

RELAXATION OF FREE-DISCONTINUITY ENERGIES WITH OBSTACLES

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Abstract. Given a Borel function ψ defined on a bounded open set Ω with Lipschitz boundary and $\varphi \in L^1(\partial\Omega, \mathcal{H}^{n-1})$, we prove an explicit representation formula for the L^1 lower semicontinuous envelope of Mumford-Shah type functionals with the obstacle constraint $u^+ \geq \psi \mathcal{H}^{n-1}$ a.e. on Ω and the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$.

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1. INTRODUCTION

Weak formulations of fracture mechanics theories for brittle hyperelastic media have been studied in the last years in the framework of free-discontinuity problems (see [1,10] and [2] for a more exhaustive list of references). In these models the state of a brittle body is described by a pair displacement-crack with total energy given by the sum of a bulk and a surface term, related to the (approximate) gradient and the set of (approximate) discontinuities of the deformation, respectively.

In particular, homogenization of brittle media with reinforcements may involve minimum problems for free-discontinuity energies with an obstacle condition. In case of bodies with a periodic distribution of perforations, intended in the sense of holes on which a Dirichlet or a unilateral obstacle condition is imposed, one is interested in analyzing the behaviour of the energy as the diameter of the perforations tends to 0.

In case of antiplane setting and selecting the Mumford-Shah energy as a prototype, one investigates the asymptotics as ε tends to 0 of Dirichlet boundary value problems for functionals of the type

$$\int_{\Omega} |\nabla u(x)|^p dx + \mathcal{H}^{n-1}(S_u) + \text{lower order terms} \quad u^+ \geq 0 \mathcal{H}^{n-1} \text{ a.e. on } \mathbf{E}_{\varepsilon} \quad (1.1)$$

where the open set $\Omega \subseteq \mathbf{R}^n$ represents a section in the cylindrical reference configuration of the body $\Omega \times \mathbf{R}$, $u \in GSBV(\Omega)$ is the antiplane displacement, and the set \mathbf{E}_{ε} is obtained periodically perforating the domain Ω with a rescaled copy of the reference hole E .

We remark that the formulation of the obstacle condition in (1.1) as an \mathcal{H}^{n-1} constraint is consistent with perforations \mathcal{L}^n negligible, and it is a natural generalization for such sets of the usual unilateral inequality in the \mathcal{L}^n sense (see Rem. 4.2 [15]).

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The homogenization problem above was addressed in the paper [15] via Γ -convergence methods. The convergence of the minimum problems associated to (1.1) to the corresponding problem for the Γ -limit is a byproduct of such analysis. The coercivity of functionals as in (1.1) is ensured by a well known result of Ambrosio (see Th. 2.1), instead the L^1 lower semicontinuity of free-discontinuity energies subject to an \mathcal{H}^{n-1} constraint has to be investigated.

In this paper we characterize the relaxed functional associated to an energy as in (1.1) under a general unilateral constraint. Namely, given a Borel function $\psi : \Omega \rightarrow \mathbf{R} \cup \{\pm\infty\}$, $p > 1$, we consider the functional

$$F_\psi(u, \Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) \quad \text{if } u \in GSBV(\Omega), u^+ \geq \psi \text{ } \mathcal{H}^{n-1} \text{ a.e. on } \Omega,$$

and $+\infty$ otherwise in $L^1(\Omega)$.

In order to deal with this problem we introduce a variational measure σ following the approach of De Giorgi for parametric Plateau problems with an obstacle (see Defs. (3.1), (3.2)). The main result proved in this paper is that the relaxed functional of F_ψ can be written in the form

$$\mathcal{F}_\psi(u, \Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) + \frac{1}{2}\sigma(\{x \in S_u : u^+(x) < \psi(x)\}) + \sigma(\{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\})$$

if $u \in GSBV(\Omega)$, $+\infty$ otherwise in $L^1(\Omega)$.

In particular, we show that the measure σ introduced above coincides with the analogous measure originally defined by De Giorgi for minimal surfaces with obstacles (see Sects. 2.3 and 3 for more exhaustive details).

An outline of the paper is as follows. In Section 2 we review some prerequisites needed in the sequel: We recall some properties of sets with finite perimeter, BV functions and De Giorgi’s measure σ . In Section 3 we introduce and study the properties of a variational measure which is naturally involved in the relaxation process. In particular, we compare it with De Giorgi’s one. In Section 4 we state and prove the main result justifying the relaxation formula above. The result is shown to be consistent with the addition of a Dirichlet boundary condition in Section 5. All results illustrated for the Mumford-Shah energy are extended in Section 6 to more general free-discontinuity energies.

2. NOTATION AND PRELIMINARIES

In the sequel $n \geq 1$ will be a fixed integer, and $p \in (1, +\infty)$ will be a fixed exponent.

2.1. Relaxation

We recall the notion of relaxation of a functional $F : X \rightarrow [0, +\infty]$ in a generic metric space (X, d) endowed with the topology induced by d (see [5,9]). The relaxed functional $\overline{F} : X \rightarrow [0, +\infty]$ is the lower semicontinuous envelope of F , that is

$$\overline{F}(u) = \sup \{G(u) : G \leq F, G \text{ } d\text{-lower semicontinuous}\}.$$

A different characterization holds for \overline{F} , namely

$$\overline{F}(u) = \inf \{ \liminf_{j \rightarrow +\infty} F(u_j) : u_j \rightarrow u \}.$$

Thus, given a candidate \mathcal{F} for the lower semicontinuous envelope of F , to show that it equals \overline{F} it suffices to prove the following two inequalities:

- (i) (*lower bound*) for every (u_j) converging to u in X , we have $\liminf_j F(u_j) \geq \mathcal{F}(u)$;
- (ii) (*upper bound*) there exists (u_j) converging to u in X such that $\limsup_j F(u_j) \leq \mathcal{F}(u)$.

2.2. BV functions

In this subsection we recall some basic definitions and results on sets of finite perimeter, BV, SBV and GSBV functions. We refer to the book [2] for all the results used throughout the whole paper, for which we will give a precise reference.

Let $A \subseteq \mathbf{R}^n$ be an open set, for every $u \in L^1(A)$ and $x \in A$, we define

$$u^+(x) = \inf \left\{ t \in \mathbf{R} : \lim_{r \rightarrow 0^+} r^{-n} \mathcal{L}^n(\{y \in B_r(x) : u(y) > t\}) = 0 \right\}$$

$$u^-(x) = \sup \left\{ t \in \mathbf{R} : \lim_{r \rightarrow 0^+} r^{-n} \mathcal{L}^n(\{y \in B_r(x) : u(y) < t\}) = 0 \right\},$$

with the convention $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. We remark that u^+ , u^- are Borel functions uniquely determined by the \mathcal{L}^n -equivalence class of u . If $u^+(x) = u^-(x)$ the common value is denoted by $\tilde{u}(x)$ or $\text{ap-}\lim_{y \rightarrow x} u(y)$ and it is said to be the *approximate limit* of u in x .

In particular, for every \mathcal{L}^n measurable set $E \subseteq \mathbf{R}^n$ it holds $(\chi_E)^+ = \chi_{E^+}$, where

$$E^+ = \{x \in \mathbf{R}^n : \limsup_{r \rightarrow 0^+} r^{-n} \mathcal{L}^n(E \cap B_r(x)) > 0\}.$$

We remark that for any $u \in L^1(A)$ and $s \leq t$, it holds

$$\{x \in A : u(x) \geq s\}^+ \supseteq \{x \in A : u^+(x) \geq t\}. \quad (2.1)$$

The set $S_u = \{x \in A : u^-(x) < u^+(x)\}$ is called the *set of approximate discontinuity points* of u and it is well known that $\mathcal{L}^n(S_u) = 0$. Let $x \in A \setminus S_u$ be such that $\tilde{u}(x) \in \mathbf{R}$, we say that u is *approximately differentiable* at x if there exists $L \in \mathbf{R}^n$ such that

$$\text{ap-}\lim_{y \rightarrow x} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{|y-x|} = 0. \quad (2.2)$$

If u is approximately differentiable at a point x , the vector L uniquely determined by (2.2), will be denoted by $\nabla u(x)$ and will be called the *approximate gradient* of u at x .

A function $u \in L^1(A)$ is said to be of *Bounded Variation* in A , in short $u \in BV(A)$, if its distributional derivative Du is a \mathbf{R}^n -valued finite Radon measure on A with mass $\|Du\|(A)$, called the *total variation* of u on A . If $u \in BV(A)$ denote by $D^a u$, $D^s u$ the absolutely and singular part of the Lebesgue's decomposition of Du w.r.t. $\mathcal{L}^n \llcorner A$, respectively. Then u turns out to be approximately differentiable a.e. on A (see Ths. 3.83 [2]), S_u to be *countably \mathcal{H}^{n-1} -rectifiable* (see Th. 3.78 [2]), and the values $u^+(x)$, $u^-(x)$ are finite and specified \mathcal{H}^{n-1} a.e. in A (see Rem. 3.79 [2]). Moreover, it holds

$$D^a u = \nabla u \mathcal{L}^n \llcorner A, \quad D^s u \llcorner S_u = (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner S_u,$$

where $\nu_u \in \mathbf{R}^n$ is an orientation for S_u .

We say that a \mathcal{L}^n measurable set $E \subseteq \mathbf{R}^n$ is of *finite perimeter* in A if $\chi_E \in BV(A)$, and we call the total variation of χ_E in A the *perimeter* of E in A , denoting it by $\text{Per}(E, A)$ and simply by $\text{Per}(E)$ if $A \equiv \mathbf{R}^n$. Setting for $t \in [0, 1]$

$$E^t = \left\{ x \in \mathbf{R}^n : \lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(E \cap B_r(x))}{\omega_n r^n} = t \right\},$$

and $\partial^* E = E \setminus (E^1 \cup E^0)$, it is well known that the set $\partial^* E$ is countably \mathcal{H}^{n-1} -rectifiable, and letting $\nu_{\partial^* E}$ be an orientation for it we have $D\chi_E = D\chi_E \llcorner \partial^* E = \nu_{\partial^* E} \mathcal{H}^{n-1} \llcorner \partial^* E$ (see Th. 3.59 [2]).

We say that $u \in BV(A)$ is a *Special Function of Bounded Variation* in A if $D^s u \equiv D^j u$ on A , in short $u \in SBV(A)$.

We say that $u \in L^1(A)$ is a *Generalized Special Function of Bounded Variation* in A , in short $u \in GSBV(A)$, if for every $M > 0$ the truncated function $(u \wedge M) \vee (-M) \in SBV(A)$.

Functions in $GSBV$ inherit from BV ones many properties: a generalized distributional derivative can be defined, they are approximately differentiable a.e. on A , and S_u turns out to be countably \mathcal{H}^{n-1} -rectifiable (see Th. 4.34 [2]).

The space $(G)SBV$ has been introduced by De Giorgi and Ambrosio [13] in connection with the weak formulation of the image segmentation model proposed by Mumford and Shah (see [18]). If $u \in GSBV(A)$ and $p \in (1, +\infty)$ the *Mumford-Shah energy* of u is defined as

$$MS_p(u, A) = \int_A |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u). \tag{2.3}$$

We recall the $GSBV$ compactness theorem due to Ambrosio in a form needed for our purposes (see Ths. 4.8 and 5.22 [2]).

Theorem 2.1. *Let $(u_j) \subset GSBV(A)$ and assume that for some $p \in (1, +\infty)$*

$$\sup_j (MS_p(u_j, A) + \|u_j\|_{L^1(A)}) < +\infty.$$

Then, there exist a subsequence (u_{j_k}) and a function $u \in GSBV(A)$ such that $u_{j_k} \rightarrow u$ a.e. in A , $\nabla u_{j_k} \rightarrow \nabla u$ weakly in $L^p(A; \mathbf{R}^n)$, $D^s u_{j_k} \llcorner S_{u_{j_k}} \rightarrow D^s u \llcorner S_u$ weakly- $$ in the sense of measures.*

In particular, if $\sup_j \|u_j\|_{L^\infty(A)} < +\infty$ then the cluster point u belongs to $SBV(A)$.

Eventually, if $\varphi : \mathbf{R}^n \rightarrow [0, +\infty)$ is a norm, then

$$\int_{S_u} \varphi(\nu_u) d\mathcal{H}^{n-1} \leq \liminf_k \int_{S_{u_{j_k}}} \varphi(\nu_{u_{j_k}}) d\mathcal{H}^{n-1}.$$

Eventually, in case $u \in GSBV(A)$ and $MS_p(u, A) < +\infty$ the values $u^+(x)$, $u^-(x)$ are finite and specified \mathcal{H}^{n-1} a.e. in A (see Th. 4.40 [2]).

To conclude the preliminaries on $GSBV$ functions we recall their characterization *via* restrictions to one-dimensional subspaces. For more details on the so called “slicing techniques” we refer both to Section 3.11 [2] and Chapter 4 [4].

Let $\xi \in \mathbf{S}^{n-1}$ be a fixed direction, denote by Π^ξ the orthogonal space to ξ . If $y \in \Pi^\xi$ and $E \subset \mathbf{R}^n$ define $E_y^\xi = \{t \in \mathbf{R} : y + t\xi \in E\}$ and $E_\xi = \{y \in \Pi^\xi : E_y^\xi \neq \emptyset\}$. Moreover, given $g : E \rightarrow \mathbf{R}$ define, for any $y \in E_\xi$, $g_{\xi,y} : E_y^\xi \rightarrow \mathbf{R}$ by $g_{\xi,y}(t) := g(y + t\xi)$.

Theorem 2.2. *Let $u \in GSBV(A)$, then $u_{\xi,y} \in GSBV(A_y^\xi)$ for all $\xi \in \mathbf{S}^{n-1}$ and \mathcal{H}^{n-1} a.e. $y \in A_\xi$. For such y we have:*

- (i) $u_{\xi,y}(t) = \nabla u(y + t\xi) \cdot \xi$ for \mathcal{L}^1 a.e. $t \in A_y^\xi$;
- (ii) $S_{u_{\xi,y}} = (S_u)_y^\xi$;
- (iii) $u_{\xi,y}^\pm(t) = u^\pm(y + t\xi)$ or $u_{\xi,y}^\pm(t) = u^\mp(y + t\xi)$ according to the cases $\langle \nu_u, \xi \rangle > 0$, $\langle \nu_u, \xi \rangle < 0$ (the case $\langle \nu_u, \xi \rangle = 0$ being negligible).

We conclude the subsection recalling a consequence of the Coarea formula (see Th. 2.93 [2]).

Proposition 2.3. *For any $u \in GSBV(A)$, for every $\xi \in \mathbf{S}^{n-1}$ and every open set $A' \subseteq A$ it holds*

$$\int_{A' \cap S_u} |\langle \nu_u(x), \xi \rangle| d\mathcal{H}^{n-1}(x) = \int_{A'_\xi} \mathcal{H}^0((S_u)_y^\xi) d\mathcal{H}^{n-1}(y). \tag{2.4}$$

2.3. De Giorgi's measure

In this subsection we recall the definition of an $(n - 1)$ -dimensional geometric measure which has been introduced in the study of obstacle problems for area-like functionals (see [6–8,12,14,19]).

Following the original definition by De Giorgi, Colombini and Piccinini [14], for any open set $A \subseteq \mathbf{R}^n$ and any set $E \subseteq \mathbf{R}^n$, we consider the set functions

$$\sigma^\varepsilon(E, A) = \inf \left\{ \text{Per}(D, A) + \frac{1}{\varepsilon} \mathcal{L}^n(D \cap A) : D = D^+, D \supseteq E \cap A \right\},$$

and

$$\sigma(E, A) = \sup_{\varepsilon > 0} \sigma^\varepsilon(E, A).$$

We collect below some properties of σ summarizing Theorems 2.3, 2.7, 2.8 and 4.10 of Chapter 4 [14].

Theorem 2.4. *Let $A \subseteq \mathbf{R}^n$ be an open set and $E \subseteq \mathbf{R}^n$.*

(a) *σ is a regular Borel measure such that*

$$c_1(n) \mathcal{H}^{n-1}(E \cap A) \leq \sigma(E, A) \leq c_2(n) \mathcal{H}^{n-1}(E \cap A) \quad (2.5)$$

for two positive constants c_1, c_2 depending only on n .

(b) *If $E \subseteq A$, then*

$$\sigma(E, A) = \sigma(E). \quad (2.6)$$

In particular, for every set $F \subseteq \mathbf{R}^n$ it holds $\sigma(F, A) = \sigma(F \cap A)$.

(c) *If E is a \mathcal{H}^{n-1} -rectifiable set, then*

$$\sigma(E, A) = 2 \mathcal{H}^{n-1}(E \cap A). \quad (2.7)$$

Remark 2.5. The papers [16,17] study in details the relationship between σ and \mathcal{H}^{n-1} . In particular, an example disproves the equality in (2.7) in general.

Moreover, the inequality $c_2(n) \leq n\omega_n/\omega_{n-1}$ is established, with ω_k the \mathcal{L}^k measure of the unit ball in \mathbf{R}^k . A further example shows the optimality of that bound for $n = 2$, and some hints are given in order to generalize such a result for arbitrary $n \geq 2$. No lower bound for $c_1(n)$ is to our knowledge explicit.

We now state alternative characterizations of σ , the first proved in [8] the others in [6].

Proposition 2.6. *For any open set $A \subseteq \mathbf{R}^n$ and any set $E \subseteq \mathbf{R}^n$, we have*

$$\begin{aligned} \sigma(E, A) &= \sup_{\varepsilon > 0} \left(\inf \left\{ \text{Per}(D, A) + \frac{1}{\varepsilon} \mathcal{L}^n(D \cap A) : D \text{ open, } D \supseteq E \cap A \right\} \right) \\ &= \sup_{\varepsilon > 0} \left(\inf \left\{ \text{Per}(D, A) + \frac{1}{\varepsilon} \mathcal{L}^n(D \cap A) : D \text{ } \mathcal{L}^n \text{ measurable, } \mathcal{H}^{n-1}(E \cap A \setminus D^+) = 0 \right\} \right) \\ &= \sup_{\varepsilon > 0} \left(\inf \left\{ \|Du\|(A) + \frac{1}{\varepsilon} \int_A |u| \, dx : u \in BV(A), u^+ \geq 1 \text{ } \mathcal{H}^{n-1} \text{ a.e. on } E \cap A \right\} \right). \end{aligned}$$

Remark 2.7. The first characterization of σ provided in Proposition 2.6 entails that for any set E for which $\sigma(E, A) < +\infty$ we can find a family of open sets (D_ε) admissible for the minimum problems $\sigma^\varepsilon(E, A)$ satisfying $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon^2)$ and

$$\sigma(E, A) = \text{Per}(D_\varepsilon, A) + o(1).$$

The following result clarifies how De Giorgi's measure σ arises in the relaxation of obstacle problems with linear growth (see Th. 3.4 in Chap. 4 [14] and Th. 6.1 [6], moreover Th. 7.1 [6] addresses the case when a Dirichlet boundary datum is added). To avoid technicalities we state it in the simplest case.

Theorem 2.8. *Given an open set $A \subseteq \mathbf{R}^n$ and a Borel function $\psi : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\pm\infty\}$, consider*

$$G_\psi(u, A) = \int_A |\nabla u| dx, \text{ if } u \in W^{1,1}(A), \tilde{u} \geq \psi \mathcal{H}^{n-1} \text{ a.e. on } A, \tag{2.8}$$

and $+\infty$ otherwise in $L^1(A)$. Then, the lower semicontinuous envelope of G_ψ in the L^1 topology is given by

$$\mathcal{G}_\psi(u, A) = \|Du\|(A) + \int_A [(\psi - u^+) \vee 0] d\sigma$$

if $u \in BV(A)$, $+\infty$ otherwise in $L^1(A)$.

3. A VARIATIONAL MEASURE

In this section we introduce a regular Borel measure on any open set $A \subseteq \mathbf{R}^n$ following one of the characterizations of the measure σ provided in Proposition 2.6.

According to the definition given by [6], for any $\varepsilon > 0$ and for any set $E \subseteq \mathbf{R}^n$ we consider the set functions $\sigma_{MS}^\varepsilon(E, A)$ defined by

$$\inf \left\{ MS_p(u, A) + \frac{1}{\varepsilon} \int_A |u|^p dx : u \in SBV(A), u^+ \geq 1 \mathcal{H}^{n-1} \text{ a.e. on } E \cap A \right\} \tag{3.1}$$

and

$$\sigma_{MS}(E, A) = \sup_{\varepsilon > 0} \sigma_{MS}^\varepsilon(E, A), \tag{3.2}$$

with the convention of dropping the dependence on A when $A = \mathbf{R}^n$.

Remark 3.1. Similarly to Remark 2.7 the very definition of σ_{MS} entails that for any set E for which $\sigma_{MS}(E, A) < +\infty$ we can find a family of functions $(v_\varepsilon) \subseteq SBV(A)$ admissible for the minimum problems $\sigma_{MS}^\varepsilon(E, A)$ satisfying $\|v_\varepsilon\|_{L^p(A)}^p = o(\varepsilon^2)$ and

$$\sigma_{MS}(E, A) = MS_p(v_\varepsilon, A) + o(1).$$

It turns out that the set function σ_{MS} introduced above coincides with the measure σ . To explain this fact we notice that the penalization of the L^p norm forces minimizing functions for $\sigma_{MS}^\varepsilon(E, A)$ to make a transition from 1 to 0 in a thinner and thinner set enclosing E . Therefore the superlinearity in the bulk term makes energetically more convenient for minimizing functions to have a discontinuity in a neighbourhood of E rather than having a high gradient energy. Finally, note that the Mumford-Shah and the total variation functionals coincide on sets of finite perimeter.

Proposition 3.2. *For any open set $A \subseteq \mathbf{R}^n$, for any set $E \subseteq \mathbf{R}^n$, we have*

$$\sigma_{MS}(E, A) = \sigma(E, A).$$

Proof. Let A be a fixed open set throughout all the proof. Given a set E , taking into account that for any measurable set $D \subseteq \mathbf{R}^n$ with finite perimeter which is admissible for $\sigma(E, A)$, the function $u = \chi_D \in SBV(A)$ is admissible for $\sigma_{MS}(E, A)$ and $MS_p(u, A) = \text{Per}(D, A)$, we have $\sigma_{MS}(E, A) \leq \sigma(E, A)$.

In order to get the opposite inequality it suffices to consider a set E such that $\sigma_{MS}(E, A) < +\infty$. Fixed a family of functions (v_ε) as in Remark 3.1, the strategy to prove the inequality $\sigma_{MS} \geq \sigma$ relies on finding suitable superlevel sets of v_ε such that their perimeters are bounded above by the Mumford-Shah energies of v_ε , their \mathcal{L}^n measures are negligible with respect to ε , and the set $E \cap A$ is contained \mathcal{H}^{n-1} a.e. in such superlevel sets.

Let $\eta > 0$, then by Remark 3.1 we can find $v_\varepsilon \in SBV(A)$ such that $v_\varepsilon^+(x) \geq 1$ \mathcal{H}^{n-1} a.e. on $E \cap A$ and

$$MS_p(v_\varepsilon, A) + \frac{1}{\varepsilon^2} \int_A |v_\varepsilon|^p dx \leq \sigma_{MS}(E, A) + \eta. \tag{3.3}$$

Up to passing to $0 \vee v_\varepsilon \wedge 1$ we may also assume that $0 \leq v_\varepsilon \leq 1$. By the BV Coarea formula (see Th. 3.40 [2]) we may choose $z_\varepsilon \in (\varepsilon^{\frac{1}{2p}}, 1)$ such that

$$(1 - \varepsilon^{\frac{1}{2p}}) \text{Per}(\{x \in A : v_\varepsilon(x) > z_\varepsilon\}, A) \leq \int_{\varepsilon^{\frac{1}{2p}}}^1 \text{Per}(\{x \in A : v_\varepsilon(x) > t\}, A) dt \leq \|Dv_\varepsilon\|(\{x \in A : v_\varepsilon(x) > \varepsilon^{\frac{1}{2p}}\}). \tag{3.4}$$

Letting $D_\varepsilon := \{x \in A : v_\varepsilon(x) > z_\varepsilon\}$ and $A_\varepsilon := \{x \in A : v_\varepsilon(x) > \varepsilon^{\frac{1}{2p}}\}$, Hölder inequality, the fact that $|v_\varepsilon^+(x) - v_\varepsilon^-(x)| \leq 1$ \mathcal{H}^{n-1} a.e., and (3.4) imply

$$(1 - \varepsilon^{\frac{1}{2p}}) \text{Per}(D_\varepsilon, A) \leq \int_{A_\varepsilon} |\nabla v_\varepsilon| dx + \mathcal{H}^{n-1}(S_{v_\varepsilon}) \leq \mathcal{L}^n(A_\varepsilon)^{\frac{p-1}{p}} \|\nabla v_\varepsilon\|_{L^p(A)} + \mathcal{H}^{n-1}(S_{v_\varepsilon}). \tag{3.5}$$

Moreover, by (3.3)

$$\mathcal{L}^n(A_\varepsilon) \varepsilon^{\frac{1}{2}} \leq \int_A |v_\varepsilon|^p dx \leq (\sigma_{MS}(E, A) + \eta) \varepsilon^2,$$

from which we infer $\mathcal{L}^n(A_\varepsilon) = o(\varepsilon)$ and $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon)$, and thus D_ε has finite perimeter in A .

In particular, using (3.3), for ε small enough (3.5) rewrites as

$$\text{Per}(D_\varepsilon, A) \leq \sigma_{MS}(E, A) + 2\eta. \tag{3.6}$$

Furthermore, $D_\varepsilon^+ \supseteq \{x \in A : v_\varepsilon^+(x) \geq 1\}$ by (2.1), and thus $\mathcal{H}^{n-1}((E \cap A) \setminus D_\varepsilon^+) = 0$. Hence, D_ε is admissible for $\sigma^\varepsilon(E, A)$ and, taking (3.6) into account, for ε small enough it holds

$$\sigma^\varepsilon(E, A) \leq \text{Per}(D_\varepsilon, A) + \frac{\mathcal{L}^n(D_\varepsilon)}{\varepsilon} \leq \sigma_{MS}(E, A) + 3\eta.$$

Taking first the supremum on ε and then letting $\eta \rightarrow 0^+$ we get the desired inequality. □

Remark 3.3. Following [8] and [6] one could equivalently define $\sigma_{MS}(E, A)$ as the supremum of the set functions

$$\inf \left\{ MS_p(u, A) + \frac{1}{\varepsilon} \int_A |u|^p dx : u \in SBV(A), u \geq 1 \text{ } \mathcal{L}^n \text{ a.e. on an open set } U \supseteq E \cap A \right\}. \tag{3.7}$$

Actually, by using the previous proposition and exploiting the equivalence between the two definitions already proven for the measure σ one gets that the final measure is the same (see Prop. 2.6).

Remark 3.4. The proof of Proposition 3.2 shows that the measure σ coincides also with the one obtained by substituting in definitions (3.1), (3.2) the Mumford-Shah energy with any of the form

$$\int_A f(\nabla u) dx + \mathcal{H}^{n-1}(S_u \cap A),$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is such that

$$c_1 |\xi|^p \leq f(\xi) \leq c_2 |\xi|^p$$

for every $\xi \in \mathbf{R}^n$, for some constants $c_1, c_2 > 0$ (see also Sect. 6).

We now introduce a Borel measure accounting for a generic obstacle. Let $\psi : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\pm\infty\}$, A an open set in \mathbf{R}^n , and $E \subseteq \mathbf{R}^n$, for any $\varepsilon > 0$ define $\sigma_{MS}^\varepsilon(E, A, \psi)$ as

$$\inf \left\{ MS_p(u, A) + \frac{1}{\varepsilon} \int_A |u|^p dx : u \in SBV(A), u^+ \geq \psi \mathcal{H}^{n-1} \text{ a.e. on } E \cap A \right\}, \tag{3.8}$$

and as usual set $\sigma_{MS}(E, A, \psi) = \sup_{\varepsilon > 0} \sigma_{MS}^\varepsilon(E, A, \psi)$. With a slight abuse of notation when $\psi(x) \equiv c$ we denote $\sigma_{MS}(\cdot, A, \psi)$ simply by $\sigma_{MS}(\cdot, A, c)$. With this notation then $\sigma_{MS}(\cdot, A, 1) = \sigma_{MS}(\cdot, A)$.

Assuming ψ to be a Borel function, one can push forward the arguments used in Proposition 3.2 and prove the following description of $\sigma_{MS}(\cdot, A, \psi)$ on Borel sets.

Proposition 3.5. *For any open set A , for any Borel function ψ , and any Borel set $E \subseteq \mathbf{R}^n$ we have*

$$\sigma_{MS}(E, A, \psi) = \sigma_{MS}(\{x \in E : \psi(x) > 0\}, A).$$

Proof. The open set A and the Borel function ψ will be fixed throughout the whole proof.

Given a Borel set E we first prove that $\sigma_{MS}(E, A, \psi) \geq \sigma_{MS}(\{x \in E : \psi(x) > 0\}, A)$. Thus, it is not restrictive to assume $\sigma_{MS}(E, A, \psi) < +\infty$. With $\lambda \in (0, 1)$ fixed, we claim that

$$\sigma_{MS}(E, A, \psi) \geq \sigma_{MS}(\{x \in E : \psi(x) > \lambda\}, A). \tag{3.9}$$

It is clear that the required inequality will easily follow letting $\lambda \rightarrow 0^+$ and using the fact already proved that $\sigma_{MS}(\cdot, A)$ is a regular Borel measure.

In order to get (3.9) we will exploit the same construction and arguments introduced in the proof of Proposition 3.2 complemented with the Borel regularity assumptions.

Let $\eta > 0$ be fixed, reasoning as in Remark 3.1 one can consider functions $w_\varepsilon \in SBV(A)$ such that $w_\varepsilon^+(x) \geq \psi(x) \mathcal{H}^{n-1}$ a.e. on $E \cap A$ and

$$MS_p(w_\varepsilon, A) + \frac{1}{\varepsilon^2} \int_A |w_\varepsilon|^p dx \leq \sigma_{MS}(E, A, \psi) + \eta. \tag{3.10}$$

Arguing as in the proof of Proposition 3.2 with respect to the functions $v_\varepsilon := 0 \vee (w_\varepsilon/\lambda) \wedge 1$, one can find superlevel sets $D_\varepsilon = \{x \in A : w_\varepsilon(x) > z'_\varepsilon\}$ with $z'_\varepsilon \in (\lambda\varepsilon^{\frac{1}{2p}}, \lambda)$ such that

$$\text{Per}(D_\varepsilon, A) + \frac{1}{\varepsilon} \mathcal{L}^n(D_\varepsilon) \leq \sigma_{MS}(E, A, \psi) + 3\eta. \tag{3.11}$$

Indeed, taking into account (3.10) and the fact that

$$\frac{1}{\varepsilon^2} \int_A |v_\varepsilon|^p dx \leq \frac{1}{\lambda^p \varepsilon^2} \int_A |w_\varepsilon|^p dx$$

if we set $D_\varepsilon = \{x \in A : v_\varepsilon(x) > z_\varepsilon\}$ with any $z_\varepsilon \in (\varepsilon^{\frac{1}{2p}}, 1)$, we get $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon)$. We now choose $z_\varepsilon \in (\varepsilon^{\frac{1}{2p}}, 1)$ such that (3.5) holds true for v_ε defined as above. Since $\mathcal{L}^n(D_\varepsilon) = o(\varepsilon)$, $\|\nabla v_\varepsilon\|_{L^p(A)} \leq \lambda^{-1} \|\nabla w_\varepsilon\|_{L^p(A)}$ and $\mathcal{H}^{n-1}(S_{v_\varepsilon}) \leq \mathcal{H}^{n-1}(S_{w_\varepsilon})$, it is enough to take $z'_\varepsilon := \lambda z_\varepsilon$ to obtain superlevel sets of the initial functions w_ε with the property (3.11).

Moreover, since for \mathcal{H}^{n-1} a.e. $x \in A \cap \{x \in E : \psi(x) > \lambda\}$ it holds $w_\varepsilon^+(x) \geq \psi(x) > \lambda$, by definition $v_\varepsilon^+(x) = 1$. Hence, taking (2.1) into account, $D_\varepsilon^+ \supseteq \{x \in A : v_\varepsilon^+(x) \geq 1\} \supseteq A \cap \{x \in E : \psi(x) > \lambda\}$ and thus the functions $u_\varepsilon = \chi_{D_\varepsilon}$ are admissible for $\sigma^\varepsilon(\{x \in E : \psi(x) > \lambda\}, A)$. Letting eventually $\eta \rightarrow 0^+$ in (3.11), we get (3.9).

Notice that the same argument implies that for any positive constant c we have

$$\sigma_{MS}(E, A, c) = \sigma_{MS}(E, A). \tag{3.12}$$

In order to prove the inverse inequality let us consider a Borel set E such that $\sigma_{MS}(\{x \in E : \psi(x) > 0\}, A) < +\infty$. This condition implies at once that the set $A \cap \{x \in E : \psi(x) = +\infty\}$ is \mathcal{H}^{n-1} negligible (see (2.5)). Setting $E_i = \{x \in E : i + 1 \geq \psi(x) > i\}$ for $i \in \mathbf{N}$, by the standard additivity property of the Borel measure $\sigma_{MS}(\cdot, A)$ we have

$$\sigma_{MS}(\{x \in E : \psi(x) > 0\}, A) = \sum_{i=0}^{+\infty} \sigma_{MS}(E_i, A).$$

For $\eta > 0$ fixed, let v_ε^i be almost optimal for $\sigma_{MS}(E_i, A, i + 1)$, that is $(v_\varepsilon^i)^+(x) \geq i + 1$ \mathcal{H}^{n-1} a.e. on E_i and

$$MS_p(v_\varepsilon^i, A) + \frac{1}{\varepsilon} \int_A |v_\varepsilon^i|^p dx \leq \sigma_{MS}(E_i, A) + \frac{\eta}{2^i}, \tag{3.13}$$

recalling that $\sigma_{MS}(E_i, A, i + 1) = \sigma_{MS}(E_i, A)$ by (3.12).

Set $u_\varepsilon^k := \sup_{0 \leq i \leq k} v_\varepsilon^i$. Then $u_\varepsilon^k \in SBV(A)$ and

$$MS_p(u_\varepsilon^k, A) + \frac{1}{\varepsilon} \int_A |u_\varepsilon^k|^p dx \leq \sum_{i=0}^k (MS_p(v_\varepsilon^i, A) + \frac{1}{\varepsilon} \int_A |v_\varepsilon^i|^p dx) \leq \sum_{i=0}^k \sigma_{MS}(E_i, A) + 2\eta. \tag{3.14}$$

In particular, (u_ε^k) is a non-decreasing sequence satisfying the hypotheses of the GSBV compactness Theorem 2.1, so that there exists a function $u_\varepsilon \in SBV(A)$ such that $u_\varepsilon^k \rightarrow u_\varepsilon$ in $L^1(A)$. Thus, from (3.14) it follows

$$MS_p(u_\varepsilon, A) + \frac{1}{\varepsilon} \int_A |u_\varepsilon|^p dx \leq \liminf_{k \rightarrow +\infty} \left(MS_p(u_\varepsilon^k, A) + \frac{1}{\varepsilon} \int_A |u_\varepsilon^k|^p dx \right) \leq \sigma_{MS}(\{x \in E : \psi(x) > 0\}, A) + 2\eta.$$

Moreover, since $A \cap \{x \in E : \psi(x) > 0\} = \cup_{i \geq 0} (A \cap E_i)$, for \mathcal{H}^{n-1} a.e. $z \in A \cap \{x \in E : \psi(x) > 0\}$ there exists $i \in \mathbf{N}$ such that $z \in A \cap E_i$, and then $u_\varepsilon^+(z) \geq (u_\varepsilon^{i+1})^+(z) \geq i + 1 \geq \psi(z)$. Eventually, u_ε is admissible as a test function for $\sigma_{MS}^\varepsilon(E, A, \psi)$ and the inequality follows as usual. \square

4. RELAXATION RESULT

Given an open bounded set Ω and a Borel function $\psi : \Omega \rightarrow \mathbf{R} \cup \{\pm\infty\}$, we study the lower semicontinuous envelope of the functional $F_\psi : L^1(\Omega) \rightarrow [0, +\infty]$ defined by

$$F_\psi(u, \Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) \quad \text{if } u \in GSBV(\Omega), u^+ \geq \psi \text{ } \mathcal{H}^{n-1} \text{ a.e. on } \Omega, \tag{4.1}$$

and $+\infty$ otherwise in $L^1(\Omega)$.

Building on what has been shown in Section 3 we are able to prove the main result of the paper.

Theorem 4.1. *Let F_ψ be as in (4.1), then its lower semicontinuous envelope in the L^1 topology is given by*

$$\mathcal{F}_\psi(u, \Omega) = \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) + \frac{1}{2} \sigma(\{x \in S_u : u^+(x) < \psi(x)\}) + \sigma(\{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}) \tag{4.2}$$

if $u \in GSBV(\Omega)$, $+\infty$ otherwise in $L^1(\Omega)$.

In the sequel it is not restrictive to presume the existence of $w \in GSBV(\Omega)$ such that $F_\psi(w, \Omega) < +\infty$, being otherwise $F_\psi \equiv \mathcal{F}_\psi \equiv +\infty$.

For such w 's we have $\{x \in \Omega : \psi(x) = +\infty\} \subseteq \{x \in \Omega : w^+(x) = +\infty\}$, which implies

$$\mathcal{H}^{n-1}(\{x \in \Omega : \psi(x) = +\infty\}) = 0 \tag{4.3}$$

(see Th. 4.40 [2]).

To prove Theorem 4.1 we address separately the lower and upper bound inequalities.

Proposition 4.2. *For every u and (u_j) in $L^1(\Omega)$ such that $u_j \rightarrow u$ in $L^1(\Omega)$ we have*

$$\liminf_j F_\psi(u_j, \Omega) \geq \mathcal{F}_\psi(u, \Omega).$$

Proof. First notice that we may assume $\liminf_j F_\psi(u_j, \Omega)$ to be finite being the result trivial otherwise; then by Ambrosio’s Theorem 2.1 we have $u \in GSBV(\Omega)$. Moreover, we may assume the inferior limit above to be a limit up to extracting a subsequence which we do not relabel for convenience.

We claim the following three estimates to hold true for every open set $A \subseteq \Omega$

$$\liminf_j F_\psi(u_j, A) \geq MS_p(u, A), \tag{4.4}$$

$$\liminf_j F_\psi(u_j, A) \geq \sigma(A \cap \{x \in S_u : u^+(x) < \psi(x)\}), \tag{4.5}$$

$$\liminf_j F_\psi(u_j, A) \geq \sigma(A \cap \{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}). \tag{4.6}$$

Given them for granted the result follows by standard measure theoretic arguments (see Prop. 1.16 [4]). Indeed, set $\Sigma_u = \{x \in \Omega : u^+(x) < \psi(x)\}$, then from (4.4)–(4.6) and taking into account (2.7), for any $\lambda, \mu \in [0, 1]$, $\lambda + \mu \leq 1$ it follows

$$\begin{aligned} \liminf_j F_\psi(u_j, A) &\geq \lambda \int_A |\nabla u|^p dx + \lambda \mathcal{H}^{n-1}(A \cap (S_u \setminus \Sigma_u)) \\ &\quad + (\lambda + 2\mu) \mathcal{H}^{n-1}(A \cap S_u \cap \Sigma_u) + (1 - \lambda - \mu) \sigma(A \cap (\Sigma_u \setminus S_u)). \end{aligned}$$

Being the left hand side above a superadditive set function on disjoint open sets of Ω and the right hand side sum of orthogonal Radon measures, we can pass to the supremum on λ, μ separately on each term and infer

$$\begin{aligned} \liminf_j F_\psi(u_j, \Omega) &\geq \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u \setminus \Sigma_u) + 2\mathcal{H}^{n-1}(S_u \cap \Sigma_u) + \sigma(\Sigma_u \setminus S_u) \\ &= \int_\Omega |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u) + \frac{1}{2} \sigma(S_u \cap \Sigma_u) + \sigma(\Sigma_u \setminus S_u), \end{aligned}$$

which gives the thesis.

Since (4.4) follows immediately by Ambrosio’s Theorem 2.1, to conclude the proof we are left with showing the validity of (4.5) and (4.6).

Step 1. *Proof of (4.5).* We begin with proving the inequality in the one-dimensional case which reads as follows

$$\liminf_j \int_\Omega |\dot{u}_j|^p dt + \mathcal{H}^0(S_{u_j}) \geq 2\mathcal{H}^0(S_u \cap \Sigma_u).$$

We notice that the approximating functions are forced to make a transition from the trace values $u^\pm(\bar{t})$ to the obstacle constraint $\psi(\bar{t})$ in any neighbourhood I of a discontinuity point \bar{t} of u where the constraint is violated, that is $u^+(\bar{t}) < \psi(\bar{t})$. Hence, to prove the estimate above we will quantify the cost of this transition, and show that it is energetically convenient for the approximating functions to have asymptotically at least 2 discontinuity points in I .

Recall that we have assumed

$$\liminf_j \int_\Omega |\dot{u}_j|^p dt + \mathcal{H}^0(S_{u_j}) = \lim_j \int_\Omega |\dot{u}_j|^p dt + \mathcal{H}^0(S_{u_j}) \leq M < +\infty, \tag{4.7}$$

which gives $u_j \in SBV(\Omega)$ for every $j \in \mathbf{N}$; moreover, for a subsequence not relabeled for convenience, we suppose $u_j \rightarrow u$ \mathcal{L}^1 a.e. in Ω .

We claim that for j sufficiently big

$$\mathcal{H}^0(A \cap S_{u_j}) \geq 2\mathcal{H}^0(A \cap S_u \cap \Sigma_u). \tag{4.8}$$

With fixed \bar{t} in the finite set $S_u \cap \Sigma_u$, there exists $\delta > 0$ such that $I_\delta = (\bar{t} - \delta, \bar{t} + \delta) \subset\subset A$ and $I_\delta \cap S_u = \{\bar{t}\}$. Furthermore, being u a Sobolev function on $(\bar{t} - \delta, \bar{t})$ and $(\bar{t}, \bar{t} + \delta)$ separately, we choose $\delta > 0$ sufficiently small such that

$$u(t) \leq u^+(\bar{t}) + \varepsilon$$

for every $t \in I_\delta$, with $\varepsilon \in (0, (\psi(\bar{t}) - u^+(\bar{t}))/4)$.

It is clear that (4.8) follows provided we show

$$\liminf_j \mathcal{H}^0(I_\delta \cap S_{u_j}) \geq 2. \tag{4.9}$$

Arguing by contradiction we first observe that, up to a subsequence, $I_\delta \cap S_{u_j} = \{t_j\}$ since by lower semicontinuity $\liminf_j \mathcal{H}^0(I_\delta \cap S_{u_j}) \geq 1$ (see Th. 2.1). In the sequel we show that this implies

$$\liminf_j \int_{I_\delta} |\dot{u}_j|^p dt \geq \delta^{1-p} \left(\frac{\psi(\bar{t}) - u^+(\bar{t})}{4} \right)^p. \tag{4.10}$$

Select two points $s_1 \in (\bar{t} - \delta, \bar{t}) \setminus \cup_j \{t_j\}$ and $s_2 \in (\bar{t}, \bar{t} + \delta) \setminus \cup_j \{t_j\}$ such that $u_j(s_i) \rightarrow u(s_i)$ for $i = 1, 2$, then for j sufficiently big and $i = 1, 2$

$$u_j(s_i) \leq u(s_i) + \varepsilon \leq u^+(\bar{t}) + 2\varepsilon.$$

Then an easy computation yields

$$\int_{I_\delta} |\dot{u}_j|^p dt \geq \max \left\{ \int_{s_1}^{\bar{t}} |\dot{u}_j|^p dt, \int_{\bar{t}}^{s_2} |\dot{u}_j|^p dt \right\} \geq \delta^{1-p} \left(\frac{\psi(\bar{t}) - u^+(\bar{t})}{4} \right)^p.$$

Indeed, either $t_j \neq \bar{t}$ and thus $u_j(\bar{t}) \geq \psi(\bar{t})$ or $t_j = \bar{t}$ and one between the one sided traces $u_j(\bar{t}^\pm) = \lim_{t \rightarrow \bar{t}^\pm} u_j(t)$ equals $u_j^+(\bar{t})$ which is bigger than or equal to $\psi(\bar{t})$. Assume for instance $t_j > \bar{t}$, then $u_j \in W^{1,p}(s_1, \bar{t})$ and by applying Jensen's inequality we infer

$$\int_{s_1}^{\bar{t}} |\dot{u}_j|^p dt \geq |\bar{t} - s_1|^{1-p} \left(\frac{u_j(\bar{t}) - u_j(s_1)}{2} \right)^p \geq \delta^{1-p} \left(\frac{\psi(\bar{t}) - u^+(\bar{t})}{4} \right)^p.$$

The remaining cases can be worked out similarly (see Fig. 1).

From (4.10) we get a contradiction since (4.7) is violated for δ sufficiently small. Thus (4.9) holds true and eventually (4.8) follows.

To recover the multi-dimensional setting $n > 1$ we use the standard integral geometric reduction technique for which we have introduced some notation in Section 2.2. With fixed $\xi \in \mathbf{S}^{n-1}$, Proposition 2.3 and Fatou's lemma imply for \mathcal{H}^{n-1} a.e. $y \in A_\xi$

$$\liminf_j \int_{A_y^\xi} |(\dot{u}_j)_{\xi,y}|^p dt + \mathcal{H}^0(A_y^\xi \cap S_{(u_j)_{\xi,y}}) < +\infty. \tag{4.11}$$

Moreover, Theorem 2.2 and Fubini's theorem imply for \mathcal{H}^{n-1} a.e. $y \in A_\xi$

$$(u_j)_{\xi,y}, u_{\xi,y} \in SBV(A_y^\xi), (u_j)_{\xi,y} \rightarrow u_{\xi,y} \text{ in } L^1(A_y^\xi). \tag{4.12}$$

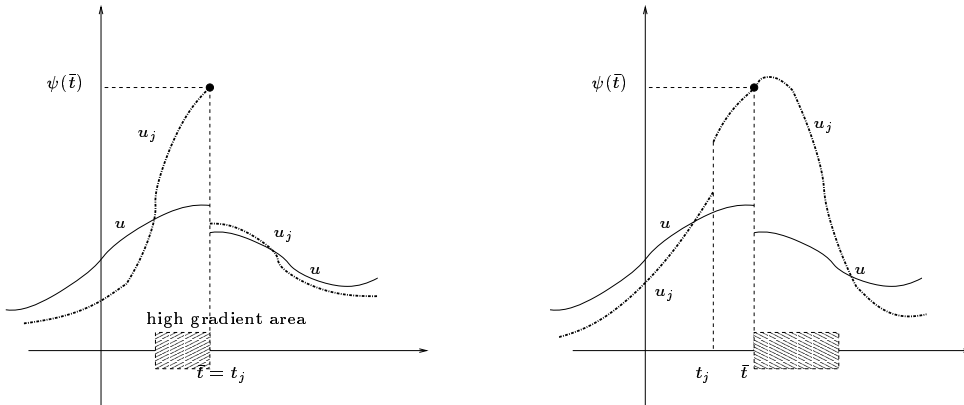


FIGURE 1. Energy concentration of u_j according to two possible configurations.

With fixed $y \in A_\xi$ satisfying (4.11) and (4.12), we may also suppose $(u_j)_{\xi,y} \rightarrow u_{\xi,y}$ \mathcal{L}^1 a.e. in A_y^ξ , up to extracting a subsequence. Then, by (4.8) we have

$$\liminf_j \mathcal{H}^0(A_y^\xi \cap S_{(u_j)_{\xi,y}}) \geq 2\mathcal{H}^0(A_y^\xi \cap S_{u_{\xi,y}} \cap (\Sigma_u)_y^\xi),$$

which, together with Fatou’s lemma and Proposition 2.3, give

$$\begin{aligned} \liminf_j F_\psi(u_j, A) &\geq \int_{A_\xi} \liminf_j \left(\int_{A_y^\xi} |(u_j)_{\xi,y}|^p dt + \mathcal{H}^0(A_y^\xi \cap S_{(u_j)_{\xi,y}}) \right) d\mathcal{H}^{n-1}(y) \\ &\geq 2 \int_{A_\xi} \mathcal{H}^0(A_y^\xi \cap (S_u)_y^\xi \cap (\Sigma_u)_y^\xi) d\mathcal{H}^{n-1}(y) = 2 \int_{A \cap S_u \cap \Sigma_u} |\langle \nu_u(x), \xi \rangle| d\mathcal{H}^{n-1}(x). \end{aligned} \tag{4.13}$$

In the last equality we took advantage of the \mathcal{H}^{n-1} -countably rectifiability of $S_u \cap \Sigma_u$ which is inherited by that of S_u .

Eventually, by passing to the supremum on a countable dense set $(\xi_k)_{k \in \mathbb{N}} \in \mathbf{S}^{n-1}$ in (4.13) and using the monotone convergence theorem we deduce

$$\liminf_j F_\psi(u_j, A) \geq 2\mathcal{H}^{n-1}(A \cap S_u \cap \Sigma_u),$$

which is equivalent to (4.5) thanks to (2.7).

Step 2. *Proof of (4.6).* We will first prove a localized “rough” version of inequality (4.6) (see (4.15)). To do that we will exploit the auxiliary obstacle function $\psi - u$ outside the jump set of u , and use $u_j - u$ as admissible test function in the formulation of the related obstacle problem for the Mumford-Shah functional. This estimate will provide straightforward the finiteness of $\sigma(A \cap (\Sigma_u \setminus S_u))$, which in turn can be used to improve the former inequality and get (4.6).

Notice that this strategy can be exploited only outside the discontinuity set S_u of the target function.

Let A' be an open set in A . An easy computation shows that for any $\varepsilon > 0$ it holds

$$\begin{aligned}
 F_\psi(u_j, A) &\geq 2^{1-p} \int_{A'} |\nabla(u_j - u)|^p dx - \int_{A'} |\nabla u|^p dx + \mathcal{H}^{n-1}(A' \cap S_{u_j}) \\
 &\geq 2^{1-p} \int_{A'} |\nabla(u_j - u)|^p dx + \mathcal{H}^{n-1}(A' \cap S_{u_j - u}) - MS_p(u, A') \\
 &\geq \inf \left\{ 2^{1-p} \int_{A'} |\nabla v|^p dx + \mathcal{H}^{n-1}(A' \cap S_v) + \frac{1}{\varepsilon} \int_{A'} |v|^p dx : v^+ \geq \psi - \tilde{u} \text{ } \mathcal{H}^{n-1} \text{ a.e. on } A' \setminus S_u \right\} \\
 &\quad - \frac{1}{\varepsilon} \int_{A'} |u_j - u|^p dx - MS_p(u, A'), \tag{4.14}
 \end{aligned}$$

where in the second inequality we used that $S_{u_j - u} \subseteq S_u \cup S_{u_j}$, and in the third that $(u_j - u)^+(x) = u_j^+(x) - \tilde{u}(x) \geq \psi(x) - \tilde{u}(x)$ for \mathcal{H}^{n-1} a.e. $x \in \Omega \setminus S_u$.

Passing first to the inferior limit as $j \rightarrow +\infty$, and then taking the supremum on $\varepsilon > 0$ in (4.14), by Remark 3.4 and Proposition 3.5 we get

$$\liminf_j F_\psi(u_j, A) \geq \sigma(\Sigma_u \setminus S_u, A') - MS_p(u, A'). \tag{4.15}$$

In particular by choosing $A' = A$ in the previous inequality we get $\sigma(A \cap (\Sigma_u \setminus S_u)) < +\infty$ so that $\mathcal{L}^n(A \cap (\Sigma_u \setminus S_u)) = 0$ by (2.5). Hence, by the outer/inner regularity of both the Lebesgue measure and the measure associated to $MS_p(u, \cdot)$ we can find a sequence of open sets $A_k \subseteq A$ such that $A_k \supseteq A \cap (\Sigma_u \setminus S_u)$ and $\lim_k MS_p(u, A_k) = 0$.

Indeed, for any $k \in \mathbb{N}$ let $C_k \subseteq S_u$ be a compact set such that $\mathcal{H}^{n-1}(S_u \setminus C_k) \leq 1/k$, and $\delta_k > 0$ such that $\int_V |\nabla u|^p dx \leq 1/k$ for any $V \mathcal{L}^n$ measurable with $\mathcal{L}^n(V) \leq \delta_k$. Since $\mathcal{L}^n(A \cap (\Sigma_u \setminus S_u)) = 0$ one can also find an open set V_k such that $A \supseteq V_k \supseteq A \cap (\Sigma_u \setminus S_u)$ and $\mathcal{L}^n(V_k) \leq \delta_k$. Then, choose in (4.15) the open sets $A_k = V_k \setminus C_k$ to get

$$\liminf_j F_\psi(u_j, A) \geq \sigma(\Sigma_u \setminus S_u, A_k) - \frac{2}{k}.$$

Then, being $A_k \cap (\Sigma_u \setminus S_u) = A \cap (\Sigma_u \setminus S_u)$, we deduce from Theorem 2.4(b)

$$\liminf_j F_\psi(u_j, A) \geq \sigma(A \cap (\Sigma_u \setminus S_u)) - \frac{2}{k}.$$

Eventually, (4.6) follows by letting $k \rightarrow +\infty$. □

We now provide the upper bound inequality.

Proposition 4.3. *For every $u \in L^1(\Omega)$ there exists $(u_j) \subset L^1(\Omega)$ such that $u_j \rightarrow u$ in $L^1(\Omega)$ and*

$$\limsup_j F_\psi(u_j, \Omega) \leq \mathcal{F}_\psi(u, \Omega).$$

Proof. We may assume $\mathcal{F}_\psi(u, \Omega) < +\infty$, from which it follows $u \in GSBV(\Omega)$ and $\sigma(\Sigma_u) < +\infty$, where we recall that $\Sigma_u = \{x \in \Omega : u^+(x) < \psi(x)\}$.

Let $w \in GSBV(\Omega)$ be such that $\mathcal{F}_\psi(w, \Omega) < +\infty$. By Remark 2.7 for every $j \in \mathbb{N}$ there exists an open set $\Omega_j \subset \Omega$ such that $\Sigma_u \cup S_w \subseteq \Omega_j$, $\mathcal{L}^n(\Omega_j) \leq \frac{1}{j^2}$ for j sufficiently big, and

$$\text{Per}(\Omega_j, \Omega) \leq \sigma(\Sigma_u \cup S_w) + \frac{1}{j}. \tag{4.16}$$

Define $u_j = u\chi_{\Omega \setminus \Omega_j} + w\chi_{\Omega_j}$, it is then easy to check that $u_j \rightarrow u$ in $L^1(\Omega)$. Furthermore, Theorem 3.84 [2] yields $u_j \in GSBV(\Omega)$ and

$$Du_j = Dw \llcorner \Omega_j^1 + (u_j^+ - u_j^-)\nu_{\partial^*\Omega_j} \mathcal{H}^{n-1} \llcorner \partial^*\Omega_j + Du \llcorner \Omega_j^0 \tag{4.17}$$

(see Ths. 3.84 and 4.34 [2]), in particular $S_{u_j} \subseteq S_w \cup \partial^*\Omega_j \cup (S_u \cap \Omega_j^0)$. Thus, $u_j^+ = w^+$ on Ω_j^1 , $u_j^+ = \tilde{w} \vee u^+$ on $\partial^*\Omega_j$, and $u_j^+ = u^+$ on Ω_j^0 . Hence, we deduce $u_j^+ \geq \psi \mathcal{H}^{n-1}$ a.e. Ω , by taking into account that $S_u \cup S_w \subseteq \Omega_j$ and $F_\psi(w, \Omega) < +\infty$. Furthermore, Lebesgue differentiation theorem gives $\mathcal{L}^n(\Omega_j^1 \setminus \Omega_j) = 0$, and then by (4.17) we have

$$\begin{aligned} F_\psi(u_j, \Omega) &= MS_p(u_j, \Omega) \leq \int_{\Omega} |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u \setminus \Sigma_u) \\ &\quad + \int_{\Omega_j} |\nabla w|^p dx + \mathcal{H}^{n-1}(S_w) + \mathcal{H}^{n-1}(\partial^*\Omega_j). \end{aligned}$$

Thus, by (4.16) it follows

$$\limsup_j F_\psi(u_j, \Omega) \leq \int_{\Omega} |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u \setminus \Sigma_u) + \sigma(\Sigma_u) + \mathcal{H}^{n-1}(S_w) + \sigma(S_w),$$

from which we infer

$$\overline{F}_\psi(u, \Omega) \leq \mathcal{F}_\psi(u, \Omega) + 3\mathcal{H}^{n-1}(S_w).$$

In order to take care of the extra energy contribution due to w we notice that $w \vee M \in GSBV(\Omega)$ is such that $F_\psi(w \vee M, \Omega) < +\infty$. Thus, we may repeat the construction performed above substituting w with $w \vee M$ in order to get

$$\overline{F}_\psi(u, \Omega) \leq \mathcal{F}_\psi(u, \Omega) + 3\mathcal{H}^{n-1}(S_{w \vee M}), \tag{4.18}$$

and thus to conclude it suffices to show that $\mathcal{H}^{n-1}(S_{w \vee M})$ is infinitesimal as $M \rightarrow +\infty$.

The BV chain rule formula implies $\mathcal{H}^{n-1}(S_{w \vee M}) = \mathcal{H}^{n-1}(\{x \in \Omega : w^+(x) > M\} \cap S_w)$ (see Th. 3.101 [2]), and thus by taking into account that $\mathcal{H}^{n-1}(\{x \in \Omega : |w^\pm(x)| = +\infty\}) = 0$ (see (4.3)) we infer

$$\lim_{M \rightarrow +\infty} \mathcal{H}^{n-1}(S_{w \vee M}) = 0. \tag{4.19} \quad \square$$

Remark 4.4. The choice to express the unilateral obstacle condition in (4.1) by the representant u^+ can be easily justified in one-dimension. Indeed, in such a case the same conclusion of Theorem 4.1 holds true even using the representant u^- .

5. RELAXATION FOR THE DIRICHLET PROBLEM WITH AN OBSTACLE

In this section we add a Dirichlet boundary value to the obstacle problem. In order to do that we suppose that Ω is a bounded open set of \mathbf{R}^n with Lipschitz boundary.

In such a case, we recall that any $u \in BV(\Omega)$ leaves an inner boundary trace on $\partial\Omega$ denoted by $\text{tr}(u)$, and that the trace operator $\text{tr} : BV(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1})$ is onto (see Th. 3.87 [2]). Moreover, for any $u \in GSBV(\Omega)$ with $MS_p(u, \Omega) < +\infty$ one can show that for \mathcal{H}^{n-1} a.e. $x \in \partial\Omega$ the value $\text{ap-lim}_{y \rightarrow x, y \in \Omega} u(y)$ exists finite (see the comments after Th. 4.34 [2] and Prop. 2.4 [11]). We call it the trace of u and denote it still by $\text{tr}(u)$.

With fixed $\varphi \in L^1(\partial\Omega, \mathcal{H}^{n-1})$ we introduce the functional $D_{\psi, \varphi} : L^1(\Omega) \rightarrow [0, +\infty]$ defined as

$$D_{\psi, \varphi}(u, \Omega) = F_\psi(u, \Omega) \text{ if } u \in GSBV(\Omega), \text{tr}(u) = \varphi \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega,$$

$+\infty$ otherwise in $L^1(\Omega)$. To ensure the problem to be non trivial we impose a compatibility condition between the trace and obstacle functions: We assume that there exists $w \in L^p \cap GSBV(\Omega)$ such that $D_{\psi,\varphi}(w, \Omega) < +\infty$, namely

$$MS_p(w, \Omega) < +\infty, w^+ \geq \psi \mathcal{H}^{n-1} \text{ a.e. in } \Omega, \text{tr}(w) = \varphi \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega. \tag{5.1}$$

In addition, given any bounded open set $\widehat{\Omega}$ such that $\widehat{\Omega} \supset \supset \Omega$, a local reflection argument shows that we may suppose $w \in GSBV(\widehat{\Omega})$, $MS_p(w, \widehat{\Omega}) < +\infty$ and $\mathcal{H}^{n-1}(\partial\Omega \cap S_w) = 0$. Hence, if $u \in GSBV(\Omega)$ we denote by \widehat{u} the function obtained extending u by w in $\mathbf{R}^n \setminus \Omega$, so that $\widehat{u} \in GSBV(\widehat{\Omega})$, and

$$MS_p(\widehat{u}, \widehat{\Omega}) = MS_p(u, \Omega) + MS_p(w, \widehat{\Omega} \setminus \overline{\Omega}) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \text{tr}(u)(x) \neq \varphi(x)\}) \tag{5.2}$$

(see Ths. 3.84 and 3.87 [2]).

Theorem 5.1. *Under the previous assumptions the lower semicontinuous envelope $\mathcal{D}_{\psi,\varphi} : L^1(\Omega) \rightarrow [0, +\infty]$ of $D_{\psi,\varphi}$ is given by*

$$\mathcal{D}_{\psi,\varphi}(u, \Omega) = \mathcal{F}_{\psi}(u, \Omega) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \text{tr}(u)(x) \neq \varphi(x)\})$$

if $u \in GSBV(\Omega)$, $+\infty$ otherwise in $L^1(\Omega)$.

In the proof below we keep the same notation of Propositions 4.2 and 4.3.

Proof. Let $(u_j) \in GSBV(\Omega)$ with $u_j \rightarrow u$ in $L^1(\Omega)$ and $\liminf D_{\psi,\varphi}(u_j, \Omega) < +\infty$ be given. Then, $u \in GSBV(\Omega)$ and $\widehat{u}_j \rightarrow \widehat{u}$ in $L^1(\widehat{\Omega})$. Furthermore, if $\widehat{\psi}$ is the extension of ψ by $-\infty$ in $\widehat{\Omega} \setminus \Omega$, we may apply Theorem 4.1 to $F_{\widehat{\psi}}$ on $\widehat{\Omega}$ and get by (5.2) above

$$\begin{aligned} \liminf_j D_{\psi,\varphi}(u_j, \Omega) &= \liminf_j F_{\widehat{\psi}}(\widehat{u}_j, \widehat{\Omega}) - MS_p(w, \widehat{\Omega} \setminus \overline{\Omega}) \\ &\geq \mathcal{F}_{\widehat{\psi}}(\widehat{u}, \widehat{\Omega}) - MS_p(w, \widehat{\Omega} \setminus \overline{\Omega}) = \mathcal{D}_{\psi,\varphi}(u, \Omega). \end{aligned}$$

To prove the upper bound inequality for $u \in GSBV(\Omega)$ such that $\mathcal{D}_{\psi,\varphi}(u, \Omega) < +\infty$, we consider a sequence (A_j) of open sets such that $E \subset A_j$, $\lim_j \mathcal{L}^n(A_j) = 0$ and $\lim_j \text{Per}(A_j) = \sigma(E)$, being $E = \{x \in \partial\Omega : \text{tr}(u)(x) \neq \varphi(x)\}$. Thanks to Theorem 2.4 (b) the sequence (A_j) can be chosen in an arbitrarily small neighbourhood of $\partial\Omega$.

Then, we define a sequence (v_j) performing the same construction of Proposition 4.3 replacing Ω_j there with $\Omega_j \cup A_j$, and w there with the function satisfying condition (5.1). Arguing as in Proposition 4.3 it is easy to check that $v_j \rightarrow u$ in $L^1(\Omega)$, and $v_j^+ \geq \psi \mathcal{H}^{n-1}$ a.e. in Ω . Furthermore, $\text{tr}(v_j) = w$ on $\partial\Omega$, so that for any open set $A \subset \subset \Omega$ we get

$$\begin{aligned} D_{\psi,\varphi}(v_j, \Omega \setminus \overline{A}) &\leq \int_{\Omega \setminus \overline{A}} |\nabla u|^p dx + \mathcal{H}^{n-1}((S_u \setminus \Sigma_u) \cap (\Omega \setminus \overline{A})) \\ &\quad + \int_{\Omega_j \cup A_j} |\nabla w|^p dx + \mathcal{H}^{n-1}(S_w \cap (\Omega \setminus \overline{A})) + \text{Per}(\Omega_j, \Omega \setminus \overline{A}) + \text{Per}(A_j, \Omega). \end{aligned}$$

By (4.16) and the definition of σ it is easy to see that if $\mathcal{H}^{n-1}(\partial A \cap (\Sigma_u \cup S_w)) = 0$ it holds

$$\lim_j \text{Per}(\Omega_j, \Omega \setminus \overline{A}) = \sigma((\Sigma_u \cup S_w) \cap (\Omega \setminus \overline{A})).$$

In addition, in Lemma 5.3 below we will show that $\lim_j \text{Per}(A_j, \Omega) = \mathcal{H}^{n-1}(E)$, taken this for granted we have

$$\limsup_j D_{\psi,\varphi}(v_j, \Omega \setminus \overline{A}) \leq \mathcal{D}_{\psi,\varphi}(u, \Omega \setminus \overline{A}) + 3\mathcal{H}^{n-1}(S_w \cap (\Omega \setminus \overline{A})). \tag{5.3}$$

To take care of the extra energy contribution due to w we refine the construction above as we did in Proposition 4.3. In this case we have to take also into account that the boundary datum cannot be changed.

To this aim choose open sets $U_1 \subset\subset U_2 \subset\subset \Omega$ with $\mathcal{H}^{n-1}(\partial U_k \cap (\Sigma_u \cup S_w)) = 0$, $k = 1, 2$, and let $\eta \in C_0^\infty(U_2)$ be a smooth cut-off function such that $\eta|_{U_1} \equiv 1$. Denote by u_j the function constructed as v_j above with $w \vee M$ in place of w . We remark that we may also assume $\mathcal{H}^{n-1}(\partial U_k \cap S_{v_j}) = \mathcal{H}^{n-1}(\partial U_k \cap S_{u_j}) = 0$ for $j \in \mathbb{N}$ and $k = 1, 2$.

Set $w_j = (1 - \eta)v_j + \eta u_j$, then $w_j \rightarrow u$ in $L^1(\Omega)$. By construction $u_j \geq v_j$ \mathcal{L}^n a.e. in Ω , then $w_j \geq v_j$ \mathcal{L}^n a.e. in Ω which in turn implies $w_j^+ \geq v_j^+ \geq \psi$ \mathcal{H}^{n-1} a.e. in Ω . Furthermore, $\text{tr}(w_j) = \varphi$ on $\partial\Omega$ and a straightforward computation yields

$$\begin{aligned} D_{\psi,\varphi}(w_j, \Omega) &\leq D_{\psi,\varphi}(v_j, \Omega \setminus \overline{U_1}) + F_\psi(u_j, U_2) + \|\nabla\eta\|_{L^\infty(\Omega)}^p \int_{U_2 \setminus \overline{U_1}} |v_j - u_j|^p dx \\ &\leq D_{\psi,\varphi}(v_j, \Omega \setminus \overline{U_1}) + F_\psi(u_j, U_2) + \|\nabla\eta\|_{L^\infty(\Omega)}^p \int_{\Omega_j \cup A_j} |w - M|^p dx. \end{aligned}$$

By assumption $w \in L^p(\Omega)$, thus by taking into account (4.18) and (5.3) we can pass to the limit in the inequality above and get

$$\overline{D}_{\psi,\varphi}(u, \Omega) \leq \mathcal{D}_{\psi,\varphi}(u, \Omega) + \mathcal{F}_\psi(u, U_2 \setminus \overline{U_1}) + 3\mathcal{H}^{n-1}(S_w \cap (\Omega \setminus \overline{U_1})) + 3\mathcal{H}^{n-1}(S_{w \vee M} \cap U_2),$$

the conclusion then follows by letting first $M \rightarrow +\infty$ (see (4.3)) and then shrinking $\Omega \setminus \overline{U_1}$ to \emptyset . □

Remark 5.2. It is easy to check that the functional $\mathcal{D}_{\psi,\varphi}$ is not coercive on $L^1(\Omega)$. In order to ensure compactness for its sublevel sets one has to add a lower order term to fulfil the assumptions of the *GSBV* compactness Theorem 2.1.

To conclude the section we are left with proving the following result.

Lemma 5.3. *For any \mathcal{H}^{n-1} measurable set $E \subseteq \partial\Omega$, and any sequence (A_j) of open sets such that $E \subset A_j$, $\lim_j \mathcal{L}^n(A_j) = 0$ and $\lim_j \text{Per}(A_j) = \sigma(E)$, we have*

$$\lim_j \text{Per}(A_j, \Omega) = \mathcal{H}^{n-1}(E).$$

Proof. First we recall that if A is an open set with Lipschitz continuous boundary the L^1 lower semicontinuous envelope of $L^1(A) \ni u \rightarrow \|Du\|(A)$ if $\text{tr}(u) = \chi_E$ \mathcal{H}^{n-1} a.e. on ∂A is given by

$$\|Du\|(A) + \int_{\partial A} |\text{tr}(u) - \chi_E| d\mathcal{H}^{n-1}$$

if $u \in BV(A)$, $+\infty$ otherwise in $L^1(A)$ (see [3]). Then, since $E \subset A_j$ and $\chi_{A_j} \rightarrow 0$ in $L^1(\mathbf{R}^n)$, choosing $A = \Omega$ yields

$$\liminf_j \text{Per}(A_j, \Omega) = \liminf_j \|D\chi_{A_j}\|(\Omega) \geq \int_{\partial\Omega} |\chi_E| d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(E).$$

We can repeat the same argument with $\mathbf{R}^n \setminus \overline{\Omega}$ in place of Ω and get

$$\liminf_j \text{Per}(A_j, \mathbf{R}^n \setminus \overline{\Omega}) \geq \mathcal{H}^{n-1}(E).$$

Hence, we deduce

$$\begin{aligned} \sigma(E) &= 2\mathcal{H}^{n-1}(E) = \lim_j \text{Per}(A_j) \\ &\geq \limsup_j \text{Per}(A_j, \Omega) + \liminf_j \text{Per}(A_j, \mathbf{R}^n \setminus \overline{\Omega}) \geq \limsup_j \text{Per}(A_j, \Omega) + \mathcal{H}^{n-1}(E), \end{aligned}$$

and the thesis follows. □

6. FURTHER RESULTS

In this section we extend the relaxation result obtained in Section 4 for the Mumford-Shah energy with an obstacle to more general free-discontinuity energies. We limit ourselves to introduce all the ingredients needed for the generalization of Theorem 4.1 in the new setting, being the proof a straightforward extension.

We consider the functional $F : L^1(\Omega) \rightarrow [0, +\infty]$

$$F(u, \Omega) = \int_{\Omega} f(\nabla u) dx + \int_{S_u} \varphi(\nu_u) d\mathcal{H}^{n-1}$$

for $u \in GSBV(\Omega)$, $+\infty$ otherwise; where $f, \varphi : \mathbf{R}^n \rightarrow \mathbf{R}$ are continuous functions satisfying:

- (a) f is convex, and there exist constants $c_1, c_2 > 0$ such that for every $\xi \in \mathbf{R}^n$

$$c_1|\xi|^p \leq f(\xi) \leq c_2|\xi|^p;$$

- (b) φ is a norm on \mathbf{R}^n .

According to Theorem 2.1 F turns out to be L^1 lower semicontinuous.

With a fixed Borel function $\psi : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\pm\infty\}$ we consider the obstacle functional $F_\psi : L^1(\Omega) \rightarrow [0, +\infty]$ related to F and ψ by

$$F_\psi(u, \Omega) = F(u, \Omega) \quad u \in GSBV(\Omega), u^+ \geq \psi \quad \mathcal{H}^{n-1} \text{ a.e. on } \Omega \tag{6.1}$$

$+\infty$ otherwise in $L^1(\Omega)$.

To determine the relaxation $\overline{F_\psi}$ of F_ψ we introduce the set functions $\sigma_\varphi^\varepsilon(\cdot, A)$ given by

$$\inf \left\{ \int_{\partial^* D \cap A} \varphi(\nu_{\partial^* D}) d\mathcal{H}^{n-1} + \frac{1}{\varepsilon} \mathcal{L}^n(D \cap A) : D \text{ } \mathcal{L}^n\text{-measurable, } \mathcal{H}^{n-1}(E \setminus D_+) = 0 \right\},$$

and $\sigma_\varphi(E, A) = \sup_{\varepsilon > 0} \sigma_\varphi^\varepsilon(E, A)$. Notice that in case φ is the Euclidean norm we recover De Giorgi's measure σ . The new set function σ_φ is obtained by replacing the usual perimeter with the anisotropic perimeter defined through φ in the definition of σ . Analogous statements to those in Proposition 2.4 hold true for σ_φ (see Props. 4.3 and 4.6 [6]); in particular σ_φ is a regular Borel measure.

Therefore, by arguing as in Section 3 and Section 4 we deduce the following result.

Theorem 6.1. *Let F_ψ be as in (6.1), then its lower semicontinuous envelope in the L^1 topology is given by*

$$\begin{aligned} \mathcal{F}_\psi(u, \Omega) &= \int_{\Omega} f(\nabla u) dx + \int_{S_u} \varphi(\nu_u) d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{2} \sigma_\varphi(\{x \in S_u : u^+(x) < \psi(x)\}) + \sigma_\varphi(\{x \in \Omega \setminus S_u : u^+(x) < \psi(x)\}) \end{aligned}$$

if $u \in GSBV(\Omega)$, $+\infty$ otherwise in $L^1(\Omega)$.

Eventually, we remark that if a Dirichlet boundary datum is imposed a result similar to Theorem 5.1 holds true.

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