverse semigroups *satisfies* $u \leq v$, with $u, v \in FI_X$ if, for every homomorphism $\varphi : FI_X \to S$ with $S \in \mathcal{V}$, we have $\varphi u \leq \varphi v$.

The following result generalizes the necessary condition of Theorem 3.3 of [28].

Proposition 2.4. *Let $a, b \in FI_X$. If $\mathcal{B}$ satisfies $b \leq a$ then the following conditions hold:*

(i) $c(a) \subseteq c(b)$;
(ii) $\delta_a \subseteq \delta_b$;
(iii) $s(a) \subseteq s(b)$;
(iv) $e(a) \subseteq e(b)$.

Proof. Let us show that $c(a) \subseteq c(b)$. If there is $x \in c(a)$ which is not in $c(b)$ then, taking the homomorphism $\varphi : FI_X \to B_2$ given by $\varphi(x) = 0$, and $\varphi(y) = (1, 1)$ for every $y \in X \setminus \{x\}$, we have $\varphi(a) = 0$ and $\varphi(b) = (1, 1)$, which contradicts our hypothesis.

Let $I = Y/\delta_b$ and let us define $\theta : X \to B_I$ such that $\theta x = (x\delta_b, x^{-1}\delta_b)$. Then there exists a homomorphism $\hat{\theta} : FI_X \to B_I$ which extends θ.

Since $\mathcal{B}$ satisfies $b \leq a$, by hypothesis, and $B_I \in \mathcal{B}$ by Lemma 2.3, we have $\hat{\theta}b \leq \hat{\theta}a$.

First we will show that $\hat{\theta}b \neq 0$. Suppose that $b = b_1 b_1^{-1} \dots b_m b_m^{-1} g$ is a canonical form of b and, for each i, $b_i = b_{i1} \dots b_{ip_i}$ is the reduced form of b_i. Let $1 \leq j < p_i$. Since $b_{i1} \dots b_{i,j+1} \in S[b]$, it follows that $(b_{ij}^{-1}, b_{i,j+1}) \in \gamma_b \subseteq \delta_b$. This allows us to obtain

$$\begin{aligned}\hat{\theta}(b_{ij} b_{i,j+1}) = (\theta b_{ij})(\theta b_{i,j+1}) &= (b_{ij}\delta_b, b_{ij}^{-1}\delta_b)(b_{i,j+1}\delta_b, b_{i,j+1}^{-1}\delta_b) \\ &= (b_{ij}\delta_b, b_{i,j+1}^{-1}\delta_b) \neq 0.\end{aligned}$$

By [28, Lem. 2.2], this implies that $\hat{\theta}b_i \neq 0$ for every i, thus $\hat{\theta}(b_i b_i^{-1}) \neq 0$, and so

$$\begin{aligned}\hat{\theta}(b_i b_i^{-1}) &= (\theta b_{i1}) \dots (\theta b_{ip_i})(\theta b_{ip_i})^{-1} \dots (\theta b_{i1})^{-1} \\ &= (b_{i1}\delta_b, b_{i1}\delta_b) \\ &= (s(b), s(b)).\end{aligned}$$

If $g \neq 1$ then g is an initial segment of b_1 so $\hat{\theta}b = \hat{\theta}g \neq 0$. If $g = 1$ then

$$\hat{\theta}(b) = \hat{\theta}(b_1 b_1^{-1})\hat{\theta}(b_2 b_2^{-1}) \dots \hat{\theta}(b_m b_m^{-1}) = (s(b), s(b)) \neq 0.$$

Since $\hat{\theta}b \leq \hat{\theta}a$ and $\hat{\theta}b \neq 0$, we have $\hat{\theta}a = \hat{\theta}b$.

Let us show that $\delta_a \subseteq \delta_b$. Let $x, y \in Y$ such that $(x, y) \in \gamma_a$. Suppose that $a = a_1 a_1^{-1} \dots a_n a_n^{-1} h$ is a canonical form of a and let $a_1 = a_{11} \dots a_{1n_1}$ $(a_{1i} \in X \cup X^{-1})$ be the reduced form of a_1. If $x \in S[a]$ then $\hat{\theta}(xx^{-1}aa^{-1}) = \hat{\theta}(aa^{-1})$ which implies that $x\delta_b = a_{11}\delta_b$. So, if $x, y \in S[a]$, then we have

$$x\delta_b = a_{11}\delta_b = y\delta_b.$$

Suppose now that there exists $u \in FI_X \cup \{1\}$ such that $ux^{-1}y \in S[a]$. We have $\hat{\theta}(ux^{-1}y(ux^{-1}y)^{-1}a) = \hat{\theta}a$ and $\hat{\theta}a \neq 0$ so

$$0 \neq \hat{\theta}(x^{-1}y) = (x^{-1}\delta_b, x\delta_b)(y\delta_b, y^{-1}\delta_b)$$

which implies that $x\delta_b = y\delta_b$. This shows that $\gamma_a \subseteq \delta_b$, thus $\delta_a \subseteq \delta_b$.

Let us show that $s(a) \subseteq s(b)$ and $e(a) \subseteq e(b)$. First, since $\hat{\theta}a = \hat{\theta}b \neq 0$, we must have $a_{11}\delta_b = b_{11}\delta_b$, which implies that $s(a) \subseteq s(b)$ by (ii).

Suppose next that $g = g_1 \dots g_r$ and $h = h_1 \dots h_s$ are reduced forms of g and h, respectively. If $g \neq 1$ and $h \neq 1$, then

$$0 \neq \hat{\theta}b = \hat{\theta}g = (g_1\delta_b, g_1^{-1}\delta_b) \dots (g_r\delta_b, g_r^{-1}\delta_b) = (g_1\delta_b, g_r^{-1}\delta_b).$$

Similarly, $\hat{\theta}a = (h_1\delta_b, h_s^{-1}\delta_b)$. Since $\hat{\theta}a = \hat{\theta}b$, we have $g_r^{-1}\delta_b = h_s^{-1}\delta_b$, so $e(a) \subseteq e(b)$ again by (ii). If $g = 1$ and $h \neq 1$ then $\hat{\theta}a = (h_1\delta_b, h_s^{-1}\delta_b)$ and $\hat{\theta}b = (b_{11}\delta_b, b_{11}\delta_b)$. Since $\hat{\theta}a = \hat{\theta}b$, we have $h_s^{-1}\delta_b = b_{11}\delta_b$, thus $e(a) \subseteq e(b)$. The case $g \neq 1$ and $h = 1$ is similar. Finally, if $g = 1 = h$ then $e(a) = s(a) \subseteq s(b) = e(b)$. □

3. Pseudoidentities satisfied by the globals of pseudovarieties containing B_2

In this section we extend some results of Section 5 of [5] from pseudovarieties of semigroups to pseudovarieties of ordered semigroups.

Let X be a finite set and $u \in \widehat{X^+}$. Let ρ_u be the equivalence relation over $Y = X \cup X^{-1}$ generated by the relation

$$\{(x^{-1}, y) : \ xy \text{ is a factor of } u\}. \tag{2}$$

Since factors of length 2 can be recognized by finite semigroups, the correspondence $u \in \widehat{X^+} \to \rho_u \in \mathcal{P}(Y^2)$ defines a continuous map, where $\mathcal{P}(Y^2)$ is viewed as a discrete space.

The *content* of u, $c(u)$, is the set of $x \in X$ such that x is a factor of u, that is $u = u_1xu_2$ for some $u_1, u_2 \in \widehat{X^*}$.

Definition 3.1. [5] For $u \in \widehat{X^+}$ and $Y = X \cup X^{-1}$, let A_u be the graph defined by

$$\begin{aligned} V(A_u) &= Y/\rho_u \\ A_u(x\rho_u, y\rho_u) &= \{z : z \in X, (x, z) \in \rho_u, (z^{-1}, y) \in \rho_u\}. \end{aligned}$$

Note that each $z \in X$ gives one edge and $E(A_u) = X$.

This definition of A_u introduces some edges $x \notin c(u)$ which do not play any role when we test pseudoidenties over A_u, but it simplifies some technical arguments.

Lemma 3.2. *Let X be a finite set. If $u \in X^+$ then $\rho_u = \delta_u$.*

Proof. Suppose that $u = x_1 \dots x_n$ with $x_i \in X$, $i = 1, \dots, n$. Then $u = uu^{-1}u$ is a canonical form of $u \in FI_X$. It follows that $(x, y) \in \gamma_u$ if and only if $x = y = x_1$ or (exactly) one the two-letter words $x^{-1}y$ and $y^{-1}x$ is a factor of u. This shows that γ_u contains the relation (2) and is contained is the reflexive symmetric closure of (2). Hence γ_u and (2) generate the same equivalence relation. □

For a finite set X and $u \in \widehat{X^+}$, we denote by $t_1(u)$ the unique letter which is a *suffix* of u, that is, $u = u_1 t_1(u)$ for some $u_1 \in \widehat{X^*}$. Dually, $i_1(u)$ is the unique letter which is a *prefix* of u, which means that $u = i_1(u)u_2$ for some $u_2 \in \widehat{X^*}$.

Theorem 3.3. *Let X be a finite set and $u, v \in \widehat{X^+}$. The following conditions are equivalent:*

(a) B_2 satisfies $u \le v$.
(b) We have $\rho_v \subseteq \rho_u$, $(i_1(u), i_1(v)) \in \rho_u$ and $(t_1(u)^{-1}, t_1(v)^{-1}) \in \rho_u$.
(c) There is a graph homomorphism $\theta : A_v \to A_u$ such that
  (i) $\theta(i_1(v)\rho_v) = i_1(u)\rho_u$ and $\theta(t_1(v)^{-1}\rho_v) = t_1(u)^{-1}\rho_u$, and
  (ii) for every $z \in A_v(x\rho_v, y\rho_v)$, $\theta(z) = z$.

Proof. First, we prove that (a) is equivalent to (b). Recall that u and v are limits of sequences of words, say (u_n) and (v_n), respectively. By continuity of ρ, i_1, and t_1, we may assume that, for every n, $\rho_{u_n} = \rho_u$, $\rho_{v_n} = \rho_v$, $i_1(u_n) = i_1(u)$, $i_1(v_n) = i_1(v)$, $t_1(u_n) = t_1(u)$, and $t_1(v_n) = t_1(v)$.

As B_2 is finite, there is p such that, for each $n \ge p$, B_2 satisfies the pseudoidentities $u = u_n$ and $v = v_n$. By Lemma 3.2 and Proposition 2.4, if B_2 satisfies the inequality $u_n \le v_n$ then Condition (b) holds for u_n and v_n. Thus if B_2 satisfies $u \le v$ then Condition (b) holds.

To prove the converse, let $\eta : \widehat{X^+} \to B_2$ be an arbitrary continuous homomorphism of ordered semigroups. If $\eta(u) = 0$, the minimum of B_2, then obviously $\eta(u) \le \eta(v)$. So, assume that $\eta(u) \ne 0$. By continuity of η, we may assume that $\eta(u_n) = \eta(u)$ for every n. It is clear that Condition (b) holds for u_n and v_n. Our goal is to show that $\eta(u_n) = \eta(v_n)$ and thus $\eta(u) = \eta(v)$, by continuity of η. Hence we may assume that $u, v \in X^+$.

Suppose that xy is a factor of v. Since $\rho_v \subseteq \rho_u$, then there are an index n and $z_1, z_2, \dots, z_{2n} \in X$ such that the words

$$z_0 z_1,\ z_2 z_1, z_2 z_3, \dots, z_{2n} z_{2n-1},\ z_{2n} z_{2n+1},$$

are factors of u, where $z_0 = x$ and $z_{2n+1} = y$. If $\eta(z_i) = (r_i, s_i)$ with $r_i, s_i \in \{1, 2\}$, then $\eta(u) \ne 0$ implies that $s_0 = r_1 = s_2 = r_3 = \dots = s_{2n} = r_{2_n+1}$. Since v is a word, it follows that $\eta(v) \ne 0$, and so $\eta(u)$ and $\eta(v)$ lie in the same $\mathcal{J}$-class. Furthermore, in this case a similar process may be used to show that $(i_1(u), i_1(v)) \in \rho_u$ implies that $\eta(u)$ and $\eta(v)$ are $\mathcal{R}$-related, and $(t_1(u)^{-1}, t_1(v)^{-1}) \in \rho_u$ implies that $\eta(u)$ and $\eta(v)$ are $\mathcal{L}$-related. Hence we have $\eta(u) = \eta(v)$.

Let us show that (b) is equivalent to (c). Suppose that Condition (b) holds. Since $\rho_v \subseteq \rho_u$, we may define a map $\varphi : A_v \to A_u$ by

$$\begin{aligned} \varphi(x\rho_v) &= x\rho_u, &&\text{for } x\rho_v \in V(A_v),\\ \varphi(z) &= z, &&\text{for } z \in A_v(x\rho_v, y\rho_v). \end{aligned}$$

It is clear that φ is a graph homomorphism. By (b), it is immediate that $\varphi(i_1(v)\rho_v) = i_1(v)\rho_u = i_1(u)\rho_u$ and $\varphi(t_1(v)^{-1}\rho_v) = t_1(v)^{-1}\rho_u = t_1(u)^{-1}\rho_u$.

Conversely, suppose that $\theta : A_v \to A_u$ is a graph homomorphism which verifies the conditions of (c). For $x \in X$, we have

$$x \in A_u(x\rho_u, x^{-1}\rho_u),\ x \in A_u(\theta(x\rho_v), \theta(x^{-1}\rho_v))$$

and x gives one edge in A_u, then $x\rho_u = \theta(x\rho_v)$. It follows that, if $(x, y) \in \rho_v$ then $\theta(x\rho_v) = \theta(y\rho_v)$, that is $x\rho_u = y\rho_u$. Thus $\rho_v \subseteq \rho_u$.

If $x \in i_1(v)\rho_v$ then $x\rho_u = \theta(x\rho_v) = \theta(i_1(v)\rho_v) = i_1(u)\rho_u$ so $i_1(v)\rho_v \subseteq i_1(u)\rho_u$. Similarly, we have $t_1(v)^{-1}\rho_v \subseteq t_1(u)^{-1}\rho_u$. □

For $u \in \widehat{X^+}$, let $\varphi_u : A_u \to X$ be the natural graph homomorphism, where the finite set X is viewed as a graph with one vertex. Then there is a unique continuous homomorphism of semigroupoids $\widehat{\varphi_u} : \widehat{A_u^+} \to \widehat{X^+}$ such that $\widehat{\varphi_u}$ extends φ_u, that is the following diagram commutes:

$$\begin{array}{ccc} A_u & \xrightarrow{\phi_u} & \widehat{A_u^+} \\ {\scriptstyle \varphi_u}\downarrow & & \downarrow{\scriptstyle \widehat{\varphi_u}} \\ X & \longrightarrow & \widehat{X^+} \end{array}$$

By Proposition 2.3 of [2], $\widehat{\varphi_u}$ is faithful. We define the *content* of $w \in \widehat{A_u^+}$ as being the content of $\widehat{\varphi_u}(w)$.

Lemma 3.4. *Let $u \in \widehat{X^+}$ and let $\widehat{\varphi_u}$ be the faithful homomorphism of semigroupoids described above. Then $\widehat{\varphi_u}$ restricted to $E(\widehat{A_u^+})$ is injective and $u = \widehat{\varphi_u}(\tilde{u})$ for some $\tilde{u} \in \widehat{A_u^+}(i_1(u)\rho_u, t_1(u)^{-1}\rho_u))$.*

Proof. Let $\varphi = \varphi_u$. Note that, if $w = a_1 \ldots a_n$ $(a_i \in X)$ is a finite path on A_u then $\hat{\varphi}(w) = w \in X^+$.

Let us show that $\hat{\varphi}$ is injective. Suppose that $\hat{\varphi}(w_1) = \hat{\varphi}(w_2)$ for some $w_1, w_2 \in E(\widehat{A_u^+})$. Let (s_n) and (r_n) be sequences of finite paths on A_u which converge to w_1 and w_2 respectively. Since the functions α, ω and content are continuous, we may assume that, for all n, $\alpha(w_1) = \alpha(s_n)$, $\alpha(w_2) = \alpha(r_n)$, $\omega(w_1) = \omega(s_n)$, $\omega(w_2) = \omega(r_n)$, $c(w_1) = c(s_n)$, and $c(w_2) = c(r_n)$. As $\hat{\varphi}$ is continuous, the sequences of words $(\hat{\varphi}(s_n))$ and $(\hat{\varphi}(r_n))$ converge to $\hat{\varphi}(w_1)$ and $\hat{\varphi}(w_2)$, respectively. Since i_1 and t_1 are continuous, we may assume that, for all n, $i_1(\hat{\varphi}(s_n)) = i_1(\hat{\varphi}(w_1))$,

$i_1(\hat\varphi(r_n)) = i_1(\hat\varphi(w_2))$, $t_1(\hat\varphi(s_n)) = t_1(\hat\varphi(w_1))$ and $t_1(\hat\varphi(r_n)) = t_1(\hat\varphi(w_2))$. Therefore, for all n, we have

$$\begin{aligned} &i_1(\hat\varphi(s_n)) = i_1(\hat\varphi(w_1)) = i_1(\hat\varphi(w_2)) = i_1(\hat\varphi(r_n)), \quad \text{and}\\ &t_1(\hat\varphi(s_n)) = t_1(\hat\varphi(w_1)) = t_1(\hat\varphi(w_2)) = t_1(\hat\varphi(r_n)). \end{aligned}$$

It follows that $\alpha(w_1) = \alpha(s_n) = i_1(\hat\varphi(s_n))\rho_u = i_1(\hat\varphi(r_n))\rho_u = \alpha(r_n) = \alpha(w_2)$ and, similarly, $\omega(w_1) = \omega(w_2)$. This shows that w_1 and w_2 are coterminal hence $w_1 = w_2$, since $\hat\varphi$ is faithful.

Finally, we show that $u \in \hat\varphi(\widehat{A_u^+})$. Suppose that $u = a_1 \dots a_n$ with $a_i \in X$. For each $i < n$, $a_i a_{i+1}$ is a factor of u so $(a_i^{-1}, a_{i+1}) \in \rho_u$, so that a_i and a_{i+1} are consecutive edges, that is $\omega(a_i) = \alpha(a_{i+1})$. Hence u may be viewed as a finite path and $\hat\varphi(u) = u$, as we have observed above. If u is the limit of a sequence (u_n) of words then, by continuity of the functions involved, we may assume that $c(u_n) = c(u)$, $i_1(u_n) = i_1(u)$, $t_1(u_n) = t_1(u)$, and $\rho_{u_n} = \rho_u$ (so $A_{u_n} = A_u$) for all n. By compactness of $\widehat{A_u^+}$, there is a subsequence of (u_n) which converges to some $\tilde u \in \widehat{A_u^+}(i_1(u)\rho_u, t_1(u)^{-1}\rho_u)$. As $\hat\varphi$ is continuous and $\hat\varphi(u_n) = u_n$ for all n, we conclude that $\hat\varphi(\tilde u) = u$. □

By Lemma 3.4, for a given $u \in \widehat{X^+}$, there is a unique $\tilde u \in E(\widehat{A_u^+})$ such that $\widehat{\varphi_u}(\tilde u) = u$ so we will abuse notation and denote $\tilde u$ by u.

Let $u, v \in \widehat{X^+}$ and let $\theta : A_v \to A_u$ be a homomorphism of graphs. By the universal property of $\widehat{A_v^+}$ there exists a unique continuous homomorphism $\hat\theta$ such that the following diagram commutes.

$$\begin{array}{ccc} A_v & \xrightarrow{\phi_v} & \widehat{A_v^+} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\hat\theta} \\ A_u & \xrightarrow[\phi_u]{} & \widehat{A_u^+} \end{array}$$

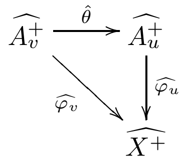
Corollary 3.5. *Let $u, v \in \widehat{X^+}$ be such that B_2 satisfies $u \le v$, and let φ_u be the homomorphism of graphs described in Lemma 3.4. Then there exists a unique edge $v' \in A_u(i_1(u)\rho_u, t_1(u)^{-1}\rho_u)$ such that $\widehat{\varphi_u}(v') = v$.*

Proof. By Lemma 3.4, there is (a unique) $\tilde v \in A_v(i_1(v)\rho_v, t_1(v)^{-1}\rho_v)$ such that $\widehat{\varphi_v}(\tilde v) = v$, where $\widehat{\varphi_v} : \widehat{A_v^+} \to \widehat{X^+}$ is a homomorphism of semigroupoids described in Lemma 3.4. By Theorem 3.3, there exists a homomorphism of graphs $\theta : A_v \to A_u$ which satisfies

$$\theta(i_1(v)\rho_v) = i_1(u)\rho_u, \quad \theta(t_1(v)^{-1}\rho_v) = t_1(u)^{-1}\rho_u \tag{3}$$

$$\theta(z) = z, \text{ for every } z \in E(A_v). \tag{4}$$

Since $\varphi_v(z) = z = \varphi_u(\theta(z))$ for every $z \in E(A_v)$, the following diagram commutes:



Let $v' = \hat\theta(\tilde v)$. Then $v' \in A_u(i_1(u)\rho_u, t_1(u)^{-1}\rho_u)$, by (3), and

$$\widehat{\varphi_u}(v') = \widehat{\varphi_u}(\hat\theta(\tilde v)) = \widehat{\varphi_v}(\tilde v) = v.$$

To complete the proof it is suffices to recall that $\widehat{\varphi_u}$ is faithful. □

Taking into account Lemma 3.4 and Corollary 3.5, we will abuse notation and denote $\hat\theta(\tilde v)$ by v.

Let S be a semigroupoid. If $|V(S)| > 1$ then the *consolidated semigroup* S_{cd} is the set $S_{cd} = E(S) \cup \{0\}$, with the multiplication defined by

$$ss' = \begin{cases} s \cdot s' & \text{if } \ \omega s = \alpha s', \\ 0 & \text{otherwise,} \end{cases}$$

and $0 \cdot a = a \cdot 0 = 0$, for every $a \in S_{cd}$. From this point, we will omit the $\cdot$ to represent the operation in the consolidated semigroup. If $|V(S)| = 1$ then the consolidated semigroup of S is S itself, viewed as a semigroup.

If S is an ordered semigroupoid, then S_{cd} is an ordered semigroup under the order given by $s \le s'$ if $s \le s'$ in S, and $0 \le a$ for every $a \in S_{cd}$ in case $0 \in S_{cd}$.

The following lemma is adapted from [5].

Lemma 3.6. *Let $u, v \in \widehat{X^+}$ and suppose that the ordered semigroup B_2 satisfies the pseudoidentity $u \le v$. Then a finite semigroupoid S satisfies $(u \le v, A_u)$ if and only if S_{cd} satisfies $u \le v$.*

Proof. By Theorem 3.3 there exists a graph homomorphism $\theta : A_v \to A_u$ that satisfies Condition (3) of Corollary 3.5. Thus, by Lemma 3.4 and Corollary 3.5, u and v represent edges of $\widehat{A_u^+}$ from $i_1(u)\rho_u$ to $t_1(u)^{-1}\rho_u$. Hence $(u \le v, A_u)$ is indeed a semigroupoid pseudoidentity.

Let S be a finite semigroupoid and let (u_n) and (v_n) be sequences of words of X^+ converging respectively to u and v in $\widehat{X^+}$. We may assume that, for all n, $i_1(u_n) = i_1(u)$, $t_1(u_n) = t_1(u)$, $i_1(v_n) = i_1(v)$, $t_1(v_n) = t_1(v)$, $A_{u_n} = A_u$, $A_{v_n} = A_v$, S_{cd} satisfies $u = u_n$, $v = v_n$, and S satisfies $(u = u_n, A_u)$ and $(v = v_n, A_u)$. This reduces the proof to the case $u, v \in X^+$. Since B_2 satisfies $u \le v$, it is clear that $c(v) \subseteq c(u)$. Without loss of generality, we may assume that $X = c(u)$.

Suppose first that $S_{cd} \models u \le v$ and consider an arbitrary semigroupoid homomorphism $\varphi : A_u^+ \to S$. Define a homomorphism $\eta : X^+ \to S_{cd}$ by taking $\eta(z) = \varphi(z)$ for each $z \in X$.

Since u and v may be viewed as paths from $i_1(u)\rho_u$ to $t_1(u)^{-1}\rho_u$ in A_u, then $\varphi(u)$ and $\varphi(v)$ are paths in S, so $\eta(u), \eta(v) \neq 0$. Since $\eta(u) \leq \eta(v)$, it follows that $\varphi(u) \leq \varphi(v)$. This shows that $S \models (u \leq v, A_u)$.

Conversely, suppose that $S \models (u \leq v, A_u)$ and let $\eta : X^+ \to S_{cd}$ be an arbitrary homomorphism. If $\eta(u) = 0$ then $\eta(u) \leq \eta(v)$.

If $\eta(u) \neq 0$ then $\eta(u)$ may be viewed as a path in S. Let us construct a graph homomorphism $\varphi : A_u \to S$ such that $\varphi(x) = \eta(x)$ for every $x \in E(A_u)$. Let $x\rho_u$ be a vertex of A_u. Suppose now that $x, y \in X^{-1}$ and $(x, y) \in \rho_u$. If $x, y \in X$ then $x = y$ or there are a positive integer n and $z_1, \ldots, z_{2n-1} \in X$ such that, for $k = 1, \ldots, n$, the words

$$z_{2k-1}z_{2k-2},\ z_{2k-1}z_{2k}$$

are factors of u, where $z_0 = x$ and $z_{2n} = y$. Since $\eta(u) \neq 0$, it follows that

$$\eta(z_{2k-1}z_{2k-2}),\ \eta(z_{2k-1}z_{2k}) \neq 0,$$

so $\eta(z_{2k-1}z_{2k-2})$ and $\eta(z_{2k-1}z_{2k})$ may be viewed as paths in S. Thus we have the chain of equalities $\alpha(\eta(z_0)) = \omega(\eta(z_1)) = \alpha(\eta(z_2)) = \omega(\eta(z_3)) = \ldots = \omega(\eta(z_{2n-1})) = \alpha(\eta(z_{2n}))$, so $\alpha(\eta(x)) = \alpha(\eta(y))$. Similarly, if $x \in X$ and $y \in X^{-1}$ then $\alpha(\eta(x)) = \omega(\eta(y^{-1}))$, and $x, y \in X^{-1}$ implies that $\omega(\eta(x^{-1})) = \omega(\eta(y^{-1}))$. This shows that

$$\varphi(x\rho_u) = \begin{cases} \alpha(\eta(x)), & \text{if } x \in X \\ \omega(\eta(x^{-1})), & \text{if } x^{-1} \in X \end{cases}$$

and $\varphi(x) = \eta(x)$, for each $x \in E(A_u)$, defines a map $\varphi : A_u \to S$. Moreover, it is immediate that φ is a graph homomorphism. Let $\overline{\varphi} : A_u^+ \to S$ be the unique semigroupoid homomorphism which extends φ. As $S \models (u \leq v, A_u)$, we have $\overline{\varphi}(u) \leq \overline{\varphi}(v)$. Since $\eta(w) = \overline{\varphi}(w)$, for every edge w of A_u^+, we have $\eta(u) \leq \eta(v)$. This proves that S_{cd} satisfies $u \leq v$. $\square$

Theorem 3.7. *Let* V *be a pseudovariety of ordered semigroups containing* B_2*. If* $\mathsf{V} = [\![(u_i \leq v_i)_{i\in I}]\!]$ *then* $g\mathsf{V} = [\![(u_i \leq v_i, A_{u_i})_{i\in I}]\!]$.

Proof. For $i \in I$, let $\mathsf{V}_i = [\![u_i \leq v_i]\!]$. In the proof of Proposition 1.2 of [27] it has been shown that $S \in g\mathsf{V}_i$ if and only if $S_{cd} \in \mathsf{V}_i$.[2] Then, by Lemma 3.6, $S \in g\mathsf{V}_i$ if and only if S satisfies $(u_i \leq v_i, A_{u_i})$, that is $g\mathsf{V}_i = [\![(u_i \leq v_i, A_{u_i})]\!]$.

Since $S \in g(\bigcap_{i\in I} \mathsf{V}_i)$ if and only if $S_{cd} \in \mathsf{V}_i$ for every $i \in I$, if and only if $S \in g\mathsf{V}_i$ for every $i \in I$, if and only if $S \in \cap_{i\in I} g\mathsf{V}_i$, we have $g(\bigcap_{i\in I} \mathsf{V}_i) = \bigcap_{i\in I} g\mathsf{V}_i$ and the result follows. $\square$

4. The category C_n

In this section we apply similar techniques to those which can be found, in the context of "unordered" semigroups, in [6]. They will serve to show that the pseudovariety of ordered semigroups $\mathsf{V}_{\frac{3}{2}} = [\![u^\omega v u^\omega \leq u^\omega : c(v) \subseteq c(u)]\!]$ [26] has infinite vertex rank.

[2]More precisely, [27] deals with monoids and categories instead of semigroups and semigroupoids, but the arguments are even simpler in our case.

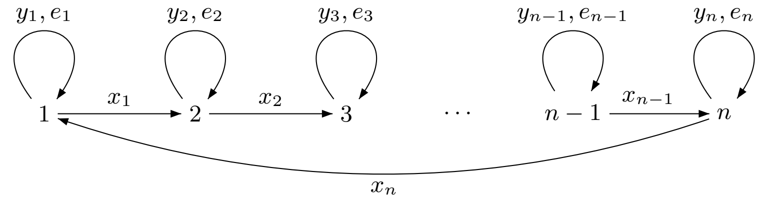


FIGURE 1. Underlying graph for the category C_n.

For $n \geq 2$, let C_n be the category generated by the graph Γ_n described by the diagram of Figure 1 subject to the following list $\mathcal{L}_n$ of relations, where $z_i = x_i \dots x_n x_1 \dots x_{i-1}$ $(1 \leq i \leq n)$, and the index addition is performed modulo n:

(R_0) $e_i x_i = x_i = x_i e_{i+1}$, $1 \leq i \leq n$;
(R_1) $e_i y_i = y_i = y_i e_i$, $1 \leq i \leq n$;
(R_2) $e_i^2 = e_i$, $1 \leq i \leq n$;
(R_3) $y_i^3 = y_i^2$, $1 \leq i \leq n$;
(R_4) $z_i^2 = z_i y_i = y_i z_i = z_i$, $1 \leq i \leq n$;
(R_5) $z_i x_i x_{i+1} = x_i x_{i+1}$, $1 \leq i \leq n$;
(R_6) $(y_i x_i \dots y_{i-1} x_{i-1})^2 = y_i x_i \dots y_{i-1} x_{i-1}$, $1 \leq i \leq n$;
(R_7) $(x_i y_{i+1} \dots x_{i-1} y_i)^2 = x_i y_{i+1} \dots x_{i-1} y_i$, $1 \leq i \leq n$;
(R_8) $y_i^2 x_i y_{i+1} x_{i+1} = x_i x_{i+1}$, $1 \leq i < n$;
(R_9) $x_{i-1} y_i x_i y_{i+1}^2 = x_{i-1} x_i$, $1 \leq i < n$;
(R_{10}) $y_n^2 x_n y_1^2 = x_n z_1$;
(R_{11}) $y_n^2 x_n y_1 x_1 \dots y_n x_n y_1^2 = x_n z_1$.

Many of these relations are naturally viewed as simplifying rules for paths over the graph Γ_n and we will therefore refer to them as rules.

To simplify notation, from hereon, when writing an expression of the form $\varepsilon_i \delta_i \dots \varepsilon_j \delta_j$ or $\varepsilon_i \dots \varepsilon_j$, we mean that the omitted factors, represented by the dots, are taken for the indices describing the shortest path from i to j in the cycle $1 \to 2 \to \cdots \to n \to 1$.

It is immediately verified that the edges of C_n have the form

$$w = \varepsilon_i^{(0)} x_i \dots \varepsilon_n^{(0)} x_n \left(\prod_{k=1}^{s} \varepsilon_1^{(k)} x_1 \varepsilon_2^{(k)} \dots x_n \right) \varepsilon_1^{(s+1)} x_1 \varepsilon_2^{(s+1)} \dots x_{j-1} \varepsilon_j^{(s+1)}$$

for some $i, j \in \{1, \dots, n\}$, $s \geq 0$ and $\varepsilon_t^{(k)} \in \{e_t, y_t, y_t^2\}$, for $k = 0, \dots, s+1$ and $t = 1, \dots, n$. The following is easy to establish as a consequence of the definition of z_i (note that $z_i x_i = x_i z_{i+1}$) and rule (R_4) of $\mathcal{L}_n$.

Lemma 4.1. *Let $i, j \in \{1, \dots, n\}$. For $w \in C_n(i, j)$, $z_i w = z_i x_i \dots x_{j-1} = w z_j$.* □

We endow C_n with the following binary relation: if u, v are coterminal edges of C_n then we set

$$u \leq v \quad \text{if} \quad u = v \text{ or } u = z_{\alpha(u)} x_{\alpha(u)} \cdots x_{\omega(u)-1}.$$

Note that the relation $\leq$ is a partial order. By Lemma 4.1, the partial order $\leq$ is compatible with multiplication (on the left and on the right), and so $(C_n, \leq)$ is an ordered category.

For $1 \leq i \leq n$, let $A_i = \{0_i, 1_i\}$, and put $A = \bigcup_{i=1}^{n} A_i$. Let P_n be the semigroup $(A \cup \{0\})^+/\Upsilon$ where Υ is the congruence generated by the set consisting of the following list $\mathcal{L}'_n$ of relations, where 2_i denotes $1_i 1_i$ and $\zeta_i = 0_i \ldots 0_n 0_1 \ldots 0_i$, for $1 \leq i \leq n$:

(R_A) $a_i b_j = 0$, $1 \leq i \leq n$, $j \neq i$, $j \neq i+1$;
(R_B) $0 a_i = 0 = a_i 0$, $1 \leq i \leq n$;
($R_{1'}$) $0_i 1_i = 1_i = 1_i 0_i$, $1 \leq i \leq n$;
($R_{2'}$) $0_i 0_i = 0_i$, $1 \leq i \leq n$;
($R_{3'}$) $1_i 1_i 1_i = 1_i 1_i$, $1 \leq i \leq n$;
($R_{4'}$) $\zeta_i^2 = \zeta_i 1_i = 1_i \zeta_i = \zeta_i$, $1 \leq i \leq n$;
($R_{5'}$) $\zeta_i 0_{i+1} 0_{i+2} = 0_i 0_{i+1} 0_{i+2}$, $1 \leq i \leq n$;
($R_{6'}$) $(1_i \ldots 1_{i-1} 0_i)^2 = 1_i \ldots 1_{i-1} 0_i$, $1 \leq i \leq n$;
($R_{7'}$) $(0_i 1_{i+1} \ldots 1_i)^2 = 0_i 1_{i+1} \ldots 1_i$, $1 \leq i \leq n$;
($R_{8'}$) $2_i 1_{i+1} 0_{i+2} = 0_i 0_{i+1} 0_{i+2}$, $1 \leq i < n$;
($R_{9'}$) $0_{i-1} 1_i 2_{i+1} = 0_{i-1} 0_i 0_{i+1}$, $1 \leq i < n$;
($R_{10'}$) $2_n 2_1 = 0_n \zeta_1$;
($R_{11'}$) $2_n 1_1 \ldots 1_n 2_1 = 0_n \zeta_1$.

Note that, if $u, v \in A^+ \setminus 0\Upsilon$ and $(u, v) \in \Upsilon$ then there are $i, j \in \{1, \ldots, n\}$ such that $i_1(u), i_1(v) \in A_i$ and $t_1(u), t_1(v) \in A_j$.

Let us define the *norm* of $w \in A^+$, denoted $\|w\|$ in the following way: if $w = a_i \in A_i$ then $\|w\| = 1$, if $w = w_1 w_2$ with $w_1, w_2 \in A^+$ then

$$\|w\| = \begin{cases} \|w_1\| + \|w_2\| - 1 & \text{if } t_1(w_1), i_1(w_2) \in A_i \text{ for some } i \\ \|w_1\| + \|w_2\| & \text{if } t_1(w_1) \in A_i,\ i_1(w_2) \in A_j \text{ with } i \neq j. \end{cases}$$

Note that $\|w\|$ counts the minimal number of factors in a factorization of w into elements of A_i^+.

We also define the norm of $u\Upsilon \in P_n \setminus \{0\Upsilon\}$, denoted by $\|u\Upsilon\|$, as being the minimum of the set $\{\|w\| : w \in u\Upsilon\}$.

An edge w of the category C_n can be completely described in terms of $y_1, \ldots, y_n$ and $e_1, \ldots, e_n$ as it is made precise in the following.

Lemma 4.2. *Let $\chi : \Gamma_n^+ \to A^+$ be the homomorphism of semigroupoids defined by*

$$\chi(e_i) = 0_i, \quad \chi(y_i) = 1_i, \quad \chi(x_i) = 0_i 0_{i+1},$$

and let $\eta_1 : \Gamma_n^+ \to C_n$ be the canonical projection. Then the following hold:

(a) *The homomorphism χ restricted to $E(\Gamma_n^+)$ is injective.*
(b) *For all paths u, v in Γ_n, if $\eta_1(u) = \eta_1(v)$ then $(\chi(u), \chi(v)) \in \Upsilon$.*
(c) *$\chi(\Gamma_n^+) \subseteq A^+ \setminus 0\Upsilon$.*

Proof. (a) Let u and v be paths in Γ_n such that $\chi(u) = \chi(v)$. Let us show that $u = v$.

If $\chi(u) \in A_i^+$ for some $i \in \{1, \ldots, n\}$, then $u = \varepsilon_1 \ldots \varepsilon_s$ and $v = \delta_1 \ldots \delta_t$ with $s, t \geq 1$ and $\varepsilon_j, \delta_k \in \{e_i, y_i\}$, for $1 \leq j \leq s$ and $1 \leq k \leq t$. Since $|\chi(e_i)| = |\chi(y_i)| = 1$, it must be $s = t$ and $\chi(\varepsilon_j) = \chi(\delta_j)$ for $1 \leq j \leq s$. But this implies that $\varepsilon_j = \delta_j$ for $1 \leq j \leq s$ and, consequently, $u = v$.

Let $k \geq 1$. Suppose that $\|\chi(u)\| = k + 1$. It follows that the paths u and v contain k edges from the set $\{x_i : i = 1, \ldots, n\}$, that is

$$u = w^{(i)} x_i w^{(i+1)} \ldots x_{i+k-1} w^{(i+k)} \quad \text{and} \quad v = w'^{(i)} x_i w'^{(i+1)} \ldots x_{i+k-1} w'^{(i+k)},$$

for some $1 \leq i \leq n$, and $w^{(j)}, w'^{(j)} \in \{e_j, y_j\}^*$ for $j = i, \ldots, i + k$. Since $\chi(u) = \chi(v)$, it must be $\chi(w^{(j)}) = \chi(w'^{(j)})$, and so $w^{(j)} = w'^{(j)}$, for every j, as it is shown above. So we conclude that $u = v$.

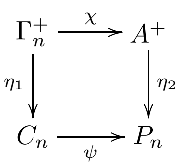
(b) Let u and v be paths in Γ_n such that $\eta_1(u) = \eta_1(v)$. Suppose that $u = v$ is the rule (R_m) of $\mathcal{L}_n$ for some $m \in \{0, \ldots, 11\}$. If $1 \leq m \leq 3$ then $\chi(u) = \chi(v)$ is the rule $(R_{m'})$ of $\mathcal{L}'_n$. If $m = 0$ then $(\chi(u), \chi(v)) \in \Upsilon$ by rule $(R_{2'})$. If $4 \leq m \leq 11$ then $(\chi(u), \chi(v)) \in \Upsilon$ by rules $(R_{1'})$, $(R_{2'})$ and $(R_{m'})$.

Suppose now, without loss of generality, that v may be obtained from u by applying a single rule $p = q$ of $\mathcal{L}_n$, that is $u = u^{(1)} p u^{(2)}$ and $v = u^{(1)} q u^{(2)}$, where $u^{(i)}$ $(i = 1, 2)$ are appropriate paths in Γ_n. Since χ is a homomorphism, Υ is a congruence, and $(\chi(p), \chi(q)) \in \Upsilon$, it follows that $(\chi(u), \chi(v)) \in \Upsilon$.

(c) Let u be a path in Γ_n. Since $\chi(u) \in (A_i^+ \ldots A_{i-1}^+)^* A_i^+ \ldots A_j^+$ for some $1 \leq i, j \leq n$, we cannot apply rules (R_A) or (R_B) to $\chi(u)$, so $\chi(u) \notin 0\Upsilon$. □

We note that in the proof of the Lemma 4.2(a), we show that if u is a path in Γ_n and $\chi(u)$ may be factorized as $\chi(u) = u'^{(1)} 0_i 0_{i+1} u'^{(2)}$ for some i, then there are appropriate paths $u^{(1)}$ and $u^{(2)}$ such that $u = u^{(1)} x_i u^{(2)}$.

Let $\eta_2 : A^+ \to P_n$ be the homomorphism defined by $\eta_2(u) = u\Upsilon$. Note that η_2 is surjective. By Lemma 4.2(b), there is a homomorphism of semigroupoids $\psi : C_n \to P_n$ such that the following diagram commutes.



Lemma 4.3. *Let $\psi : C_n \to P_n$ be the homomorphism of semigroupoids defined above. The following hold:*

(a) $\psi_{|E(C_n)}$ *is injective;*
(b) $\psi(E(C_n)) = P_n \setminus \{0\}$*, where* 0 *also denotes* 0Υ*.*

Proof. (a) Let us show that $\psi_{|E(C_n)}$ is injective. First, we consider a new list $\mathcal{L}''_n$ of rules which contains rules (R_A) and (R_B) of $\mathcal{L}'_n$ and all rules $\chi(p) = \chi(q)$ where each $p = q$ is a rule of $\mathcal{L}_n$. It is easy to see that the rules of the list $\mathcal{L}''_n$ also generate the congruence Υ.

Let u and v be paths in Γ_n such that $\psi(\eta_1(u)) = \psi(\eta_1(v))$, that is $(\chi(u), \chi(v)) \in \Upsilon$. Note that $\chi(u), \chi(v) \notin 0\Upsilon$, by Lemma 4.2(c). Let us show that $\eta_1(u) = \eta_1(v)$. If $\chi(u) = \chi(v)$ is a rule of $\mathcal{L}''_n$ then there is a rule $p = q$ of $\mathcal{L}_n$ such that $\chi(u) = \chi(p)$ and $\chi(v) = \chi(q)$. By Lemma 4.2(a), we have $u = p$ and $v = q$. Since $\eta_1(p) = \eta_1(q)$, it follows that $\eta_1(u) = \eta_1(v)$.

Let us assume, without loss of generality, that $\chi(v)$ may be obtained from $\chi(u)$ by applying a single rule $\chi(p) = \chi(q)$ of $\mathcal{L}''_n$, say with $i_1(\chi(p)) \in A_i$ and $t_1(\chi(p)) \in A_j$ for some i and j. This means that we may factorize $\chi(u) = u'^{(1)}a'\chi(p)b'u'^{(2)}$ and $\chi(v) = u'^{(1)}a'\chi(q)b'u'^{(2)}$ where $t_1(u'^{(1)}) = 0_{i-1}$ or $u'^{(1)}$ is the empty word, $a' \in A_i^*$, $b' \in A_j^*$, and $i_1(u'^{(2)}) = 0_{j+1}$ or $u'^{(2)}$ is the empty word. Let us assume that $|u'^{(1)}|, |u'^{(2)}| > 1$. The other cases are similar.

If $a' \in A_i^+$ and $b' \in A_j^+$, by the definition of χ, we must have $\chi(u) = u''^{(1)}0_{i-1}0_i a''\chi(p)b''0_j 0_{j+1}u''^{(2)}$ and $\chi(v) = u''^{(1)}0_{i-1}0_i a''\chi(q)b''0_j 0_{j+1}u''^{(2)}$, with $a'' \in A_i^*$ and $b'' \in A_j^*$. Arguing as in proof of Lemma 4.2(a) we obtain paths a and b in Γ_n such that $\chi(a) = a''$ and $\chi(b) = b''$. As it has been observed above, there are appropriate paths $u^{(1)}$ and $u^{(2)}$ such that $u = u^{(1)}x_{i-1}apbx_j u^{(2)}$ and $v = u^{(1)}x_{i-1}aqbx_j u^{(2)}$. Since $\eta_1(p) = \eta_1(q)$, it follows that $\eta_1(u) = \eta_1(v)$.

If a' is the empty word then $i_1(\chi(p)) = i_1(\chi(q)) = 0_i$. Since $\chi(u)$ and $\chi(v)$ do not have factors of the form $0_{i-1}0_i a_{i+1}$ with $a_{i+1} \in A_{i+1}$, then p and q must begin with the same edge e_i, that is $p = q$ is the rule (R_2) of $\mathcal{L}_n$. Without loss of generality, we may assume that

$$\chi(u) = u''^{(1)}0_{i-1}0_i 0_i b' 0_{i+1} u''^{(2)} \quad \text{and} \quad \chi(v) = u''^{(1)}0_{i-1}0_i b' 0_{i+1} u''^{(2)}$$

with $b' = b''0_i \in A_i^* 0_i$. So there are appropriate paths $b, u^{(1)}$ and $u^{(2)}$ such that

$$u = u^{(1)}x_{i-1}e_i b x_i u^{(2)} \quad \text{and} \quad v = u^{(1)}x_{i-1}b x_i u^{(2)}.$$

By rule (R_0) of $\mathcal{L}_n$, we have $\eta_1(u) = \eta_1(v)$. If b' is the empty word, the same argument serves to prove that $\eta(u) = \eta(v)$.

(b) Let us show that $\psi(E(C_n)) = P_n \setminus \{0\}$. Let w be a representative of a Υ-class of $P_n \setminus \{0\}$. Suppose that $\|w\| = k + 1$. By rules $(R_{1'})$ and $(R_{2'})$, w is Υ-related with the word v which is obtained from w by introducing, between consecutive letters $b_i \in A_i$ and $b_{i+1} \in A_{i+1}$ such that $b_i b_{i+1}$ is a factor of w, the factor $0_i 0_{i+1}$. It is obvious that $\|v\| = k + 1$ and so we may write $v = v^{(i)}0_i 0_{i+1} v^{(i+1)} \dots 0_{i+k-1}0_{i+k} v^{(i+k)}$ with $v^{(r)} \in A_r^+$. Proceeding as above, we are able to find paths $u^{(r)}$ such that $\chi(u^{(r)}) = v^{(r)}$. It follows that $v = \chi(u^{(i)}x_i u^{(i+1)} \dots x_{i+k-1}u^{(i+k)})$. Finally, we have $\eta_2(w) = \eta_2(v) = \eta_2(\chi(u^{(i)}x_i u^{(i+1)} \dots x_{i+k-1}u^{(i+k)})) = \psi(\eta_1(u^{(i)}x_i u^{(i+1)} \dots x_{i+k-1}u^{(i+k)}))$. $\square$

Note that the homomorphism ψ defined in Lemma 4.3 does not induce a homomorphism $(C_n)_{cd} \to P_n$ since $y_i y_{i+1} = 0$ in $(C_n)_{cd}$ while $\psi(y_i)\psi(y_{i+1}) = (1_i 1_{i+1})/\Upsilon \neq 0$ in P_n.

Throughout this paper we are almost always identifying each Υ-class with a representative of that Υ-class.

Let $u', v' \in P_n$. We let $u' \leq v'$ if either $u' = 0$ or there exist $u, v \in C_n$ such that $u' = \psi(u)$, $v' = \psi(v)$ and $u \leq v$. Taking into account the definition of $\leq$ in C_n, we obtain $u' \leq v'$ if and only if $u' = 0$ or there are $i, j \in \{1, \ldots, n\}$ such that $i_1(u'), i_1(v') \in A_i$, $t_1(u'), t_1(v') \in A_j$ and

$$u' = \zeta_i 0_{i+1} \ldots 0_j \quad \text{or} \quad u' = v'.$$

By Lemma 4.3, the relation $\leq$ is a partial order compatible with multiplication. Therefore $(P_n, \leq)$ is an ordered semigroup and $\psi : C_n \to P_n$ is an order-faithful homomorphism of semigroupoids.

For $i, j \in \{1, \ldots, n\}$, let $P_n(i, j)$ be the set

$$\{w \in P_n \setminus \{0\} : 0_i w 0_j = w\}.$$

If $i = j$, then we write $P_n(i)$ instead of $P_n(i, i)$. Note that $P_n(i)$ is a submonoid of P_n with identity 0_i.

Some basic properties of the relation Υ are stated in the next lemma.

Lemma 4.4. *Let* $1 \leq i, j \leq n,\ w \in P_n(i),\ t \in P_n(i, j)$. *Then*

(a) $\zeta_i t = \zeta_i 0_{i+1} \ldots 0_j = 0_i \ldots 0_{j-1} \zeta_j = t \zeta_j$;

(b) $\zeta_i w = \zeta_i = w \zeta_i$;

(c) $w 0_{i+1} 0_{i+2} = 0_i 0_{i+1} 0_{i+2}$ *and* $0_{i-2} 0_{i-1} w = 0_{i-2} 0_{i-1} 0_i$;

(d) $1_i 0_{i+1} 1_{i+2} = 0_i 0_{i+1} 0_{i+2}$.

Proof. All the properties can be easily proven from the definition of P_n. To illustrate this, we prove (c) and (d).

(c) We have

$$w 0_{i+1} 0_{i+2} = w 0_i 0_{i+1} 0_{i+2} = w \zeta_i 0_{i+1} 0_{i+2} = \zeta_i 0_{i+1} 0_{i+2} = 0_i 0_{i+1} 0_{i+2},$$

by rule $(R_{5'})$ and (b). The other equality is similar.

(d) By rule $(R_{1'})$ and (c), we have

$$1_i 0_{i+1} 1_{i+2} = 1_i 0_{i+1} 0_{i+2} 1_{i+2} = 0_i 0_{i+1} 0_{i+2} 1_{i+2} = 0_i 0_{i+1} 0_{i+2}. \qquad \square$$

For $i = 1, \ldots, n$, let $B_i = 0_i^* 1_i 0_i^*$ and $C_i = A_i^* 1_i A_i^* 1_i A_i^*$. The next result describes two rational languages recognized by the natural homomorphism from A^+ to P_n.

Lemma 4.5. *Let $\eta_2 : A^+ \to P_n$ be the homomorphism defined in Lemma 4.3. Let L_1 and L_2 be the rational languages*

$$L_1 = (B_1 \cup 0_1^+)B_2 \dots B_n(B_1 \dots B_n)^* A_1^+ \quad \text{and}$$
$$L_2 = (B_1 \cup 0_1^+)B_2 \dots B_n(B_1 \dots B_n)^* C_1 \dots C_n(B_1 \dots B_n)^*(B_1 \cup 0_1^+).$$

Then $\eta_2^{-1}\eta_2(L_i) = L_i$, for $i = 1, 2$.

Proof. Let $L \in \{L_1, L_2\}$ and $w \in L$. Then we cannot apply rules (R_A), (R_B), $(R_{4'})$, $(R_{5'})$, $(R_{8'})$, $(R_{9'})$, $(R_{10'})$ and $(R_{11'})$ to w. On the other hand, if we apply rules $(R_{1'})$, $(R_{2'})$, $(R_{3'})$, $(R_{6'})$ and $(R_{7'})$, then we obtain another word of L.

Now, if $u \in \eta_2^{-1}\eta_2(L)$ then there exists $w \in L$ such that $(w, u) \in \Upsilon$, hence $u \in L$. The inclusion $L \subseteq \eta_2^{-1}\eta_2(L)$ is clear. □

By abuse of language, we say that $u \in A^+$ is a *factor* of $w \in P_n \setminus \{0\}$ when $u\Upsilon$ is a factor of w.

For $a, b, c, d \in \{0, 1, 2\}$, we define the following elements of $P_n(1)$:

$$w_1(a, b) = a_1 1_2 \dots 1_n b_1, \qquad \text{and} \qquad w_2(c, d) = c_1 2_2 \dots 2_n d_1.$$

Some other elements of $P_n(1)$ may be obtained as products of $w_1(a', b')$ and $w_2(c', d')$ for some $a', b', c', d' \in \{0, 1, 2\}$:

$$\begin{aligned} w' &= 0_1 1_2 \dots 1_n 1_1 \dots 1_n 0_1 = w_1(0, 1) w_1(0, 0); \\ w_3(a, d) &= a_1 1_2 \dots 1_n 2_1 \dots 2_n d_1 = w_1(a, 1) w_2(1, d); \\ w_4(c, d) &= c_1 2_2 \dots 2_n 1_1 \dots 1_n d_1 = w_2(c, 1) w_1(0, d), \quad \text{and} \\ w_5(a, d) &= a_1 1_2 \dots 1_n 2_1 \dots 2_n 1_1 \dots 1_n d_1 = w_1(a, 0) w_2(2, 1) w_1(0, d). \end{aligned}$$

Corollary 4.6. *Let $a, d \in \{0, 1\}$ and $b, c \in \{0, 1, 2\}$. In $P_n(1)$, we have*

(a) $w_5(a, d) \neq \zeta_1$ and $w_2(c, d) \neq \zeta_1$;

(b) $w_1(a, b) \neq \zeta_1$ and $w_1(a, b) \neq w_2(c, d)$;

(c) If 2_i is a factor of $w_1(a, b)$ then $i = 1$ and $b = 2$.

Proof. (a) The word $\zeta_1 \in A^+$ is not in L_2 and the word $w_5(a, d)$ belongs to L_2 so $w_5(a, d) \neq \zeta_1$ in $P_n(1)$, by Lemma 4.5. On the other hand, we have $w_5(a, d) = w_1(a, b) w_2(c, d) w_1(a', d)$ for some $b, a' \in \{0, 1, 2\}$ such that $b+c = 2$ and $d+a' = 1$. Hence, if $w_2(c, d) = \zeta_1$ then $w_5(a, d) = \zeta_1$, by Lemma 4.4(b), and the latter we have already shown to fail.

(b) Since the word $w_1(a, b)$ belongs to L_1 and the words $w_2(c, d)$ and ζ_1 are not in L_1, we have $w_1(a, b) \neq w_2(c, d)$ and $w_1(a, b) \neq \zeta_1$, by Lemma 4.5.

(c) If 2_i is a factor of $w_1(a, b)$ then there are $u^{(1)}, u^{(2)} \in A^*$ such that $w_1(a, b) = u^{(1)} 2_i u^{(2)}$ in $P_n(i)$. Since the word $w_1(a, b)$ belongs to L_1, the same happens with the word $u^{(1)} 2_i u^{(2)}$ by Lemma 4.5. So it must be $i = 1$, $2_i u^{(2)} \in 2_1 A_1^*$, and $b = 2$. □

For $i = 1, \dots, n$, let $F_i = \{0_i, 1_i, 2_i\}$.

Lemma 4.7. *Let m be a positive integer. The number of Υ-classes which contain words w such that $\|w\| \leq m$ is finite.*

Proof. Suppose that $w\Upsilon \neq 0$. Then by rules $(R_{1'})$, $(R_{2'})$ and $(R_{3'})$ there are $i, j \in \{1, \ldots, n\}$ and a word $u \in (F_i \ldots F_{i-1})^* F_i \ldots F_j$ such that $\|w\Upsilon\| = \|u\|$ and $(w, u) \in \Upsilon$. As the set $\{u \in (F_i \ldots F_{i-1})^* F_i \ldots F_j : \|u\| \leq m\}$ is finite, the lemma follows. □

In the following, we characterize all elements of $P_n(1)$.

Notice that, for every $b \in \{0, 1, 2\}$, $k = 1, 3, 5$, and $l = 2, 3, 4, 5$,

$$w_k(2, b) = \zeta_1 = w_l(b, 2).$$

In the set $\{0, 1, 2\}$ we define an operation $\oplus$ in the following way: for $t \in \{0, 1, 2\}$, let $0 \oplus t = t = t \oplus 0$, $1 \oplus 1 = 2$ and $2 \oplus t = t \oplus 2 = 2$. Then $\{0, 1, 2\}$ is a commutative monoid with zero 2 and identity 0.

Lemma 4.8. *The monoid $P_n(1)$ consists of the elements*

$$0_1,\ 1_1,\ 2_1,\ \zeta_1,\ w_1(a, b),\ w_2(c, d),\ w',\ w_3(a, d),\ w_4(c, d),\ w_5(a, d) \qquad (5)$$

with $a, d \in \{0, 1\}$ and $b, c \in \{0, 1, 2\}$, where 0_1 is the identity and ζ_1 is the zero.

Proof. Let M be the set of the elements (5) of $P_n(1)$. From the definitions of P_n and $P_n(1)$, and Lemma 4.4(b), we obtain the following equalities: for every $a, b \in \{0, 1, 2\}$ and $c \in \{1, 2\}$,

$$\begin{aligned}
&0_1 w = w 0_1 = w,\ \zeta_1 w = w \zeta_1 = \zeta_1,\ \text{for every } w \in M,\\
&a_1 b_1 = (a \oplus b)_1,\ c_1 w' = w_1(c, 0),\ w' c_1 = w_1(0, c),\\
&c_1 w_k(a, b) = w_k(a \oplus c, b) \ \text{ and } \ w_k(a, b) c_1 = w_k(a, b \oplus c),\ \text{for } 1 \leq k \leq 5.
\end{aligned}$$

Let $a, d, a', d' \in \{0, 1\}$ and $b, c, b', c' \in \{0, 1, 2\}$. Then we may establish the partial multiplication table indicated in Table 1.

Since w', $w_3(a', d')$, $w_4(c', d')$ and $w_5(a', d')$ can be written as products of $w_1(a, b)$ and $w_2(c, d)$, the above shows that M is closed under multiplication and thus M is a monoid.

Let $u\Upsilon \in P_n(1)$. By rules $(R_{1'})$, $(R_{2'})$, and $(R_{3'})$, there is a word $v \in (F_1 \ldots F_n)^* F_1$ such that $v = u$ in $P_n(1)$, so we may assume that $u \in (F_1 \ldots F_n)^* F_1$. Note that $\|u\| = kn + 1$ for some $k \geq 0$. If $k = 0$ then $u = a_1$ with $a \in \{0, 1, 2\}$, thus $u \in M$. Suppose that $\|u\| = n + 1$, and $u \neq \zeta_1$ in $P_n(1)$. Then $u = a_1 (c^{(2)})_2 \cdots (c^{(n)})_n b_1$ with $a, b, c^{(i)} \in \{0, 1, 2\}$ for $2 \leq i \leq n$. If there is j such that $c^{(j)} = 0$ then $0_{j-1} 0_j 0_{j+1}$ is a factor of u, so $u = \zeta_1$ in $P_n(1)$, by Lemma 4.4. It follows that all c_i are non-zero. If $c^{(j)} = 1$ then

$$u = w_1(a, b) = a_1 (1)_2 \ldots (1)_n b_1$$

Table 1. Partial multiplication table.

	$w_1(a,b)$	$w_2(c,d)$
$w_1(a',b')$	$w_1(a',b)$, if $b' \oplus a = 1, a' \oplus b \neq 0$ w', if $b' \oplus a = 1, a' = b = 0$ ζ_1, otherwise	$w_3(a',d)$, if $b' \oplus c = 2$ ζ_1, otherwise
$w_2(c',d')$	$w_4(c',b)$, if $d' \oplus a = 1, b \neq 2$ ζ_1, otherwise	ζ_1
w'	$w_1(0,b)$, if $a = 1, b \neq 0$ w', if $a = 1, b = 0$ ζ_1, otherwise	$w_3(0,d)$, if $c = 2$ ζ_1, otherwise
$w_3(a',d')$	$w_5(a',b)$, if $d' \oplus a = 1, b \neq 2$ ζ_1, otherwise	ζ_1
$w_4(c',d')$	$w_4(c',b)$ if $d' \oplus a = 1, b \neq 2$ ζ_1, otherwise	ζ_1
$w_5(a',d')$	$w_5(a',b)$ if $d' \oplus a = 1, b \neq 2$ ζ_1, otherwise	ζ_1

with $a \in \{0,1\}$, by rules $(R_{6'})$, $(R_{7'})$, and Lemma 4.4. If $c^{(j)} = 2$ then

$$u = w_2(a,b) = a_1 2_2 \dots 2_n b_1.$$

By rule $(R_{9'})$, it must be $b \neq 2$.

If $k > 1$, then u may be written as a product of factors of $P_n(1)$ with length $n+1$, namely factors of the form ζ_1, $w_1(a,b)$, and $w_2(c,d)$. By the first part of the proof, $P_n(1) = M$. □

The following lemma is a consequence of Lemmas 4.7 and 4.8.

Lemma 4.9. *For all $i,j \in \{1,\dots,n\}$, the set $P_n(i,j)$ is finite.*

Proof. If $w \in P_n(i,j)$ then either $\|w\| < 2n$ or $w = uw'v$ with $\|u\|, \|v\| < n$ and $w' \in P_n(1)$. Since $P_n(1)$ is finite by Lemma 4.8, it follows that $P_n(i,j)$ is finite, by Lemma 4.7. □

Lemma 4.10. *Let $u \in P_n(1) \setminus \{2_1\}$. If there is $1 \leq i \leq n$ such that 2_i is a factor of u then $u^2 = \zeta_1$.*

Proof. First we claim that w has a factor of the form 2_i if and only if $w \in K$ where

$$K = \{2_1, \zeta_1, w_1(a,2), w_2(c,d), w_3(a,d), w_4(c,d), w_5(a,d) : \\ a,d \in \{0,1\};\ c \in \{0,1,2\}\}.$$

In fact, it is clear that all elements of K have a factor 2_i for some $i \in \{1,\dots,n\}$. On the other hand, if $w \notin K$ then $w \in \{0_1, 1_1, w', w_1(a,b) : a,b \in \{0,1\}\}$. It is obvious that 0_1 and 1_1 do not have the factor 2_1 and, by Corollary 4.6(c), there is no i such that 2_i is a factor of $w_1(a,b)$ (with $a,b \in \{0,1\}$). Finally if w' has a factor 2_i for some i then the same happens with $1_1 w' = w_1(1,0)$, which contradicts the above.

Now, using Table 1 given in the proof of Lemma 4.8, it is easy to verify that $w^2 = \zeta_1$ for every $w \in K \setminus \{2_1\}$. □

Lemma 4.11. *Suppose $u\Upsilon \in P_n(k) \setminus \{\zeta_k\}$. Then 2_j is a factor of $u\Upsilon$ if and only if 2_j is a factor of the word u.*

Proof. Since $u\Upsilon \in P_n(k) \setminus \{\zeta_k\}$, none of the rules (R_A), (R_B), $(R_{4'})$, $(R_{5'})$, $(R_{8'})$, $(R_{9'})$, $(R_{10'})$, $(R_{11'})$ may be applied to an element of the congruence class $u\Upsilon$. For the remaining rules, it suffices to observe that, if one side has 2_j as a factor, then so does the other. □

The next result plays a useful role in the sequel.

Lemma 4.12. *Let $1 \le j \le n$ and $w \in P_n(j) \setminus \{2_j\}$. If there is $1 \le i \le n$ such that 2_i is a factor of w then $w^3 = \zeta_j$.*

Proof. For $j = 1$, this is a consequence of Lemmas 4.10 and 4.4. Suppose that $j \neq 1$. Let $u\Upsilon \in P_n(j)$ such that the word 2_j does not belong to $u\Upsilon$. Since 2_i is a factor of $u\Upsilon$, there are words $u^{(1)}, u^{(2)} \in u\Upsilon$ such that $u^{(1)}2_i u^{(2)} \in u\Upsilon$. We may write the word $u^{(1)}2_i u^{(2)}$ as $v^{(1)}v^{(2)}v^{(3)}$ with $v^{(1)} \in A_j^+ \dots A_n^+, v^{(2)} \in (A_1^+ \dots A_n^+)^* A_1^+$, and $v^{(3)} \in A_2^+ \dots A_j^+$. It is obvious that 2_i is a factor of $v^{(1)}, v^{(2)}$, or $v^{(3)}$ so, by Lemmas 4.4 and 4.10, and rules $(R_{1'})$, $(R_{2'})$, and $(R_{4'})$ we have

$$\begin{aligned} u^3\Upsilon &= (v^{(1)}0_1 v^{(2)}v^{(3)})^3\Upsilon \\ &= (v^{(1)}(v^{(2)}v^{(3)}v^{(1)}0_1 v^{(2)}v^{(3)}v^{(1)}0_1)v^{(2)}v^{(3)})\Upsilon \\ &= (v^{(1)}\zeta_1 v^{(2)}v^{(3)})\Upsilon = \zeta_j\Upsilon, \end{aligned}$$

which completes the proof. □

As a consequence of Lemma 4.12, the only idempotents of $P_n(i)$ which contain 2_i as a factor are 2_i and ζ_i.

Lemma 4.13. *The set of idempotents of $P_n(i)$ is $\{u^3 : u \in P_n(i)\}$. Moreover, for every $u \in P_n(i) \setminus \{0_i, 1_i, 2_i\}$, we have $u^3 = \zeta_i$ or $u = a_i 1_{i+1} \dots 1_{i-1} b_i$ with $a + b = 1$.*

Proof. Let $u \in P_n(i)$. If there is j such that 2_j is a factor of u then either $u^3 = u = 2_i$ or $u^3 = \zeta_i$, by Lemma 4.12.

If there is $j \in \{1, \dots, n\}$ such that $0_j 0_{j+1} 0_{j+2}$ is a factor of u^2 then $u^2 = \zeta_i$ by Lemma 4.4, and so $u^3 = \zeta_i$. If u^2 does not have factors of the form 2_j or $0_j 0_{j+1} 0_{j+2}$ then, either $u \in \{0_i, 1_i\}$, or there is a word $v \in a_i 1_{i+1} \dots 1_{i-1}(1_i \dots 1_{i-1})^* b_i$, for some $a, b \in \{0, 1\}$ with $a + b = 1$, such that $u = v$ in $P_n(i)$. In the first case, $u^3 \in \{0_i, 2_i\}$, and in the latter case, $u^2 = u = a_i 1_{i+1} \dots 1_n 1_1 \dots 1_{i-1} b_i$ in $P_n(i)$, by rules $(R_{1'})$, $(R_{6'})$ and $(R_{7'})$.

Finally, suppose that $\|u\| > 1$, u does not have factors of form 2_k, u^2 does not have factors of the form $0_k 0_{k+1} 0_{k+2}$, but there is j such that 2_j is a factor of u^2.

Under these conditions, there is a word $v \in a_i 1_{i+1} \dots 1_{i-1}(1_i \dots 1_{i-1})^* b_i$, for some $a, b \in \{0,1,2\}$ with $a + b \in \{1,2\}$, such that $u = v$ in $P_n(i)$. Let us show that $a+b$ cannot be 1. Indeed, if $a+b=1$ then $v = v^2 = a_i 1_{i+1} \dots 1_n 1_1 \dots 1_{i-1} b_i$ in $P_n(i)$, and so there are words $1_1 \dots 1_{i-1} a'_i$ and $b'_i 1_{i+1} \dots 1_n 0_1$, with $a + a' = 1$ and $b + b' = 1$, such that

$$1_1 \dots 1_{i-1} a'_i v b' i 1_{i+1} \dots 1_n 0_1 = w_1(1,0)$$

in $P_n(1)$, by rules $(R_{1'})$, $(R_{6'})$, and $(R_{7'})$. Now, by Lemma 4.4, since $u^6 = v^6 = v$ in $P_n(i)$,

$$\begin{aligned} w_1(1,0) &= 1_1 \dots 1_{i-1} a'_i v b' i 1_{i+1} \dots 1_n 0_1 \\ &= 1_1 \dots 1_{i-1} a'_i u^6 b' i 1_{i+1} \dots 1_n \\ &= 0_1 1_1 \dots 1_{i-1} a'_i (u^2)^3 b'_i 1_{i+1} \dots 1_n 0_1 \\ &= 1_1 \dots 1_{i-1} a'_i \zeta_i b'_i 1_{i+1} \dots 1_n 0_1 = \zeta_1, \end{aligned}$$

which contradicts Corollary 4.6. If $a+b=2$ then $u^2 = v^2 = \zeta_i$, by rule $(R_{8'})$, and so $u^3 = \zeta_i$. This shows that $\{u^3 : u \in P_n(i)\}$ is a set of idempotents. It is immediate that this set contains all idempotents of $P_n(i)$. □

In the following, we list the idempotents of the monoid $P_n(i)$.

Corollary 4.14. *Let $1 \le i \le n$. Then the idempotents of $P_n(i)$ are*

$$0_i,\ 2_i,\ \zeta_i \text{ and } a_i 1_{i+1} \dots 1_n 1_1 \dots 1_{i-1} b_i \tag{6}$$

with $a + b = 1$.

Proof. By rules $(R_{1'})$, $(R_{2'})$, $(R_{3'})$, $(R_{4'})$, $(R_{6'})$, and $(R_{7'})$, and by definition of $P_n(i)$, the elements of (6) are idempotents. By Lemma 4.13, (6) lists all idempotents of $P_n(i)$. □

We recall that $\mathsf{V}_{\frac{3}{2}}$ denotes the pseudovariety of ordered semigroups defined by the pseudoidentities

$$u^\omega v u^\omega \le u^\omega$$

where $c(v) \subseteq c(u)$. It is clear that $B_2 \in \mathsf{V}_{\frac{3}{2}}$ so, by Theorem 3.7, the global of $\mathsf{V}_{\frac{3}{2}}$ is defined by the pseudoidentities of the form

$$(u^\omega v u^\omega \le u^\omega, A_{u^\omega v u^\omega}),$$

with $c(v) \subseteq c(u)$. We next prove that C_n fails a specific pseudoidentity of this form.

Proposition 4.15. *Let $n \ge 2$. Then C_n does not belong to $g\mathsf{V}_{\frac{3}{2}}$.*

Proof. Consider the graph Γ_n described in Figure 1 and let $u = y_1x_1 \dots y_nx_n$ and $v = y_1^2x_1 \dots y_n^2x_n$ be finite paths in Γ. Note that the graphs $A_{u^\omega vu^\omega}$ and Γ_n are identical up to the name of the vertices. Since $c(v) \subseteq c(u)$, it follows that $g\mathsf{V}_{\frac{3}{2}}$ satisfies the pseudoidentity $(u^\omega vu^\omega \leq u^\omega, \Gamma_n)$, by Theorem 3.7.

Let us show that C_n fails this pseudoidentity, and consequently, C_n does not lie in $g\mathsf{V}_{\frac{3}{2}}$. Arguing by contradiction, suppose that C_n satisfies the pseudoidentity in question. Evaluate the graph Γ_n in C_n through a graph homomorphism which maps the edges x_i and y_i of Γ_n to the corresponding edges x_i and y_i of the category C_n. We obtain two edges u and v of C_n such that

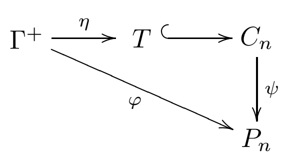
$$\psi(u^\omega) = \psi(u) = w_1(1,0) \quad \text{and} \quad \psi(u^\omega vu^\omega) = \psi(uvu) = w_5(1,0),$$

where ψ is the order-faithful homomorphism defined in Lemma 4.3. Therefore we must have $w_5(1,0) \leq w_1(1,0)$. By definition of $\leq$ in P_n, it must be either $w_5(1,0) = \zeta_1$ or $w_5(1,0) = w_1(1,0)$. Since $w_5(1,0) \neq \zeta_1$ by Corollary 4.6(a), we deduce that $w_5(1,0) = w_1(1,0)$. As a consequence we have $\zeta_1 = w_5(1,0)w_4(2,0) = w_1(1,0)w_4(2,0) = w_5(1,0)$, which contradicts Corollary 4.6. $\square$

The following result is a decisive property of C_n related with the pseudovariety $g\mathsf{V}_{\frac{3}{2}}$.

Proposition 4.16. *Let Γ be a finite graph and let u, v be loops at the same vertex of $\widehat{\Gamma^+}$ such that $c(v) \subseteq c(u)$. Then every subcategory of C_n with $n-1$ vertices satisfies $(u^\omega vu^\omega \leq u^\omega, \Gamma)$.*

Proof. Let T be a subcategory of C_n with $n-1$ vertices. Let u and v be loops at the same vertex of $\widehat{\Gamma^+}$ such that $c(v) \subseteq c(u)$. By Lemmas 4.3 and 4.9, C_n is finite and so there are edges u' and v' of Γ^+ such that $u = u'$ and $v = v'$ in T, and we may assume that u and v are edges of Γ^+. Let $\eta : \Gamma^+ \to T$ be a homomorphism of semigroupoids and let ψ be the order-faithful homomorphism defined in Lemma 4.3. We denote by φ the homomorphism $\psi \circ \eta$ as depicted in the following commutative diagram:



First, we recall that, by Lemma 4.13, $\varphi(u^\omega) = \varphi(u^3)$. Since u^3vu^3 is a path in Γ, we have $\varphi(u^3vu^3) \neq 0$. Let $k = \alpha(\eta u)$, so $k \in \{1, \dots, n\}$. Since T has $n-1$ vertices, there is $j \in \{1, \dots, n\}$, with $j \neq k$, such that, for all $x \in c(u)$, neither $i_1(\varphi(x))$ nor $t_1(\varphi(x))$ belongs to A_j. If $\varphi(u^3) = \zeta_k$ then $\varphi(u^3vu^3) = \zeta_k$. If $\varphi(u^3) \neq \zeta_k$ then, by Corollary 4.14, the idempotent $\varphi(u^3)$ satisfies the relation

$$\varphi(u^3) \in \{0_k, 2_k, a_k1_{k+1} \dots 1_n1_1 \dots 1_{k-1}b_k : a + b = 1\}.$$

If $\varphi(u^3) \in \{0_k, 2_k\}$ then $\varphi(u^3vu^3) = \varphi(u^3)$ since $c(v) \subseteq c(u)$. Suppose that $\varphi(u^3) = a_k 1_{k+1} \cdots 1_n 1_1 \ldots 1_{k-1} b_k$, with $a + b = 1$, and $\varphi(u^3vu^3) \neq \zeta_k$. By Lemma 4.13, $\varphi(u^3) = \varphi(u)$ and, by Corollary 4.6(c), $\varphi(u)$ has no factors 2_i for every $i \in \{1, \ldots, n\}$, since there are a' and b' such that $1_1 \ldots 1_{k-1} a'_k \varphi(u) b'_k 1_{k+1} \ldots 1_n 0_1 = w_1(1, 0)$. Hence for every $x \in c(u)$ there is no $i \in \{1, \ldots, n\}$ such that 2_i is a factor of $\varphi(x)$.

Now, by Lemma 4.11, 2_i is a factor of $\varphi(uvu)$ if and only if 2_i is a factor of every representative of the Υ-class $\varphi(uvu)$. Since $\varphi(uvu)$ is a product of factors of the form $\varphi(x)$, with $x \in c(u)$, none of which has 2_i as a factor by the preceding paragraph, the only way 2_i may appear as a factor of $\varphi(uvu)$ is if uvu has a factor xy, with $x, y \in c(u)$, such that $t_1(\varphi(x)) = 1_i = i_1(\varphi(y))$. Since j is a not a vertex of T, this cannot happen for $i = j$. Hence 2_j is not a factor of $\varphi(uvu)$.

Let $z = 1_1 \ldots 1_{k-1} b_k \varphi(v) a_k 1_{k+1} \ldots 1_n 1_1 \in P_n(1)$. Since

$$\zeta_k \neq \varphi(uvu) = a_k 1_{k+1} \ldots 1_n z 1_2 \ldots 1_{k-1} b_k,$$

then $z \neq \zeta_1$ and 2_j is not a factor of z, so $z \in \{1_1, w_1(1, 1)\}$ by Lemma 4.8. Therefore $\varphi(uvu) = a_k 1_{k+1} \ldots 1_n 1_1 1_2 \ldots 1_{k-1} b_k$, by rules $(R_{1'})$, $(R_{6'})$, and $(R_{7'})$.

We have shown that $\varphi(u^3vu^3) = \zeta_k$ or $\varphi(u^3vu^3) = \varphi(u^3)$. It follows that $\varphi(u^3vu^3) \leq \varphi(u^3)$ in P_n. As ψ is an order-faithful homomorphism, we conclude that T satisfies the pseudoidentity $(u^\omega v u^\omega \leq u^\omega, \Gamma)$. □

We may now prove the main result of this section.

Theorem 4.17. *The pseudovariety $g\mathsf{V}_{\frac{3}{2}}$ has infinite v-rank.*

Proof. Let us assume that $g\mathsf{V}_{\frac{3}{2}}$ has finite v-rank r, that is, $g\mathsf{V}_{\frac{3}{2}}$ admits a basis of pseudoidentities Σ over graphs with at most r vertices. We claim that the category C_{r+1} belongs to $g\mathsf{V}_{\frac{3}{2}}$. Indeed, when we verify the pseudoidentities in Σ, we only work with subcategories of C_{r+1} with at most r vertices. By Proposition 4.16, such subcategories belong to $g\mathsf{V}_{\frac{3}{2}}$, hence C_{r+1} satisfies the pseudoidentities of Σ and then it lies in $g\mathsf{V}_{\frac{3}{2}}$, which is in contradiction with Proposition 4.15. □

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