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STRONG STABILIZATION OF CONTROLLED VIBRATING SYSTEMS

Jean-François Couchouron[1]

Abstract. This paper deals with feedback stabilization of second order equations of the form

$$y_{tt} + A_0 y + u(t) B_0 y(t) = 0, \quad t \in [0, +\infty[,$$

where A_0 is a densely defined positive selfadjoint linear operator on a real Hilbert space H, with compact inverse and B_0 is a linear map in diagonal form. It is proved here that the classical sufficient ad-condition of Jurdjevic-Quinn and Ball-Slemrod with the feedback control $u = \langle y_t, B_0 y\rangle_H$ implies the strong stabilization. This result is derived from a general compactness theorem for semigroup with compact resolvent and solves several open problems.

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1. Introduction

This paper is concerned by the question of strong feedback stabilization of the following controlled equations

$$S\left(A_0, B_0, y^0, z^0\right) = \begin{cases} y_{tt} + A_0 y + u(t) B_0 y(t) = 0, & t \in [0, +\infty[, \\ y(0) = y^0, \; y_t(0) = z^0, \end{cases} \tag{1.1}$$

where A_0 is a densely defined positive selfadjoint linear operator on a real Hilbert space H with inner product $\langle , \rangle_H$; B_0 is a bounded linear map from $H_{A_0} = D\left(A_0^{\frac{1}{2}}\right)$ endowed with the graph norm into H.

The problem of strong stabilization consists in finding a control feedback $u = F(y)$ in (1.1) such that the state $w(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ strongly in $H_{A_0} \times H$ when $t \longrightarrow +\infty$.

The aim of this paper is to prove that with diagonal operators B_0 and under the following Jurjevic-Quinn ad-condition

$$\left[(\forall t \geq 0) \left\langle z_t(t), B_0(z(t))\right\rangle_H = 0\right] \Longrightarrow \left[z(0) = z_t(0) = 0\right],$$

where z stands for a solution of $z_{tt} + A_0 z = 0, \; t \geq 0$, the strong stabilization problem of (1.1) can be solved by choosing the control feedback as

$$u(t) = \left\langle y_t(t), B_0 y(t)\right\rangle_H. \tag{1.2}$$

Keywords and phrases. Precompactness, compact resolvent, almost periodic functions, Fourier series, mild solution, integral solution, Control Theory, Stabilization.

[1] Université Paul Verlaine de Metz, LMAM et INRIA Lorraine, Île du Saulcy, 57045 Metz, France. `couchour@univ-metz.fr`

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The problem was studied in finite dimension by Jurdjevic and Quinn in 1978 (see [11]) and in infinite dimension by Ball and Slemrod in 1979 (see [1,2]). But Ball and Slemrod obtained only weak stabilization with moreover some restrictions on B_0. Either B_0 is sum of a linear compact operator and a symmetric linear operator with a separation property about the diagonal exponents in the trigonometric Fourier expansion of $\langle z_t(t), B_0(z(t))\rangle_H$, where z stands for a solution of $z_{tt} + A_0 z = 0,\ t \geq 0$; or B_0 is a linear bounded operator with a uniform separation property about the previous exponents. These assumptions on the Fourier exponents imply the Jurdjevic-Queen ad-condition. But even with these constraints the question of strong stabilization remained an open problem.

In this paper the strong stabilization has been obtained for diagonal operators B_0. No compactness condition on B_0 is required. Such a B_0 satisfies the uniform separation exponents property of Ball and Slemrod. Consequently, our strong stabilization result applies to examples for which weak stabilization was already known (by using for instance results of [1,2]).

For the strong stabilization problem the crucial point consists in showing the precompactness of the range $w(\mathbb{R}^+)$, where $w = \begin{pmatrix} y \\ y_t \end{pmatrix}$ is the solution of the first order system associated to (1.1) and (1.2) in the state space. This compactness theorem will be obtained as a consequence of an abstract result proved in [4] which plays the role of an Ascoli-Arzelà theorem for evolution equations. The lack of dissipation in the first order system relative to the stabilization problem makes harder the study of precompactness and does not allow to apply techniques such as multipliers methods or theorems such as Dafermos-Slemrod or Pazy theorems (see [7,12]). In applications to vibrating problems this lack of dissipation is often provided by lack of damping (see for instance [3]).

The paper is organized as follows: assumptions and notations are given in Section 1; the main results are stated in Section 2; the proofs are regrouped in Section 3; different examples are given in Section 4.

2. Assumptions and notations

Let H be an Hilbert space with inner product $\langle .,.\rangle_H$ and its associated norm $\|.\|_H$. If A is an operator, the notation $D(A)$ stands for the domain of A. We suppose throughout this paper:

(A1) A_0 is a densely defined positive selfadjoint linear operator on H;

(A2) A_0^{-1} is everywhere defined and compact;

(A3) the eigenvalues $\mu_1, \mu_2, \dots$ of A_0 are simple;

Thanks to (A2) and classical properties of linear compact operators, we will take an orthonormal basis $(e_k)_{k\in\mathbb{N}^*}$ in H satisfying for all $k \in \mathbb{N}^*$,

$$e_k \in D(A_0) \text{ and } A_0 e_k = \mu_k e_k \text{ with } \mu_k > 0,\ \mu_k < \mu_{k+1}, \text{ and } \mu_j \longrightarrow +\infty.$$

Moreover, $H_{A_0} = D\left(A_0^{\frac{1}{2}}\right)$ is an Hilbert space with the inner product

$$\langle y, z\rangle_{H_{A_0}} = \left\langle A_0^{\frac{1}{2}} y, A_0^{\frac{1}{2}} z\right\rangle_H,$$

for all $y, z \in H_{A_0}$.

In view of (1.2) equation (1.1) can be removed into a first order system in $X = H_{A_0} \times H$ as follows

$$Q\left(A, g, w^0\right) = \begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = Aw + g(w), \quad t \in [0, +\infty[, \\ w(0) = w^0 = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in H_{A_0} \times H, \end{cases} \tag{2.1}$$

where we have set

$$w = \begin{pmatrix} y \\ z \end{pmatrix} \in H_{A_0} \times H,\ A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix},\ g(w) = \begin{pmatrix} 0 \\ -\langle z, B_0 y\rangle_H B_0 y \end{pmatrix}$$

and

$$D(A) = D(A_0) \times D\left(A_0^{\frac{1}{2}}\right) = D(A_0) \times H_{A_0}.$$

Let us note that the feedback control u takes the following form

$$u(t) = \langle z(t), B_0 y(t)\rangle_H = \langle w(t), Bw(t)\rangle_{H_{A_0}\times H},$$

with

$$Bw = \begin{pmatrix} 0 \\ B_0 y \end{pmatrix}.$$

Note also that $X = H_{A_0} \times H$ is an Hilbert space with the following inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_{H_{A_0}\times H} = \langle y, \widetilde{y}\rangle_{H_{A_0}} + \langle z, \widetilde{z}\rangle_H,$$

for all $y, \widetilde{y} \in H_{A_0}$ and $z, \widetilde{z} \in H$.

It follows from our assumptions that A is skew adjoint and generates a C^0 group e^{tA} of linear isometries on $H_{A_0} \times H$.

Introduce now the following conditions.

(A4) $B_0 : H_{A_0} \longrightarrow H$ is linear and bounded and

$$B_0(e_k) = \lambda_k e_k.$$

for all $k \in \mathbb{N}^*$.

(A5) $\lambda_k \neq 0$ for all $k \in \mathbb{N}^*$.

Notations. If $\xi \in H$ we will set $\xi_k = \langle \xi, e_k\rangle_H$ for all $k \in \mathbb{N}^*$ and thus $\xi = \sum_{k=1}^{+\infty} \xi_k e_k$. We will denote in the sequel $\|w\|$ instead of $\|w\|_{H_{A_0}\times H}$ and $\langle w, \widetilde{w}\rangle$ instead of $\langle w, \widetilde{w}\rangle_{H_{A_0}\times H}$ (if there is no ambiguity).

Remark 2.1. Let $w = \begin{pmatrix} y \\ z \end{pmatrix} \in H_{A_0} \times H$ and $\widetilde{w} = \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} \in H_{A_0} \times H$. It comes

$$\langle w, \widetilde{w}\rangle = \sum_{k=1}^{+\infty} \mu_k y_k \widetilde{y_k} + z_k \widetilde{z}_k., \text{ and } \|w\|^2 = \sum_{k=1}^{+\infty} \mu_k y_k^2 + z_k^2. \tag{2.2}$$

Remark 2.2. A simple computation gives an explicit expression of $\mathrm{e}^{\tau A}$:

$$\mathrm{e}^{\tau A}\begin{pmatrix} y \\ z \end{pmatrix} = \sum_{k=1}^{+\infty} \begin{pmatrix} \left(z_k \dfrac{\sin(\sqrt{\mu_k}\tau)}{\sqrt{\mu_k}} + y_k \cos(\sqrt{\mu_k}\tau)\right) e_k \\ \left(z_k \cos(\sqrt{\mu_k}\tau) - y_k\sqrt{\mu_k}\sin(\sqrt{\mu_k}\tau)\right) e_k \end{pmatrix} \tag{2.3}$$

for $\tau \in \mathbb{R},\ y \in H_{A_0},\ z \in H$.

Remark 2.3. The resolvent $(I - \lambda A)^{-1}$ is compact for all $\lambda > 0$, since the injection $D(A) \hookrightarrow H_{A_0} \times H$ is compact ($D(A)$ endowed with the graph norm). This compactness property follows from the compactness assumption (A2).

Remark 2.4. According to (A4), assumption (A5) is equivalent (see Lem. 4.7) to the following ad-condition

$$\left[(\forall t \geq 0) \left\langle \mathrm{e}^{tA} w^0, B\left(\mathrm{e}^{tA} w^0\right)\right\rangle_{H_{A_0} \times H} = 0\right] \Longrightarrow \left[w^0 = 0\right]. \tag{2.4}$$

3. The main results

Theorem 3.1. *Suppose* [(A1),...,(A5)] *holds. Then* (2.1) *has a unique mild solution* w *on* $[0,+\infty[$ *and* $w(t) \underset{t\longrightarrow+\infty}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *in* $H_A \times H$*. In other words the feedback control given by* (1.2) *strongly stabilizes* (1.1)*.*

Remark 3.1. Assumption (A3) is necessary in Theorem 3.1 because analogously to the proof given in [1] we can show that if (A3) does not hold equation (1.1) is not weakly stabilizable by the feedback (1.2).

In order to prove Theorem 3.1 the essential part of the work will consist in showing that the orbit $w(\mathbb{R}^+)$ is precompact. In this direction we will need a differential topological tool, namely the following theorem.

Theorem 3.2. *Let* A_1 *be a m-dissipative operator with compact resolvent on the Banach space* X*. Suppose* $G \in \mathbb{L}^2([0,+\infty[, X)$ *and suppose that the mild solution* w *of* $\frac{\mathrm{d}w}{\mathrm{d}t} = A_1 w + G(t)$, $t \in [0,+\infty[$, $w(0) = w^0 \in \overline{D(A_1)}$, *is bounded and uniformly continuous on* $[0,+\infty[$*. Then the positive orbit* $w(\mathbb{R}^+)$ *is precompact in* X*.*

We refer the reader to [4] for the proof of Theorem 3.2 and for more general statements: really, Theorem 3.2 is a corollary of Corollary 5.2, p. 18, of [4] since the set denoted by $\mathbb{L}_\alpha(X)$ (with $\alpha > 0$) in this Corollary 5.2 contains all $\mathbb{L}^p([0,+\infty[, X)$ for $1 \leq p < +\infty$.

Remark 3.2. The precompactness of the orbit $w(\mathbb{R}^+)$ for the solution of the quasi-autonomous problem $\frac{\mathrm{d}w}{\mathrm{d}t} = A_1 w + G(t)$ is a classical result (see [7]) when $G \in \mathbb{L}^1([0,+\infty[, X)$. But here, in our stabilization problem this situation does not hold. It just happens $G \in \mathbb{L}^2([0,+\infty[, X)$ as we will see later. That explains why we will use Theorem 3.2 and why we will focus on the uniform continuity.

Some comments about precompactness and uniform continuity

If w is a function defined on $[0,+\infty[$ with values in the Banach space X, the uniform continuity of w implies a (uniform) equicontinuity of the sequel $(w_n)_n$ in $C([0,T], X)$ defined for any fixed $T > 0$ by $w_n(t) = w(nT + t)$, for each $t \in [0,T]$. And this equicontinuity can be viewed as a suitable (uniform in n) continuity property with respect to translation operators. Moreover, the precompactness of the orbit $w(\mathbb{R}^+)$ implies the precompactness of the sections $\{w_n(t);\ n \in \mathbb{N}\}$, with $t \in [0,T]$. In a general way, several precompactness theorems for subsets of a Banach space of functions $v : J \to X$, where J is an interval of $\mathbb{R}$, connect the two following ingredients:

(1) a precompactness property in the space values X such as, precompactness of the sections or compact embedding or compact resolvent...;

(2) a translation-continuity property such as uniform continuity, equicontinuity...

Theorem 3.2 quoted and stated previously provides an example of this kind of results. Let us cite also: the Ascoli-Arzelà Theorem in $Y = C([0,T], X)$, or the Riesz-Fréchet-Kolmogorov Theorem in $Y = \mathbb{L}^p([0,T], \mathbb{R})$ (for instance), or extensions due to Simon to $Y = \mathbb{L}^p([0,T], \mathcal{B})$, where $\mathcal{B}$ is a Banach space (see [14]), or Theorem 4.1 and Lemma 4.3 given by Haraux in [9]. Lemma 4.3 in [9] concerns not only solutions of differential problems but functions in $C([0,+\infty[, X)$. It is a pure topological result. It could be used also like Theorem 3.2 to deduce the precompactness of $w(\mathbb{R}^+)$ from the uniform continuity of w, where w is the solution of (2.1). But this application is not direct and ask some constructions and computations (using in particular the explicit form of the semigroup, the Duhamel's formula, and the compact embedding $D(A_0) \times H_{A_0} \hookrightarrow H_{A_0} \times H$).

4. Proofs

Proof of Theorem 3.1. Let $w = \begin{pmatrix} y \\ z \end{pmatrix}$ and $g(w) = \begin{pmatrix} 0 \\ -\langle z, B_0 y\rangle_H B_0 y \end{pmatrix}$. Since g is Lipschitz on bounded subsets of X, and A skew adjoint the following equation

$$Q\left(A, g, w^0\right) = \begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = Aw + g(w), & t \in [0, +\infty[, \\ w(0) = w^0 = \begin{pmatrix} y^0 \\ z^0 \end{pmatrix} \in H_{A_0} \times H, \end{cases}$$

has a unique local mild solution w.

Set

$$G(t) = g(w(t)) = -u(t) \begin{pmatrix} 0 \\ B_0(y(t)) \end{pmatrix} = -u(t) Bw(t).$$

Recall the following result on the Cauchy problem (which can be deduced from [13], p. 185, or [5] for more general multivalued versions). In [13], locally Lipschitz means Lipschitz on bounded subsets (as in the present application). In the theorem below locally Lipschitz has the wider sense that each point has a neighbourhood where the map is Lipschitz. Set

$$CP(A_1, B_1, T) = \begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = A_1 w + B_1 w, \ t \in [0, T[\\ w(0) = w^0. \end{cases}$$

Lemma 4.1. *Let A_1 be the infinitesimal generator of a strongly continuous semi-group of bounded linear operators $\left(\mathrm{e}^{tA_1}\right)_{t\geq 0}$ in the Banach space E and let B_1 be a nonlinear continuous operator on E locally Lipchitz and bounded on bounded subsets of E. Then the Cauchy problem $CP(A_1, B_1, T)$ has a unique mild solution $w_T(.)$ on $[0, T]$ for $T > 0$ sufficiently small. If there is an* a priori *upper bound for local solutions, that is a non decreasing real valued function $M(.)$ everywhere defined on $[0, +\infty[$ satisfying $\sup_{t\in[0,T]} \|w_T(t)\| \leq M(T)$ the Cauchy problem $CP(A_1, B_1, T)$ has a unique mild solution for every $T > 0$.*

Lemma 4.2. *The mild solution w of $Q\left(A, g, w^0\right)$ is defined and bounded on the whole interval $[0, +\infty[$ and we have*

$$\|w(t)\| \leq \left\|w^0\right\|, \ u \in \mathbb{L}^2([0, +\infty[) \cap \mathbb{L}^\infty([0, +\infty[) \tag{4.1}$$

and,

$$G \in \mathbb{L}^2([0, +\infty[, H_{A_0} \times H) \cap \mathbb{L}^\infty([0, +\infty[, H_{A_0} \times H). \tag{4.2}$$

Proof. Since A is skew-adjoint it follows

$$\frac{1}{2}\|w(t)\|^2 - \frac{1}{2}\|w(s)\|^2 = -\int_s^t \langle z, B_0 y\rangle_H^2 (t)\,\mathrm{d}t, \ 0 \leq s \leq t.$$

From this last relation we deduce w is defined on the whole interval $[0, +\infty[$, bounded in $H_A \times H$ and $t \longmapsto \|w(t)\|$ is a decreasing function. In particular we deduce $\|w(t)\| \leq \left\|w^0\right\|$. Because B is bounded relations $u(t) = \langle z(t), B_0 y(t)\rangle_H = \langle w(t), Bw(t)\rangle$, yield (4.1) and (4.2). The proof of Lemma 4.2 is now complete. □

Remark 4.1. With the previous notations for the mild solution $w = (y, z)^T$ we have $z = y_t$ in H and thus the control can be written $u = \langle y_t, B_0 y\rangle_H$.

In view of Theorem 3.2 the precompactness of $w([0, +\infty[)$ will be a consequence of the following proposition.

Proposition 4.1. *The solution w is uniformly continuous on $[0, +\infty[$.*

In order to prove this proposition we have to compute $\|w(t+h) - w(t)\|$ for $t \geq 0$ and $h \geq 0$.

Notation. In the sequel let $t \geq 0$, $h \geq 0$, $h \leq 1$ and $t - 2h \geq 0$ and denote by K the norm of the bounded linear operator $B_0 : H_A \longrightarrow H$.

Remark 4.2. With the previous notations we have

$$\|Bw\| = \|B_0 y\|_H \leq K\|y\|_{H_{A_0}} \leq K\|w\|.$$

Lemma 4.3. *We have*

$$\begin{array}{rcl} \|G(t)\| & \leq & K^2 \left\|w^0\right\|^3, \\ \|G(t)\| & \leq & K \left\|w^0\right\| |u(t)| \ \textit{and} \\ \left\|w(t+h) - \mathrm{e}^{hA} w(t)\right\|^2 & \leq & hK^2 \left\|w^0\right\|^2 \int_0^h u^2(t+\tau)\,\mathrm{d}\tau. \end{array}$$

Proof. According to the Duhamel's formula we have

$$w(t+h) - \mathrm{e}^{hA} w(t) = \int_0^h \mathrm{e}^{(h-\tau)A} G(t+\tau)\,\mathrm{d}\tau, \tag{4.3}$$

and thus

$$\left\|w(t+h) - \mathrm{e}^{hA} w(t)\right\|^2 \leq h \int_0^h \|G(t+\tau)\|^2\,\mathrm{d}\tau, \tag{4.4}$$

thanks to the Cauchy-Schwarz inequality and the isometric aspect of e^{hA}. Now from Lemma 4.2 and the definitions of G and K it comes

$$\left\|w(t+h) - \mathrm{e}^{hA} w(t)\right\|^2 \leq hK^2 \left\|w^0\right\|^2 \int_0^h u^2(t+\tau)\,\mathrm{d}\tau.$$

That ends the proof of Lemma 4.3. □

Notation. Set in the sequel

$$w(t) = \sum_{k=1}^{+\infty} \begin{pmatrix} y_k(t) e_k \\ z_k(t) e_k \end{pmatrix}, \ w_k(t) = \begin{pmatrix} y_k(t) e_k \\ z_k(t) e_k \end{pmatrix}.$$

Lemma 4.4. *We have*

$$\left\|\mathrm{e}^{hA} w(t) - w(t)\right\|^2 = 4 \sum_{k=1}^{+\infty} \left(\mu_k y_k^2(t) + z_k^2(t)\right) \sin^2\left(\sqrt{\mu_k}\frac{h}{2}\right)$$

and

$$\left| \|w(t+h) - w(t)\|^2 - \left\|\mathrm{e}^{hA} w(t) - w(t)\right\|^2 \right| \leq 4\sqrt{h} K \left\|w^0\right\|^2 \sqrt{\int_0^h u^2(t+\tau)\,\mathrm{d}\tau}.$$

Proof. An immediate computation gives

$$\|w_k(t)\|^2 = \mu_k y_k^2(t) + z_k^2(t)$$

and

$$\mathrm{e}^{hA} w_k(t) = \begin{pmatrix} \left(z_k(t) \dfrac{\sin\left(\sqrt{\mu_k} h\right)}{\sqrt{\mu_k}} + y_k(t) \cos\left(\sqrt{\mu_k} h\right)\right) e_k \\ \left(z_k(t) \cos\left(\sqrt{\mu_k} h\right) - y_k(t) \sqrt{\mu_k} \sin\left(\sqrt{\mu_k} h\right)\right) e_k \end{pmatrix}.$$

So on one hand, from (2.3) it follows

$$\begin{aligned}\left\|e^{hA}w(t)-w(t)\right\|^2 &= 2\left(\|w(t)\|^2-\left\langle e^{hA}w(t),w(t)\right\rangle\right)\\ &= 4\sum_{k=1}^{+\infty}\left(\mu_k y_k^2(t)+z_k^2(t)\right)\sin^2\left(\sqrt{\mu_k}\tfrac{h}{2}\right).\end{aligned}$$

And on the other hand, by using Lemma 4.3 and $\|w(\tau)-w(\sigma)\|\le 2\left\|w^0\right\|$, we obtain

$$\begin{aligned}&\left|\|w(t+h)-w(t)\|^2-\left\|e^{hA}w(t)-w(t)\right\|^2\right|\\ &\le\left(\|w(t+h)-w(t)\|+\left\|e^{hA}w(t)-w(t)\right\|\right)\left|\,\|w(t+h)-w(t)\|-\left\|e^{hA}w(t)-w(t)\right\|\,\right|\\ &\le\left(\|w(t+h)-w(t)\|+\left\|e^{hA}w(t)-w(t)\right\|\right)\left\|w(t+h)-e^{hA}w(t)\right\|\\ &\le 4\sqrt{h}K\left\|w^0\right\|^2\sqrt{\textstyle\int_0^h u^2(t+\tau)\,d\tau}\end{aligned}$$

and the lemma is proved. □

Lemma 4.5. *One has*

$$\begin{aligned}&\left|u(t+h)+u(t-h)-2\sum_{k=1}^{+\infty}\lambda_k z_k(t)\,y_k(t)\cos\left(2\sqrt{\mu_k}h\right)\right|\\ &\qquad\le 2\sqrt{h}K^2\left\|w^0\right\|^2\left(\sqrt{\textstyle\int_0^h u^2(t+\tau)\,d\tau}+\sqrt{\textstyle\int_{-h}^0 u^2(t+\tau)\,d\tau}\right).\end{aligned}$$

Proof. The triangle inequality gives

$$\begin{aligned}\left|u(t+h)-\left\langle e^{hA}w(t),B\left(e^{hA}w(t)\right)\right\rangle\right| &\le \left|\langle w(t+h),B(w(t+h))\rangle-\left\langle e^{hA}w(t),B\left(e^{hA}w(t)\right)\right\rangle\right|\\ &\le \left|\left\langle w(t+h)-e^{hA}w(t),B(w(t+h))\right\rangle\right|\\ &\quad+\left|\left\langle e^{hA}w(t),B\left(w(t+h)-e^{hA}w(t)\right)\right\rangle\right|.\end{aligned}$$

Thanks to Lemma 4.3 we conclude

$$\left|u(t+h)-\left\langle e^{hA}w(t),B\left(e^{hA}w(t)\right)\right\rangle\right|\le 2\sqrt{h}K^2\left\|w^0\right\|^2\sqrt{\int_0^h u^2(t+\tau)\,d\tau}. \tag{4.5}$$

In the same way we obtain

$$\left|u(t-h)-\left\langle e^{-hA}w(t),B\left(e^{-hA}w(t)\right)\right\rangle\right|\le 2\sqrt{h}K^2\left\|w^0\right\|^2\sqrt{\int_0^h u^2(t-\tau)\,d\tau}. \tag{4.6}$$

According to assumption (A4) on the operator B_0, it comes

$$\begin{aligned}\left\langle e^{hA}w(t),B\left(e^{hA}w(t)\right)\right\rangle &= \sum_{k=1}^{+\infty}\lambda_k\left(z_k(t)\cos\left(\sqrt{\mu_k}h\right)-y_k(t)\sqrt{\mu_k}\sin\left(\sqrt{\mu_k}h\right)\right)\\ &\qquad\times\left(z_k(t)\frac{\sin\left(\sqrt{\mu_k}h\right)}{\sqrt{\mu_k}}+y_k(t)\cos\left(\sqrt{\mu_k}h\right)\right)\end{aligned}$$

thus,

$$\left\langle e^{hA}w,B\left(e^{hA}w\right)\right\rangle=\sum_{k=1}^{+\infty}\lambda_k z_k y_k\cos\left(2\sqrt{\mu_k}h\right)+\frac{1}{2}\lambda_k\left(\frac{z_k^2}{\sqrt{\mu_k}}-y_k^2\sqrt{\mu_k}\right)\sin\left(2\sqrt{\mu_k}h\right). \tag{4.7}$$

Let us write

$$\left|u(t+h)+u(t-h)-\left\langle \mathrm{e}^{hA}w(t),B\left(\mathrm{e}^{hA}w(t)\right)\right\rangle-\left\langle \mathrm{e}^{-hA}w(t),B\left(\mathrm{e}^{-hA}w(t)\right)\right\rangle\right|$$
$$\leq\left|u(t+h)-\left\langle \mathrm{e}^{hA}w(t),B\left(\mathrm{e}^{hA}w(t)\right)\right\rangle\right|+\left|u(t-h)-\left\langle \mathrm{e}^{-hA}w(t),B\left(\mathrm{e}^{-hA}w(t)\right)\right\rangle\right|.$$

Then we end the proof of Lemma 4.5 by applying (4.5), (4.6) and (4.7) in the last inequality. □

Lemma 4.6. *Let $0\leq 2h\leq s\leq t$, and*

$$\begin{aligned}\xi(t,h) &= \frac{1}{2}\left\|w(t+h)-w(t)\right\|^2 \text{ and}\\ \delta(s,t,h) &= \xi(t,h)+\xi(t,-h)-\xi(s,h)-\xi(s,-h).\end{aligned}$$

There is a constant $M>0$ satisfying

$$\begin{aligned}|\delta(s,t,2h)| \leq\ & M\int_s^t\left(\left(u^2(\tau)+u^2(\tau+h)+u^2(\tau-h)\right)+\int_{-2h}^{2h}u^2(\tau+\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau\\ &+M\left(\sqrt{\int_{-2h}^{2h}u^2(t+\tau)\,\mathrm{d}\tau}+\sqrt{\int_{-2h}^{2h}u^2(s+\tau)\,\mathrm{d}\tau}\right).\end{aligned}$$

Proof. Define the function ε by

$$\begin{aligned}\delta(s,t,2h) =\ & \frac{1}{2}\left\|\mathrm{e}^{2hA}w(t)-w(t)\right\|^2-\frac{1}{2}\left\|\mathrm{e}^{2hA}w(s)-w(s)\right\|^2\\ &+\frac{1}{2}\left\|\mathrm{e}^{-2hA}w(t)-w(t)\right\|^2-\frac{1}{2}\left\|\mathrm{e}^{-2hA}w(s)-w(s)\right\|^2+\varepsilon(s,t,h).\end{aligned}$$

Then for $\sigma\geq 2h$, the following inequality

$$\sqrt{\int_0^{2h}u^2(\sigma+\tau)\,\mathrm{d}\tau}+\sqrt{\int_{-2h}^{0}u^2(\sigma+\tau)\,\mathrm{d}\tau}\leq\sqrt{2}\sqrt{\int_{-2h}^{2h}u^2(\sigma+\tau)\,\mathrm{d}\tau}$$

and Lemma 4.4 yield

$$|\varepsilon(s,t,h)|\leq 4\sqrt{h}K\left\|w^0\right\|^2\left(\sqrt{\int_{-2h}^{2h}u^2(t+\tau)\,\mathrm{d}\tau}+\sqrt{\int_{-2h}^{2h}u^2(s+\tau)\,\mathrm{d}\tau}\right). \tag{4.8}$$

Now, Lemma 4.4 gives

$$\delta(s,t,2h)=4\sum_{k=1}^{+\infty}\left(\mu_k y_k^2(t)+z_k^2(t)-\mu_k y_k^2(s)-z_k^2(s)\right)\sin^2\left(\sqrt{\mu_k}h\right)+\varepsilon(s,t,h). \tag{4.9}$$

But we have

$$\frac{1}{2}\left(\mu_k y_k^2(t)+z_k^2(t)-\mu_k y_k^2(s)-z_k^2(s)\right)=-\int_s^t\lambda_k z_k(\tau)\,y_k(\tau)\,u(\tau)\,\mathrm{d}\tau. \tag{4.10}$$

Indeed $w_k=\begin{pmatrix}y_k e_k\\ z_k e_k\end{pmatrix}$ is solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}w_k=Aw_k-\lambda_k u(t)\begin{pmatrix}0\\ y_k e_k\end{pmatrix}$$

and therefore the function $\|w_k\|$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_k\|^2(t) = -\lambda_k u(t) z_k(t) y_k(t). \tag{4.11}$$

Consequently, (4.10) and (4.9) and the definition of u imply

$$\begin{aligned}\sum_{k=1}^{+\infty}\left(\mu_k y_k^2(t) + z_k^2(t) - \mu_k y_k^2(s) - z_k^2(s)\right)\sin^2\left(\sqrt{\mu_k}h\right) &= -2\int_s^t \sum_{k=1}^{+\infty}\lambda_k z_k(\tau) y_k(\tau) u(\tau)\sin^2\left(\sqrt{\mu_k}h\right)\mathrm{d}\tau \\ &= -\int_s^t \sum_{k=1}^{+\infty}\lambda_k z_k(\tau) y_k(\tau) u(\tau)\,\mathrm{d}\tau + \int_s^t \sum_{k=1}^{+\infty}\lambda_k z_k(\tau) y_k(\tau) u(\tau)\cos\left(2\sqrt{\mu_k}h\right)\mathrm{d}\tau\end{aligned}$$

and then

$$\begin{aligned}&\sum_{k=1}^{+\infty}\left(\mu_k y_k^2(t) + z_k^2(t) - \mu_k y_k^2(s) - z_k^2(s)\right)\sin^2\left(\sqrt{\mu_k}h\right) \\ &\qquad = -\int_s^t u^2(\tau)\,\mathrm{d}\tau + \int_s^t u(\tau)\sum_{k=1}^{+\infty}\lambda_k z_k(\tau) y_k(\tau)\cos\left(2\sqrt{\mu_k}h\right)\mathrm{d}\tau.\end{aligned}$$

Set

$$\gamma(\tau,h) = u(\tau+h) + u(\tau-h) - 2\sum_{k=1}^{+\infty}\lambda_k z_k(\tau) y_k(\tau)\cos\left(2\sqrt{\mu_k}h\right). \tag{4.12}$$

By using (4.12), (three times) the inequality $|ab| \leq \frac{1}{2}\left(a^2+b^2\right)$ and finally Lemma 4.5 combining with $(a+b)^2 \leq 2\left(a^2+b^2\right)$, we obtain successively

$$\begin{aligned}\left|2\int_s^t u(\tau)\sum_{k=1}^{+\infty}\lambda_k z_k(\tau) y_k(\tau)\cos\left(2\sqrt{\mu_k}h\right)\mathrm{d}\tau\right| &= \left|\int_s^t u(\tau)\left(u(\tau+h) + u(\tau-h) - \gamma(\tau,h)\right)\mathrm{d}\tau\right| \\ &\leq \frac{1}{2}\int_s^t \left(3u^2(\tau) + u^2(\tau+h) + u^2(\tau-h) + \gamma^2(\tau,h)\right)\mathrm{d}\tau \\ &\leq \frac{1}{2}\int_s^t \left(3u^2(\tau) + u^2(\tau+h) + u^2(\tau-h) + C\int_{-h}^{h} u^2(\tau+\sigma)\,\mathrm{d}\tau\right)\mathrm{d}\tau, \end{aligned}\tag{4.13}$$

where we have set

$$C = 2\left(2\sqrt{h}K^2\left\|w^0\right\|^2\right)^2.$$

Consequently, from (4.8), (4.9), (17), (4.13) we see that there is a constant $M > 0$ satisfying

$$\begin{aligned}|\delta(s,t,2h)| \quad &\leq \quad M\int_s^t \left(\left(u^2(\tau) + u^2(\tau+h) + u^2(\tau-h)\right) + \int_{-h}^{h} u^2(\tau+\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau \\ &\quad + M\left(\sqrt{\int_{-2h}^{2h} u^2(t+\tau)\,\mathrm{d}\tau} + \sqrt{\int_{-2h}^{2h} u^2(s+\tau)\,\mathrm{d}\tau}\right).\end{aligned}$$

Lemma 4.6 is now proved. □

End of proof of Proposition 4.1

The function w is uniformly continuous on $[0,+\infty[$ if and only if

$$\limsup_{h\to 0, h\geq 0}\left(\sup_{t\geq h}\left(\xi\left(t,h\right)+\xi\left(t,-h\right)\right)\right)=0. \tag{4.14}$$

Let $s>0$. Since w is uniformly continuous on $[0,s]$ it follows

$$\limsup_{h\to 0, h\geq 0}\left(\sup_{t\geq h}\left(\xi\left(t,h\right)+\xi\left(t,-h\right)\right)\right)=\limsup_{\substack{h\to 0\\ 0\leq h\leq s}}\left(\sup_{t\geq s}\delta\left(s,t,h\right)\right). \tag{4.15}$$

Using then Lemma 4.6 and the continuity of the translation in $\mathbb{L}^2\left([0,+\infty[\right)$, we find

$$0\leq \limsup_{h\to 0, h\geq 0}\left(\sup_{t\geq h}\left(\xi\left(t,h\right)+\xi\left(t,-h\right)\right)\right)\leq 3M\int_s^{+\infty}u^2\left(\tau\right)\mathrm{d}\tau. \tag{4.16}$$

Now, letting $s\to+\infty$ in (4.16), we obtain (4.14) and thus the required uniform continuity. The proof of Proposition 4.1 is now complete. □

It remains to prove Theorem 3.1.

Proof of Theorem 3.1

Let $t_n\longrightarrow+\infty$ in $\mathbb{R}^+$. Since the orbit $w\left(\mathbb{R}^+\right)$ is precompact in $X=H_{A_0}\times H$ using Proposition 4.1 we can suppose by taking a cluster point w_∞ of $\left(w\left(t_n\right)\right)_n$ and a suitable subsequence that we have $w\left(t_n\right)\longrightarrow w_\infty$. Then for $h\in\mathbb{R}^+$ Lemma 4.3 and the semigroup continuity imply $w\left(t_n+h\right)\longrightarrow \mathrm{e}^{hA}w_\infty$. Now, from the continuity of B_0 it follows

$$u\left(t_n+h\right)\longrightarrow\left\langle \mathrm{e}^{hA}w_\infty, B\left(\mathrm{e}^{hA}w_\infty\right)\right\rangle. \tag{4.17}$$

Let $v_n\left(\tau\right)=u\left(t_n+\tau\right)$. Since $u\in\mathbb{L}^2\left(\mathbb{R}^+\right)$, the sequence $\left(v_n\right)_n$ converges to zero in $\mathbb{L}^1\left([0,T[\right)$ for each $T>0$. Consequently, considering again a suitable subsequence we obtain $v_{n_q}\left(\tau\right)\longrightarrow 0$, a.e. $\tau\in[0,T[$. Thanks to (4.17) we then conclude

$$\left\langle \mathrm{e}^{hA}w_\infty, B\left(\mathrm{e}^{hA}w_\infty\right)\right\rangle=0, \tag{4.18}$$

for all $h\in\mathbb{R}^+$, because $T>0$ is arbitrary and $h\longmapsto\left\langle \mathrm{e}^{hA}w_\infty, B\left(\mathrm{e}^{hA}w_\infty\right)\right\rangle$ is continuous.

Lemma 4.7. *Assumptions* (A4), (A5) *provide the following implication:*

$$\left[(\forall t\geq 0)\ \left\langle \mathrm{e}^{tA}w^0, B\left(\mathrm{e}^{tA}w^0\right)\right\rangle=0\right]\Longrightarrow\left[w^0=0\right]. \tag{4.19}$$

Proof of Lemma 4.7. Set $w^0=\sum_{k=1}^{+\infty}\begin{pmatrix} y_k e_k\\ z_k e_k\end{pmatrix}$ and $w_N^0=\sum_{k=1}^{N}\begin{pmatrix} y_k e_k\\ z_k e_k\end{pmatrix}$ for each $N\in\mathbb{N}^*$. Then the function defined by $f\left(t\right)=\left\langle \mathrm{e}^{tA}w^0, B\left(\mathrm{e}^{tA}w^0\right)\right\rangle$ is almost periodic (see [2,8]) as uniform limit on $\mathbb{R}$ of trigonometric polynomials $f_N\left(t\right)=\left\langle \mathrm{e}^{tA}w_N^0, B\left(\mathrm{e}^{tA}w_N^0\right)\right\rangle$. Indeed, this result can be deduced from the following inequalities which use the isometric aspect of e^{tA} and Remark 4.2:

$$\begin{aligned}|f\left(t\right)-f_N\left(t\right)|&\leq\left|\left\langle \mathrm{e}^{tA}\left(w^0-w_N^0\right), B\left(\mathrm{e}^{tA}w^0\right)\right\rangle\right|+\left|\left\langle \mathrm{e}^{tA}w_N^0, B\left(\mathrm{e}^{tA}\left(w^0-w_N^0\right)\right)\right\rangle\right|\\ &\leq\|w^0-w_N^0\|K\|w^0\|+\|w_N^0\|K\|w^0-w_N^0\|\\ &\leq 2K\|w^0\|\|w^0-w_N^0\|,\end{aligned}$$

for all $t \geq 0$. Moreover, from (4.7) it follows that the coefficients of the Fourier series $\sum_{k=1}^{+\infty} c_k e^{2i\sqrt{\mu_k}t} + c_{-k} e^{-2i\sqrt{\mu_k}t}$ associated to f are given by

$$\overline{c_{-k}} = c_k = \frac{1}{2}\lambda_k \left(y_k z_k - \frac{i}{2} \left(\frac{z_k^2}{\sqrt{\mu_k}} - y_k^2 \sqrt{\mu_k} \right) \right), \tag{4.20}$$

and thus $|c_k|^2 = \frac{|\lambda_k^2|}{16\mu_k} \left(z_k^2 + \mu_k y_k^2 \right)^2$. Consequently if f is identically zero on $\mathbb{R}^+$, using the Bohr transform we find

$$c_k = \lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{-2i\sqrt{\mu_k}t} f(t)\, dt = 0, \tag{4.21}$$

for all k. Finally (A5) gives $w^0 = 0$, and Lemma 4.7 is proved.

Now, (4.18) and Lemma 4.7 imply $w_\infty = 0$. Therefore the unique strong cluster point of the precompact orbit $w(\mathbb{R}^+)$ is zero, that is $\lim_\infty w = 0$. The proof of Theorem 3.1 is then complete. □

5. Applications

Example 1 (wave equation)**.** Let Ω be a bounded open subset of $\mathbb{R}^n$. Consider the system

$$\begin{cases} y_{tt} - \Delta y + u(t)\, y = 0, \ x \in \Omega, \ t \in [0, +\infty[, \\ y|_{\partial\Omega} = 0. \end{cases} \tag{5.1}$$

The stabilization problem of the wave equation (5.1) with feedback $u(t) = \langle y_t(t), y(t) \rangle$ has the form (1.1)–(1.2) with,

$A_0 = -\Delta$, $D(A_0) = H^2(\Omega) \cap H_0^1(\Omega)$, $H_{A_0} = H_0^1$ and $H = \mathbb{L}^2(\Omega)$, $B_0 = Id_{H_{A_0}}$.

The feedback control is given by

$$u(t) = \int_\Omega y_t y \, dx. \tag{5.2}$$

Assumptions (A1)-…-(A5) hold if and only if the eigenvalues of $-\Delta$ with Dirichlet boundary conditions are simple. For instance, this geometric condition on Ω is fulfilled if $n = 1$ and Ω is an open interval or $n = 2$ and Ω is an open rectangle with irrational proportion. See [6] for various examples or counterexamples for such spectral geometric properties in $\mathbb{R}^n$. So by using Theorem 3.1, whenever the eigenvalues are simple the previously defined feedback control strongly stabilizes the wave equation for all initial data in $X = H_0^1 \times \mathbb{L}^2(\Omega)$.

Let us notice that in this example the operator B is compact since it has its values in $D(A)$.

Example 2 (vibrating beam with hinged ends)**.** Let Ω be a bounded open subset of $\mathbb{R}^n$. Consider the system

$$\begin{cases} y_{tt} + \Delta^2 y + u(t)\, \Delta y = 0, & x \in \Omega, \ t \in [0, +\infty[, \\ y = \Delta y = 0, & x \in \partial\Omega. \end{cases} \tag{5.3}$$

In this example we have $H = \mathbb{L}^2(\Omega)$, and $A_0 = \Delta^2$, $D(A_0) = \left\{ y \in H^4(\Omega);\ y, \Delta y \in H_0^1(\Omega) \right\}$. The stabilization problem has the form (1.1)–(1.2) with $B_0 = \Delta$, and $H_{A_0} = H^2(\Omega) \cap H_0^1(\Omega)$, the control being defined as,

$$u(t) = \langle y_t(t), y(t) \rangle = \int_\Omega y_t \Delta y \, dx. \tag{5.4}$$

If $n = 1$ and $\Omega =]0, 1[$ equations is a model for the transverse deflection of a beam with hinged ends and u denotes the axial load on the beam. As previously, whenever the eigenvalues are simple (this is the case for instance if $n = 1$) Theorem 3.1 guarantees that the feedback control u strongly stabilizes the beam equation for all initial data in X.

Let us notice that in this example the operator B is not compact.

Remark 5.1. Examples 1 and 2 are given in [1,2]. But in these papers only weak stabilization was proved and the question of strong stabilization remained an open problem. The previous developments give thus a positive answer to the strong stabilization problem for these systems.

Remark 5.2. The approach developed in this paper does not run for the following example in [1,2] which concerns a vibrating beam with clamped ends.

$$\left\{\begin{array}{ll} y_{tt}+y_{xxxx}+u\left(t\right)y_{xx}=0, & x\in\left]0,1\right[,\ t\in\left[0,+\infty\right[,\\ y=y_x=0, & x\in\left\{0,1\right\}. \end{array}\right. \tag{5.5}$$

Indeed in this situation assumption (A4) does not hold.

Example 3 (a rotating body beam and a generalization)**.** In [3] the authors consider a stabilization problem by a torque control of a rotating body beam without damping. It is about a disk with a beam (an antenna for instance) attached to its center and perpendicular to the disk's plane. The beam is confined to another plane which is perpendicular to the disk and rotates with the disk. The dynamics (after scaling simplifications) of the motion y is

$$\left\{\begin{array}{l} y_{tt}+y_{xxxx}-\omega^2\left(t\right)y=0, \qquad x\in\left]0,1\right[,\ t\in\left[0,+\infty\right[,\\ y\left(0,t\right)=y_x\left(0,t\right)=y_{xx}\left(1,t\right)=y_{xxx}\left(1,t\right)=0,\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\omega\left(t\right)\right)=\gamma\left(t\right) \end{array}\right. \tag{5.6}$$

with,

$$\gamma\left(t\right)=\frac{\Gamma\left(t\right)-2\omega\left(t\right)\int_0^1 yy_t\mathrm{d}x}{I_d+\int_0^1 y^2\mathrm{d}x}\cdot \tag{5.7}$$

The torque control applied to the disk Γ appears throughout γ, and $\omega=\dot{\theta}$ is the angular velocity. Coron and d'Andréa Novel have constructed a feedback torque control law which strongly stabilizes the equilibrium point $\left(0,\omega_0\right)$, where ω_0 satisfies $\left|\omega_0\right|<\omega_c$ for a critical angular velocity ω_c. The authors propose a control of the form

$$\omega\left(t\right)=\omega_0+\sigma\left(\int_0^1 yy_t\mathrm{d}x\right). \tag{5.8}$$

With suitable assumptions on σ the following relations hold

$$\omega^2-\omega_0^2\in\mathbb{L}^2\left(0,+\infty\right),\ v\in\mathbb{L}^2\left(0,+\infty\right), \tag{5.9}$$

where v stands for $\int_0^1 yy_t\mathrm{d}x$.

Of course the situation of this example is different from the framework described in this paper, since we have to stabilize system (5.6) at a prescribed non zero equilibrium point for $\left(y,\omega\right)$. But the method developed here can be adapted to solve one of the difficulties of this problem, namely the precompactness of the orbit in the state space, in order to obtain strong stabilization. In this goal, let us state the following theorem. Consider again equation (1.1), but this time with a general feedback control u. Thus we do not assume (1.2). Let

$$v=\left\langle z,B_0y\right\rangle_H. \tag{5.10}$$

Consider again the system $Q\left(A,g,w^0\right)$ associated to (1.1). Now, we have

$$g\left(w\right)=\begin{pmatrix}0\\-uB_0y\end{pmatrix},\ w=\begin{pmatrix}y\\z\end{pmatrix}. \tag{5.11}$$

Theorem 5.1. *Suppose* [(A1),...,(A5)] *holds. Suppose that* $Q\left(A,g,w^0\right)$ *with* g *given by* (5.11) *has a global mild solution* w. *Assume in addition that* $u(.) \in \mathbb{L}^2(0,+\infty)$ *and* $v(.) \in \mathbb{L}^2([0,+\infty[)$ *as function of* t, *where* v *is defined in* (5.10). *Then the feedback control* u *strongly stabilizes* (1.1).

Theorem 5.1 applies in the example of the rotating body beam by setting $u(t) = \omega^2(t) - \omega_0^2$ and removing A_0 into $A_0 + \omega_0^2 B_0$. With such modifications the new unbounded operator A governing the system $Q\left(A,g,w^0\right)$ is linear skew adjoint with the following inner product in $X = H_{A_0} \times H$ defined by

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \breve{y} \\ \breve{z} \end{pmatrix} \right\rangle_X = \langle y, \breve{y}\rangle_{H_{A_0}} - \frac{\omega_0^2}{2}\left(\langle z, B_0\breve{y}\rangle_H + \langle \breve{z}, B_0 y\rangle_H\right) + \langle z, \breve{z}\rangle_H . \tag{5.12}$$

Really we obtain a new inner product and an equivalent norm on X if we have

$$\left(\mu_k - \lambda_k\omega_0^2\right) \uparrow +\infty \text{ and } \inf_k \frac{\mu_k - \lambda_k\omega_0^2}{\mu_k} > 0. \tag{5.13}$$

Of course due to the definition of ω_c the choice of $\omega_0 \in\,]-\omega_c, \omega_c[$ and the relations $\lambda_k = 1$, these conditions hold in the present example.

Sketch of proof of Theorem 5.1

We have just to prove again the uniform continuity of the mild solution w of $Q\left(A,g,w^0\right)$, since the end of the proof of Theorem 3.1 remains valid in the present case (by changing $u(t_n+\tau)$ into $v(t_n+\tau)$ in (4.17)). Lemmas 4.3 and 4.4 remain true after changing $\|w^0\|$ into $\|w\|_\infty$, where $\|w\|_\infty$, means $\sup_{t\geq 0}\|w(t)\|$. Lemma 4.5 must be replaced by the following lemma.

Lemma 5.1. *One has*

$$\begin{aligned} &\left| v(t+h) + v(t-h) - 2\sum_{k=1}^{+\infty} \lambda_k z_k(t)\, y_k(t) \cos\left(2\sqrt{\mu_k}h\right)\right| \\ &\quad \leq 2\sqrt{h}K^2\|w\|_\infty^2 \left(\sqrt{\int_0^h u^2(t+\tau)\,\mathrm{d}\tau} + \sqrt{\int_{-h}^0 u^2(t+\tau)\,\mathrm{d}\tau}\right). \end{aligned}$$

Consequently the bound for δ in Lemma 4.6 becomes

$$\begin{aligned} |\delta(s,t,2h)| \;\leq\; & M\int_s^t \left(\left(u^2(\tau) + v^2(\tau+h) + v^2(\tau-h)\right) + \int_{-2h}^{2h} u^2(\tau+\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau \\ & + M\left(\sqrt{\int_{-2h}^{2h} u^2(t+\tau)\,\mathrm{d}\tau} + \sqrt{\int_{-2h}^{2h} u^2(s+\tau)\,\mathrm{d}\tau}\right). \end{aligned}$$

And as we have seen previously, owing to the $\mathbb{L}^2$ assumption upon u and v, such an estimate yields the uniform continuity of w (see (4.15) and (4.16)). Consequently, Theorem 5.1 is established. □

Remark 5.3. In fact no Ingham gap condition (see [3,10]) is required for these different theorems. So, one of the interest of Theorem 5.1 is to show that the precompactness property (which constitutes the fundamental technical task in [3]) can be derived from a general approach and that the gap condition used in this paper can be dropped.

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