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RESTRICTED ADMISSIBILITY OF BATCHES INTO AN $M/G/1$ TYPE BULK QUEUE WITH MODIFIED BERNOULLI SCHEDULE SERVER VACATIONS

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Abstract. We investigate the steady state behavior of an $M/G/1$ queue with modified Bernoulli schedule server vacations. Batches of variable size arrive at the system according to a compound Poisson process. However, all arriving batches are not allowed into the system. The restriction policy differs when the server is available in the system and when he is on vacation. We obtain in closed form, the steady state probability generating functions for the number of customers in the queue for various states of the server, the average number of customers as well as their average waiting time in the queue and the system. Many special cases of interest including complete admissibility, partial admissibility and no server vacations have been discusssed. Some known results are derived as particular cases of our model.

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1. Introduction

Bulk queues is an important area of queueing theory. This has received considerable attention from researchers and practitioners interested in its applications. In recent years literature on bulk queues has grown tremendously. In a bulk queueing model, the arrivals or departures or both may happen in bulk, *i.e.* in batches of fixed or variable size. Among numerous researchers who worked on bulk queues, we mention Bailey [1], Conolle [7], Bhat [2], Jaiswal [15], Cohen [6], Harris [13], Borthakur [3], Medhi and Borthakur [24], Kleinrock [18], Levy and Yechiali [20], Medhi [25–27], Neuts [29, 30], Nadarajan [28], Van Hoorn [40], Chaudhry and Templeton [4], Gross and Harris [12], Kashyap and Chaudhry [16].

In this paper we study the steady state behavior of a bulk queue, an $M^{[x]}/G/1$ queue, with modified Bernoulli schedule server vacations. Vacation queues including bulk queues with vacation have been studied by many authors including Scholl and Kleinrock [34], Lee [19], Fuhrman [11], Doshi [9, 10], Servi [35, 36], Keilson and Servi [17], Shanthikumar [37], Shanthikumar and Sumita [38], Cramer [8], Choi and Park [5], Takagi [39] and Madan [21–23]. These authors and many others have considered different vacation policies including Bernoulli schedules. In Bernoulli schedules, they assume that at the completion of each service, the server may take a vacation (with vacation times having an exponential distribution or an arbitrary distribution) with probability p or may continue to stay in the system with probability $1 - p$. This policy further assumes that whenever the server becomes idle after serving the last customer in the queue, he must take a vacation at such instants of

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time. However, in this paper, we assume modified Bernoulli schedules, modified in the sense that whenever the server completes a service, including the case when it is the service of the last customer in the queue, he may take a vacation with probability p or may continue staying in the system with probability $1-p$. Thus, unlike the policy of usual Bernoulli schedules considered by other authors, ours is a policy of uniform Bernoulli schedules under which the server, whether idle or not, may or may not take a vacation with the same probabilities p and $1-p$ respectively, just after completeing a service. In addition to this, we further assume restricted admissibility of arriving batches in our model. In existing queueing literature, one finds some papers such as Rue and Rosenshine [32], Stidham [33], Neuts [31] and Huang and McDonald [14] which deal with control policies of arrivals into queues and queueing networks. However, these papers deal with control policies different from the one considered by us. Moreover, unlike our paper, they neither deal with batch arrivals nor with server vacations. In our restriction policy, we assume that, not all arriving batches are allowed to join the system at all times. Such a restriction may be necessary in many real-life queueing situations, particularly in the over-saturated queues where arrivals occur faster than departures. One may encounter such situations in telecommunications, transportation, computer networks, traffic highways, dams and airports. And surely, there could be many more such situations. We further assume different restriction policies for the different states of the server, that is, when the server is available in the system and when he is away on vacation. We consider the following example:

1. Variable size batches of packets consisting of messages arrive at a communication system for onward transmission (service). If the administrator feels that the messages are arriving faster than they can be transmitted, then he may adopt a policy of restricting the arriving batches. This will help him to prevent the system from becoming over-loaded. Further, this system may undergo routine maintenance from time to time. This is analogous to vacations considered in our model. If he accepts all arriving batches, particularly all batches that arrive during the period of vacation, then this will, all the more, over-load the system. Therefore, he may decide that he will accept, say, r_1 percent of the arriving batches when the system is working and r_2 percent of the arriving batches when the system is down (vacation). Without loss of generality, it is reasonable to assume that $0 \le r_2 < r_1 \le 1$. Here, we may note that although assuming $r_1 = 0$ will not be reasonable, r_2 may, however, be assumed to be zero, which implies that all batches, which arrive during the downtime of the system are rejected. Further, $r_2 < r_1$ is also justified, which means that a fewer number of batches are accepted during the downtime than when the system is working. Of course, the administrator has always the option to choose $r_1 = r_2 = 1$, which is the normal case with no restriction at all. Our mathematical model is briefly described by the following underlying assumptions.

2. The mathematical model

(a) Customers arrive at the system in batches of variable size according to a compound Poisson process with mean arrival rate $\lambda(>0)$. Let π_i be the probability that a batch of size I arrives at the system where $\sum_{i=1}^{\infty} \pi_i = 1$.

(b) Not all arriving batches are allowed to join the system at all times. Let r_i $(0 < r_i \le 1)$ be the probability that an arriving batch will be allowed to join the system while the server is available in the system and let r_2 $(0 \le r_2 \le 1)$ be the probability that an arriving batch will be allowed to join the system during server's vacation period.

(c) The service to customers is provided one by one on a first come, first served basis and their service time, S follows a general (arbitrary) distribution with distribution function $B(s)$ and the density function $b(s)$. Without loss of generality it is assumed that the customers arriving in a batch are pre-ordered for the purpose of service. Further, let $\mu(x)\mathrm{d}x$ be the conditional probability of completion of a service during the interval $(x,\, x+\mathrm{d}x)$ given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)} \tag{2.1}$$

and therefore,

$$b(s) = \mu(s)\exp\left[-\int_0^s \mu(x)\mathrm{d}x\right]\cdot \tag{2.2}$$

(d) As soon as the service of a customer is complete then with probability p the server may decide to go on a vacation or with probability $1-p$ he may continue to be available for the next service.

(e) The server's vacation periods follow an exponential distribution with mean vacation time $1/\alpha$ ($\alpha > 0$).

(f) Various stochastic processes involved in the system are independent of each other.

3. Definitions and notations

We assume that $W_n\ (x,t)$ is the steady state probability that at time t, the server is busy and there are n ($\geq$) customers in the queue excluding the one in service and the elapsed service time of this customer is x. Accordingly, $W_n(t) = \int_0^\infty W_n(x,t)\mathrm{d}x$ denotes the probability that at time t, the server is busy and there are n customers in the queue excluding the one in service irrespective of the value of x. Next, we let $V_n(t)$ be the probability that at time t, the server is away on vacation and there are n ($\geq$) customers in the queue. Finally let $Q(t)$ denote the probability that at time t, there is no customer in the system and the server is idle but available in the system.

Then, assuming that the steady state exists, we let $\mathrm{Lim}_{t\to\infty} W_n(x,t) = W_n(x)$, $\mathrm{Lim}_{t\to\infty} W_n(t) = W_n = \int_0^\infty W_n(x)\mathrm{d}x$, $\mathrm{Lim}_{t\to\infty} V_n(t) = V_n$ and $\mathrm{Lim}_{t\to\infty} Q(t) = Q$.

Thus $W_n(x)$, W_n, V_n and Q denote the corresponding steady state probabilities.

In addition, we define the following steady state probability generating functions:

$$W(x,z) = \sum_0^\infty W_n(x)z^n, \tag{3.1}$$

$$W(z) = \sum_0^\infty W_n z^n, \tag{3.2}$$

$$V(z) = \sum_0^\infty W_n z^n, \tag{3.3}$$

$$\pi(z) = \sum_0^\infty \pi_i z^i, \quad |z| \leq 1. \tag{3.4}$$

4. Steady state equations

Using usual probability arguments we obtain the following set of forward difference- differential equations for the system:

$$\frac{\partial}{\partial x} W_n(x) + (\lambda + \mu(x))W_n(x) = \sum_1^n \lambda\pi_i r_1 W_{n-i}(x) + \lambda(1-r_1)W_n(x), n \geq 1, \tag{4.1}$$

$$\frac{\partial}{\partial x}W_0(x)+(\lambda+\mu(x))W_0(x)=\lambda(1-r_1)W_0(x), \tag{4.2}$$

$$(\lambda+\alpha)V_n=\sum_{i=1}^{n}\lambda\pi_i r_2 V_{n-i}+\lambda(1-r_2)V_n+p\int_0^\infty W_n(x)\mu(x)\mathrm{d}x, n\geq 1, \tag{4.3}$$

$$(\lambda+\alpha)V_n=\lambda(1-r_2)V_0+p\int_0^\infty W_0(x)\mu(x)\mathrm{d}x, \quad n\geq 1, \tag{4.4}$$

$$\lambda Q=\lambda(1-r_1)Q+(1-p)\int_0^\infty W_0(x)\mu(x)\mathrm{d}x+\alpha V_0. \tag{4.5}$$

The above equations are to be solved subject to the boundary conditions

$$W_n(0)=(1-p)\int_0^\infty W_{n+1}(x)\mu(x)\mathrm{d}x+\alpha V_{n+1}, \quad n\geq 1, \tag{4.6}$$

$$W_n(0)=(1-p)\int_0^\infty W_1(x)\mu(x)\mathrm{d}x+\alpha V_1+\lambda r_1 Q. \tag{4.7}$$

5. Steady state probability generating functions

We multiply (4.1) by z^n, sum over n from 1 to ∞ and add the result to (4.2), use (3.1) and (3.4) and simplify. We thus have

$$\frac{\partial}{\partial x}W(x,z)+[\lambda r_1\pi(z)+\mu(x)]W(x,z)=0. \tag{5.1}$$

A similar operation on (4.3) and (4.4) yields

$$[\lambda r_2-\lambda r_2\pi(z)+\alpha]V(z)=p\int_0^\infty W(x,z)\mu(x)\mathrm{d}x. \tag{5.2}$$

And yet again a similar operation on (4.6) and (4,7) gives

$$zW(0,z)=(1-p)\int_0^\infty W(x,z)\mu(x)\mathrm{d}x+\alpha V(z)+(1-p)\int W_0(x)\mu(x)\mathrm{d}x-\alpha V_0+\lambda r_1 Qz. \tag{5.3}$$

Using (4.5) we can re-write (5.3) as

$$zW(0,z)=(1-p)\int_0^\infty W(x,z)\mu(x)\mathrm{d}x+\alpha V(z)+\lambda r_1 Q(z-1). \tag{5.4}$$

Next, we integrate (5.1) with respect to x and obtain

$$W(x,z)=W(0,z)\exp\left[-\lambda r_1(1-\pi(z))-\int_0^x\mu(t)\mathrm{d}t\right]. \tag{5.5}$$

Now, we consider the integral $\int_0^\infty W(x,z)\mu(x)\mathrm{d}x$ which appears in the right hand sides of (5.2) and (5.4). In order to determine this integral we substitute for $W(x,z)$ from (5.5) into this integral and have

$$\int_0^\infty W(x,z)\mu(x)\mathrm{d}x = W(0,z)\int_0^\infty \exp\left[\lambda r_1(1-\pi(z))x - \int_0^x \mu(t)\mathrm{d}t\right]\mu(x)\mathrm{d}x, \tag{5.6}$$

which, on using (2.2) yields

$$\int_0^\infty W(x,z)\mu(x)\mathrm{d}x = W(0,z)B^*\left[\lambda r_1(1-\pi(z))\right], \tag{5.7}$$

where $B^*[\lambda r_1(1-\pi(z))] = \int_0^\infty \exp[\lambda r_1(1-\pi(z))x]\mathrm{d}B(x)$ is the Laplace–Stieljes transform of the service time.

Using (5.7) in (5.2), we obtain

$$[\alpha + \lambda r_2(1-\pi(z))]V(z) = pW(0,z)B^*[\lambda r_1(1-\pi(z))]. \tag{5.8}$$

Then using (5.7) in (5.4) we have

$$zW(0,z) = (1-p)W(0,z)B^*[\lambda r_1(1-\pi(z))] + \alpha V(z) + \lambda r_1 Q(z-1). \tag{5.9}$$

Again (5.9) can be re-written as

$$[z-(1-p)B^*[\lambda r_1(1-\pi(z))]]\,W(0,z) = \alpha V(z) + \lambda r_1 Q(z-1). \tag{5.10}$$

Next, we integrate (5.5) with respect to x and use (2.2). Thus we have

$$W(z) = W(0,z)\left[\frac{1-B^*[\lambda r_1(1-\pi(z))]}{\lambda r_1(1-\pi(z))}\right]. \tag{5.11}$$

Further, equation (5.8) gives

$$W(0,z) = \left[\frac{\alpha+\lambda r_2(1-\pi(z))}{pB^*[\lambda r_1(1-\pi(z))]}\right]V(z). \tag{5.12}$$

Using (5.12) into (5.10) we obtain

$$[z-(1-p)B^*[\lambda r_1(1-\pi(z))]]\left[\frac{\alpha+\lambda r_2(1-\pi(z))}{pB^*[\lambda r_1(1-\pi(z))]}\right]V(z) = \alpha V(z) + \lambda r_1 Q(z-1). \tag{5.13}$$

Then (5.13) yields, on simplifying,

$$V(z) = \frac{p\lambda r_1 B^*[\lambda r_1(1-\pi(z))]Q(z-1)}{[z-(1-p)B^*[\lambda r_1(1-\pi(z))][\alpha+\lambda r_2(1-\pi(z))]] - p\alpha B^*[\lambda r_1(1-\pi(z))]}. \tag{5.14}$$

Using equation (5.12) into equation (5.11) we obtain

$$W(z) = \left[\frac{1-B^*[\lambda r_1(1-\pi(z))]}{\lambda r_1(1-\pi(z))}\right]\left[\frac{\alpha+\lambda r_2(1-\pi(z))}{pB^*[\lambda r_1(1-\pi(z))}\right]V(z), \tag{5.15}$$

Which, on using equation (5.14), further simplifies to

$$W(z) = \left[\frac{1-B^*[\lambda r_1(1-\pi(z))}{(1-\pi(z))]}\right]\left[\frac{[\alpha+\lambda r_2(1-\pi(z))]Q(z-1)}{z-(1-p)B^*[\lambda r_1(1-\pi(z))]][\alpha+\lambda r_2(1-\pi(z))] - p\alpha B^*[\lambda r_1(1-\pi(z))]}\right]. \tag{5.16}$$

Now we shall determine the only unknown constant Q which appears in the numerators of the right hand sides of (5.14) and (5.16). Clearly at $z = 1$, equations (5.14) and (5.16) are indeterminate of the 0/0 form. Therefore, we resort to the use of L'Hopital's rule on equations (5.14) and (5.16). Then on simplifying we obtain

$$V = V(1) = \mathrm{Lim}_{z\to 1} V(z) = \frac{p\lambda r_1 Q}{\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)}, \tag{5.17}$$

$$W = W(1) = \mathrm{Lim}_{z\to 1} W(z) = \mathrm{Lim}_{z\to 1}\left[\frac{1 - B^*[\lambda r_1(1-\pi(z))]}{(1-\pi(z))}\right]$$

$$\times \mathrm{Lim}_{z\to 1}\left[\frac{[\alpha + \lambda r_2(1-\pi(z))]Q(z-1)}{[z-(1-p)B^*[\lambda r_1(1-\pi(z))]][\alpha+\lambda r_2(1-\pi(z))] - p\alpha B^*[\lambda r_1(1-\pi(z))]}\right]$$

$$= \left[\frac{\lambda r_1 E(S)}{E(I)}\right]\left[\frac{\alpha Q}{\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)}\right], \tag{5.18}$$

where V and W now respectively denote the steady state probability that the server is away on vacation and he is busy, without regard to the number of customers in the queue or system.

We note that in obtaining (5.17) and (5.18) we have used the facts that $B^*(0) = \lambda r_1 E(S)$ and $\pi'(1) = E(I)$ where $E(S)$ is the average service time and $E(I)$ is the average size of the arriving batch.

To determine Q, we use (5.17) and (5.18) into the normalizing condition $W(1) + V(1) + Q = 1$ and obtain on simplifying

$$Q = \frac{E(I)[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)]}{p\lambda r_1 E(I) + \alpha\lambda r_1 E(S) + E(I)[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)]}\,. \tag{5.19}$$

Having thus determined Q, we substitute its value into (5.14) and (5.16) and the desired steady state probability generating functions are now completely determined.

We note that for the steady state to exist, the necessary stability condition, which emerges from (5.19) is

$$\lambda < \frac{\alpha}{\alpha r_1 E(S) + p r_2 E(I)}\,. \tag{5.20}$$

Further, we see that the utilization factor, ρ of the system is simply the proportion of the time when the server is busy. Thus ρ is given by W found in (5.18) which on using the value of Q from (5.19) yields

$$\rho = W == \frac{\alpha\lambda r_1 E(S)}{p\lambda r_1 E(I) + \alpha\lambda r_1 E(S) + E(I)[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)]}\,. \tag{5.21}$$

Then using (5.19, 5.17) yields

$$V = \frac{p\lambda r_1 E(I)}{p\lambda r_1 E(I) + \alpha\lambda r_1 E(S) + E(I)[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)]}\,. \tag{5.22}$$

6. The average queue size and the system size

Let $P_q(z) = V(z) + W(z)$ be the steady state probability generating function of the queue size irrespective of whether the server is available in the system or away on vacation. Then adding (5.14) and (5.16) we write

$$P_q(z) = \frac{N(z)}{D(Z)}, \tag{6.1}$$

where

$$N(z) = p\lambda r_1 B^*[\lambda r_1(1-\pi(z))]Q(z-1) + \left[\frac{1-B^*[\lambda r_1(1-\pi(z))]}{(1-\pi(z))}\right][\alpha + \lambda r_2(1-\pi(z))](z-1)Q, \tag{6.2}$$

$$D(z) = [z-(1-p)B^*[\lambda r_1(1-\pi(z))]][\alpha+\lambda r_2(1-\pi(z))] - p\alpha B^*[\lambda r_1(1-\pi(z))]. \tag{6.3}$$

Then the average queue size is given by $L_q = \frac{\mathrm{d}}{\mathrm{d}z}P_q(z)\big|_{z=1}$. However, since $P_q(z)$ in (6.1) is indeterminate of the 0/0 form at $z=1$, we resort to L'Hopital's rule twice and then we obtain

$$L_q = \frac{\mathrm{d}}{\mathrm{d}z}P_q(z)\bigg|_{z=1} = \mathrm{Lim}_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{N(z)}{D(z)}\right] = \frac{D'1N''(1)-N'(1)D''(1)}{2[D'(1)]^2}, \tag{6.4}$$

where primes denote derivatives with respect to z at $z=1$.

The result in (6.4) is well known and is used frequently in queueing theory (see Kashyap and Chaudhry [15]).

Carrying out the derivatives and simplifying some algebra, we obtain

$$N'(1) = \left[p\lambda r_1 + \frac{\alpha\lambda r_1 E(S)}{E(I)}\right]Q, \tag{6.5}$$

$$N''(1) = \left[p\lambda^2 r_1^2 E(I^2) + \alpha\left[\frac{E(I)\lambda^2 r_1^2 E(S^2) - \lambda r_1 E(S)E(I(I-1))}{(E(I)^2)}\right]\right]Q, \tag{6.6}$$

$$D'(1) = \alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I), \tag{6.7}$$

$$D''(1) = -p\lambda r_2 E(I(I-1)) - 2\lambda r_2 E(I) + 2(1-p)\lambda^2 r_1 r_2 E(I)E(S) - \alpha\lambda^2 r_1^2 E(S^2), \tag{6.8}$$

where I denotes the number of customers in a batch and therefore, $E(I)$ and $(EI(I-1))$ are respectively the mean and the second factorial moment of I. Similarly, $E(S)$ and $E(S^2)$ are the mean and second moment of the service time, S. We note that in obtaining above results we have used the following facts:

$$B^*[\lambda r_1(1-\pi(z))]|_{z=1} = \frac{\mathrm{d}}{\mathrm{d}z}B^*[\lambda r_1(1-\pi(z))]|_{z=1} = \lambda r_1 E(S),$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}z^2}[\lambda r_1(1-\pi(z))]|_{z=1} = (\lambda r_1)^2 E(S^2), \pi'(z) = E(I), \pi''(z) = E(I(I-1)).$$

Further, in obtaining above results we have also used

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1-B^*[\lambda r_1(1-\pi(z))]}{1-\pi(z)}\right]\bigg|_{z=1} = \frac{E(I)\lambda^2 r_1^2 E(S^2) - \lambda r_1 E(S)E(I(I-1))}{2(E(I))^2},$$

which has been obtained by separately using L'Hopital's rule.

Using (6.5–6.7) and (6.8) into (6.4), we obtain

$$L_q = \frac{[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)][p\lambda^2 r_1^2 E(I^2) + \alpha \left[\frac{E(I)\lambda^2 r_1^2 E(S^2) - \lambda r_1 E(S)E(I(I-1))}{(E(I)^2)}\right] Q}{2[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)]^2}$$
$$+ \frac{\left[p\lambda r_1 + \frac{\alpha\lambda r_1 E(S)}{E(I)}\right]\left[p\lambda r_2 E(I(I-1)) + 2\lambda r_2 E(I) - 2(1-p)\lambda^2 r_1 r_2 E(I)E(S) + \alpha\lambda^2 r_1^2 E(S^2)\right] Q}{2[\alpha - \alpha\lambda r_1 E(S) - p\lambda r_2 E(I)]^2}. \tag{6.9}$$

Further, we can also find L, the average number of customers in the system as

$$L = L_q + \rho, \tag{6.10}$$

where L_q and ρ have been found in (6.9) and (5.21) respectively.

All results obtained in this section are also new and more general.

7. The average waiting time in the queue and the system

Let W_q and W_s respectively denote the average waiting time in the queue and the system. Then we have

$$W_q = \frac{L_q}{\lambda_a}, \tag{7.1}$$

where L_q has been found in (6.9) and λ_a is the actual arrival rate into the system and is given by

$$\lambda_a = \lambda r_1 (W + Q) + \lambda r_2 V, \tag{7.2}$$

where Q, W and V have been found in (5.19, 5.21) and (5.22) respectively.

Therefore, using (6.9) and (7.2, 7.1) explicitly gives W_q.

Finally, we can also find W_s as

$$W_s = \frac{L}{\lambda_a}, \tag{7.3}$$

where L and λ_a are found (6.10) and (7.2) respectively.

8. Special cases

We note that many particular cases for various service distributions can be derived from the above results by substituting various appropriate values of $B^*[\lambda r_1(1-\pi(z))]$. However, we shall consider here some special cases.

Case 1: Single arrivals with restricted admissibility

In this case we let $\pi_1 = 1$ and $\pi_i = 0$ for $i \neq 1$ and consequently we have $E(I) = 1$, $E(I^2) = 1$, $E(I(I-1)) = 0$ and $\pi(z) = z$. With these substitutions, the results found in (5.14, 5.16, 5.19) and (5.20) and (5.21, 5.22) and (6.9) reduce to

$$V(z) = \frac{p\lambda r_1 B^*[\lambda r_1(1-z)]Q(z-1)}{[z - 1(1-p)B^*[\lambda r_1(1-z)]][\alpha + \lambda r_2(1-z)] - p\alpha B^*[\lambda r_1(1-z)]}, \tag{8.1}$$

$$W(z)=\left[\frac{1-B^*[\lambda r_1(1-z)]}{(1-z)}\right]\left[\frac{[\alpha+\lambda r_2(1-z))]Q(z-1)}{[z-(1-p)B^*[\lambda r_1(1-z)]][\alpha+\lambda r_2(1-z)]-p\alpha B^*[\lambda r_1(1-z)]}\right], \quad (8.2)$$

$$Q=\frac{[\alpha-\alpha\lambda r_1E(S)-p\lambda r_2]}{p\lambda r_1+[\alpha-p\lambda r_2]}, \quad (8.3)$$

$$\lambda<\frac{\alpha}{\alpha r_1E(S)+pr_2}, \quad (8.4)$$

$$W=\rho=\frac{\alpha\lambda r_1E(S)}{p\lambda r_1+\alpha\lambda r_1E(S)+[\alpha-\alpha\lambda r_1E(S)-p\lambda r_2]}, \quad (8.5)$$

$$V=\frac{p\lambda r_1}{p\lambda r_1+\alpha\lambda r_1E(S)+[\alpha-\alpha\lambda r_1E(S)-p\lambda r_2]}, \quad (8.6)$$

$$\begin{aligned}L_q &= \frac{[\alpha-\alpha\lambda r_1E(S)-p\lambda r_2][p\lambda^2r_1^2+\alpha\lambda^2r_1^2E(S^2)]Q}{2[\alpha-\alpha\lambda r_1E(S)-p\lambda r_2]^2}\\ &+\frac{[p\lambda r_1+\alpha\lambda r_1E(S)][2\lambda r_2-2(1-p)\lambda^2r_1r_2E(S)+\alpha\lambda^2r_1^2E(S^2)]Q}{2[\alpha-\alpha\lambda r_1E(S)-p\lambda r_2]^2}. \quad (8.7)\end{aligned}$$

We can also find W_q, L and W_s using relations (6.10, 7.1, 7.2) and (7.3) under the conditions of this case.

Case 2: Single arrivals with restricted admissibility, exponential service and exponential vacations (M/M/M/1 queue)

In addition to the conditions of Case 1, we let $E(S)=1/\mu$, $E(S^2)=2/\mu^2$ $(\mu>0)$ and $B^*[\lambda r_1(1-z)]=\frac{\mu}{\lambda r_1(1-z)+\mu}$ in the results of Case 1 and obtain

$$V(z)=\frac{p\lambda r_1\left(\frac{\mu}{\lambda r_1(1-z)+\mu}\right)Q(z-1)}{\left[z-(1-p)\left(\frac{\mu}{\lambda r_1(1-z)+\mu}\right)\right][\alpha+\lambda r_2(1-z)]-p\alpha\left(\frac{\mu}{\lambda r_1(1-z)+\mu}\right)}, \quad (8.8)$$

$$W(z)=\left[\frac{1-\left(\frac{\mu}{\lambda r_1(1-z)+\mu}\right)}{(1-z)}\right]\left[\frac{[\alpha+\lambda r_2(1-z)]Q(z-1)}{\left[z-(1-p)\left(\frac{\mu}{\lambda r_1(1-z)+\mu}\right)\right][\alpha+\lambda r_2(1-\pi(z))]-p\alpha\left(\frac{\mu}{\lambda r_1(1-z)+\mu}\right)}\right], \quad (8.9)$$

$$Q=\frac{\left[\alpha-\frac{\alpha\lambda r_1}{\mu}-p\lambda r_2\right]}{p\lambda r_1+[\alpha-p\lambda r_2]}, \quad (8.10)$$

$$\lambda<\frac{\alpha}{\frac{\alpha r_1}{\mu}+pr_2}, \quad (8.11)$$

$$\rho = \frac{(\alpha\lambda r_1/\mu)}{p\lambda r_1 + (\alpha\lambda r_1/\mu) + [\alpha - (\alpha\lambda r_1/\mu) - p\lambda r_2]}, \tag{8.12}$$

$$V = \frac{p\lambda r_1}{p\lambda r_1 + (\alpha\lambda r_1/\mu) + [\alpha - (\alpha\lambda r_1/\mu) - p\lambda r_2]}, \tag{8.13}$$

$$\begin{aligned} L_q &= \frac{[\alpha - (\alpha\lambda r_1/\mu) - p\lambda r_2][p\lambda^2 r_1^2 + (2\alpha\lambda^2 r_1^2/\mu^2)]Q}{2[\alpha - (\alpha\lambda r_1/\mu) - p\lambda r_2]^2} \\ &+ \frac{[p\lambda r_1 + (\alpha\lambda r_1/\mu)][2\lambda r_2 - (2(1-p)\lambda^2 r_1 r_2/\mu) + (2\alpha\lambda^2 r_1^2/\mu^2)]Q}{2[\alpha - (\alpha\lambda r_1/\mu) - p\lambda r_2]^2}. \end{aligned} \tag{8.14}$$

Again, W_q, L and W_s can also be found using relations (6.10, 7.1, 7.2) and (7.3) under the conditions of this case.

Case 3: Restricted admissibility only during vacation periods (partial admissibility)
In this case we let $r_1 = 1$, in the main results (5.14, 5.16, 5.19, 5.20) and (5.21, 5.22) and (6.9). Thus we obtain

$$V(z) = \frac{p\lambda B^*[\lambda r_2(1-\pi(z))]Q(z-1)}{[z-(1-p)B^*[\lambda(1-\pi(z))]][\alpha + \lambda r_2(1-\pi(z))] - p\alpha B^*[\lambda(1-\pi(z))]}, \tag{8.15}$$

$$W(z) = \left[\frac{1 - B^*[\lambda(1-\pi(z))]}{(1-\pi(z))}\right]\left[\frac{[\alpha + \lambda r_2(1-\pi(z))]Q(z-1)}{[z-(1-p)B^*[\lambda(1-\pi(z))]][\alpha + \lambda r_2(1-\pi(z))] - p\alpha B^*[\lambda(1-\pi(z)]}\right], \tag{8.16}$$

$$Q = \frac{E(I)[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)]}{p\lambda E(I) + \alpha\lambda E(S) + E(I)[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)]}, \tag{8.17}$$

$$\lambda < \frac{\alpha}{\alpha E(S) + p r_2 E(I)}, \tag{8.18}$$

$$\rho = \frac{\alpha\lambda E(S)}{p\lambda E(I) + \alpha\lambda E(S) + E(I)[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)]}, \tag{8.19}$$

$$V = \frac{p\lambda E(I)}{p\lambda E(I) + \alpha\lambda E(S) + E(I)[\alpha - \alpha E(S) - p\lambda r_2 E(I)]}, \tag{8.20}$$

$$\begin{aligned} L_q &= \frac{[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)][p\lambda^2 E(I^2)] + \alpha\left[\frac{E(I)\lambda^2 E(S^2) - \lambda E(S)E(I(I-1))}{(E(I)^2)}\right]Q}{2[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)]^2} \\ &+ \frac{\left[p\lambda + \frac{\alpha\lambda E(S)}{E(I)}\right][p\lambda r_2 E(I(I-1)) + 2\lambda r_2 E(I) - 2(1-p)\lambda^2 r_2 E(I)E(S) + \alpha\lambda^2 E(S^2)]Q}{2[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)]^2}. \end{aligned} \tag{8.21}$$

Case 4: All arrivals are allowed to join at all times (complete admissibility)
In this case, we let $r_1 = r_2 = 1$ in the main results, equations (5.14, 5.16, 5.19–5.22) and (6.9) or simply $r_2 = 1$ in the results of Case 3. We then have

$$V(z) = \frac{p\lambda B^*[\lambda(1-\pi(z))]Q(z-1)}{[z-(1-p)B^*[\lambda(1-\pi(z))]][\alpha+\lambda(1-\pi(z))] - p\alpha B^*[\lambda(1-\pi(z)]}, \tag{8.22}$$

$$W(z) = \left[\frac{1-B^*[\lambda(1-\pi(z))]}{(1-\pi(z))}\right]\left[\frac{[\alpha+\lambda(1-\pi(z))]Q(z-1)}{[z-(1-p)B^*[\lambda(1-\pi(z)]][\alpha+\lambda(1-\pi(z))] - p\alpha B^*[\lambda(1-\pi(z))]}\right], \tag{8.23}$$

$$Q = \frac{E(I)[\alpha - \alpha\lambda E(S) - p\lambda E(I)]}{p\lambda E(I) + \alpha\lambda E(S) + E(I)[\alpha - \alpha\lambda E(S) - p\lambda E(I)]}, \tag{8.24}$$

$$\lambda < \frac{\alpha}{\alpha E(S) + pE(I)}, \tag{8.25}$$

$$\rho = \frac{\alpha\lambda E(S)}{p\lambda E(I) + \alpha\lambda E(S) + E(I)[\alpha - \alpha\lambda E(S) - p\lambda r_2 E(I)]}, \tag{8.26}$$

$$V = \frac{p\lambda E(I)}{p\lambda E(I) + \alpha\lambda E(S) + E(I)[\alpha - \alpha E(S) - p\lambda E(I)]}, \tag{8.27}$$

$$\begin{aligned} L_q \quad = \quad & \frac{[\alpha - \alpha\lambda E(S) - p\lambda E(I)][p\lambda^2 E(I^2) + \alpha\left[\frac{E(I)\lambda^2 E(S^2) - \lambda E(S)E(I(I-1))}{(E(I)^2)}\right]Q}{2[\alpha - \alpha\lambda E(S) - p\lambda E(I)]^2} \\ & + \frac{\left[p\lambda + \frac{\alpha\lambda E(S)}{E(I)}\right][p\lambda E(I(I-1)) + 2\lambda E(I) - 2(1-p)\lambda^2 E(I)E(S) + \alpha\lambda^2 E(S^2)]Q}{2[\alpha - \alpha\lambda E(S) - p\lambda E(I)]^2}. \end{aligned} \tag{8.28}$$

Case 5: Single arrivals with complete admissibility and no vacations (M/G/1 queue)
In this case, we let $p = 0$, $\pi(z) = z$, $\lambda E(S) = \rho$, $E(S^2) = \sigma_s^2 + (1/\mu^2)$, $E(I) = 1$, $E(I^2) = 1$ and $E(I(I-1)) = 0$ in the results of Case 4. Then we have $V(z) = 0$ as it should be and further

$$p_q(z) = W(z) = \frac{[B^*[\lambda(1-z)-1]]Q}{z - B^*[\lambda(1-z)]}, \tag{8.29}$$

where $Q = 1 - \rho$.

Let $P(z)$ denote the probability generating function of the number of customers in the system. Then we have

$$P(z) = Q + P_q(z) = (1-\rho) + \frac{[B^*[\lambda(1-z)-1]](1-\rho)}{z - B^*[\lambda(1-z)]} = \frac{(1-\rho)B^*[\lambda(1-z)][z-1]}{z - B^*[\lambda(1-z)]}. \tag{8.30}$$

We note that (8.30) is a known result of the M/G/1 queue (see Kashyap and Chaudhry [15], Eq. (28), p. 45).

Further, using the above substitutions and taking limit as $(1/\alpha) \to 0$ and simplifying, equation (8.28) yields

$$L_q = \frac{\lambda^2\sigma_s^2 + \rho^2}{2[1-\rho]^2} \cdot \tag{8.31}$$

Again (8.31) is a known result of the M/G/1 queue (see Kashyap and Chaudhry [16], Eq. (45), p. 47).

Remark. As far as the authors are aware, the results obtained in Sections 5, 6 and 7 and also the results of all special cases (except Case 5) are all new.

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