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INTEGER PARTITIONS, TILINGS OF 2D-GONS AND LATTICES

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Abstract. In this paper, we study two kinds of combinatorial objects, generalized integer partitions and tilings of 2D-gons (hexagons, octagons, decagons, etc.). We show that the sets of partitions, ordered with a simple dynamics, have the distributive lattice structure. Likewise, we show that the set of tilings of a 2D-gon is the disjoint union of distributive lattices which we describe. We also discuss the special case of linear integer partitions, for which other dynamical models exist.

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1. Preliminaries

A *generalized integer partition problem*, or simply a *partition problem* [8,17], is defined by a (possibly infinite) Directed Acyclic Graph (DAG) $G = (V, E)$ and a positive integer h. A solution of such a problem, called a *partition*, is a function $a : V \to \{0, 1, \ldots, h\}$ such that $a(v) \geq a(w)$ for all v and w in V such that there is an edge from v to w in G. The integer $a(v)$ is usually denoted by a_v, and the set of all the solutions of a given partition problem (G, h) is $P(G, h)$. The graph G is called the *base* of the partition problem, and h is the height of the problem. We extend here the usual definition by allowing $h = \infty$, which means that the parts may be unbounded.

A *tiling problem* is defined by a finite set of tiles T, called *prototiles*, and a polygon P. A solution of the problem is a *tiling*: an arrangement of translated copies of prototiles, called *tiles*, which covers exactly P with no gap and no overlap. We are

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concerned here with tilings of $2D$-gons with parallelograms. A $2D$*-gon* Z is defined from a family of positive, plane, pairwise independent vectors $\{v_1, v_2, \ldots, v_D\}$, and a set of positive integers $\{l_1, l_2, \ldots, l_D\}$ by:

$$Z = \left\{ \sum_{i=1}^{D} \alpha_i v_i, 0 \leq \alpha_i \leq l_i \right\}.$$

When $D = 2$, one obtains parallelograms, when $D = 3$, one obtains hexagons, when $D = 4$, one obtains octagons, etc. Each prototile is a parallelogram obtained as the Minkowski sum of 2 vectors among the ones which generate the $2D$-gon we want to tile. A $2D$-gon can be viewed as the projection onto the plane of a part of a D-dimensional grid, and a tiling of this $2D$-gon is nothing but the projection of a particular set of faces of this part of grid [18]. Figure 1 shows a tiling of an hexagon and one of an octagon. These tilings appear in various contexts. In particular, they are one of the mains models used in physics to study quasicrystals [10].

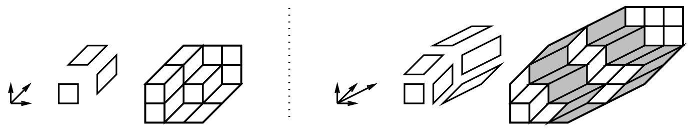

FIGURE 1. Examples of tilings together with the vectors which define the $2D$-gons and the induced prototiles. Notice that removing the shaded tiles in the rightmost tiling gives back the leftmost one.

In this paper, we will use dynamical models and order theory to prove some strong structural properties of the set of tilings of a $2D$-gon. In particular, some special kinds of orders, namely distributive lattices, will appear. An order is a *lattice* if any two elements have an *infimum*, *i.e.* a greatest lower element, and a *supremum*, *i.e.* a lowest greater element. The infimum of two elements a and b in a lattice L is denoted by $\inf_L(a, b)$ or $a \wedge_L b$, and their supremum is denoted by $\sup_L(a, b)$ or $a \vee_L b$. A lattice is *distributive* if for all a, b and c: $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ and $(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$. Lattices in general, and distributive lattices in particular, are strongly structured sets. Many general results and algorithms are known about them. For more details, see [5].

2. The lattices of integers partitions

2.1. Generalized integer partitions

One can obtain all the solutions of a partition problem $(G = (V, E), h)$ with the following discrete dynamical model. We consider a partition a as a *state* of the model, and a_v as a number of grains stored at $v \in V$. The initial state of the model is the empty one: $a_v = 0$ for all $v \in V$. Then, the following rule is iterated:

a grain can be added at $v \in V$ if and only if the obtained configuration is a partition.

In other words, if a is the state of the model, then the transition $a \longrightarrow b$ is possible if there is a vertex v such that $\forall \nu \neq v$, $b_\nu = a_\nu$, $b_v = a_v + 1$ and b is a partition. The state b is then called a *successor* of a. See for example Figure 2.

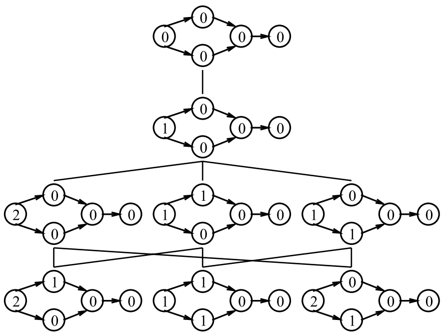

FIGURE 2. The first partitions of the set $P(G, 2)$ for a given graph G, with the possible transitions (not all the reachable partitions are displayed in this diagram).

Since there can be no cycle in a sequence of states of the model, this transition rule induces a partial order over the possible states, *i.e.* over the set of partitions $P(G, h)$. This definition generalizes the well known Young lattice, which is obtained when G is a directed linear chain of vertices infinite on its right, and with $h = \infty$. We can now state the main result of this section, which will be useful during the study of tiling problems in Section 3.

Theorem 1. *Given a partition problem $(G = (V, E), h)$, the set $P(G, h)$ equipped with the order induced by the transition rule is a* distributive lattice*. Moreover, the infimum (resp. supremum) of two given partitions a and b in this set is the partition c (resp. d) defined by:*

$$\forall v \in V,\ c_v = \max(a_v, b_v),$$

$$\forall v \in V,\ d_v = \min(a_v, b_v).$$

Proof. It is clear that c is a partition. Consider now a partition γ. If for one vertex $\nu \in V$, $\gamma_\nu < c_\nu = \max(a_\nu, b_\nu)$ then γ is clearly unreachable from a or b by iteration of the transition rule. Likewise, if for all $\nu \in V$, $\gamma_\nu > c_\nu$ then γ is clearly reachable from c. Therefore, $c = \inf(a, b)$. The proof for $d = \sup(a, b)$ is similar. Therefore, $P(G, h)$ is a lattice. Now, it is easy from these formula to verify that the properties required for a lattice to be distributive are fulfilled. $\square$

If $h = \infty$, one obtains an infinite lattice which contains all the possible partitions over the base graph of the problem. Moreover, it is easy to verify that the sets $P(G, h)$ with $h < \infty$ are sub-lattices of the infinite one.

We will now use Theorem 1 to study special classes of partition problems, the *linear* partitions. The reader mostly interested in tilings may directly go to Section 3.

2.2. Linear partitions

When one considers a partition problem defined over a linear directed graph ($V = \mathbb{N}$, $E = \{(i, i+1)\}$), then the solutions are called *linear partitions*, and the obtained distributive lattice is known as the Young lattice. A linear partition of an integer n is nothing but a decreasing sequence of integers, called *parts*, such that the sum of the parts is exactly n. The order induced by the transition rule is nothing but the componentwise order: $a \leq b$ if and only if $a_i \leq b_i$ for all i. Moreover, the infimum c of two partitions a and b is given by $c_i = \max(a_i, b_i)$ for all i (the supremum is defined dually). Linear partitions have been widely studied as a fundamental combinatorial object [1]. A linear partition is usually represented by its Ferrer's diagram, a sequence of columns such that if the i-th part is equal to k then the i-th column contains exactly k stacked squares, called grains.

In 1973, Brylawski proposed a dynamical model to study these partitions [4]: given a partition a, a grain can *fall* from column i to column $i+1$ if $a_i - a_{i+1} \geq 2$, and a grain can *slip* from column i to column $j > i+1$ if for all $i < k < j$, $a_k = a_i - 1 = a_j + 1$. Brylawski showed that the iteration of these rules from the partition (n) gives the lattice of all the linear partitions of n ordered with respect to the dominance ordering, defined by:

$$a \geq b \text{ if and only if } \sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i \text{ for all } j,$$

i.e. the prefix sums of a are greater than or equal to the prefix sums of b. This lattice is denoted by $L_B(n)$. See Figure 3 (left) for an example. If one iterates only the first rule defined by Brylawski, one obtains the Sand Pile Model and the set of linear partitions obtained from (n) is again a lattice, denoted by $SPM(n)$, with respect to the dominance ordering [12]. See Figure 3 (right) for an example. In [16] and [14], it is proved that when these models are started with one infinite first column the sets of reachable configurations are infinite lattices, denoted by $SPM(\infty)$ and $L_B(\infty)$. It is also shown in these papers that, if one considers a and b in $SPM(\infty)$ or $L_B(\infty)$ then their infimum c is defined by:

$$c_i = \max\left(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j\right) - \sum_{j > i} c_j \quad \text{for all } i. \tag{1}$$

The lattice $L_B(\infty)$ contains all the linear partitions, just as the Young lattice L_Y. We will now study the connection between the dynamical model defined by Brylawski and the one defined in Section 2.1.

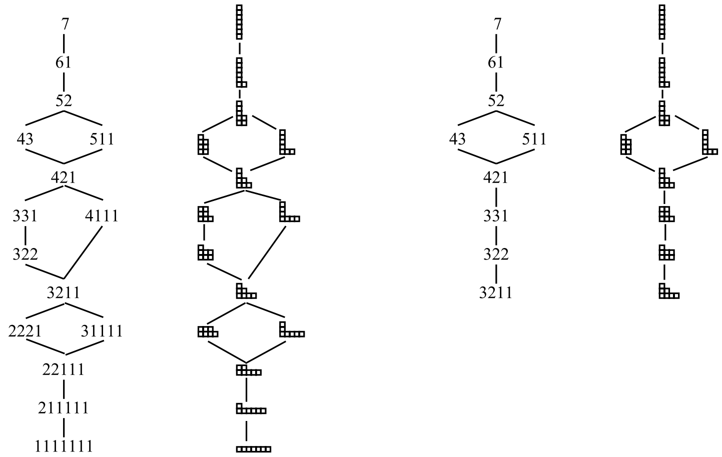

FIGURE 3. The lattices $L_B(7)$ (left) and $SPM(7)$ (right), together with the Ferrer's diagrams representations.

Theorem 2. *The application:*

$$\pi_{L_B} : \quad L_B(\infty) \quad \longrightarrow \quad L_Y$$

such that $\pi_{L_B}(a)_i$ *is equal to* $\sum_{j \geq i} a_j$ *is an order embedding which preserves the infimum.*

Proof. To clarify the notations, let us denote by π the application π_{L_B} in this proof. Let a and b be two elements of $L_B(\infty)$. We must show that $\pi(a)$ and $\pi(b)$ belong to L_Y, that $a \geq_{L_B(\infty)} b$ is equivalent to $\pi(a) \geq_{L_Y} \pi(b)$ and that $\inf_{L_Y}(\pi(a), \pi(b))$ $= \pi(\inf_{L_B(\infty)}(a, b))$. The two first points are easy: $\pi(x)$ is obviously a decreasing sequence of integers for any x, and the order is preserved. Now, let $u = \inf(a, b)$. Then,

$$\begin{array}{rcl} \pi(u)_i & = & \sum_{j \geq i} u_j \\ & = & \max\left(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j\right) \quad \text{from (1)} \\ & = & \max(\pi(a)_i, \pi(b)_i) \\ & = & \inf(\pi(a), \pi(b))_i \end{array}$$

which proves the claim. $\square$

Notice that if one considers the restriction of π_{L_B} to $SPM(\infty)$, denoted by π_{SPM}, a similar proof shows that π_{SPM} is an order embedding which preserves the infimum. However, these orders embeddings are not *lattices* embeddings, since they do not preserve the supremum: if $a = (2,2)$ and $b = (1,1,1)$, then $\pi_{L_B}(a) = (4,2)$, $\pi_{L_B}(b) = (3,2,1)$, $c = \sup_{L_B(\infty)}(a,b) = (2,1)$ but $\pi_{L_B}(c) = (3,1)$ and $\sup_{L_Y}((4,2),(3,2,1)) = (3,2)$. Actually, there can be no lattice embedding from $L_B(\infty)$ to L_Y since the fact that L_Y is a *distributive* lattice would imply that $L_B(\infty)$ would be distributive, which is not true.

3. Tilings of $2D$-gons

A dynamical transformation is usually defined over tilings of $2D$-gons: when three tiles form a small hexagon in a tiling t, then one can locally rearrange them in order to obtain a new tiling t' of the same $2D$-gon. We then write $t \longrightarrow t'$, and t' is said to be obtained from t by a *flip*. The fact that we start with a given tiling gives an (arbitrary) orientation to the notion of flips: the transformation of t' into t is called an *inverse* flip. In the following we will only be concerned with flips (not inverse ones), unless explicitly specified. See Figure 5 for some examples.

Integer partitions and tilings of $2D$-gons have been brought together in [9, 10]. Some special cases were already known from [2, 3, 8] but the correlations between partitions and tilings are treated in general for the first time in [10]. We will now describe this correspondence shortly, since we will widely use it in the following. For more details, we refer to the original papers.

Let us first recall the classical notions of de Bruijn lines and families [6,7]. Given a $2D$-gon tiling t and a vector v used in the definition of the $2D$-gon, the i-th family of tiles of t is the set of all tiles defined using v. A line of this family is a subset of this family obtained as the set of tiles crossed when one goes from one side of the $2D$-gon to the opposite side crossing only edges of tiles which correspond to v. Each tile is crossed by exactly 2 de Bruijn lines, and there is no intersection of 3 lines. On the other hand, the lines in a given family can never intersect. For example, in Figure 1 (right) the tiles which belong to the third family of the rightmost tiling are shaded. It contains two lines. Notice that removing the D-th family of tiles in a $2D$-gon tiling t gives a $2(D-1)$-gon tiling, which we will denote by $\bar{t}$.

Given a $2D$-gon tiling, one can always [10] give an orientation to the de Bruijn lines such that for all family, each line in this family have the same orientation and one can not go from a tile back to itself by following lines with respect to this orientation. In the following, we will always suppose that such an orientation is given. This leads to the definition of the graph of a tiling t: the vertices of this graph are the tiles of t, and there is an edge between two vertices if the corresponding tiles are adjacent, if they are in the same line, and if the orientation of the line is the same as the orientation of the edge. See Figure 4 (left) for an example.

First, let us see how one can associate a partition to a tiling t of a $2D$-gon Z. For any tile τ which is not in the D-th family (*i.e.* any tile in $\bar{t}$), and any de Bruijn line l which is not in the D-th family, let $w_{\tau,l}$ be the number of de Bruijn lines of

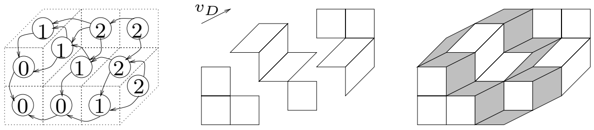

FIGURE 4. From left to right: a partition which is solution of a partition problem over the graph of a hexagon tiling (dotted), the tiles of this tiling translated with respect to the values of the corresponding parts and the vector v_D, and the obtained octagon tiling after completion. This gives an example of the function $\mathcal{T}$. Notice that one can do the way back from right to left and obtain this way an example for $\mathcal{P}$.

the D-th family crossed when one goes from the tile τ to the end of l (following the de Bruijn line). One can always choose the orientations to have $w_{\tau,l_1} = w_{\tau,l_2}$ where l_1 and l_2 are the two de Bruijn lines which cross τ. Therefore, one can denote this value by w_τ. Now, consider the graph of $\bar{t}$, $G = (V, E)$, and the function p defined for all v in V by $p(v) = w_\tau$, where τ is the tile associated to the vertex v. This function is a solution to the partition problem (G, h) where h is the total number of de Bruijn lines in the D-th family in t. In the following, given a tiling t, we will denote by $\mathcal{P}(t)$ the partition associated this way to t.

Conversely, given a partition p solution of the partition problem (G, h) where G is the graph of a tiling t, one wants to define a tiling t' associated to p. Let Z be the $2D$-gon tiled by t. Let Z' be the $2D$-gon generated by the same family of vectors than Z with an additional one: v_D with $l_D = h$. Let us consider the following partition of the set of vertices of G (and dually of the tiles of t): $V_i = \{v \in V \text{ such that } p_v = i\}$. One can now construct t' by insertion of a line of the D-th family in t between the tiles corresponding to the sets V_i and V_{i+1} for all i. One obtains this way t', the tiling of Z' associated to p. In the following, given a partition p, we will denote by $\mathcal{T}(p)$ the tiling associated this way to p. See Figure 4 for an example.

This process gives a method to generate all the tilings of a given $2D$-gon Z. One starts from an hexagon tiling, which is nothing but the projection of the Ferrer's diagram of a planar partition [8]. See Figure 1 (left) for an example. From the graph of this tiling, one defines a partition problem, the solutions of which are equivalent to octagon tilings, as explained above. Likewise, one can construct a $2(D + 1)$-gon tiling from a $2D$-gon tiling for any D, and so obtains a way to generate $2D$-gon tilings for any D. It is shown in [9, 10] that the application $\mathcal{T}$, which generate the tiling associated to a partition, is a bijection between the set of the partitions solutions to (G_t, h) for all graph G_t of a $2D$-gon tiling and the set of tilings of a $2(D + 1)$-gon. Moreover, it is shown in these papers that this bijection is an order isomorphism.

As already mentionned, it was shown in [13] and [11] that one can obtain all the tilings of a $2D$-gon from a given one by iterating the flip operation. A sequence of such transformations is denoted by $\xrightarrow{*}$, which is equivalent to the transitive and reflexive closure of $\rightarrow$, also denoted by $\geq$, depending on the emphasis given to the dynamical aspect or to the order theoretical approach. We denote by $T(Z, D)$ the set of all the tilings of the $2D$-gon Z ordered by $\geq$. An example is given in Figure 5. Notice that all sequences of flips (with no inverse flips) from a tiling to another one have the same length [9], which will be useful in the following:

Theorem 3. *The set $T(Z, D)$ is the disjoint union of distributive lattices L_i such that a flip transforms a tiling in L_i into another one in L_i if and only if it involves at least one tile of the D-th family. Moreover, for all $t \in L_i$ and $u \in L_j$, $\bar{t} = \bar{u} \Leftrightarrow i = j$.*

Proof. Let us consider the maximal subsets L_i of $T(Z, D)$ such that a flip goes from a tiling in L_i to another one in L_i if and only if it involves at least one tile of the D-th family. It is shown in [10] and [9] that such a set, equipped with the transition rule described above (flip), is isomorphic to the set of the solutions of a partition problem, depending on Z and D, equipped with the transition rule described in Section 2 (addition of one grain). We know from Theorem 1 that this set is a distributive lattice. Therefore, we obtain the first part of the claim.

It is then clear that if s and t are in L_i then $\bar{s} = \bar{t}$: it suffices to notice that if $t \longrightarrow t'$ such that this flip involves at least one tile in the D-th family then $\bar{t} = \bar{t'}$. Moreover if $\bar{s} = \bar{t}$ then s can not be obtained from t with a flip involving three tiles with none of them belonging to the D-th family: such a flip changes the position of the tiles in $\bar{s}$ and $\bar{t}$. This ends the proof. $\square$

Theorem 4. *Let a, b and c be in a L_i (L_i being one of the sets partitioning $T(Z, D)$ defined in Th. 3) such that a is the unique maximal element of L_i and b is its unique minimal element. If a flip involving three tiles none of them belonging to the D-th family is possible from c then it is possible from a and b.*

Proof. First notice that, since a flip inside L_i involves tiles which are in the D-th family, $\bar{a} = \bar{b} = \bar{c}$. Therefore, the flip from c is possible from a and b if the three tiles it involves are neighbours in a and b. From Theorem 3, $\mathcal{P}(a)$ and $\mathcal{P}(b)$, the partitions which correspond to the tilings a and b, are respectively the maximal and minimal elements of the set $P(G, h)$ of solutions to a partition problem (G, h). Therefore, from Theorem 1, $\mathcal{P}(a)$ and $\mathcal{P}(b)$ have all their parts equal to respectively 0 and h. Then, from the definition of $\mathcal{T} = \mathcal{P}^{-1}$, all the tiles which do not belong to the D-th family tile an hexagon included in Z, and so all the flips involving three such tiles are possible from a and b. $\square$

We can now define the order $\overline{T(Z, D)}$ as the quotient of $T(Z, D)$ with respect to the equivalence relation $s \sim t \Leftrightarrow \bar{s} = \bar{t}$, *i.e.* defined by the lattices L_i. In other words, we consider the set of the lattices L_i as the set of vertices of $\overline{T(Z, D)}$, and there is one edge from L_i to L_j in $\overline{T(Z, D)}$ if and only if there is at least a transition from one element of L_i to one element of L_j in $T(Z, D)$. Also notice that if, given a tiling of a $2D$-gon Z, we delete the tiles in the D-th family, we obtain a new tiling of a $2(D-1)$-gon. This $2(D-1)$-gon only depends on Z and

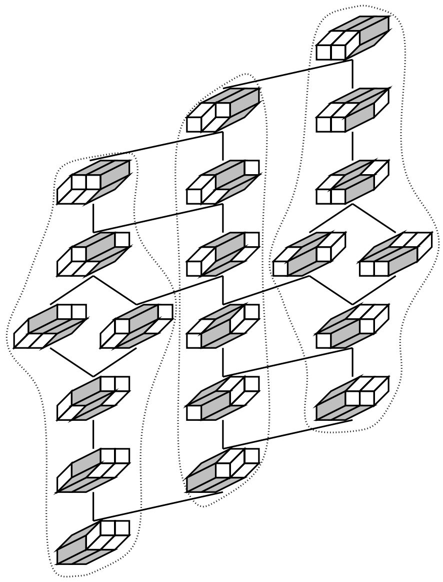

FIGURE 5. $T(Z,4)$ for a given octagon Z. The possible transitions (flips) are represented. The shaded tiles show the 4-th family, and the dotted sets are the distributive lattices L_1, L_2 and L_3, as stated by Theorem 3.

does not depend of the considered tiling, since, as one can easily verify, if $t \longrightarrow t'$ then $\bar{t}$ and $\bar{t'}$ tile the same $2(D-1)$-gon. Let us denote by $\overline{Z}$ this $2(D-1)$-gon.

Theorem 5. *The order $\overline{T(Z,D)}$ is isomorphic to the order $T(\overline{Z}, D-1)$.*

Proof. From Theorem 3, one can associate to each $L_i \in \overline{T(Z,D)}$ a tiling t_i such that for all a in L_i, $\bar{a} = t_i$. It is clear that t_i is in $T(\overline{Z}, D-1)$. Conversely, if one has a tiling of $\overline{Z}$, then one can use the construction of a $2(D+1)$-gon tiling from a $2D$-gon tiling described above to obtain a tiling t of Z. Therefore, there is a bijection between $\overline{T(Z,D)}$ and $T(\overline{Z}, D-1)$. We will now see that it is an order isomorphism. From Theorem 3, if there exists a flip $a \rightarrow b$ between two tilings $a \in L_i$ and $b \in L_j$ with $i \neq j$, then it does not involve any tile of the D-th family, and so there exists a flip $t_i \rightarrow t_j$ in $T(\overline{Z}, D-1)$. Conversely, if there is a flip $t_i \rightarrow t_j$ in $T(\overline{Z}, D-1)$, then there exists $a \in L_i$ and $b \in L_j$ such that $a \rightarrow b$: from Theorem 4, it suffices for example to take the maximal elements of L_i and L_j respectively for a and b. Therefore, $\overline{T(Z,D)}$ is isomorphic to $T(\overline{Z}, D-1)$. $\square$

With these theorems, one has much information about any set $T(Z,D)$: it is the disjoint union of distributive lattices, and its quotient with respect to these lattices has itself the same structure, since it is isomorphic to $T(\overline{Z}, D-1)$. This shows that the sets $T(Z,D)$ are strongly structured, and makes it possible to write efficient algorithms based on this structure.

4. Conclusion and perspectives

We gave structural results on generalized integer partitions and tilings of $2D$-gons. Our main tools were dynamical models and order theory, which allows a simple presentation of the topic.

There are two immediate directions in which it seems promising to extend the results presented here. The first one is to study generalizations of $2D$-gon tilings in higher dimensions, namely *zonotope* tilings. In dimension 3, it is not even known wether all the tilings of a given zonotope can be obtained from a particular one by flipping tiles. The introduction of a new definition of the tilings of zonotopes, based on the notions of orders and lattices, may help in understanding this. The other important remark is that the choice of the D-th family all along our work is arbitrary. This means that one could choose any family in place of the D-th, and so there are many ways to decompose $T(Z, D)$ into a disjoint union of distributive lattices. This is a strong and surprinsing fact, which has to be fully explored.

Finally, one may wonder if the results presented here always stands when the support of the tiling is not a $2D$-gon. It would be interesting to know the limits of our structural results. They may be very general, and lead to sub-lattices properties of the obtained sets of tilings. Likewise, it would be useful to study what happens when the size of the $2D$-gon grows to infinity. Some results about that are presented in [10] and [9] but a lot of work remains to be done.

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Note. This paper is an improved version of the extended abstract [15], which contains no proof.

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