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AN ALGEBRAIC FRAMEWORK FOR LINEAR IDENTIFICATION

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Abstract. A closed loop parametrical identification procedure for continuous-time constant linear systems is introduced. This approach which exhibits good robustness properties with respect to a large variety of additive perturbations is based on the following mathematical tools:

(1) module theory;
(2) differential algebra;
(3) operational calculus.

Several concrete case-studies with computer simulations demonstrate the efficiency of our on-line identification scheme.

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1. Introduction

This article on a closed loop parametrical identification procedure for continuous-time constant linear systems is a direct continuation of [12]. In order to better understand its aims let us begin with a first order input-output system. It might be convenient to employ as often done in practice (see, *e.g.* [1], and also [12]), the notations of operational calculus:

$$sy = ay + u + y(0) + \frac{\gamma}{s}\cdot \tag{1.1}$$

The constant parameter a is unknown as well as γ in the constant load perturbation $\frac{\gamma}{s}$. For regulating the output y from an arbitrary initial condition $y(0)$, towards a desired equilibrium value Y, we are proposing the PI-like controller

$$u = -a_e y - k_2(y - Y) - \frac{k_1}{s}(y - Y) \tag{1.2}$$

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where

- k_1, k_2 are appropriate constant gains;
- a_e is an on-line identifier of a.

Multiplying both sides of (1.1) by s yields:

$$s^2 y = asy + su + sy(0) + \gamma.$$

Taking derivatives, twice, with respect to s permits to ignore the load perturbation and the initial condition:

$$\left[s\frac{\mathrm{d}^2y}{\mathrm{d}s^2} + 2\frac{\mathrm{d}y}{\mathrm{d}s}\right] a = s^2\frac{\mathrm{d}^2y}{\mathrm{d}s^2} + 4s\frac{\mathrm{d}y}{\mathrm{d}s} + 2y - \left(s\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + 2\frac{\mathrm{d}u}{\mathrm{d}s}\right)\cdot$$

Multiply both sides by s^{-2} for avoiding derivations with respect to time:

$$\left[s^{-1}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} + 2s^{-2}\frac{\mathrm{d}y}{\mathrm{d}s}\right] a = \frac{\mathrm{d}^2y}{\mathrm{d}s^2} + 4s^{-1}\frac{\mathrm{d}y}{\mathrm{d}s} + 2s^{-2}y - \left(s^{-1}\frac{\mathrm{d}^2u}{\mathrm{d}s^2} + 2s^{-2}\frac{\mathrm{d}u}{\mathrm{d}s}\right)\cdot$$

The well-known rule of operational calculus (see, *e.g.* [8, 27, 28, 41]) which is associating to $\frac{\mathrm{d}^\nu}{\mathrm{d}s^\nu}$, $\nu \geq 0$, the multiplication by $(-1)^\nu t^\nu$, yields the following on-line identifier, *i.e.*, a time-domain representation with no derivatives but only integrations with respect to time:

$$a_e = \frac{t^2y(t) - \int_0^t \left(4\sigma y(\sigma) + \sigma^2 u(\sigma)\right)\mathrm{d}\sigma + 2\int_0^t\int_0^\sigma \left(\lambda u(\lambda) + y(\lambda)\right)\mathrm{d}\lambda\mathrm{d}\sigma}{\int_0^t \sigma^2 y(\sigma)\mathrm{d}\sigma - 2\int_0^t\int_0^\sigma \lambda y(\lambda)\mathrm{d}\lambda\mathrm{d}\sigma}\cdot \tag{1.3}$$

For the simulations we are using the following values

$$a = 2, \quad \gamma = 0.5, \quad k_1 = b^2, \quad k_2 = 2b, \quad b = 3, \quad Y = 2, \quad y(0) = 0.2.$$

In the vicinity of $t = 0$, we are setting $a_e = 0$ in the controller (1.2). In Figure 1, the peak for a_e at about $t = 1.2$ is due to the fact that both the numerator and the denominator of the identifier (1.3) are equal to 0 for that value of t. This should not be considered as a drawback since the exact value of a is obtained almost instantaneously.

This paper aims at giving a theoretical framework of this identification scheme for constant linear systems by further utilising the algebraic tools of [10, 12], *i.e.*, module theory and Mikusiński's operational calculus, which will therefore not be reviewed here. Our treatment of identifiability, identification and adaptive control may be subsumed as follows:

- the approach of non-linear identifiability *via* differential algebra [6, 7, 24, 30, 34] permits to introduce in Section 2.2 *linear identifiability* and *weak linear identifiability*, which are often encountered in practice;
- operational calculus allows another differentially algebraic definition in Section 2.5.5 of (weak) linear identifiability[4], that is based on the notion of *algebraic derivative* [27, 28, 41];
- most usual structured perturbations may be eliminated thanks to their torsion property;
- by calculating the unknown parameters in a small time interval, we are able to achieve identification in closed loop.

In spite of some relationship with the huge literature on the subject (see, *e.g.* [2, 4, 5, 15, 17, 18, 20–23, 25, 29, 31, 33, 37–40] and the references therein), a major difference lies in the absence of any least squares and/or probabilistic methods.

[4]Although we believe that subsequent works will demonstrate the vast superiority of operational calculus, we are leaving the two standpoints for the convenience of the readers who are familiar with today's setting of identifiability *via* differential algebra. Note moreover that the two approaches coincide in all our case-studies. The following open problem might nevertheless be of some interest: compare the two definitions of (weak) linear identifiability.

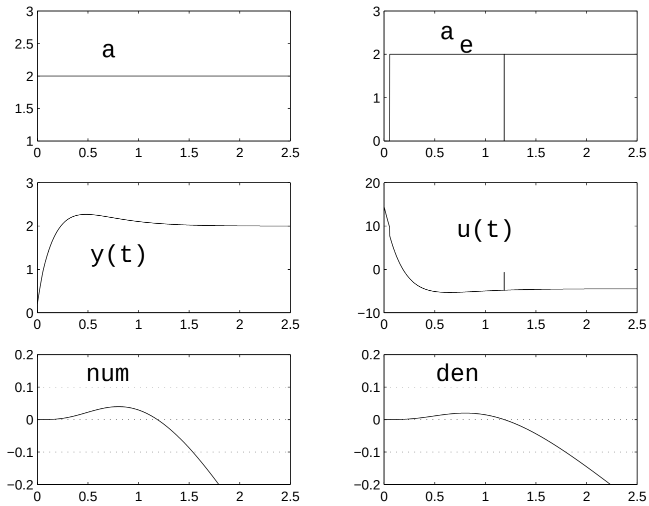

FIGURE 1. Closed loop response of the first order system with on-line identification.

Our paper ends with three more complex case-studies:

(1) a first order system over the field $\mathbb{C}$ of complex numbers, which corresponds to the visual based control of planar manipulators [16];
(2) a second order SISO system where we show with the help of a low pass filter the robustness of our approach with respect to high frequency perturbations;
(3) a multivariable simplified version of a heat exchanger [14].

Future publications (see, *e.g.* [36]) will be devoted to the parametrical identification of discrete-time linear systems along the lines of [9,11], as well as to specific applications such as to the control of externally perturbed rotating beams, the regulation and tracking in parametric uncertain dc to dc power converters [35] and other uncertain linear control problems of practical interest. Connections with delay systems, partial differential equations, non-linear systems, and signal processing will also be developed.

2. An algebraic setting for linear identifiability

2.1. **Linear systems**[5]

Let k be the field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers. Denote by K a finite algebraic extension of the field $k(\boldsymbol{\Theta})$ generated by a finite set $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_r)$ of *unknown parameters.* Consider the ring $K[\frac{\mathrm{d}}{\mathrm{d}t}]$ of linear differential operators of the form $\sum_{\text{finite}} c_\nu \frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu}$, $c_\nu \in K$. The parameters are assumed to be constant, *i.e.*, $\frac{\mathrm{d}\theta_\iota}{\mathrm{d}t} = 0$, $\iota = 1, \ldots, r$. Thus $K[\frac{\mathrm{d}}{\mathrm{d}t}]$ is a commutative principal ideal domain. A *linear system* with a set $\boldsymbol{\Theta}$ of constant unknown parameters is a finitely generated free $K[\frac{\mathrm{d}}{\mathrm{d}t}]$-module Λ. We distinguish in Λ a set $\underline{\boldsymbol{\pi}} = (\underline{\pi}_1, \ldots, \underline{\pi}_q)$ of *perturbations.* The short exact sequence

$$0 \to \mathrm{span}_{K[\frac{\mathrm{d}}{\mathrm{d}t}]}(\underline{\boldsymbol{\pi}}) \to \Lambda \to \Lambda^{\mathrm{nom}} \to 0$$

defines the *nominal*, or *unperturbed*, system $\Lambda^{\mathrm{nom}} = \Lambda/\mathrm{span}_{K[\frac{\mathrm{d}}{\mathrm{d}t}]}(\underline{\boldsymbol{\pi}})$. The canonical image of any element $\lambda \in \Lambda$ in Λ^{nom} is written λ^{nom}.

[5] We slightly change the setting of [10] in order to take into account the unknown parameters.

A *linear dynamics* is a linear system Λ which is equipped with a *control*, *i.e.*, a finite subset $\boldsymbol{u} = (u_1, \ldots, u_m)$ of Λ such that

- $\mathrm{span}_{K[\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{u}) \cap \mathrm{span}_{K[\frac{\mathrm{d}}{\mathrm{d}t}]}(\underline{\boldsymbol{\pi}}) = \{0\}$;
- the quotient module $\Lambda^{\mathrm{nom}}/\mathrm{span}_{K[\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{u}^{\mathrm{nom}})$ is torsion.

If $\boldsymbol{u} = \emptyset$, then Λ^{nom} is torsion. An *input-output system* is a linear dynamics Λ which is equipped with an *output*, *i.e.*, a finite subset $\boldsymbol{y} = (y_1, \ldots, y_p)$ of Λ. From now on Λ will be an input-output system.

2.2. **Identifiability**

The symmetric K-algebra $\mathbf{Sym}(\Lambda^{\mathrm{nom}})$ generated by Λ^{nom}, viewed as a K-vector space, may be endowed with a canonical structure of *differential ring* [3, 19] by setting for $\xi, \eta \in \Lambda^{\mathrm{nom}}$, $\frac{\mathrm{d}\xi\eta}{\mathrm{d}t} = \frac{\mathrm{d}\xi}{\mathrm{d}t}\eta + \xi\frac{\mathrm{d}\eta}{\mathrm{d}t}$. The quotient field $\mathbf{Q}(\mathbf{Sym}(\Lambda^{\mathrm{nom}}))$ becomes thus a *differential field* [3, 19].

Let Œ $\subseteq \mathbf{Q}(\mathbf{Sym}(\Lambda^{\mathrm{nom}}))$ be the differential overfield of k generated by $\boldsymbol{u}^{\mathrm{nom}}$ and $\boldsymbol{y}^{\mathrm{nom}}$. The set $\boldsymbol{\Theta}$ of unknown parameters is said to be *algebraically identifiable* (resp. *rationally identifiable*) if, and only if, any component of $\boldsymbol{\Theta}$ is algebraic over (resp. belongs to) Œ. It is said to be *linearly identifiable* if, and only if,

$$P \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = Q \tag{2.1}$$

where

- P and Q are respectively $r \times r$ and $r \times 1$ matrices;
- the entries of P and Q belong to $\mathrm{span}_{k[\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{u}^{\mathrm{nom}}, \boldsymbol{y}^{\mathrm{nom}})$;
- $\det(P) \neq 0$.

The set $\boldsymbol{\Theta}$ of unknown parameters is said to be *weakly linearly identifiable* if, and only if, there exists a finite set $\boldsymbol{\Theta}' = (\theta'_1, \ldots, \theta'_{q'})$ of *related unknown parameters* such that

- the components of $\boldsymbol{\Theta}'$ (resp. $\boldsymbol{\Theta}$) are algebraic over $k(\boldsymbol{\Theta})$ (resp. $k(\boldsymbol{\Theta}')$);
- $\boldsymbol{\Theta}'$ is linearly identifiable.

The next result is obvious.

Proposition 2.1. *Linear (resp. rational) identifiability implies rational (resp. algebraic) identifiability. Weak linear identifiability implies algebraic identifiability.*

Example 2.1. Consider the two nominal systems $(\frac{\mathrm{d}}{\mathrm{d}t} - a)y^{\mathrm{nom}} = u^{\mathrm{nom}}$ and $(\frac{\mathrm{d}}{\mathrm{d}t} - b^2)y^{\mathrm{nom}} = u^{\mathrm{nom}}$, $m = p = 1$, where a and b are the unknown parameters. It is clear that a is linearly identifiable, but not b which is weakly linearly identifiable.

Example 2.2. Consider the nominal system

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - a\right) y^{\mathrm{nom}} = bu^{\mathrm{nom}} \tag{2.2}$$

$m = p = 1$, where a and b are constant unknown parameters. The couple (a, b) which satisfies

$$\begin{pmatrix} y^{\mathrm{nom}} & u^{\mathrm{nom}} \\ \dot{y}^{\mathrm{nom}} & \dot{u}^{\mathrm{nom}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \dot{y}^{\mathrm{nom}} \\ \ddot{y}^{\mathrm{nom}} \end{pmatrix} \tag{2.3}$$

is linearly identifiable.

2.3. Perturbations

2.3.1. *A Weyl algebra*

Introduce the Weyl algebra [26] $K[t, \frac{\mathrm{d}}{\mathrm{d}t}]$. Let

$$\Lambda_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]} = K\left[t, \frac{\mathrm{d}}{\mathrm{d}t}\right] \otimes_{K[\frac{\mathrm{d}}{\mathrm{d}t}]} \Lambda$$

be the left $K[t, \frac{\mathrm{d}}{\mathrm{d}t}]$-module obtained by extending the ring of scalars. For the purpose of dealing with peculiar perturbations we will be defining from now on a system $\mathbf{L}$ as a finitely generated left $K[t, \frac{\mathrm{d}}{\mathrm{d}t}]$-module in the following way:

$$\mathbf{L} = \Lambda_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]}/M$$

where M is a finitely generated module spanned by elements of $\mathrm{span}_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\underline{\boldsymbol{\pi}})$. Call again perturbation the q-tuple $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_q) \subset \mathbf{L}$, which is the canonical image of $\underline{\boldsymbol{\pi}} = (\underline{\pi}_1, \ldots, \underline{\pi}_q)$. As in Section 2.1 the short exact sequence

$$0 \to \mathrm{span}_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{\pi}) \to \mathbf{L} \to \mathbf{L}^{\mathrm{nom}} \to 0$$

defines the nominal, or unperturbed, system $\mathbf{L}^{\mathrm{nom}} = \mathbf{L}/\mathrm{span}_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{\pi})$. The canonical image of any element $\ell \in \mathbf{L}$ in $\mathbf{L}^{\mathrm{nom}}$ is written ℓ^{nom}. The next property is obvious.

Proposition 2.2. $\mathbf{L}^{\mathrm{nom}} \simeq K[t, \frac{\mathrm{d}}{\mathrm{d}t}] \otimes_{K[\frac{\mathrm{d}}{\mathrm{d}t}]} \Lambda^{\mathrm{nom}}$. *The canonical mapping* $\Lambda^{\mathrm{nom}} \to \mathbf{L}^{\mathrm{nom}}$, $\lambda^{\mathrm{nom}} \mapsto \ell^{\mathrm{nom}} = 1 \otimes \lambda^{\mathrm{nom}}$, *is injective.*

With a slight abuse of notations Λ^{nom} will be considered as a subset of $\mathbf{L}^{\mathrm{nom}}$.

2.3.2. *Structured perturbations*

Equation (2.1) becomes

$$P \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = Q + Q' \tag{2.4}$$

where

- P and Q are respectively $r \times r$ and $r \times 1$ matrices;
- the entries of P and Q belong to $\mathrm{span}_{k[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{u}, \boldsymbol{y})$;
- $\det(P) \neq 0$;
- the entries of the $r \times 1$ matrix Q' belong to $\mathrm{span}_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{\pi})$.

The perturbation $\boldsymbol{\pi}$ is assumed to be *structured*, *i.e.*, the module $\mathrm{span}_{k[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{\pi})$ is torsion.

Example 2.3. The perturbation $\kappa H(t)$, $\kappa \in k$, where

$$H(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

is the Heaviside function, is structured: it is annihilated by $t\frac{\mathrm{d}}{\mathrm{d}t}$.

We may state, since $k[t, \frac{\mathrm{d}}{\mathrm{d}t}]$ is a principal left ideal ring [26], the following useful property:

Proposition 2.3. *The set of* $\Delta \in k[t, \frac{\mathrm{d}}{\mathrm{d}t}]$ *such that equation (2.4) becomes*

$$\Delta P \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = \Delta Q \tag{2.5}$$

is the principal left ideal $\operatorname{ann}(\mathcal{Q}') \subseteq k[t, \frac{\mathrm{d}}{\mathrm{d}t}]$, $\operatorname{ann}(\mathcal{Q}') \neq \{0\}$, *where* $\mathcal{Q}'$ *is the submodule of* $\operatorname{span}_{K[t,\frac{\mathrm{d}}{\mathrm{d}t}]}(\boldsymbol{\pi})$ *spanned by the entries of* Q'.

Equation (2.5) is called a *(linear) identifier.* The next property is clear:

Proposition 2.4. *Any entry of the matrices* ΔP, ΔQ *may be written*

$$\sum_{\text{finite}} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} t^\beta x$$

where $x \in \operatorname{span}_k(\boldsymbol{u}, \boldsymbol{y})$, $\alpha, \beta \geq 0$.

2.4. Persistent trajectories

To the submodule $\Upsilon \subseteq \Lambda$ spanned by $\boldsymbol{u}$ and $\boldsymbol{y}$ corresponds a linear differential algebraic variety $\mathcal{T} \subset \mathcal{U}^{(m+p)}$ where $\mathcal{U}$ is a universal differential overfield of k (see [3, 19]). Any element of $\mathcal{T}$ will be called a *trajectory.* A trajectory τ is called *persistent* (resp. *non-persistent*) for the identifier (2.5) if, and only if, $\det(\Delta P)(\tau) \neq 0$ (resp. $\det(\Delta P)(\tau) = 0$). The next result, which is obvious, means in plain words that most trajectories are persistent.

Proposition 2.5. *The set of persistent trajectories of a linear identifier is open with respect to the differential Zariski topology.*

Example 2.4. Consider again Example 2.2 where equation (2.3) may be considered as a linear identifier without perturbations. The non-persistent trajectories satisfy

$$\det \begin{pmatrix} y^{\mathrm{nom}} & u^{\mathrm{nom}} \\ \dot{y}^{\mathrm{nom}} & \dot{u}^{\mathrm{nom}} \end{pmatrix} = y^{\mathrm{nom}} \dot{u}^{\mathrm{nom}} - \dot{y}^{\mathrm{nom}} u^{\mathrm{nom}} = 0.$$

Remark 2.1. This setting is ignoring the fact[6] that $\det(\Delta P)$ may only be equal to 0 for some values[7] of t.

2.5. Operational calculus

2.5.1. *Basics*[8]

A *linear system*, with a set $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_r)$ of *unknown parameters*, is a finitely generated free $K[s]$-module Λ, where the field K is a finite algebraic extension of $\mathbb{C}(\boldsymbol{\Theta})$. The element s is of course assumed to be $\mathbb{C}$-algebraically independent of $\boldsymbol{\Theta}$. We distinguish in Λ a set $\underline{\boldsymbol{\pi}} = (\underline{\pi}_1, \ldots, \underline{\pi}_q)$ of *perturbations.* The short exact sequence

$$0 \to \operatorname{span}_{K[s]}(\underline{\boldsymbol{\pi}}) \to \Lambda \to \Lambda^{\mathrm{nom}} \to 0$$

[6] Note however that differential algebra may be enriched in order to encompass usual numerical values (see, *e.g.* [32] and [30]).

[7] See in the example of the introduction the short discussion about the numerical simulations.

[8] We slightly change the setting of [12] in order to take into account the unknown parameters. Note also that many definitions of Sections 2.1, 2.2 and 2.3 are repeated almost verbatim.

defines the *nominal*, or *unperturbed*, system $\Lambda^{\text{nom}} = \Lambda/\text{span}_{K[s]}(\underline{\boldsymbol{\pi}})$. The canonical image of any element $\lambda \in \Lambda$ in Λ^{nom} is written λ^{nom}.

A *linear dynamics* is a linear system Λ which is equipped with a *control*, *i.e.*, a finite subset $\boldsymbol{u} = (u_1, \ldots, u_m)$ of Λ such that

- $\text{span}_{K[s]}(\boldsymbol{u}) \cap \text{span}_{K[s]}(\underline{\boldsymbol{\pi}}) = \{0\}$;
- the quotient module $\Lambda^{\text{nom}}/\text{span}_{K[s]}(\boldsymbol{u}^{\text{nom}})$ is torsion.

If $\boldsymbol{u} = \emptyset$, then Λ^{nom} is torsion. An *input-output system* is a linear dynamics Λ which is equipped with an *output*, *i.e.*, a finite subset $\boldsymbol{y} = (y_1, \ldots, y_p)$ of Λ. From now on Λ will be an input-output systems.

2.5.2. *The algebraic derivative*

Endow the set $\mathcal{C}$ of continuous functions $[0, +\infty) \to \mathbb{C}$ with a structure of commutative ring with respect to the addition $+$

$$(f + g)(t) = f(t) + g(t)$$

and to the convolution (product) $\star$

$$(f \star g)(t) = (g \star f)(t) = \int_0^t f(\tau)g(t - \tau)\mathrm{d}\tau = \int_0^t g(\tau)f(t - \tau)\mathrm{d}\tau.$$

Any element of the *Mikusiński field* $\mathcal{M}$, *i.e.*, the quotient field of $\mathcal{C}$, is called an *operator*. Any function $f : \mathbb{R} \to \mathbb{C}$, which belongs to $\mathcal{M}$, may be written $\{f\}$. Note that in general the product of two elements $a, b \in \mathcal{M}$ will be written ab and not $a \star b$.

For any $f \in \mathcal{C}$, it is known (see [27, 28, 41]) that the mapping $f \mapsto \frac{\mathrm{d}f}{\mathrm{d}s} = \{-tf\}$ satisfies the properties of a derivation, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}s}(f + g) = \frac{\mathrm{d}f}{\mathrm{d}s} + \frac{\mathrm{d}g}{\mathrm{d}s}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}(f \star g) = \frac{\mathrm{d}f}{\mathrm{d}s} \star g + f \star \frac{\mathrm{d}g}{\mathrm{d}s}\cdot$$

It can be trivially extended to a derivation, called the *algebraic derivative*, of $\mathcal{M}$ by setting, if $g \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\{f\} \star \{g\}^{-1}\right) = \frac{\frac{\mathrm{d}f}{\mathrm{d}s} \star g - f \star \frac{\mathrm{d}g}{\mathrm{d}s}}{\{g\}^2}.$$

Endowed with the algebraic derivative, $\mathcal{M}$ becomes a *differential field* [3, 19], whose subfield of constants[9] is $\mathbb{C}$.

2.5.3. $K[s, \frac{\mathrm{d}}{\mathrm{d}s}]$*-modules*

The set of differential operators $\sum_{\text{finite}} \gamma_\nu \frac{\mathrm{d}^\nu}{\mathrm{d}s^\nu}$, $\gamma_\nu \in K[s]$, is the Weyl algebra [26] $K[s, \frac{\mathrm{d}}{\mathrm{d}s}]$. Write

$$\Lambda_{K[s, \frac{\mathrm{d}}{\mathrm{d}s}]} = K[s, \frac{\mathrm{d}}{\mathrm{d}s}] \otimes_{K[s]} \Lambda$$

the left $K[s, \frac{\mathrm{d}}{\mathrm{d}s}]$-module obtained by extending the ring of scalars. For the purpose of dealing with peculiar perturbations we will be defining from now on a system $\mathbf{L}$ as a finitely generated left $K[s, \frac{\mathrm{d}}{\mathrm{d}s}]$-module in the following way:

$$\mathbf{L} = \Lambda_{K[s, \frac{\mathrm{d}}{\mathrm{d}s}]}/M$$

where M is a finitely generated module spanned by elements of $\text{span}_{K[s, \frac{\mathrm{d}}{\mathrm{d}s}]}(\underline{\boldsymbol{\pi}})$. Call again perturbation the q-tuple $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_q) \subset \mathbf{L}$, which is the canonical image of $\underline{\boldsymbol{\pi}}$. As in Section 2.1 the short exact sequence

$$0 \to \text{span}_{K[s, \frac{\mathrm{d}}{\mathrm{d}s}]}(\boldsymbol{\pi}) \to \mathbf{L} \to \mathbf{L}^{\text{nom}} \to 0$$

[9] A *constant* is an element c such that $\frac{\mathrm{d}c}{\mathrm{d}s} = 0$.

defines the nominal, or unperturbed, system $\mathbf{L}^{\mathrm{nom}} = \mathbf{L}/\mathrm{span}_{K[s,\frac{\mathrm{d}}{\mathrm{d}s}]}(\boldsymbol{\pi})$. The canonical image of any element $\ell \in \mathbf{L}$ in $\mathbf{L}^{\mathrm{nom}}$ is written ℓ^{nom}. The next property is obvious.

Proposition 2.6. $\mathbf{L}^{\mathrm{nom}} \simeq K[s,\frac{\mathrm{d}}{\mathrm{d}s}] \otimes_{K[s]} \Lambda^{\mathrm{nom}}$. *The canonical mapping* $\Lambda^{\mathrm{nom}} \to \mathbf{L}^{\mathrm{nom}}$, $\lambda^{\mathrm{nom}} \mapsto \ell^{\mathrm{nom}} = 1 \otimes \lambda^{\mathrm{nom}}$*, is injective.*

With a slight abuse of notations Λ^{nom} will be considered as a subset of $\mathbf{L}^{\mathrm{nom}}$.

2.5.4. *Structured perturbations*

The perturbation $\boldsymbol{\pi}$ is said to be *structured* if, and only if, the module $\mathrm{span}_{\mathbb{C}[s,\frac{\mathrm{d}}{\mathrm{d}s}]}(\boldsymbol{\pi})$ is torsion.

Example 2.5. The perturbation $\kappa\frac{\mathrm{e}^{-Ls}}{s^n}$, $L \geq 0$, $\kappa \in \mathbb{C}$, $n \geq 0$, which is annihilated by $(\frac{\mathrm{d}}{\mathrm{d}s} - L)s^n = s^n(\frac{\mathrm{d}}{\mathrm{d}s} - L) + ns^{n-1}$, is structured.

2.5.5. *Identifiability*

The symmetric $K[s]$-algebra $\mathbf{Sym}(\mathbf{L}^{\mathrm{nom}})$ generated by $\mathbf{L}^{\mathrm{nom}}$, viewed as a $K[s]$-module, may be endowed with a canonical structure of *differential ring* [3, 19] with respect to the algebraic derivative $\frac{\mathrm{d}}{\mathrm{d}s}$ by setting for $\xi, \eta \in \Lambda^{\mathrm{nom}}$, $\frac{\mathrm{d}\xi\eta}{\mathrm{d}s} = \frac{\mathrm{d}\xi}{\mathrm{d}t}\eta + \xi\frac{\mathrm{d}\eta}{\mathrm{d}t}$. The quotient field $\mathbf{Q}(\mathbf{Sym}(\mathbf{L}^{\mathrm{nom}}))$ becomes thus a *differential field* [3, 19].

Let $\mathbf{S} \subseteq \mathbf{Q}(\mathbf{Sym}(\mathbf{L}^{\mathrm{nom}}))$ be the differential overfield of $\mathbb{C}$ generated by $\boldsymbol{u}^{\mathrm{nom}}$ and $\boldsymbol{y}^{\mathrm{nom}}$. The set $\boldsymbol{\Theta}$ of unknown parameters is said to be *algebraically identifiable* (resp. *rationally identifiable*) if, and only if, any component of $\boldsymbol{\Theta}$ is algebraic over (resp. belongs to) $\mathbf{S}$. It is said to be *linearly identifiable* if, and only if,

$$P\begin{pmatrix}\theta_1\\ \vdots\\ \theta_r\end{pmatrix} = Q$$

where

- P and Q are respectively $r \times r$ and $r \times 1$ matrices;
- the entries of P and Q belong to $\mathrm{span}_{\mathbb{C}[s,\frac{\mathrm{d}}{\mathrm{d}s}]}(\boldsymbol{u}^{\mathrm{nom}}, \boldsymbol{y}^{\mathrm{nom}})$;
- $\det(P) \neq 0$.

Example 2.6. Write the Example 2.2 with the notation of operational calculus

$$(s-a)y^{\mathrm{nom}} = bu^{\mathrm{nom}}.$$

The couple (a, b) which satisfies

$$\begin{pmatrix} y^{\mathrm{nom}} & u^{\mathrm{nom}} \\ \frac{\mathrm{d}y^{\mathrm{nom}}}{\mathrm{d}s} & \frac{\mathrm{d}u^{\mathrm{nom}}}{\mathrm{d}s} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} sy^{\mathrm{nom}} \\ y^{\mathrm{nom}} + s\frac{\mathrm{d}y^{\mathrm{nom}}}{\mathrm{d}s} \end{pmatrix}$$

is again linearly identifiable.

The set $\boldsymbol{\Theta}$ of unknown parameters is said to be *weakly linearly identifiable* if, and only if, there exists a finite set $\boldsymbol{\Theta}' = (\theta'_1, \ldots, \theta'_{q'})$ of *related unknown parameters* such that

- the components of $\boldsymbol{\Theta}'$ (resp. $\boldsymbol{\Theta}$) are algebraic over $\mathbb{C}(\boldsymbol{\Theta})$ (resp. $\mathbb{C}(\boldsymbol{\Theta}')$);
- $\boldsymbol{\Theta}'$ is linearly identifiable.

Remark 2.2. The derivation of the operational analogue of the linear identifier (2.5) is an immediate exercise which is left to the reader.

3. Three additional case-studies

3.1. A remark of practical interest

By taking, for an appropriate trajectory, iterated time integrals of both sides of equation (2.5) it is possible thanks to Proposition 2.4 to compute the unknown parameters by only utilising time integrals of the products of t^α, $\alpha \geq 0$, with the input and output signals. Note moreover the smoothing effects of those iterated time integrals with respect to high frequency perturbations.

3.2. Non-calibrated visual based control of a planar manipulator[10]

A visual based control of a two link planar manipulator, partially controlled by a fast velocity feedback loop, can be formulated as follows (see [16]). Given the visual flow dynamics, written in complex notation, as

$$\dot{z} = \xi v, \quad \xi \in \mathbb{C} \tag{3.1}$$

find the complex control input v such that z follows as closely as possible a desired reference trajectory z^* in the visual frame space, regardless of the unknown, but constant, value of $\xi = a\mathrm{e}^{\theta\sqrt{-1}}$, $a, \theta \in \mathbb{R}$, $a > 0$. Note that

- $a = |\xi|$ is a scaling factor of the camera;
- $\theta = \arg(\xi)$ is the fixed angular orientation of the camera around its visual axis, assumed to be perpendicular to the robot workspace.

The complex state variable $z = y_1 + y_2\sqrt{-1}$ represents the visual frame coordinates. The complex control variable $v = u_1 + u_2\sqrt{-1}$ generates the required transformed robot joints velocity references.

3.2.1. *A certainty equivalence PI controller*

A global trajectory tracking controller of the PI type is given by

$$v = \frac{1}{\xi}\left[\dot{z}^* - k_1(z - z^*) - k_0\int_{t_0}^{t}(z - z^*)\mathrm{d}\sigma\right]. \tag{3.2}$$

The closed loop complex-valued tracking error dynamics $e = z - z^*$ is given by

$$\dot{e} + k_1 e + k_0\int_{t_0}^{t} e\mathrm{d}\sigma = 0 \tag{3.3}$$

where the design coefficients k_0, k_1 may be chosen to be real for ensuring $\lim_{t\to+\infty} e(t) = 0$.

3.2.2. *An algebraic identifier for the calibration parameters*

As in the first order system of the introduction, we obtain the following identifier $\widehat{\xi}$ of ξ:

$$\widehat{\xi} = \frac{(t - t_0)z - \int_{t_0}^{t} z\mathrm{d}\sigma}{\int_{t_0}^{t}(\sigma - t_0)v\mathrm{d}\sigma}\cdot \tag{3.4}$$

Once the complex parameter ξ is obtained by accurately evaluating the previous expression after a time interval of the form $[t_0, t_0 + \delta]$ has elapsed, with δ being an arbitrarily small strictly positive real number, we use such an on-line computed value, $\widehat{\xi}$, in the proposed PI certainty equivalence controller (3.2).

[10] See [13].

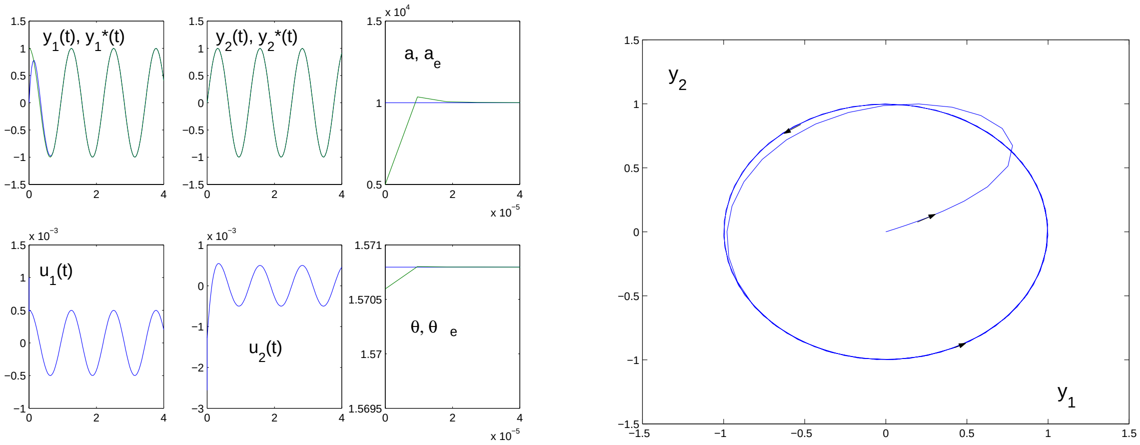

FIGURE 2. Trajectory tracking for visual servoing problem with PI controller based on the algebraic identifier.

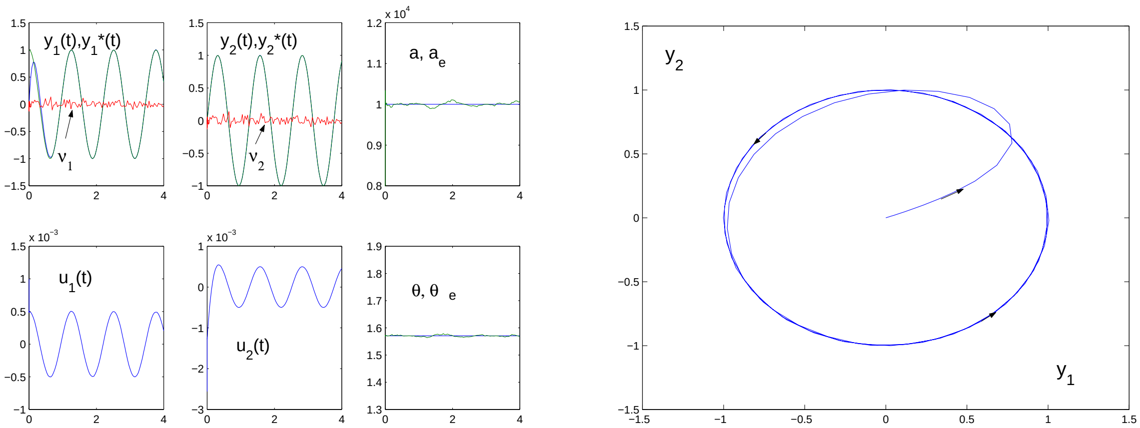

FIGURE 3. Trajectory tracking for stochastically perturbed visual servo problem with PI controller based on the algebraic identifier.

3.2.3. *Simulation results*

It is desired to track a circular trajectory in the visual frame space given by $y_1^*(t) = R\cos(\omega t)$, $y_2^*(t) = R\sin(\omega t)$, $R = 1$, $\omega = 5$. The unknown orientation angle for the camera is $\theta = \pi/2$ and the unknown scaling parameter is $a = 10^4$. Figure 2 shows the performance of the proposed controller-identifier given in equations (3.2–3.4). The true value of the unknown complex parameter $\xi = \xi_1 + \xi_2\sqrt{-1}$ is determined in approximately $\delta = 10^{-5}$ seconds. The values of ξ_1 and ξ_2 used in the controller during the calculation period $[t_0, t_0 + \delta]$, with $t_0 = 0$, is arbitrarily set to be $\xi_1 = 1$ and $\xi_2 = 5000$. The actual values of such parameters are $\xi_1 = 0$ and $\xi_2 = 10\,000$. The design coefficients, k_1, k_0, were chosen to be real and of the classical form $k_1 = 2\zeta\omega_n$, $k_0 = \omega_n^2$, with $\zeta = 0.8$ and $\omega_n = 8$.

We have tested the robustness of the proposed identifier-controller scheme by introducing a stochastic perturbation $\nu = \nu_1 + \nu_2\sqrt{-1}$ to the flow dynamics (3.1), *i.e.*, $\dot{z} = ae^{\theta\sqrt{-1}}v + \nu$. The real and imaginary components ν_1 and ν_2 are obtained by means of a computer-generated zero mean random process of amplitude 0.05 approximately. The results of using the proposed controller scheme on the same tracking task defined before yield the trajectories shown in Figure 3. Samples of the computer generated noise inputs to the system are also shown.

3.3. A second order system

In order to illustrate that the proposed approach is not foreign to robust behaviour with respect to high frequency perturbations, consider the second order system

$$\left(s^2 + as + b\right) y = cu + sy(0) + \dot{y}(0) + \frac{A_0}{s} + \varpi \tag{3.5}$$

where the constant parameters $a, b, c \in \mathbb{R}$ are unknown. The additive perturbations consist of

(1) a constant load perturbation $\frac{A_0}{s}$ of unknown magnitude $A_0 \in \mathbb{R}$;
(2) a high frequency perturbation ϖ, which will be made precise for the computer simulations.

It is desired to perform a desired rest-to-rest maneuver for the output variable y, from any arbitrary initial condition $y(0)$, and for an unknown initial velocity $\dot{y}(0)$. The maneuver is to be accomplished *via* the tracking a given reference trajectory y^*, defined on the finite time interval $[0, T]$.

Introduce a second order low pass filter, with a measurable output y_f, given by

$$y_f = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\, y \tag{3.6}$$

where the known constant parameters $\zeta, \omega_n \in \mathbb{R}$ are strictly positive. Thus

$$(s^2 + as + b)(s^2 + 2\zeta\omega_n s + \omega_n^2) y_f = \omega_n^2 \left[cu + sy(0) + \dot{y}(0) + \frac{A_0}{s} + \varpi \right]. \tag{3.7}$$

3.3.1. *On-line identifier*

The on-line identifier is obtained by setting $\varpi = 0$ in (3.7)

$$(s^2 + as + b)(s^2 + 2\zeta\omega_n s + \omega_n^2) y_f = \omega_n^2 \left[cu + sy(0) + \dot{y}(0) + \frac{A_0}{s} \right].$$

Multiplying both sides by s yields

$$s(s^2 + as + b)(s^2 + 2\zeta\omega_n s + \omega_n^2) y_f = \omega_n^2 \left[csu + s^2 y(0) + s\dot{y}(0) + A_0 \right]. \tag{3.8}$$

Derive both sides three times with respect to s, in order to eliminate the unknown parameter A_0, and the initial conditions $y(0)$, $\dot{y}(0)$. Then multiply both sides of the resulting expression by s^{-5}. We obtain in the time domain a relation of the form

$$a\pi_1(t) + b\pi_2(t) + c\pi_3(t) = q(t) \tag{3.9}$$

with[11]

$$
\begin{aligned}
\pi_1(t) = & \Bigg[- \left(\int t^3 y\right) + \left(\int^{(2)} \left(12t^2 - 2\zeta\omega_n t^3\right) y\right) + \left(\int^{(3)} \left(-36t + 18\zeta\omega_n t^2 - \omega_n^2 t^3\right) y\right) \\
& + \left(\int^{(4)} \left(24 - 36\zeta\omega_n t + 6\omega_n^2 t^2\right) y\right) + \left(\int^{(5)} \left(12\zeta\omega_n - 6\omega_n^2 t\right) y\right) \Bigg] \\
\pi_2(t) = & \Bigg[- \left(\int^{(2)} t^3 y\right) + \left(\int^{(3)} \left(9t^2 - 2\zeta\omega_n t^3\right) y\right) + \left(\int^{(4)} \left(-18t + 12\zeta\omega_n t^2 - \omega_n^2 t^3\right) y\right) \\
& + 6 \left(\int^{(5)} (1 - 12\zeta\omega_n t + 3\omega_n^2 t^2) y\right) \Bigg] \\
\pi_3(t) = & -\omega_n^2 \left[3 \left(\int^{(5)} t^2 u\right) - \left(\int^{(4)} t^3 u\right) \right] \\
q(t) = & -\Bigg[- t^3 y + \left(\int \left(15t^2 - 2\zeta\omega_n t^3\right) y\right) + \left(\int^{(2)} \left(-60t + 24\zeta\omega_n t^2 - \omega_n^2 t^3\right) y\right) \\
& + \left(\int^{(3)} \left(60 - 72\zeta\omega_n t + 9\omega_n^2 t^2\right) y\right) + \left(\int^{(4)} \left(48\zeta\omega_n - 18\omega_n^2 t\right) y\right) + 6\omega_n^2 \left(\int^{(5)} y\right) \Bigg].
\end{aligned}
$$

Integrating the expression (3.9) one and two times, respectively, leads to the following system of linear equations for the estimates a_e, b_e, c_e of the unknown parameters

$$
\begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) \\ P_{21}(t) & P_{22}(t) & P_{23}(t) \\ P_{31}(t) & P_{32}(t) & P_{33}(t) \end{bmatrix} \begin{bmatrix} a_e \\ b_e \\ c_e \end{bmatrix} = \begin{bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{bmatrix}
$$

with

$$
\begin{aligned}
& P_{11}(t) = \pi_1(t), \quad P_{12}(t) = \pi_2(t), \quad P_{13}(t) = \pi_3(t) \\
& P_{21}(t) = \int \pi_1(t), \quad P_{22}(t) = \int \pi_2(t), \quad P_{23}(t) = \int \pi_3(t) \\
& P_{31}(t) = \int^{(2)} \pi_1(t), \quad P_{32}(t) = \int^{(2)} \pi_2(t), \quad P_{33}(t) = \int^{(2)} \pi_3(t)
\end{aligned}
$$

and

$$
Q_1(t) = q(t), \quad Q_2(t) = \int q(t), \quad Q_3(t) = \int^{(2)} q(t).
$$

[11]We have denoted by $\int^{(\alpha)} \phi(t)$ the quantity $\int_0^t \int_0^{\beta_1} \int_0^{\beta_2} \cdots \int_0^{\beta_{\alpha-1}} \phi(\beta_\alpha) \mathrm{d}\beta_\alpha \cdots \mathrm{d}\beta_2 \mathrm{d}\beta_1$. Moreover, $\int^{(1)} \phi(t) = \int \phi(t) = \int_0^t \phi(\beta_1) \mathrm{d}\beta_1$.

3.3.2. *A certainty equivalence GPI controller*

Following [12] *integral reconstructors* of derivatives of the filtered output y_f are given by

$$
\begin{aligned}
\underline{sy_f} &= -\left[(2\zeta\omega_n + a)y_f + (\omega_n^2 + 2a\zeta\omega_n + b)s^{-1}y_f + (a\omega_n^2 + 2b\zeta\omega_n)s^{-2}y_f + b\omega_n^2 s^{-3}y_f\right] + c\omega_n^2 s^{-3}u \\
\underline{s^2y_f} &= -\left[(2\zeta\omega_n + a)\underline{sy_f} + (\omega_n^2 + 2a\zeta\omega_n + b)y_f + (a\omega_n^2 + 2b\zeta\omega_n)s^{-1}y_f + b\omega_n^2 s^{-2}y_f\right] + c\omega_n^2 s^{-2}u \\
\underline{s^3y_f} &= -\left[(2\zeta\omega_n + a)\underline{s^2y_f} + (\omega_n^2 + 2a\zeta\omega_n + b)\underline{sy_f} + (a\omega_n^2 + 2b\zeta\omega_n)y_f + b\omega_n^2 s^{-1}y_f\right] + c\omega_n^2 s^{-1}u. \qquad (3.10)
\end{aligned}
$$

Define the following *generalized proportional integral*, or *GPI*, controller [12], which is of course independent of A_0

$$
\begin{aligned}
u = \frac{1}{\omega_n^2 c}\Big\{&(2\zeta\omega_n + a)\underline{s^3y_f} + (\omega_n^2 + 2a\zeta\omega_n + b)\underline{s^2y_f} + (a\omega_n^2 + 2b\zeta\omega_n)\underline{sy_f} \\
&+ b\omega_n^2 y_f + s^4 y_f^* - k_3(\underline{s^3y_f} - s^3y_f^*) - k_2(\underline{s^2y_f} - s^2y_f^*) - k_1(\underline{sy_f} - sy_f^*) - k_0(y_f - y_f^*) + \xi\Big\} \qquad (3.11)
\end{aligned}
$$

with

$$\xi = -\left[k_{-1}s^{-1} + k_{-2}s^{-2} + k_{-3}s^{-3} + k_{-4}s^{-4}\right]e_f \qquad (3.12)$$

and $e_f = y_f - y_f^*$. The closed loop system tracking error satisfies

$$\left[s^8 + k_3s^7 + k_2s^6 + k_1s^5 + k_0s^4 + k_{-1}s^3 + k_{-2}s^2 + k_{-3}s + k_{-4}\right]e_f = 0. \qquad (3.13)$$

An asymptotically exponentially stable behaviour is obtained for the tracking error e_f by choosing the design coefficients $k_3, k_2, \ldots, k_{-2}, k_{-3}$ so that the corresponding characteristic polynomial is Hurwitz.

3.3.3. *Simulation results*

Simulations were performed for an unstable system (3.5), with

$$a = -1, \quad b = -2, \quad c = 1, \quad A_0 = 0.5.$$

The second order low pass filter was characterised by the parameters

$$\zeta = 0.8, \quad \omega_n = 1.$$

The desired trajectory, y^*, for the rest-to-rest maneuver on the system output y was set to be an interpolating polynomial function, of Bézier type, of the following form

$$y^*(t) = y^{\text{init}}(t_0) + (y^{\text{final}}(T) - y^{\text{initial}}(t_0))\psi(t, t_0, T)$$

where $\psi(t, t_0, T)$ smoothly interpolates between the values 0 and 1:

$$\psi(t, t_0, T) = \Delta^8\left[r_1 - r_2\Delta^2 + \cdots - r_8\Delta^7 + r_9\Delta^8\right]$$

with $\Delta = (t - t_0)/(T - t_0)$ and,

$$
\begin{gathered}
r_1 = 12\,870,\ r_2 = 91\,520,\ r_3 = 288\,288,\ r_4 = 524\,160, \\
r_5 = 600\,600,\ r_6 = 443\,520,\ r_7 = 205\,920,\ r_8 = 54\,912,\ r_9 = 6435.
\end{gathered}
$$

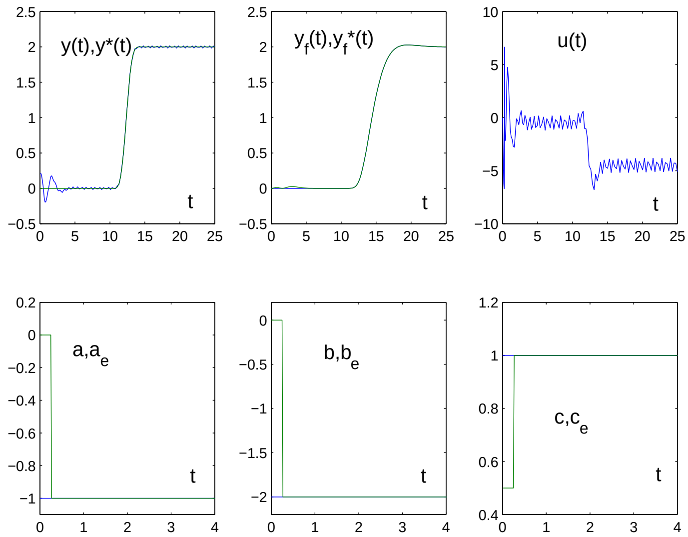

FIGURE 4. Closed loop trajectory tracking response of second order unstable system depicting the unknown parameters identification.

The initial and final values of the desired transfer, for the output variable y, and the required finite time interval for accomplishing the desired maneuver were defined by

$$y^{\text{init}}(t_0) = 0, \quad y^{\text{final}}(T) = 2, \quad t_0 = 10, \quad T = 15.$$

In order to test the response of the designed control system to external high-frequency perturbations, set $\varpi = A_1 \cos(\omega t)$, with $A_1 = 1.5$, $\omega = 10$ [rad/s]. The closed loop poles for the 8^{th}-order characteristic polynomial were all set to -3.

Figure 4 shows the obtained digital computer simulations depicting closed loop performance of the proposed certainty equivalence controller (3.10–3.12). The parameters a_e and b_e were arbitrarily initialized to be zero while the proposed identifier computed the actual value of the parameters on the basis of on-line gathered data. In order to avoid divisions by zero, the estimated parameter c_e was arbitrary initialized to the value of 0.5.

3.4. **The heat exchanger**

Consider with [14] the following simple model of a heat exchanger

$$\begin{aligned} V_c \frac{\mathrm{d}T_c}{\mathrm{d}t} &= f_c\,(T_{c_i} - T_c) + \beta\,(T_h - T_c) \\ V_h \frac{\mathrm{d}T_h}{\mathrm{d}t} &= f_h\,(T_{h_i} - T_h) - \beta\,(T_h - T_c) \end{aligned} \tag{3.14}$$

T_{c_i} is the cold water inflow temperature acting as an input; T_{h_i} is the hot water inflow temperature acting as a second input to the system; T_c is the cold water outlet temperature while T_h is the hot water outlet temperature. The parameters f_c, f_h represent the volumetric input flows which are assumed to be constant.

Let the volumetric input flows be equal, *i.e.*, $f_c = f_h = f$ and take the inflow temperatures as control inputs, $T_{c_i} = u_1$, $T_{h_i} = u_2$. We have,

$$\dot{x} = \begin{bmatrix} -\dfrac{f+\beta}{V_c} & \dfrac{\beta}{V_c} \\ \dfrac{\beta}{V_h} & -\dfrac{f+\beta}{V_h} \end{bmatrix} x + \begin{bmatrix} \dfrac{f}{V_c} & 0 \\ 0 & \dfrac{f}{V_h} \end{bmatrix} u \tag{3.15}$$

$$x = \begin{bmatrix} T_c \\ T_h \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We also let $V_c = V_h = V$ and denote $a = \frac{f+\beta}{V}$, $b = \frac{\beta}{V}$ and $c = \frac{f}{V}$. The system output vector, denoted by y, consists of the states x_1 and x_2. We include external constant perturbation inputs of unknown values, denoted by ξ_1 and ξ_2, to the cold and hot water temperature dynamics, respectively. We, thus, have the following model:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} x + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} u + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \\ y_1 &= x_1 \\ y_2 &= x_2. \end{aligned} \tag{3.16}$$

3.4.1. *An on-line identifier*

Taking derivatives twice with respect to s in

$$\begin{aligned} s^2 y_1 - s y_1(0) + a s y_1 &= b s y_2 + c s u_1 + \xi_1 \\ s^2 y_2 - s y_2(0) + a s y_2 &= b s y_1 + c s u_2 + \xi_2 \end{aligned}$$

permits to eliminate the perturbations and the initial conditions:

$$\begin{aligned} s^2\frac{\mathrm{d}^2 y_1}{\mathrm{d}s^2} + 4s\frac{\mathrm{d}y_1}{\mathrm{d}s} + 2y_1 + a\left[s\frac{\mathrm{d}^2 y_1}{\mathrm{d}s^2} + 2\frac{\mathrm{d}y_1}{\mathrm{d}s}\right] - b\left[s\frac{\mathrm{d}^2 y_2}{\mathrm{d}s^2} + 2\frac{\mathrm{d}y_2}{\mathrm{d}s}\right] &= c\left[s\frac{\mathrm{d}^2 u_1}{\mathrm{d}s^2} + 2\frac{\mathrm{d}u_1}{\mathrm{d}s}\right] \\ s^2\frac{\mathrm{d}^2 y_2}{\mathrm{d}s^2} + 4s\frac{\mathrm{d}y_2}{\mathrm{d}s} + 2y_2 + a\left[s\frac{\mathrm{d}^2 y_2}{\mathrm{d}s^2} + 2\frac{\mathrm{d}y_2}{\mathrm{d}s}\right] - b\left[s\frac{\mathrm{d}^2 y_1}{\mathrm{d}s^2} + 2\frac{\mathrm{d}y_1}{\mathrm{d}s}\right] &= c\left[s\frac{\mathrm{d}^2 u_2}{\mathrm{d}s^2} + 2\frac{\mathrm{d}u_2}{\mathrm{d}s}\right]. \end{aligned}$$

We obtain

$$\begin{aligned} \pi_{11}(t)a + \pi_{12}(t)b + \pi_{13}(t)c &= q_1(t) \\ \pi_{21}(t)a + \pi_{22}(t)b + \pi_{23}(t)c &= q_2(t) \end{aligned} \tag{3.17}$$

with

$$\begin{aligned}
\pi_{11}(t) &= \int t^2 y_1 - 2\int\int t y_1\\
\pi_{12}(t) &= -\int t^2 y_2 + 2\int\int t y_2\\
\pi_{13}(t) &= -\int t^2 u_1 + 2\int\int t u_1\\
q_1(t) &= -t^2 y_1 + 4\int t y_1 - 2\int\int y_1\\
\pi_{21}(t) &= \int t^2 y_2 - 2\int\int t y_2\\
\pi_{22}(t) &= -\int t^2 y_1 + 2\int\int t y_1\\
\pi_{23}(t) &= -\int t^2 u_2 + 2\int\int t u_2\\
q_2(t) &= -t^2 y_2 + 4\int t y_2 - 2\int\int y_2.
\end{aligned}$$

A possible choice to complete a system of equations for the unknown parameters a, b and c, which we now denote by a_e, b_e and c_e respectively, is obtained by considering the integral of any one of the two equations in (3.17). We obtain the linear system

$$\begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t)\\ P_{21}(t) & P_{22}(t) & P_{23}(t)\\ P_{31}(t) & P_{32}(t) & P_{33}(t) \end{bmatrix} \begin{bmatrix} a_e\\ b_e\\ c_e \end{bmatrix} = \begin{bmatrix} Q_1(t)\\ Q_2(t)\\ Q_3(t) \end{bmatrix}$$

with

$$\begin{aligned}
&P_{11}(t) = \pi_{11}(t),\quad P_{12}(t) = \pi_2(t),\quad P_{13}(t) = \pi_{13}(t),\quad Q_1(t) = q_1(t)\\
&P_{21}(t) = \int \pi_{11}(t),\ P_{22}(t) = \int \pi_{12}(t),\ P_{23}(t) = \int \pi_{13}(t),\\
&Q_2(t) = \int q_1(t)\\
&P_{31}(t) = \pi_{21}(t),\quad P_{32}(t) = \pi_{22}(t),\quad P_{33}(t) = \pi_{23}(t),\quad Q_3(t) = q_2(t).
\end{aligned}$$

3.4.2. *A certainty equivalence controller*

The estimated values of the system parameters are used in the proposed multivariable GPI controller:

$$\begin{aligned}
u_1 &= \frac{1}{c_e}\left[a_e y_1 - b_e y_2 + \dot{x}_1^*(t) - k_{21}(y_1 - x_1^*(t)) - k_{11}\int_0^t (y_1(\sigma) - x_1^*(\sigma))\mathrm{d}\sigma\right]\\
u_2 &= \frac{1}{c_e}\left[a_e y_2 - b_e y_1 + \dot{x}_2^*(t) - k_{22}(y_2 - x_2^*(t)) - k_{12}\int_0^t (y_2(\sigma) - x_2^*(\sigma))\mathrm{d}\sigma\right].
\end{aligned} \tag{3.18}$$

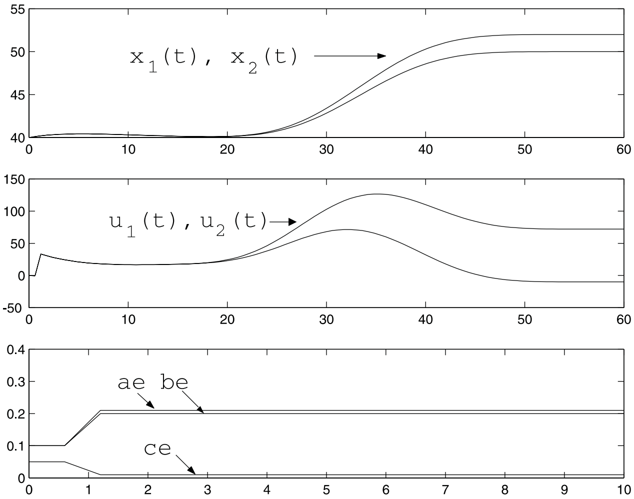

FIGURE 5. Heat exchanger closed loop tracking responses with parameter identification.

3.5. Simulation results

The heat exchanger data provided in [14] was used for simulation purposes

$$a = 0.21, \quad b = 0.2, \quad c = 0.01$$

with the constant perturbation parameters set to be

$$\xi_1 = 0.2 \quad \xi_2 = 0.2.$$

A tracking maneuver, entitling a set point change, from equilibrium to equilibrium, was set for the cold and hot water temperature variables x_1 and x_2. The simulations in Figure 5 depict a smooth transfer between an initial condition $x(t_0) = (40, 40)$ at time $t_0 = 10$ min and a final desired state $x(T) = (50, 52)$ at time $T = 60$ min. The system parameters a_e and b_e were arbitrarily set to be 0.1 while, to avoid divisions by zero, the value for c_e was set to be 0.05, at the beginning of the computations. Once stable values of the parameter estimations were obtained, these were replaced on the certainty equivalence controller (3.18).

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