RAIRO-Oper. Res. 48 (2014) 521–543
DOI: 10.1051/ro/2014022

RAIRO Operations Research
www.rairo-ro.org

FINITE BUFFER *GI*/*Geo*/1 BATCH SERVICING QUEUE WITH MULTIPLE WORKING VACATIONS

P. Vijaya Laxmi[1] and Kanithi Jyothsna[1]

Abstract. This paper analyzes a discrete-time finite buffer renewal input queue with multiple working vacations where services are performed in batches of maximum size "b". The service times both during a regular service period and vacation period and vacation times are geometrically distributed. Employing the supplementary variable and imbedded Markov chain techniques, we derive the steady-state queue length distributions at pre-arrival, arbitrary and outside observer's observation epochs. Based on the queue length distributions, some performance measures and waiting time distribution in the queue have been discussed. Finally, numerical results showing the effect of model parameters on the key performance measures are presented.

Keywords. Discrete-time, finite buffer, batch service, multiple working vacations, waiting time.

Mathematics Subject Classification. 60K25, 90B22.

1. Introduction

Discrete-time queueing models have received significant interest during the last few decades owing to their wide applications in many areas such as digital computers, communication networks, *etc.*, because of their clock-driven operations. These models are more accurate and efficient than their continuous-time counterparts to analyze and design digital transmitting systems. Modeling of discrete-time queues is more involved and quite different from the corresponding continuous-time queueing models. Further, the advantage of analyzing discrete-time queues is that one

Received May 3, 2013. Accepted March 27, 2014.

[1] Department of Applied Mathematics, Andhra University, Visakhapatnam 530 003, India. vijaya_iit2003@yahoo.co.in; mail2jyothsnak@yahoo.co.in

can obtain the continuous-time results from it as a limiting case but the converse is not true. However, from an application point of view, both discrete- and continuous-time queues have equal importance. In discrete-time queueing systems, the arrivals and departures can occur simultaneously at a slot boundary. Their order may be taken care of by either arrival-first (AF) or departure-first (DF) management policies, which are commonly known as late arrival system with delayed access (LAS-DA) and early arrival system (EAS), respectively. Extensive analysis of a wide variety of discrete-time queueing models have been reported in Bruneel and Kim [1], Gravey and Hébuterne [5], Hunter [8], Takagi [15] and Woodward [16].

During the past two decades, discrete-time queues with server vacations have been widely used in the performance analysis of communication systems. In the classical vacation models, authors often assume that the server stops serving completely during the vacation period. However, there are numerous situations where the server remains active during the vacation period and serves the customers at a different service rate. Motivated by the analysis of a reconfigurable wavelength-division multiplexing optical access network, Servi and Finn [13] introduced a kind of working vacation (WV) policy: the sever will not completely remain inactive during the vacation period rather it will render service to the queue with a slower rate. When a vacation ends and if there are customers in the queue, a regular service period begins and the server serves the queue with its original service rate, otherwise it takes another vacation and continues to do so till it finds at least one waiting customer at a vacation termination epoch. Such a vacation policy is called multiple working vacations (MWV). The discrete-time $Geo/Geo/1$ queue with MWV has been discussed by Tian *et al.* [14]. They have obtained the distributions for the number of customers in the system using matrix-geometric method. Li *et al.* [11] studied an infinite buffer $GI/Geo/1$ queue with MWV under EAS and LAS schemes. Using the matrix-geometric solution method, they have obtained the steady-state distribution of the number of customers in the system and presented the stochastic decomposition property of the queue length and waiting time. Goswami an Mund [3] obtained the system length distributions of a finite buffer $GI/Geo/1$ queue with MWV using supplementary variable and imbedded Markov chain techniques. Li and Tian [12] studied the $GI/Geo/1$ queue with WV and vacation interruptions. Yu *et al.* [17] analyzed a $GI/Geo/1/N$ queue with MWV and changeover times.

All the above studies on discrete-time MWV queues have been carried out under the assumption that the server serves the customers one at a time. However, there are many instances where services are carried out in batches to enhance the performance of the system. Such a service mechanism is called bulk/batch service. For a wide variety of bulk service queues, see Chaudhry and Templeton [2]. Gupta and Goswami [6] have considered a finite buffer $Geo/Geo/1$ queue with bulk service under AF and DF management policies. Goswami and Vijaya Laxmi [4] have analyzed a finite buffer $GI/Geo/1$ batch service queue with multiple vacations (MV) using supplementary variable and embedded Markov chain techniques. Recently, Jiang *et al.* [9] studied the $Geo/Geo/1$ queue with bulk service and MWV for an EAS policy.

The present literature shows that the analysis of a finite buffer bulk service $GI/Geo/1$ queue with MWV has not been carried out so far, to the best of our knowledge. The model has potential applications in many areas such as in telecommunication networks, manufacturing systems, computer and switching systems, etc, where jobs are processed in batches. For example, consider Broadband Integrated Services Digital Network (B-ISDN) based on the Asynchronous Transfer Mode (ATM) technology that provides a common interface for multimedia service including data, voice and video. The ATM is a multiplexing and switching technology that transfers information through the network in fixed size units, called cells. It is assumed that only a limited number of packets are transmitted during a slot and that a well-defined polling protocol is used to serve input packets. Since ATM is based on packet switching principle, all events such as arrivals and transmission of packets are allowed only at regularly spaced points in time. Therefore, the underlying mechanism of such system is modeled adequately by the discrete-time queues. Motivated by such situations, we analyze a finite buffer discrete-time bulk service queue with MWV wherein arrivals occur according to discrete-time renewal input. One may note that the uncorrelated arrival process generally gives a reasonably good approximation than geometric distribution and also it can include the special cases of geometric, deterministic and other discrete distributions. The service times both during regular service period and WV period and vacation times are assumed to be independent and geometrically distributed. The model is analyzed using supplementary variable and imbedded Markov chain techniques. The steady-state distributions of the number of customers at pre-arrival, arbitrary and outside observer's observation epochs have been obtained under an EAS policy. Some performance measures and the analysis of waiting time distribution in the queue have been discussed. Numerical results have been presented in the form of table and graphs to show the effect of model parameters on the performance indices.

The rest of the paper is organized as follows. Section 2 presents the description of the model. In Sections 3 and 4, the supplementary variable and imbedded Markov chain techniques are employed to analyze the steady-state distributions at different time epochs. Various performance measures of the model and waiting time analysis is carried out in Section 5. Using some numerical results, we demonstrate the parameter effect on the performance measures of the system in Section 6. Finally, Section 7 concludes the paper.

2. Model description

Let us consider a discrete-time finite buffer bulk service queue with MWV for an EAS. Customers are served by a single server in batches of maximum size "b". The server is allowed to take WV whenever the system becomes empty. On return from a WV if the system is non-empty, it switches to a regular service period;

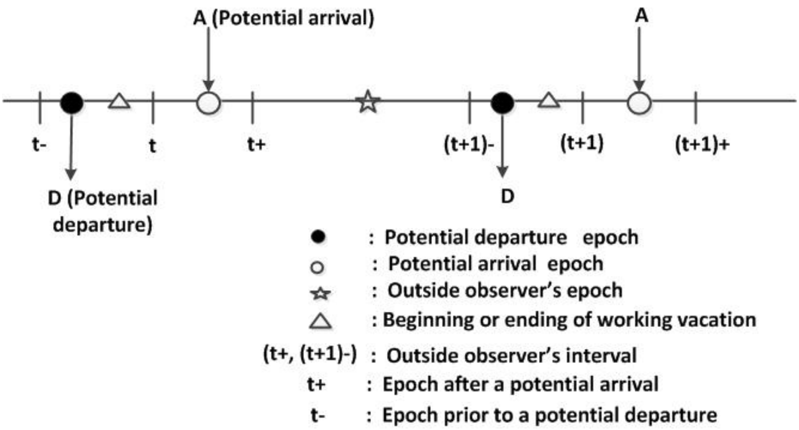

FIGURE 1. Various time epochs in EAS.

otherwise another vacation follows until it finds at least one waiting customer in the queue at a vacation completion epoch. During vacation, customers are served in batches of maximum size "b" but with a slower rate. The system has finite buffer capacity of size N ($> b$) that is, the maximum number of customers allowed in the system at any time is $(N + b)$.

The inter-arrival times A of two successive arrivals are assumed to be independent and identically distributed (i.i.d.) random variables (r.v.s) with common probability mass function (p.m.f.) $a_i = P(A = i),\ i \geq 1$, probability generating function (p.g.f.) $A^*(z) = \sum_{i=1}^{\infty} a_i z^i$ and mean inter-arrival time $1/\lambda = A^{*(1)}(1)$, where $A^{*(1)}(1)$ is the first derivative of $A^*(z)$ with respect to z evaluated at $z = 1$. The service times of the batches are assumed to be independent and geometrically distributed with probability mass function (p.m.f.) $P(S = i) = \mu\bar{\mu}^{i-1},\ i \geq 1$ and mean service time $1/\mu$, where for any real number $x \in [0, 1]$, we denote $\bar{x} = 1 - x$. The service times during a WV period and the vacation times are also assumed to be independent and geometrically distributed r.v.s with rates η and θ, respectively. Further, it is also assumed that the inter-arrival times, service times during regular service period and service times during WV are independent. The traffic intensity is given by $\rho = \lambda/b\mu$.

Let us assume that the time axis is slotted into intervals of equal length with the length of a slot being unity. Further, let the time axis be marked by $0, 1, 2, \ldots, t, \ldots$, and assume that a potential arrival takes place in $(t, t+)$ and a potential departure occurs in $(t-, t)$. The WV can only start or end in $(t-, t)$ just after the departure, see Figure 1. The state of the system at time t, is described by the following r.v.s:

- $N_q(t)$ = number of customers in the queue (excluding the batch in service),
- $N_s(t)$ = number of customers in service,

- $U(t)$ = the remaining inter-arrival time for the next arrival,
- $\xi(t) = \begin{cases} 0, & \text{if the server is on working vacation,} \\ 1, & \text{if the server is in regular service period.} \end{cases}$

Let us define the joint probabilities as

$$\vartheta_{0,0}(u,t) = P\Big\{N_q(t) = 0, N_s(t) = 0,\ U(t) = u,\ \xi(t) = 0\Big\},$$

$$\vartheta_{i,j}(u,t) = P\Big\{N_q(t) = i, N_s(t) = j,\ U(t) = u,\ \xi(t) = 0\Big\},$$
$$0 \le i \le N,\ 1 \le j \le b,\ u \ge 0,$$

$$\pi_{i,j}(u,t) = P\Big\{N_q(t) = i, N_s(t) = j,\ U(t) = u,\ \xi(t) = 1\Big\},$$
$$0 \le i \le N,\ 1 \le j \le b,\ u \ge 0.$$

At steady-state, let $\vartheta_{i,j}(u) = \lim_{t\to\infty} \vartheta_{i,j}(u,t)$ and $\pi_{i,j}(u) = \lim_{t\to\infty} \pi_{i,j}(u,t)$.

3. Steady-state distribution at arbitrary epoch

To obtain the queue length distribution at arbitrary epoch, we develop the difference equations using the remaining inter-arrival time as the supplementary variable. Observing the state of the system at two consecutive time epochs t and $(t+1)$ and using the probabilistic arguments, we have the following system of difference equations at steady-state, for $u \ge 1$:

$$\vartheta_{0,0}(u-1) = \vartheta_{0,0}(u) + \mu \sum_{k=1}^{b} \pi_{0,k}(u) + \eta \sum_{k=1}^{b} \vartheta_{0,k}(u) + \eta a_u \vartheta_{0,0}(0), \tag{3.1}$$

$$\begin{aligned}\vartheta_{0,j}(u-1) = {} & \bar{\theta}\bar{\eta}\vartheta_{0,j}(u) + \bar{\theta}\eta \sum_{k=1}^{b} \vartheta_{j,k}(u) + \bar{\theta}\eta a_u \sum_{k=1}^{b} \vartheta_{j-1,k}(0) + I(j=1) \\ & \times \bar{\theta}\bar{\eta} a_u \vartheta_{0,0}(0),\ 1 \le j \le b,\end{aligned} \tag{3.2}$$

$$\vartheta_{i,j}(u-1) = \bar{\theta}\bar{\eta}\vartheta_{i,j}(u) + \bar{\theta}\bar{\eta} a_u \vartheta_{i-1,j}(0),\ 1 \le i \le N-1,\ 1 \le j \le b-1, \tag{3.3}$$

$$\vartheta_{N,j}(u-1) = \bar{\theta}\bar{\eta}\vartheta_{N,j}(u) + \bar{\theta}\bar{\eta} a_u \left(\vartheta_{N-1,j}(0) + \vartheta_{N,j}(0)\right), 1 \le j \le b, \tag{3.4}$$

$$\begin{aligned}\vartheta_{i,b}(u-1) = {} & \bar{\theta}\bar{\eta}\left(\vartheta_{i,b}(u) + a_u \vartheta_{i-1,b}(0)\right) + \bar{\theta}\eta \sum_{k=1}^{b} (a_u(\vartheta_{i+b-1,k}(0) \\ & + I(i=N-b)\vartheta_{N,k}(0)) + \vartheta_{i+b,k}(u)),\ 1 \le i \le N-b,\end{aligned} \tag{3.5}$$

$$\vartheta_{i,b}(u-1)=\bar{\theta}\bar{\eta}\vartheta_{i,b}(u)+\bar{\theta}\bar{\eta}a_u\vartheta_{i-1,b}(0),\ N-b+1\le i\le N-1, \tag{3.6}$$

$$\begin{aligned}\pi_{0,j}(u-1)&=\bar{\mu}\pi_{0,j}(u)+\mu\sum_{k=1}^{b}\pi_{j,k}(u)+\mu a_u\sum_{k=1}^{b}\pi_{j-1,k}(0)+\theta\bar{\eta}\vartheta_{0,j}(u)\\&\quad+\theta\eta\sum_{k=1}^{b}\vartheta_{j,k}(u)+\theta\eta a_u\sum_{k=1}^{b}\vartheta_{j-1,k}(0)+I(j=1)\theta\bar{\eta}a_u\vartheta_{0,0}(0),\\&\qquad 1\le j\le b,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\pi_{i,j}(u-1)&=\bar{\mu}\pi_{i,j}(u)+\bar{\mu}a_u\pi_{i-1,j}(0)+\theta\bar{\eta}\vartheta_{i,j}(u)\\&\quad+\theta\bar{\eta}a_u\vartheta_{i-1,j}(0),\ 1\le i\le N-1,\ 1\le j\le b-1,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\pi_{N,j}(u-1)&=\bar{\mu}\pi_{N,j}+\bar{\mu}a_u(\pi_{N-1,j}(0)+\pi_{N,j}(0))+\theta\bar{\eta}\vartheta_{N,j}(u)\\&\quad+\theta\bar{\eta}a_u\left(\vartheta_{N-1,j}(0)+\vartheta_{N,j}(0)\right),1\le j\le b,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\pi_{i,b}(u-1)&=\bar{\mu}\pi_{i,b}(u)+\mu\sum_{k=1}^{b}\pi_{i+b,k}(u)+\bar{\mu}a_u\pi_{i-1,b}(0)\\&\quad+\mu a_u\sum_{k=1}^{b}(\pi_{i+b-1,k}(0)+I(i=N-b)\pi_{N,k}(0))\\&\quad+\theta\bar{\eta}\left(\vartheta_{i,b}(u)+a_u\vartheta_{i-1,b}(0)\right)+\eta\sum_{k=1}^{b}(a_u(\vartheta_{i+b-1,k}(0)\\&\quad+I(i=N-b)\vartheta_{N,k}(0))+\vartheta_{i+b,k}(u)),\ 1\le i\le N-b,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\pi_{i,b}(u-1)&=\bar{\mu}\pi_{i,b}(u)+\bar{\mu}a_u\pi_{i-1,b}(0)+\theta\bar{\eta}\vartheta_{i,b}(u)\\&\quad+\theta\bar{\eta}a_u\vartheta_{i-1,b}(0),\ N-b+1\le i\le N-1,\end{aligned} \tag{3.11}$$

where $I(x)$ is an indicator function which yields 1 if the expression x is true, otherwise, it takes the value 0.

Let us define the z-transforms of $\vartheta_{i,j}(u)$ and $\pi_{i,j}(u)$ as

$$\begin{aligned}\vartheta^*_{0,0}(z)&=\sum_{u=0}^{\infty}\vartheta_{0,0}(u)z^u,\\\vartheta^*_{i,j}(z)&=\sum_{u=0}^{\infty}\vartheta_{i,j}(u)z^u,\ 0\le i\le N,\ 1\le j\le b,\\\pi^*_{i,j}(z)&=\sum_{u=0}^{\infty}\pi_{i,j}(u)z^u,\ 1\le i\le N,\ 1\le j\le b.\end{aligned}$$

It follows that $\vartheta^*_{0,0}(1) = \vartheta_{0,0}$ is the probability that the server is idle during WV at an arbitrary epoch, $\vartheta^*_{i,j}(1) = \vartheta_{i,j}$ and $\pi^*_{i,j}(1) = \pi_{i,j}$, where $\vartheta_{i,j}$ $(\pi_{i,j})$ denotes the probability that there are i customers in the queue when the server is busy with a batch of j customers during WV (regular service) at an arbitrary epoch. Multiplying the system of difference equations (3.1) to (3.11) by z^u and summing over u from 1 to ∞, we obtain

$$
\begin{aligned}
(z-1)\vartheta^*_{0,0}(z) = {}&\eta\sum_{k=1}^{b}\left(\vartheta^*_{0,k}(z)-\vartheta_{0,k}(0)\right)\\
&+\mu\sum_{k=1}^{b}\left(\pi^*_{0,k}(z)-\pi_{0,k}(0)\right)+\eta A^*(z)\vartheta_{0,0}(0)-\vartheta_{0,0}(0), \quad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
(z-\bar\theta\bar\eta)\vartheta^*_{0,j}(z) = {}&\bar\theta\eta\sum_{k=1}^{b}\left(\vartheta^*_{j,k}(z)-\vartheta_{j,k}(0)\right)+\bar\theta\eta A^*(z)\sum_{k=1}^{b}\vartheta_{j-1,k}(0)\\
&+I(j=1)\bar\theta\bar\eta A^*(z)\vartheta_{0,0}(0)-\bar\theta\bar\eta\vartheta_{0,j}(0),\ 1\le j\le b, \quad (3.13)
\end{aligned}
$$

$$
(z-\bar\theta\bar\eta)\vartheta^*_{i,j}(z) = \bar\theta\bar\eta A^*(z)\vartheta_{i-1,j}(0)-\bar\theta\bar\eta\vartheta_{i,j}(0), 1\le i\le N-1,\ 1\le j\le b-1, \quad (3.14)
$$

$$
(z-\bar\theta\bar\eta)\vartheta^*_{N,j}(z) = \bar\theta\bar\eta A^*(z)\left(\vartheta_{N-1,j}(0)+\vartheta_{N,j}(0)\right)-\bar\theta\bar\eta\vartheta_{N,j}(0),\ 1\le j\le b, \quad (3.15)
$$

$$
\begin{aligned}
(z-\bar\theta\bar\eta)\vartheta^*_{i,b}(z) = {}&\bar\theta\eta\sum_{k=1}^{b}\left(\vartheta^*_{i+b,k}(z)-\vartheta_{i+b,k}(0)\right)\\
&+\bar\theta\bar\eta A^*(z)\vartheta_{i-1,b}(0)-\bar\theta\bar\eta\vartheta_{i,b}(0)+\bar\theta\eta A^*(z)\sum_{k=1}^{b}(\vartheta_{i+b-1,k}(0)\\
&+I(i=N-b)\vartheta_{N,k}(0)),\ 1\le i\le N-b, \quad (3.16)
\end{aligned}
$$

$$
(z-\bar\theta\bar\eta)\vartheta^*_{i,b}(z) = \bar\theta\bar\eta A^*(z)\vartheta_{i-1,b}(0)-\bar\theta\bar\eta\vartheta_{i,b}(0),\ N-b+1\le i\le N-1, \quad (3.17)
$$

$$
\begin{aligned}
(z-\bar\mu)\pi^*_{0,j}(z) = {}&\mu\sum_{k=1}^{b}\left(\pi^*_{j,k}(z)-\pi_{j,k}(0)\right)+\mu A^*(z)\sum_{k=1}^{b}\pi_{j-1,k}(0)\\
&+\theta\bar\eta\left(\vartheta^*_{0,j}(z)-\vartheta_{0,j}(0)\right)+\theta\eta\sum_{k=1}^{b}(\vartheta^*_{j,k}(z)-\vartheta_{j,k}(0))\\
&+\theta\eta A^*(z)\sum_{k=1}^{b}\vartheta_{j-1,k}(0)+I(j=1)\theta\bar\eta A^*(z)\vartheta_{0,0}(0)\\
&-\bar\mu\pi_{0,j}(0),\ 1\le j\le b, \quad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
(z-\bar\mu)\pi^*_{i,j}(z) = {}&\bar\mu A^*(z)\pi_{i-1,j}(0)+\theta\bar\eta A^*(z)\vartheta_{i-1,j}(0)+\theta\bar\eta(\vartheta^*_{i,j}(z)-\vartheta_{i,j}(0))\\
&-\bar\mu\pi_{i,j}(0), 1\le i\le N-1, 1\le j\le b-1, \quad (3.19)
\end{aligned}
$$

$$\begin{aligned}(z-\bar{\mu})\pi^*_{N,j}(z) = {} & \bar{\mu}A^*(z)\left(\pi_{N-1,j}(0)+\pi_{N,j}(0)\right)+\theta\bar{\eta}\left(\vartheta^*_{N,j}(z)-\vartheta_{N,j}(0)\right)\\ & +\theta\bar{\eta}A^*(z)\left(\vartheta_{N-1,j}(0)+\vartheta_{N,j}(0)\right)-\bar{\mu}\pi_{N,j}(0), \end{aligned} \tag{3.20}$$

$$\begin{aligned}(z-\bar{\mu})\pi^*_{i,b}(z) = {} & \mu\sum_{k=1}^{b}\left(\pi^*_{i+b,k}(z)-\pi_{i+b,k}(0)\right)+\bar{\mu}A^*(z)\pi_{i-1,b}(0)\\ & +\theta\bar{\eta}A^*(z)\vartheta_{i-1,b}(0)+\mu A^*(z)\sum_{k=1}^{b}(\pi_{i+b-1,k}(0)\\ & +I(i=N-b)\pi_{N,k}(0))\\ & +\theta\bar{\eta}\left(\vartheta^*_{i,b}(z)-\vartheta_{i,b}(0)\right)+\theta\eta\sum_{k=1}^{b}\left(\vartheta^*_{i+b,k}(z)-\vartheta_{i+b,k}(0)\right)\\ & +\theta\eta A^*(z)\sum_{k=1}^{b}(\vartheta_{i+b-1,k}(0)+I(i=N-b)\vartheta_{N,k}(0))\\ & -\bar{\mu}\pi_{i,b}(0),\ 1\le i\le N-b, \end{aligned} \tag{3.21}$$

$$\begin{aligned}(z-\bar{\mu})\pi^*_{i,b}(z) = {} & \bar{\mu}A^*(z)\pi_{i-1,b}(0)+\theta\bar{\eta}\left(\vartheta^*_{i,b}(z)-\vartheta_{i,b}(0)\right)+\theta\bar{\eta}A^*(z)\vartheta_{i-1,b}(0)\\ & -\bar{\mu}\pi_{i,b}(0),\ N-b+1\le i\le N-1. \end{aligned} \tag{3.22}$$

Using equations (3.12) to (3.22), one important result is presented below in the form of a theorem.

Theorem 3.1. *The mean number of entrances into the system per unit time equals the mean arrival rate* i.e.,

$$\vartheta_{0,0}(0)+\sum_{i=0}^{N}\sum_{j=1}^{b}\left(\vartheta_{i,j}(0)+\pi_{i,j}(0)\right)=\lambda. \tag{3.23}$$

Proof. Adding equations (3.12) to (3.22), we get

$$\begin{aligned}\vartheta^*_{0,0}(z)+\sum_{i=0}^{N}\sum_{j=1}^{b}\left(\vartheta^*_{i,j}(z)+\pi^*_{i,j}(z)\right) & \\ = \frac{A^*(z)-1}{z-1}\left\{\vartheta_{0,0}(0)+\sum_{i=0}^{N}\sum_{j=1}^{b}\left(\vartheta_{i,j}(0)+\pi_{i,j}(0)\right)\right\}. & \end{aligned}$$

Taking limit as $z\to 1$ and using the normalization condition

$$\vartheta_{0,0}+\sum_{i=0}^{N}\sum_{j=1}^{b}\left(\vartheta_{i,j}+\pi_{i,j}\right)=1, \tag{3.24}$$

we get the desired result. □

3.1. Relation between queue length distributions at arbitrary pre-arrival epochs

In order to obtain the relations between queue length distribution at arbitrary and pre-arrival epochs we first connect pre-arrival epoch probabilities $\vartheta^-_{0,0},\ \vartheta^-_{i,j}$ and $\pi^-_{i,j}$ with the rate probabilities $\vartheta_{0,0}(0),\ \vartheta_{i,j}(0),\ 0 \le i \le N,\ 1 \le j \le b$ and $\pi_{i,j}(0),\ 0 \le i \le N,\ 1 \le j \le b$ as follows:

$$\vartheta^-_{0,0} = \vartheta_{0,0}(0)/\lambda; \qquad \vartheta^-_{i,j} = \vartheta_{i,j}(0)/\lambda,\ 0 \le i \le N,\ 1 \le j \le b;$$
$$\pi^-_{i,j} = \pi_{i,j}(0)/\lambda, 0 \le i \le N,\ 1 \le j \le b, \quad (3.25)$$

where λ is given by (3.23). Our main objective is to obtain the distribution of number of customers in the queue at an arbitrary epoch when the server is on vacation, $\vartheta_{0,0},\ \vartheta_{i,j}$ $(0 \le i \le N,\ 1 \le j \le b)$ or busy $\pi_{i,j}$ $(0 \le i \le N,\ 1 \le j \le b)$. This is discussed in the following theorem.

Theorem 3.2. *The relation between the pre-arrival epoch probabilities* $\left\{\vartheta^-_{i,j},\ \pi^-_{i,j}\right\}$ *and arbitrary epoch probabilities* $\left\{\vartheta_{i,j},\ \pi_{i,j}\right\}$ *are given by*

$$\vartheta_{N,j} = \frac{\lambda\bar{\theta}\bar{\eta}}{1-\bar{\theta}\bar{\eta}}\vartheta^-_{N-1,j},\ 1 \le j \le b, \quad (3.26)$$

$$\vartheta_{i,j} = \frac{\lambda\bar{\theta}\bar{\eta}}{1-\bar{\theta}\bar{\eta}}\left(\vartheta^-_{i-1,j} - \vartheta^-_{i,j}\right),\ 1 \le i \le N-1,\ 1 \le j \le b-1, \quad (3.27)$$

$$\vartheta_{i,b} = \frac{\lambda\bar{\theta}\bar{\eta}}{1-\bar{\theta}\bar{\eta}}\left(\vartheta^-_{i-1,b} - \vartheta^-_{i,b}\right),\ N-b+1 \le i \le N-1, \quad (3.28)$$

$$\vartheta_{i,b} = \frac{\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\left[\lambda\left(\bar{\eta}(\vartheta^-_{i-1,b} - \vartheta^-_{i,b}) + \eta\sum_{k=1}^{b}(\vartheta^-_{i+b-1,k} - I(i \ne N-b)\vartheta^-_{i+b,k})\right)\right.$$
$$\left. + \eta\sum_{k=1}^{b}\vartheta_{i+b,k}\right],\ 1 \le i \le N-b, \quad (3.29)$$

$$\vartheta_{0,j} = \frac{\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\left[\lambda\left(\eta\sum_{k=1}^{b}(\vartheta^-_{j-1,k} - \vartheta^-_{j,k}) + \bar{\eta}(I(j=1)\vartheta^-_{0,0} - \vartheta^-_{0,j})\right) + \eta\sum_{k=1}^{b}\vartheta_{j,k}\right],$$
$$1 \le j \le b, \quad (3.30)$$

$$\pi_{N,j} = \frac{1}{\mu}\left[\bar{\mu}\lambda\pi^-_{N-1,j} + \theta\bar{\eta}\left(\vartheta_{N,j} + \lambda\vartheta^-_{N-1,j}\right)\right],\ 1 \le j \le b, \quad (3.31)$$

$$\pi_{i,j} = \frac{1}{\mu}\Big[\bar{\mu}\lambda(\pi^-_{i-1,j} - \pi^-_{i,j}) + \theta\bar{\eta}\left(\vartheta_{i,j} + \lambda(\vartheta^-_{i-1,j} - \vartheta^-_{i,j})\right)\Big],\ 1 \le i \le N-1,$$
$$1 \le j \le b-1, \tag{3.32}$$

$$\pi_{i,b} = \frac{1}{\mu}\Big[\bar{\mu}\lambda\left(\pi^-_{i-1,b} - \pi^-_{i,b}\right) + \theta\bar{\eta}\left(\vartheta_{i,b} + \lambda(\vartheta^-_{i-1,b} - \vartheta^-_{i,b})\right)\Big],$$
$$N-b+1 \le i \le N-1, \tag{3.33}$$

$$\begin{aligned}\pi_{i,b} = {} & \sum_{k=1}^{b}\left(\pi_{i+b,k} + \lambda\left(\pi_{i+b-1,k} - I(i \neq N-b)\pi^-_{i+b,k}\right)\right) + \frac{\lambda\bar{\mu}}{\mu}\Big[\pi^-_{i-1,b} - \pi^-_{i,b}\Big] \\ & + \frac{\theta}{\mu}\Big[\bar{\eta}\big(\vartheta_{i,b} + \lambda\big(\vartheta^-_{i-1,b} - \vartheta^-_{i,b}\big)\big) \\ & + \eta\sum_{k=1}^{b}\left(\vartheta_{i+b,k} + \lambda\big(\vartheta^-_{i+b-1,k} - I(i \neq N-b)\vartheta^-_{i+b,k}\big)\right)\Big],\ 1 \le i \le N-b,\end{aligned} \tag{3.34}$$

$$\begin{aligned}\pi_{0,j} = {} & \sum_{k=1}^{b}\left(\pi_{j,k} + \lambda\left(\pi^-_{j-1,k} - \pi^-_{j,k}\right)\right) - \frac{\lambda\bar{\mu}}{\mu}\pi^-_{0,j} \\ & + \frac{\theta}{\mu}\Big[\bar{\eta}\big(\vartheta_{0,j} + \lambda\big(I(j=1)\vartheta^-_{0,0} - \vartheta^-_{0,j}\big)\big) + \eta\sum_{k=1}^{b}\left(\vartheta_{j,k} + \lambda\big(\vartheta^-_{j-1,k} - \vartheta^-_{j,k}\big)\right)\Big], \\ & 1 \le j \le b,\end{aligned} \tag{3.35}$$

$$\vartheta_{0,0} = 1 - \left(\sum_{i=0}^{N}\sum_{j=1}^{b}(\vartheta_{i,j} + \pi_{i,j})\right). \tag{3.36}$$

Proof. Setting $z = 1$ in (3.13) to (3.22) and using (3.25), we get $\vartheta_{i,j}$, $(0 \le i \le N,\ 1 \le j \le b)$ and $\pi_{i,j}$, $(0 \le i \le N,\ 1 \le j \le b)$. Finally the only unknown $\vartheta_{0,0}$ is obtained using the normalization condition (3.24). $\square$

3.2. Outside observer's distribution

Steady-state queue length distribution at an outside observer's observation epoch plays an important role in evaluating various performance measures of the model. In EAS, an outside observer's observation epoch falls in a time interval after a potential arrival and before a potential departure. Let $\vartheta^o_{0,0}$, $\vartheta^o_{i,j}$, $\pi^o_{i,j}$ $(0 \le i \le N,\ 1 \le j \le b)$ denote the probabilities that an outside observer finds the server idle during vacation, i customers in the queue and j customers in service when the server is in WV and regular service period, respectively. These probabilities can be obtained by observing the arbitrary and outside observer's observation epochs presented in Figure 1 and are given by

$$\vartheta^o_{N,j} = \frac{1}{\bar{\theta}\bar{\eta}}\vartheta_{N,j}, \quad 1 \le j \le b,$$

$$\vartheta^o_{i,j} = \frac{1}{\bar{\theta}\bar{\eta}}\vartheta_{i,j}, \quad 1 \le i \le N-1,\ 1 \le j \le b-1,$$

$$\vartheta^o_{i,b} = \frac{1}{\bar{\theta}\bar{\eta}}\vartheta_{i,b}, \quad N-b+1 \le i \le N-1,$$

$$\vartheta^o_{i,b} = \frac{1}{\bar{\theta}\bar{\eta}}\left(\vartheta_{i,b} - \bar{\theta}\eta\sum_{k=1}^{b}\vartheta^o_{i+b,k}\right), \quad 1 \le i \le N-b,$$

$$\vartheta^o_{0,j} = \frac{1}{\bar{\theta}\bar{\eta}}\left(\vartheta_{0,j} - \bar{\theta}\eta\sum_{k=1}^{b}\vartheta^o_{j,k}\right), \quad 1 \le j \le b,$$

$$\pi^o_{N,j} = \frac{1}{\bar{\mu}}\left(\pi_{N,j} - \theta\bar{\eta}\vartheta^o_{N,j}\right),\ 1 \le j \le b,$$

$$\pi^o_{i,j} = \frac{1}{\bar{\mu}}\left(\pi_{i,j} - \theta\bar{\eta}\vartheta^o_{i,j}\right),\ 1 \le i \le N-1,\ 1 \le j \le b-1,$$

$$\pi^o_{i,b} = \frac{1}{\bar{\mu}}\left(\pi_{i,b} - \theta\bar{\eta}\vartheta^o_{i,b}\right),\ N-b+1 \le i \le N-1,$$

$$\pi^o_{i,b} = \frac{1}{\bar{\mu}}\left(\pi_{i,b} - \mu\sum_{k=1}^{b}\pi^o_{i+b,k} - \theta\left(\bar{\eta}\vartheta^o_{i,b} + \eta\sum_{k=1}^{b}\vartheta^o_{i+b,k}\right)\right),\ 1 \le i \le N-b,$$

$$\pi^o_{0,j} = \frac{1}{\bar{\mu}}\left(\pi_{0,j} - \mu\sum_{k=1}^{b}\pi^o_{j,k} - \theta\left(\bar{\eta}\vartheta^o_{0,j} + \eta\sum_{k=1}^{b}\vartheta^o_{j,k}\right)\right),\ 1 \le j \le b,$$

$$\vartheta^o_{0,0} = \vartheta_{0,0} - \mu\sum_{k=1}^{b}\pi^o_{0,k} - \eta\sum_{k=1}^{b}\vartheta^o_{0,k}.$$

4. Imbedded Markov chain analysis

In this section, we obtain the queue length distribution at pre-arrival epoch. Let f_n $(n \ge 0)$ and g_n $(n \ge 0)$ be the conditional probabilities that n customers have been served during an inter-arrival time in the regular service period and in a WV period, respectively. Let h_n $(n \ge 0)$ be the conditional probability that there are n service completions during an inter-arrival time when the WV terminates and the server enters into the regular service period. Hence, for all $n \ge 0$, we have

$$f_n = \sum_{i=n}^{\infty} a_i \binom{i}{n}\mu^n\bar{\mu}^{i-n}, \quad g_n = \sum_{i=n}^{\infty} a_i\bar{\theta}^i\binom{i}{n}\eta^n\bar{\eta}^{i-n},$$

$$h_n = \sum_{u=\max(1,n)}^{\infty} a_u \sum_{j=0}^{n}\sum_{k=\max(1,j)}^{u}\theta\bar{\theta}^{k-1}\binom{k}{j}\eta^j\bar{\eta}^{k-j}\binom{u-k}{n-j}\mu^{n-j}\bar{\mu}^{u-k-n+j}.$$

The probability generating functions of f_n, g_n and h_n are given by

$$F(z) = \sum_{i=0}^{\infty} f_i z^i = A^*(\bar{\mu} + \mu z), \quad G(z) = \sum_{i=0}^{\infty} g_i z^i = A^* \left\{ \bar{\theta}(\bar{\eta} + \eta z) \right\},$$
$$H(z) = \sum_{i=0}^{\infty} h_i z^i = \frac{\theta(\bar{\eta} + \eta z)\Big[A^*(\bar{\mu} + \mu z) - A^* \left\{ \bar{\theta}(\bar{\eta} + \eta z) \right\}\Big]}{\bar{\mu} + \mu z - \bar{\theta}(\bar{\eta} + \eta z)}.$$

Let $t_0, t_1, \ldots$ be the time epochs at which an arrival occurs and t_n^- denote the time epoch just before the arrival instant t_n. The inter-arrival times $T_{n+1} = t_{n+1} - t_n$, $n = 0, 1, 2, \ldots$ are i.i.d. r.v.s with common distribution function $A(u)$. The state of the system at t_i^- is defined as $\{N_q(t_i^-),\ N_s(t_i^-),\ \xi(t_i^-)\}$, where $N_q(t_i^-),\ N_s(t_i^-)$ denote the number of customers in the queue and in service, respectively, and $\xi(t_i^-)$ indicates whether the server is in WV ($\xi(t_i^-) = 0$) or busy with regular service ($\xi(t_i^-) = 1$). The process $\{N_q(t_i^-), N_s(t_i^-), \xi(t_i^-)\}$ forms an imbedded Markov chain with finite state space $\Omega = \{(0,0,0)\} \cup \{(i,j,k);\ 0 \le i \le N,\ 1 \le j \le b,\ k = 0, 1\}$. At steady-state assume that

$$\begin{aligned}
\vartheta_{0,0}^- &= \lim_{n\to\infty} \Pr\{N_q(t_n^-) = 0,\ N_s(t_n^-) = 0, \\
&\qquad\qquad \xi(t_n^-) = 0\}, \\
\vartheta_{i,j}^- &= \lim_{n\to\infty} \Pr\{N_q(t_n^-) = i,\ N_s(t_n^-) = j,\ \xi(t_n^-) = 0\},\ 0 \le i \le N, \\
&\qquad\qquad 1 \le j \le b, \\
\pi_{i,j}^- &= \lim_{n\to\infty} \Pr\{N_q(t_n^-) = i,\ N_s(t_n^-) = j,\ \xi(t_n^-) = 1\},\ 0 \le i \le N, \\
&\qquad\qquad 1 \le j \le b,
\end{aligned}$$

where $\vartheta_{i,j}^-$ ($\pi_{i,j}^-$) represents the probability of i customers in the queue prior to an arrival epoch when the server is busy with a batch of j customers in WV period (regular service period). Observing the state of the system at two consecutive imbedded points, we have the one step transition probability matrix (TPM) $\boldsymbol{\mathcal{P}}$ of dimension $(2b(N+1)+1) \times (2b(N+1)+1)$ with four block matrices of the form

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} \boldsymbol{\Delta}_{((N+1)b+1)\times((N+1)b+1)} & \boldsymbol{\Phi}_{((N+1)b+1)\times((N+1)b)} \\ \boldsymbol{\Lambda}_{((N+1)b)\times((N+1)b+1)} & \boldsymbol{\Upsilon}_{((N+1)b)\times((N+1)b)} \end{bmatrix}.$$

The block $\boldsymbol{\Delta}$ refers to the transitions from vacation state to vacation state and the block $\boldsymbol{\Phi}$ refers to the transitions from vacation state to service state of the TPM.

The elements of these blocks can be obtained from the following expressions:

$$\mathbf{\Delta}_{(k,i)(m,j)} = \begin{cases} g_{\left(\frac{k+1-m}{b}\right)} & : \ b \le k \le N-1,\ 1 \le m \le N-b,\ j=b,\ 1 \le i \le b, \\ & \quad (k+1) \ge (m+j),\ \frac{k+1-m}{b} \text{ is an integer}, \\ g_{\left[\frac{k}{b}\right]+1} & : \ 0 \le k \le N-1,\ m=0,\ 1 \le i,\ j \le b, j = \lfloor k/b \rfloor + 1, \\ g_0 & : \ 0 \le k \le N-1,\ 1 \le m \le N, 1 \le i, j \le b,\ i=j, \\ & \quad k+1=m, \\ g_0 & : \ k=m=i=0,\ j=1, \\ \mathbf{\Delta}_{(k-1,i)(m,j)} & : \ k=N,\ 0 \le m \le N,\ 1 \le i,\ j \le b, \\ \psi(k,i) & : \ k=m=i=j=0, \\ 0 & : \ \text{otherwise}, \end{cases}$$

$$\mathbf{\Phi}_{(k,i)(m,j)} = \begin{cases} h_{\left(\frac{k+1-m}{b}\right)} & : \ b \le k \le N-1,\ 1 \le m \le N-b,\ j=b,\ 1 \le i \le b, \\ & \quad (k+1) \ge (m+j),\ \frac{k+1-m}{b} \text{ is an integer}, \\ h_{\left[\frac{k}{b}\right]+1} & : \ 0 \le k \le N-1,\ m=0,\ 1 \le i,\ j \le b, j = \lfloor k/b \rfloor + 1, \\ h_0 & : \ 0 \le k \le N-1,\ 1 \le m \le N,\ 1 \le i, j \le b,\ i=j, \\ & \quad k+1=m, \\ h_0 & : \ k=m=i=0, j=1, \\ \mathbf{\Phi}_{(k-1,i)(m,j)} & : \ k=N,\ 0 \le m \le N,\ 1 \le i,\ j \le b, \\ 0 & : \ \text{otherwise}, \end{cases}$$

where $\psi(k,i) = 1 - \sum_{m=0}^{N} \sum_{j=1}^{b} \big(\mathbf{\Delta}_{(k,i)(m,j)} + \mathbf{\Phi}_{(k,i)(m,j)}\big)$.

The block $\mathbf{\Lambda}$ refers to the transitions from service state to vacation state and the block $\mathbf{\Upsilon}$ refers to the transitions from busy state to busy state. The structure of these blocks is as follows:

$$\mathbf{\Lambda}_{(k,i)(m,j)} = \begin{cases} \varphi_{k,i} & : \ 0 \le k \le N-1,\ 1 \le i \le b,\ m=j=0, \\ \mathbf{\Lambda}_{(k-1,i)(m,j)} & : \ k=N,\ 0 \le m \le N,\ 1 \le i,\ j \le b, \\ 0 & : \ \text{otherwise}, \end{cases}$$

$$\mathbf{\Upsilon}_{(k,i)(m,j)} = \begin{cases} f_{\left(\frac{k+1-m-b}{b}\right)} & : \ b \le k \le N-1,\ 1 \le m \le N-b,\ j=b,\ 1 \le i \le b, \\ & \quad (k+1) \ge (m+j),\ \frac{k+1-m-b}{b} \text{ is an integer}, \\ f_{\left[\frac{k}{b}\right]+1} & : \ 0 \le k \le N-1,\ m=0,\ 1 \le i,\ j \le b, j = \lfloor k/b \rfloor + 1, \\ f_0 & : \ 0 \le k \le N-1,\ 1 \le m \le N,\ i=j,\ k+1=m, \\ \mathbf{\Upsilon}_{(k-1,i)(m,j)} & : \ k=N,\ 0 \le m \le N,\ 1 \le i,\ j \le b, \\ 0 & : \ \text{otherwise}, \end{cases}$$

where $\varphi(k,i) = 1 - \sum_{m=0}^{N} \sum_{j=1}^{b} \mathbf{\Upsilon}_{(k,i)(m,j)}$.

One may note that here $[x]$ and $\lfloor x/y \rfloor$ denote the greatest integer contained in x and remainder of x when both x and y are integers, respectively.

Now the pre-arrival epoch probabilities $\vartheta^-_{0,0},\ \vartheta^-_{i,j},\ (0 \le i \le N,\ 1 \le j \le b)$ and $\pi^-_{i,j}\ (0 \le i \le N,\ 1 \le j \le b)$ can be obtained by solving the system of equations

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}\boldsymbol{\mathcal{P}},\ \boldsymbol{\Pi}\mathbf{e} = 1,$$

where $\boldsymbol{\Pi} = (\vartheta^-_{0,0},\ \vartheta^-_{0,1},\dots,\vartheta^-_{0,b},\vartheta^-_{1,1},\dots,\vartheta^-_{1,b},\dots,\ \vartheta^-_{N,1},\dots,\vartheta_{N,b},\ \pi^-_{0,1},\dots,\pi^-_{0,b},$ $\pi^-_{1,1},\dots,\pi^-_{1,b},\dots,\ \pi^-_{N,1},\dots,\pi_{N,b})$ and $\mathbf{e}$ is $(2b(N+1)+1)$ dimensional column vector with all its components being unity. For solving such system of equations we have used the GTH algorithm given in Latouche and Ramaswami [10].

Remark 4.1. By taking batch size $b = 1$, our model reduces to $GI/Geo/1$ queue with MWV. In this case, the joint probabilities $\vartheta_{i,j},\ \pi_{i,j}\ (0 \le i \le N,\ 1 \le j \le b)$ become $\vartheta_{i,1},\ \pi_{i,1}\ (0 \le i \le N)$. To obtain the relations between pre-arrival and arbitrary epoch probabilities, let us define $P_{0,0} = \vartheta_{0,0},\ P_{i+1,0} = \vartheta_{i,1},\ 0 \le i \le N$ and $P_{i+1,1} = \pi_{i,1},\ (0 \le i \le N)$ where $P_{i+1,0}\ (P_{i+1,1})$ represents the probability of $i+1$ customers in the system when the server in WV (regular service) period. The equations (3.27), (3.28), (3.32) and (3.33) do not exist and the remaining equations reduce to

$$\begin{aligned}
P_{N+1,0} =& \frac{\lambda\bar{\theta}\bar{\eta}}{\theta+\eta-\theta\eta} P^-_{N,0},\\
P_{i,0} =& \frac{\bar{\theta}}{\theta+\eta-\theta\eta}\big[\eta P_{i+1,0} + \lambda\eta P^-_{i,0} - \lambda\bar{\eta}P^-_{i,0} + \lambda\bar{\eta}P^-_{i-1,0} - I(i\neq N)\lambda\eta P^-_{i+1,0}\big],\\
& i = N,\ N-1,\dots,1,\\
P_{N+1,1} =& \frac{\theta\bar{\eta}}{\mu}P_{N+1,0} + \frac{\lambda}{\mu}\big[\bar{\mu}P^-_{N,1} + \theta\bar{\eta}P^-_{N,0}\big],\\
P_{i,1} =& P_{i+1,1} + \frac{\lambda}{\mu}\big[\mu P^-_{i,1} + I(i\neq 1)\bar{\mu}P^-_{i-1,1} + \theta\eta P^-_{i,0} + \theta\bar{\eta}P^-_{i-1,0} - \bar{\mu}P^-_{i,1}\\
& - I(i\neq N)\left(\mu P^-_{i+1,1} + \theta\eta P^-_{i+1,0}\right) - \theta\bar{\eta}P^-_{i,0}\big]\\
& + \frac{\theta}{\mu}\big[\bar{\eta}P_{i,0} + \eta P_{i+1,0}\big], i = N,\ N-1,\dots,1,\\
P_{0,0} =& 1 - \sum_{i=1}^{N+1}(P_{i,0} + P_{i,1}).
\end{aligned}$$

The above results match with Goswami and Mund [3] by taking the number in system as $N+1$ in their paper.

Remark 4.2. $\eta = 0$, that is, the model reduces to $GI/Geo^{[b]}/1$ queue with MV and our results match numerically with those of Goswami and Vijaya Laxmi [4].

5. Performance measures

Once the distribution of number of customers in the queue at different epochs is known, various performance measures of the model can be evaluated. The expected queue length (L_q^o) at an outside observer's observation epoch, the expected queue length (L_q) at an arbitrary epoch are given by

$$L_q^o = \sum_{i=1}^{N}\sum_{j=1}^{b} i\left(\vartheta_{i,j}^o + \pi_{i,j}^o\right), \quad L_q = \sum_{i=1}^{N}\sum_{j=1}^{b} i\left(\vartheta_{i,j} + \pi_{i,j}\right).$$

The probability of blocking or loss is given by $P_{\text{loss}} = \sum_{j=1}^{b}\left(\vartheta_{N,j}^- + \pi_{N,j}^-\right)$. Using Little's rule the expected waiting time in the queue (W_q) of a customer is given by $W_q = L_q^o/\lambda^{'}$, where $\lambda^{'} = \lambda(1 - P_{\text{loss}})$ is the effective arrival rate.

5.1. Waiting time analysis

In this section, we obtain waiting time (in the queue) distribution (measured in slots) of an arriving customer under first come first served (FCFS) discipline. Let us define W_{qa} and $W_{qa}^*(z)$ as the steady-state waiting time and its p.g.f., respectively. An arriving customer may observe the system in any one of the following cases.

Case 1. The server is in WV and $nb + j,\ (0 \le n \le m-1,\ 0 \le j \le b-1)$, where $m = [N/b]$, customers are in the queue. Then, on arrival a customer waits for the service completion of $(n+1)$ batches. The server serves the customers during WV and/or service period.

Case 2. The server is in WV and $nb+j,\ (n = m,\ 0 \le j \le N-1-mb)$, customers are in the queue. Then, on arrival a customer waits for the service completion of $(m+1)$ batches. The server serves the customers during WV and/or service period.

Case 3. The server is busy and $nb+j,\ (0 \le n \le m-1,\ 0 \le j \le b-1)$ customers are in the queue. On arrival a customer waits for the service completion of $(n+1)$ groups.

Case 4. The server is busy and $nb + j,\ (n = m,\ 0 \le j \le N-1-mb)$, customers are in the queue. An arriving customer waits for the service completion of $(m+1)$ groups.

Combining all the above cases, the p.g.f of the waiting time is obtained as

$$W_{qa}^*(z) = \frac{1}{1-P_{\text{loss}}}\left[\sum_{n=1}^{m-1}\left(\frac{\mu z}{1-\bar{\mu}z}\right)^{n+1}\sum_{i=1}^{b}\sum_{j=0}^{b-1}\pi_{nb+j,i}^-\right.$$
$$+\left(\frac{\mu z}{1-\bar{\mu}z}\right)^{m+1}\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\pi_{mb+j,i}^-$$

$$+\sum_{n=0}^{m-1}\frac{1}{\bar{\theta}}\left(\frac{\eta\bar{\theta}z}{1-\bar{\theta}\bar{\eta}z}\right)^{n+1}\sum_{i=1}^{b}\sum_{j=0}^{b-1}\vartheta^{-}_{nb+j,i}+\frac{1}{\bar{\theta}}\left(\frac{\eta\bar{\theta}z}{1-\bar{\theta}\bar{\eta}z}\right)^{m+1}$$

$$\times\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\vartheta^{-}_{mb+j,i}$$

$$+\sum_{n=0}^{m-1}\left(\frac{\theta\bar{\eta}z}{1-\bar{\theta}\bar{\eta}z}\right)\left(\frac{\mu z}{1-\bar{\mu}z}\right)^{n+1}\sum_{i=1}^{b}\sum_{j=0}^{b-1}\vartheta^{-}_{nb+j,i}$$

$$+\left(\frac{\theta\bar{\eta}z}{1-\bar{\theta}\bar{\eta}z}\right)\left(\frac{\mu z}{1-\bar{\mu}z}\right)^{m+1}\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\vartheta^{-}_{mb+j,i}$$

$$+\sum_{n=1}^{m-1}\sum_{k=1}^{n}\frac{1}{\bar{\theta}}\left(\frac{\eta\bar{\theta}z}{1-\bar{\theta}\bar{\eta}z}\right)^{k}\left(\frac{\theta}{1-\bar{\theta}\bar{\eta}z}\right)\left(\frac{\mu z}{1-\bar{\mu}z}\right)^{n+1-k}$$

$$\times\sum_{i=1}^{b}\sum_{j=0}^{b-1}\vartheta^{-}_{nb+j,i}$$

$$+\sum_{k=1}^{m}\frac{1}{\bar{\theta}}\left(\frac{\eta\bar{\theta}z}{1-\bar{\theta}\bar{\eta}z}\right)^{k}\left(\frac{\theta}{1-\bar{\theta}\bar{\eta}z}\right)\left(\frac{\mu z}{1-\bar{\mu}z}\right)^{m+1-k}$$

$$\left.\times\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\vartheta^{-}_{mb+j,i}\right].$$

Therefore, the expected waiting time in the queue is given by

$$W_{qa}=\frac{1}{1-P_{\text{loss}}}\left[\sum_{n=1}^{m-1}\left(\frac{n+1}{\mu}\right)\sum_{i=1}^{b}\sum_{j=0}^{b-1}\pi^{-}_{nb+j,i}+\left(\frac{m+1}{\mu}\right)\right.$$

$$\times\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\pi^{-}_{mb+j,i}+\sum_{n=0}^{m-1}\left(\frac{n+1}{\eta\bar{\theta}^{2}}\right)\left(\frac{\eta\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\right)^{n+2}$$

$$\times\sum_{i=1}^{b}\sum_{j=0}^{b-1}\vartheta^{-}_{nb+j,i}+\left(\frac{m+1}{\eta\bar{\theta}^{2}}\right)\left(\frac{\eta\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\right)^{m+2}\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\vartheta^{-}_{mb+j,i}$$

$$+\sum_{n=0}^{m-1}\left\{\frac{\theta\bar{\eta}}{\left(1-\bar{\theta}\bar{\eta}\right)^{2}}+\left(\frac{\theta\bar{\eta}}{1-\bar{\theta}\bar{\eta}}\right)\left(\frac{n+1}{\mu}\right)\right\}\sum_{i=1}^{b}\sum_{j=0}^{b-1}\vartheta^{-}_{nb+j,i}$$

$$+\left\{\frac{\theta\bar{\eta}}{\left(1-\bar{\theta}\bar{\eta}\right)^{2}}+\left(\frac{\theta\bar{\eta}}{1-\bar{\theta}\bar{\eta}}\right)\left(\frac{m+1}{\mu}\right)\right\}\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\vartheta^{-}_{mb+j,i}$$

$$+\sum_{n=1}^{m-1}\sum_{k=1}^{n}\left\{\frac{k(\eta\bar{\theta})^{k}\theta}{\bar{\theta}(1-\bar{\theta}\bar{\eta})^{k+2}}+\left(\frac{\eta\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\right)^{k}\frac{\theta\bar{\eta}}{(1-\bar{\theta}\bar{\eta})^{2}}+\left(\frac{\eta\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\right)^{k}\right.$$

$$\times\left(\frac{\theta}{1-\bar{\theta}\bar{\eta}}\right)\left(\frac{n+1-k}{\mu\bar{\theta}}\right)\Bigg\}\sum_{i=1}^{b}\sum_{j=0}^{b-1}\vartheta^{-}_{nb+j,i}$$
$$+\sum_{k=1}^{m}\left\{\frac{k(\eta\bar{\theta})^{k}\theta}{\bar{\theta}(1-\bar{\theta}\bar{\eta})^{k+2}}+\left(\frac{\eta\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\right)^{k}\frac{\theta\bar{\eta}}{(1-\bar{\theta}\bar{\eta})^{2}}\right.$$
$$\left.+\left(\frac{\eta\bar{\theta}}{1-\bar{\theta}\bar{\eta}}\right)^{k}\left(\frac{\theta}{1-\bar{\theta}\bar{\eta}}\right)\left(\frac{m+1-k}{\mu\bar{\theta}}\right)\right\}\sum_{i=1}^{b}\sum_{j=0}^{N-mb-1}\vartheta^{-}_{mb+j,i}\Bigg]. \tag{5.1}$$

It may be noted that the expected waiting time in the queue W_{qa} obtained from (5.1) matches numerically with the waiting time W_q obtained using Little's rule.

6. Numerical results and discussions

To demonstrate the applicability of the formulae obtained in the previous sections, we present some numerical results in the form of table and graphs. We fix the capacity of the system as $N = 15$ and batch size as $b = 2$. The various parameters of the model are chosen to be $\lambda = 0.2,\ \mu = 0.3,\ \eta = 0.2$ and $\theta = 0.1$, unless they are considered as variables or their values are mentioned in the respective table and figures. All the calculations have been done on Mathematica software package. Distribution of number of customers in the queue at various epochs for geometric inter-arrival time distribution is given in Table 1. It can be seen from the table that the pre-arrival and arbitrary epoch probabilities are same due to Bernoulli arrivals.

Figure 2 illustrates the dependence of blocking probability (P_{loss}) on traffic intensity (ρ) for various inter-arrival time distributions. The inter-arrival time distributions are taken as geometric, deterministic ($a_5 = 1.0$) and arbitrary ($a_1 = 0.3,\ a_4 = 0.4,\ a_9 = 0.2,\ a_{13} = 0.1$) with same mean $\lambda = 0.2$. As one would intuitively expect, it is observed that the blocking probability increases as traffic load increases for any inter-arrival time distribution. Further, deterministic inter-arrival time distribution minimizes the blocking probability as shown in the proof of a folk theorem on queueing delay by Hajek [7].

The effect of η on expected queue length (L_q) for different values of vacation parameter (θ) when inter-arrival times are geometric is presented in Figure 3. For fixed θ, L_q decreases as η increases, which is consistent with our intuition that larger the value of η, more the number of service completions. Further, it can be observed that for $\eta \le \mu$ the expected queue lengths decrease with the increase of θ and when η crosses μ this trend is reversed. This strengthens our choice of the value of η being less than μ.

Figure 4 depicts the effect of ρ on the expected queue length (L_q) for $Geo/Geo^{[2]}/1/15$ queue with MWV and MV. It is observed that as ρ increases L_q increases in both the models. Further, the expected queue lengths are lower in the case of MWV model when compared to MV model. This shows that MWV models perform better than MV models.

TABLE 1. Queue length distribution at various epochs when inter-arrival time is geometric.

i	$\sum_{j=1}^{b} \vartheta_{i,j}^{-}$	$\sum_{j=1}^{b} \pi_{i,j}^{-}$	$\sum_{j=1}^{b} \vartheta_{i,j}$	$\sum_{j=1}^{b} \pi_{i,j}$	$\sum_{j=1}^{b} \vartheta_{i,j}^{o}$	$\sum_{j=1}^{b} \pi_{i,j}^{o}$
0	0.193531	0.141781	0.193531	0.141781	0.236065	0.113425
1	0.071230	0.077665	0.071230	0.077665	0.095690	0.090488
2	0.026217	0.037807	0.026217	0.037807	0.035219	0.045779
3	0.009649	0.017252	0.009649	0.017252	0.012962	0.021363
4	0.003551	0.007557	0.003551	0.007557	0.004771	0.009496
5	0.001307	0.003218	0.001307	0.003218	0.001756	0.004086
6	0.000481	0.001342	0.000481	0.001342	0.000646	0.001718
7	0.000177	0.000551	0.000177	0.000551	0.000237	0.000709
8	0.000065	0.000223	0.000065	0.000223	0.000087	0.000289
9	0.000023	0.000089	0.000023	0.000089	0.000032	0.000116
10	0.000008	0.000035	0.000008	0.000035	0.000011	0.000046
11	0.000003	0.000014	0.000003	0.000014	0.000004	0.000018
12	0.000001	0.000005	0.000001	0.000005	0.000001	0.000007
13	0.000000	0.000002	0.000000	0.000002	0.000000	0.000002
14	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
15	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
Sum	0.712450	0.287550	0.712450	0.287550	0.712450	0.287550

$\vartheta_{0,0}^{-} = 0.406203, \ \vartheta_{0,0} = 0.406203, \ \vartheta_{0,0}^{o} = 0.324962$

$L_q^o = 0.563628, \ L_q = 0.444869, \ W_q = 2.81814, \ W_{qa} = 2.81814.$

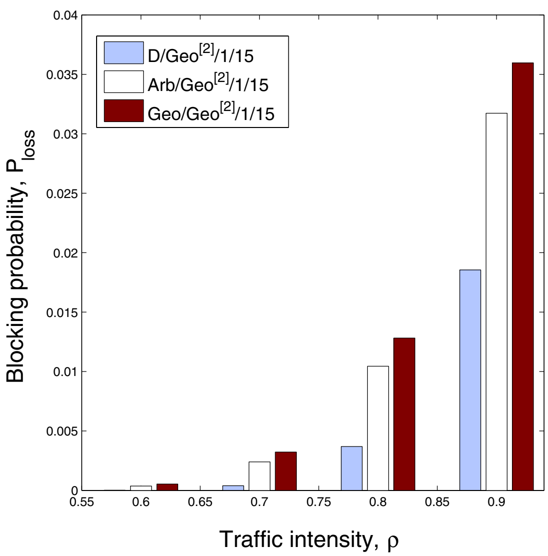

FIGURE 2. Effect of ρ on blocking probability.

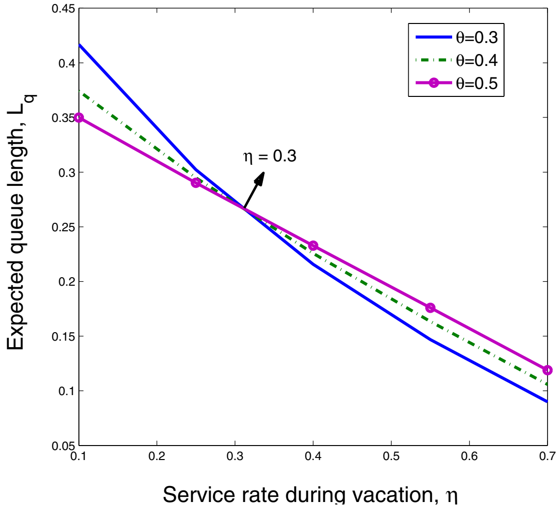

FIGURE 3. Effect of η on L_q.

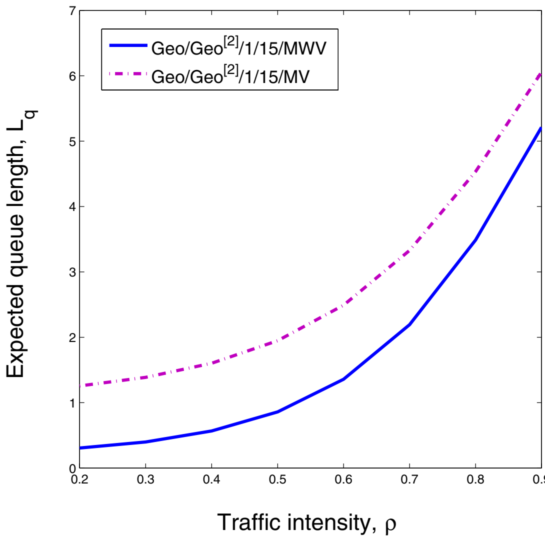

FIGURE 4. Effect of ρ on L_q.

Figure 5 illustrates the dependence of the blocking probability (P_{loss}) on the buffer size (N) varying from 10 to 15 and batch size (b) varying from 3 to 10 in a $Geo/Geo/1$ queue with MWV. We observe that for a fixed batch size (b) the blocking probability decreases as N increases. Further, for small values of N, P_{loss} decreases with increase of b and when N becomes large the effect of b on P_{loss} is

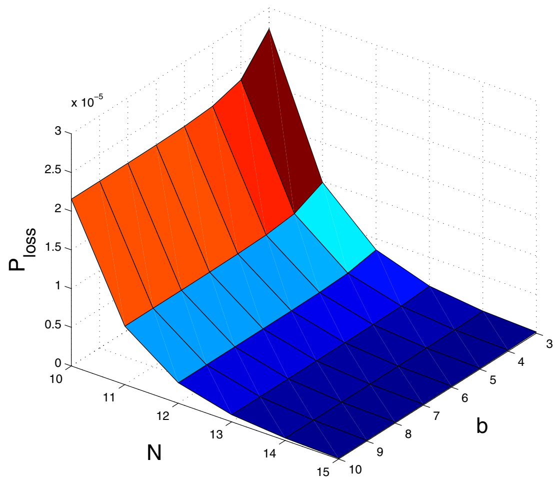

FIGURE 5. Impact on P_{loss} for various b and N.

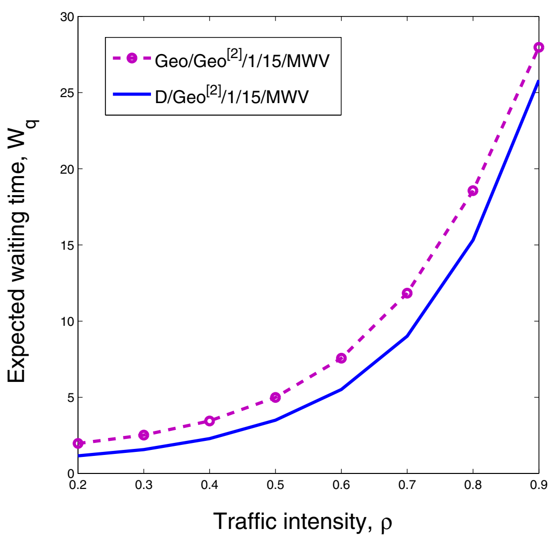

FIGURE 6. Effect of ρ on W_q.

insignificant. Hence, we can setup an admissible batch size and buffer capacity of the system in order to have lower blocking probabilities.

The effect of traffic load (ρ) on the expected waiting time in the queue (W_q) is shown in Figure 6 for geometric ($\lambda = 0.2$) and deterministic ($a_5 = 1.0$) inter-arrival time distributions with MWV. The other parameters of the model are $\mu = 0.3,\ \eta = 0.2$ and $\theta = 0.1$. As expected, the expected waiting time increases

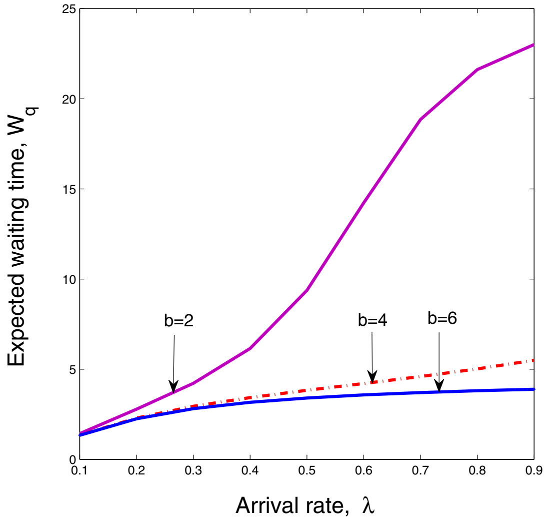

FIGURE 7. Effect of λ on W_q with varying b.

with the increase of ρ. Further, the expected waiting time is lower in the case of deterministic inter-arrival time distribution.

Figure 7 shows the effect of arrival rate (λ) on the expected waiting time in the queue (W_q) for various batch sizes b in a $Geo/Geo/1/15$/MWV queue. The other parameters of the model are same as in Figure 6. From the figure, one may observe that for fixed b, the expected waiting time increases as λ increases. But for fixed λ, as the batch size b increases the expected waiting time in the queue decreases due to the fact that larger the value of b, more the number of customers served in batches.

Figures 8 and 9, respectively, show the dependence of expected queue length (L_q) and expected waiting time (W_q) on N and b for $Geo/Geo/1$ queue with MWV. Observe that L_q and W_q decrease with the increase of b. This is due to the fact as the batch size (b) increases the number of customers served in batches increase resulting in the decrease of L_q and W_q. Further, the expected queue length and expected waiting time increase with the increase of buffer capacity (N). However, the effect of N on L_q and W_q is insignificant when compared to the effect of b.

7. CONCLUSIONS

In this paper, we have carried out an analysis of discrete-time finite buffer bulk service queue with MWV that has potential applications in many areas such as telecommunication systems, manufacturing systems, computer networks, *etc.*, where jobs are processed in batches. The inter-arrival times of customers are arbitrarily distributed while the service times during regular busy period, during WV

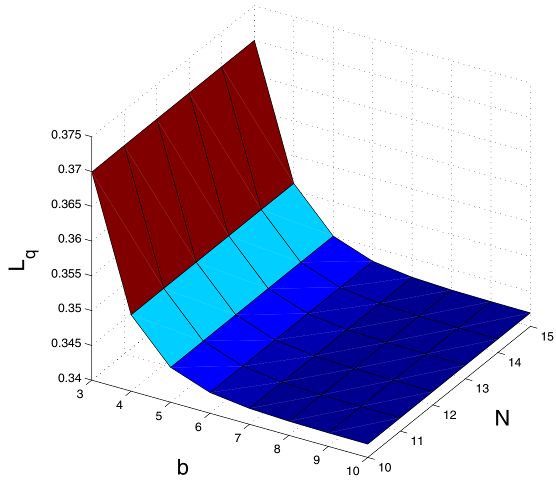

FIGURE 8. Impact of b and N on L_q.

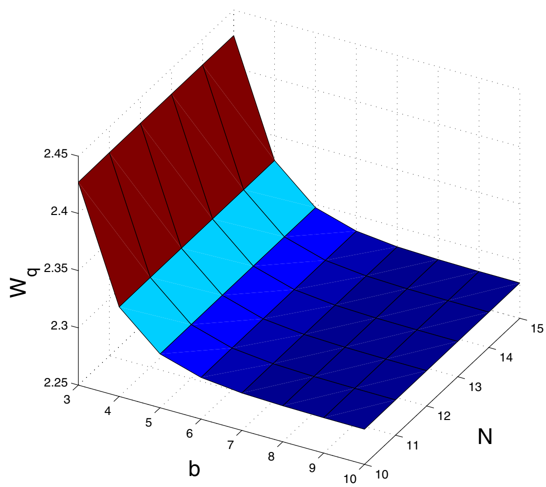

FIGURE 9. Impact of b and N on W_q.

and vacation times are geometrically distributed. We have obtained the queue length distributions at different time epochs using supplementary variable and imbedded Markov chain techniques. Utilizing these distributions, some important performance measures of the model such as expected queue length, blocking probability, *etc.*, have been derived. The analysis of actual waiting time in the queue is also carried out. Computational experiences are demonstrated with a variety of numerical results in the form table and graphs. Extension of our results to $DMAP/Geo^{[b]}/1/N$ and $GI^{[X]}/Geo^{[b]}/1/N$ queues with WV is left for future research.

Acknowledgements. The authors would like to thank the anonymous referees for their valuable comments and suggestions which have helped in improving the quality of the paper.

References

[1] H. Bruneel and B.G. Kim, *Discrete-time models for communication systems including ATM.* Kluwer Academic Publishers, Boston (1983).

[2] M.L. Chaudhry and J.G.C. Templeton, *A first course in bulk queues.* Wiley, New York (1983).

[3] V. Goswami and G.B. Mund, Analysis of discrete-time $GI/Geo/1/N$ queue with multiple working vacations. *J. Sys. Sci. Sys. Eng.* **19** (2011) 367–384.

[4] V. Goswami and P. Vijaya Laxmi, Analysis of finite-buffer discrete-time batch-service queue with multiple vacations. *Int. J. Inform. Manage. Sci.* **22** (2011) 291–310.

[5] A. Gravey and G. Hébuterne, Simultaneity in discrete-time single server queues with Bernoulli inputs. *Perform. Eval.* **14** (1992) 123–131.

[6] U.C. Gupta and V. Goswami, Performance analysis of finite buffer discrete-time queue with bulk service. *Comput. Oper. Res.* **29** (2002) 1331–1341.

[7] B. Hajek, The proof of a folk theorem on queueing delay with applications to routing in networks. *J. Assoc. Comput. Mach.* **30** (1983) 834-851.

[8] J.J. Hunter, *Mathematical techniques of applied probability. Volume 2. Discrete-time models: Techniques and applications.* Academic Press, New York (1983).

[9] C. Jiang, T. Yinghui and Y. Miaomiao, The discrete-time bulk service $Geo/Geo/1$ queue with multiple working vacations. *J. App. Math.* (2013) doi.org/10.1155/2013/587269.

[10] G. Latouche and V. Ramaswami, *Introdction to matrix analytic method in stochastic modelling.* SIAM and ASA, Philadelphia (1990).

[11] J. Li, N. Tian and W. Liu, Discrete-time $GI/Geo/1$ queue with multiple working vacations. *Queueing Sys.* **56** (2007) 53–63.

[12] J. Li and N. Tian, The discrete-time $GI/Geo/1$ queue with working vacations and vacation interruption. *App. Math. Comput.* **185** (2007) 1–10.

[13] L.D. Servi and S.G. Finn, $M/M/1$ queue with working vacations $M/M/1/N/WV$. *Perform. Eval.* **50** (2002) 41–52.

[14] N. Tian, Z. Ma and M. Liu, The discrete time $Geom/Geom/1$ queue with multiple working vacations. *App. Math. Modell.* **32** (2008) 2941–2953.

[15] H. Takagi, *Queueing analysis - A foundation of performance evaluation. Volume 3. Discrete-time systems.* North Holland, Amsterdam (1993).

[16] M.E. Woodward, *Communication and computer networks: Modelling with discrete-time queues.* Los Alamitos, CA: IEEE Computer Society, Press (1994).

[17] M. Yu, Y. Tang, Y. Fu and L. Pan, $GI/Geom/1/N/MWV$ queue with changeover time and searching for the optimum service rate in working vacation period. *J. Comput. App. Math.* **235** (2011) 2170–2184.