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# THE OUTPUT LEAST SQUARES IDENTIFIABILITY OF THE DIFFUSION COEFFICIENT FROM AN $H^1$–OBSERVATION IN A 2–D ELLIPTIC EQUATION *

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**Abstract**. Output least squares stability for the diffusion coefficient in an elliptic equation in dimension two is analyzed. This guarantees Lipschitz stability of the solution of the least squares formulation with respect to perturbations in the data independently of their attainability. The analysis shows the influence of the flow direction on the parameter to be estimated. A scale analysis for multi-scale resolution of the unknown parameter is provided.

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## 1. Introduction

We consider in this paper the identification of the diffusion coefficient $a$ in an elliptic equation on a bounded, two-dimensional domain $\Omega$ with appropriate boundary conditions:

$$-\text{div}\ (a \text{ grad } u) = f \ \text{ in } \ \Omega. \tag{1.1}$$

Known are the set $C$ of admissible parameters for $a$, and a measurement

$$z = \mathcal{O}u \tag{1.2}$$

of $u$, where $\mathcal{O}$ denotes the observation operator.

The *identifiability* of the diffusion coefficient $a$ in $C$ is usually understood as the injectivity of the $a \to \mathcal{O}u_a$ mapping on $C$, where $u_a$ denotes the solution to (1.1) as a function of $a$.

The stronger notion of parameter *stability* is used for the continuous dependence of the inverse of $a \to \mathcal{O}u_a$.

Identifiability and stability of $a$ in (1.1) have been treated in several papers. We mention [17] and [10], where the analysis is based on the method of characteristics for the hyperbolic equation for $a$, which arises from (1.1) when $f$ and $u$ are given functions. The analysis in [15] is based on variational techniques. All these results refer to the case of distributed observation. Many current research efforts focus on identifiability in the case of boundary observations. Relevant references can be found in [14], for example.

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In practice however the measurement $z$ may typically not belong to the set of attainable outputs $D = \{\mathcal{O}u_a : a \in C\}$, and one has to estimate $a$ in the least squares sense:

$$\text{minimize } \frac{1}{2} |\mathcal{O}u_a - z|^2 \text{ over } C. \tag{1.3}$$

Identifiability and stability have of course to hold if one wants the optimization problem (1.3) to be well behaved, but these properties are far from being sufficient.

A still stronger definition was introduced in [2]: the parameter $a \in C$ is called *Output Least Squares (OLS-) identifiable* if there exists a neighborhood $\mathcal{V}$ of $D$ such that for every $z \in \mathcal{V}$ the least squares problem (1.3) has no local minima and a unique solution in $C$ depending Lipschitz-continuously on $z \in \mathcal{V}$. The precise definition of this concept is recalled in Section 2. OLS-identifiability takes into account the situation relevant in practice where due to errors in the data and in the model, the data $z$ may not be contained in the attainable set $D$. As a consequence the problem of unique and continuous projection of $z$ onto the nonconvex set $D$ must be considered. This makes OLS-identifiability quite difficult to establish. It was analysed so far only for the determination of coefficients in lower order terms by means of regularized least squares formulations and observation in $L^2(\Omega)$ [6], and for the diffusion coefficient in a one-dimensional version of equation (1.1) with distributed $H^1$ observation [5].

We consider in this paper the OLS-identifiability of the diffusion coefficient $a$ in the two-dimensional equation (1.1) in the case of a distributed $H^1$ observation, where $\mathcal{O}u_a = \text{grad } u_a$. Of course, it can seem unrealistic that one can observe or measure the gradient of $u$ throughout $\Omega$. However, the results of this paper can be combined with the technique of state-space regularization of [7] to handle the somewhat more realistic case of a distributed $L^2$ observation. Let us mention also that our results can easily be extended to equations containing lower order terms.

OLS-identifiability with boundary measurement is a completely open problem.

The paper and the results are organized as follows:

In Section 2 we define the inverse problem under consideration, and set up conditions on $f$ and $\Omega$ which will ensure the identifiability of $a$.

We introduce in Section 3 a div/rot decomposition of vector fields in $\mathbb{L}^2(\Omega)$ which will be used throughout the paper.

Sections 4 and 5 are devoted to the stability analysis of $a$ in $C \subset L^2(\Omega)$ from $u_a$ in $H^1(\Omega)$: in Section 4 we shall observe that even such a strong observation does not control the variations of $a$ perpendicular to the flow lines, and conclude that stability does not hold when $C$ is infinite dimensional. Then we show in Section 5 that stability is restored in finite dimensional subsets $C_n$ of $C$, with a stability constant which blows up to infinity when the dimension of $C_n$ increases.

In Section 6, we establish the sensitivity, deflection and curvature estimates which are needed for the theory of strictly quasiconvex sets [3, 4] to guarantee that the projection on the sets $D_n$ of attainable outputs is well behaved. When the dimension of $C_n$ increases, the sensitivity decreases toward zero, the deflection remains bounded, and the curvature tends to infinity. This generalizes to the two dimensional case the results of [16].

The main OLS-identifiability result for $a \in C_n \subset L^2(\Omega)$ from a measure $z$ of grad $u$ in $H^1(\Omega)$ is stated and proved in Section 7: at each scale $n$, the diffusion coefficient $a$ is OLS-identifiable in $C_n$ over a neighborhood $\mathcal{V}_n$ of the output set $D$. When the parameterization of $a$ is refined, *i.e.* when $n$ goes to infinity, the neighborhood $\mathcal{V}_n$ shrinks around $D$, and the Lipschitz constant of the $z \to \hat{a}$ mapping explodes. These theoretical results are coherent with the nice properties of multiscale parameterization which have been observed numerically in [16].

## 2. The inverse problem

We consider a domain $\Omega \subset \mathbb{R}^2$ such that its boundary $\partial\Omega$ is partitioned into $\Gamma_D$, $\Gamma_N$, and $\Gamma_i$, $i = 1, \cdots, N$. Here $\Gamma_i$ represent the boundaries of holes, which will be used to model source and sink terms, and $\Gamma_D$ and $\Gamma_N$ are a partition of the outer boundary of $\Omega$ corresponding to Dirichlet and Neumann boundary conditions.

We define the Hilbert space

$$\begin{array}{l} V = \{v \in H^1(\Omega)\colon v|_{\Gamma_D} = 0,\ v|_{\Gamma_i} = v_i = \text{ const },\ i = 1, \cdots, N\} \\ \|v\|_V = |\nabla v|_{\mathbb{L}^2}, \end{array} \tag{2.1}$$

where in case $\Gamma_D = \phi$ the condition $v|_{\Gamma_D} = 0$ is replaced by $\int_\Omega v = 0$. It is supposed that $\Omega$ satisfies

$$\begin{array}{l} \Omega \text{ is bounded and connected; } \partial\Omega \in C^{1,1};\ \bar{\Gamma}_N, \bar{\Gamma}_D \text{ and} \\ \bar{\Gamma}_i, i = 1, \cdots, N \text{ are pairwise disjoint.} \end{array} \tag{2.2}$$

On $V$ we define the linear form $L$ by

$$\begin{array}{l} L(v) = \int_\Omega f\,v + \int_{\Gamma_N} g\,v + \sum_{i=1}^N Q_i v_i, \quad \text{for } v \in V, \\ \text{where } f \in L^p(\Omega), g \in L^p(\Gamma_N), \text{for some } p > 2, Q_i \in \mathbb{R}, i = 1, \cdots, N, \end{array} \tag{2.3}$$

with the additional condition that

$$\int_\Omega f \ + \int_{\Gamma_N} g \ + \sum_{i=1}^N Q_i = 0 \ \text{ if } \ \Gamma_D = \phi.$$

Henceforth we denote by $C$ the set of admissible parameters $a$. The precise conditions on $C$ we be given further below. In particular they will allow to associate to every $a \in C$ the solution $u = u_a \in V$ defined by

$$\text{(Q)} \qquad \int_\Omega a \nabla u \nabla v = L(v) \ \text{ for all } \ v \in V,$$

which is the variational formulation of the elliptic equation

$$\begin{cases} -\text{div } (a \text{ grad } u) = f \ \text{ in } \ \Omega \\ u|_{\Gamma_D} = 0,\ a\dfrac{\partial u}{\partial n}|_{\Gamma_N} = g \\ \int_{\Gamma_i} a\dfrac{\partial u}{\partial n} = Q_i,\ u = \text{ unknown constant on } \Gamma_i, i = 1, \cdots, N. \end{cases} \tag{2.4}$$

We shall be concerned with the inversion of the mapping $a \to \text{grad } u_a$ from $L^2(\Omega)$ to $\mathbb{L}^2(\Omega)$ in the least-squares sense:

$$\text{(P)} \qquad \text{minimize } \frac{1}{2}|\text{ grad } u_a - z|^2_{\mathbb{L}^2} \ \text{ over } \ a \in C,$$

where $z \in \mathbb{L}^2(\Omega)$ is a given observation.

**Definition 2.1.** The parameter $a$ is OLS-identifiable in $C$ from the measurement $z$ of grad$u$ if and only if the nonlinear least squares problem (P) is quadratically (Q-) wellposed in the sense that:

$$D = \{\text{grad} u_a \in \mathbb{L}^2(\Omega)\colon a \in C\}$$

possesses a neighborhood $\mathcal{V}$ such that

1. for every $z \in \mathcal{V}$ problem (P) has a unique solution $\hat{a}$;
2. for every $z \in \mathcal{V}$ problem (P) has no parasitic local minima;
3. the mapping $z \to \hat{a}$ is Lipschitz continuous from $\mathbb{L}^2(\Omega)$ to $L^2(\Omega)$.

Our objective is to find conditions on $C$ such that the parameter $a$ is output least squares OLS-identifiable on $C$. For this purpose we require first that the parameters belong to the space

$$\mathcal{E} = \{a \in C^{0,1}(\overline{\Omega})\colon a|_{\Gamma_i} = \text{ unknown constant } a_i = i = 1, \cdots, N\}, \tag{2.5}$$

equipped with the norm $\|\cdot\|_{C^{0,1}}$. The set of admissible parameters $C$ is assumed to satisfy:

$$C \subset \{a \in \mathcal{E}\colon a_m \leq a(x) \text{ a.e. in } \Omega, \|a\|_{C^{0,1}} \leq a_M\}, \tag{2.6}$$

and

$$C \text{ is convex and closed in } L^2(\Omega), \tag{2.7}$$

where $0 < a_m \leq a_M$ are given constants. Note that the image in $C(\subset \mathcal{E})$ of the mapping $z \to \hat{a}$ is considered in the $L^2$-norm, whereas the set $C$ is endowed with the norm of $\mathcal{E}$. Condition (2.6) ensures that (Q) has a unique solution for all $a \in C$. Moreover, the requirement that $a$ is Lipschitz continuous, together with the modelling of source and sink terms by holes, and the regularity hypotheses (2.2, 2.3) for $\Omega$, $f$, and $g$ imply that $\{|u_a|_{W^{2,p}}\colon a \in C\}$ is bounded, (see [18], p. 180). Since $W^{2,p}(\Omega)$ is continuously embedded into $C^1(\overline{\Omega})$ for every $p > 2$, then exist $u_M$ and $\gamma_M$ such that

$$|u_a|_{L^\infty} \leq u_M, |\text{grad } u_a|_{\mathbb{L}^\infty} \leq \gamma_M \text{ for all } a \in C. \tag{2.8}$$

The hypothesis that the parameters satisfy $a|_{\Gamma_i} = a_i$ = unknown constant is the 2-D generalization of the hypothesis that $a$ is constant on some neighborhood of each Dirac source term, which was required in the 1-D case to ensure OLS–identifiability [5]. It is also physically not too restrictive, as one can assume that the size of $\Gamma_i$'s, which model the well boundaries, are small compared to the size of $\Omega$. The convexity and closedness condition (2.7) are required for the study of OLS–identifiability by the geometric techniques for nonlinear least squares developed in [3, 4].

As a first step towards OLS–identifiability we shall analyse in Section 3 inverse stability estimates of the form

$$\text{(S)} \qquad |(a-b) \text{ grad } u_a|_{\mathbb{L}^2} \leq k\, |b(\text{grad } u_a - \text{ grad } u_b)|_{\mathbb{L}^2},$$

for $k \geq 0$. As this stability estimate ought to hold for perturbations $a - b$ in any direction $b \in \mathcal{E}$, we attempt to prove (S) only at points $a \in C$ which are identifiable:

**Definition 2.2.** The parameter $a \in C$ is identifiable in $\mathcal{E}$ if, for every $b \in \mathcal{E}$ which admits a solution $u_b \in V$ to (Q) one has

$$u_b = u_a \text{ implies } b = a. \tag{2.9}$$

However, we shall see in Section 3 that one cannot find for any $k > 0$ an infinite set $C$ satisfying (2.5–2.7) on which (S) holds uniformly. Therefore we reduce in Section 4 our attention to finite dimensional parameter sets: for this purpose let $\mathcal{E}_n$, $n \in \mathbb{N}$ be a family of subspaces such that

$$\begin{cases} \mathcal{E}_0 = \{\text{constant functions}\} \subset \mathcal{E}_1 \subset \mathcal{E}_2 \cdots \subset \mathcal{E} \subset C^{0,1}(\bar{\Omega}) \\ \dim \mathcal{E}_n < \infty \text{ for each } n \\ \overline{\bigcup_{n \in \mathbb{N}} \mathcal{E}_n} = L^2(\Omega), \end{cases} \tag{2.10}$$

where the closure is taken in $L^2(\Omega)$, and define for all $n$:

$$C_n = C \cap \mathcal{E}_n. \tag{2.11}$$

In order to have a chance for (S) to hold uniformly on $C_n$ we require that the data of the problem, *i.e.* $(\Omega, \Gamma_D, \Gamma_N, \Gamma_i, f, g, Q_i, a_m, a_M, \mathcal{E}_n)$, are chosen such that

$$\text{(H)} \qquad \begin{cases} \text{for all } \ n \in \mathbb{N} \ \text{ one has} \\ a \in C_n \ \text{ implies that } a \text{ is identifiable in } \mathcal{E}_n. \end{cases}$$

For the definition of identifiability of $a \in C_n$ in $\mathcal{E}_n$ one simply replaces $C, \mathcal{E}$ in Definition 2.2 by $C_n, \mathcal{E}_n$. Under condition (H) we shall be able to prove in Section 4 the inverse stability estimate (S) on $C_n$ for all $n \in \mathbb{N}$, for some $k = k_n$, with $\lim_{n\to\infty} k_n = \infty$, and to estimate, in Section 5, sensitivity, deflection and curvature of the $a \to \text{grad}\, u_a$ mapping. These estimates will be used to prove, in Section 6, that $a$ is OLS–identifiable on $C_n$ for each $n$, provided the diameter of $C$ in $L^\infty(\Omega)$, denoted by $\text{diam}_\infty$, is small enough. The last section will be devoted to the analysis of the advantages and disadvantages of parameterizing the problem by $b = 1/a$ instead of $a$.

Before proceeding to the next section we make sure that our theory is not empty by giving an example for sufficient conditions which ensure that (H) holds. We require two technical lemmas.

**Lemma 2.1.** *The parameter $a \in C$ is identifiable in $\mathcal{E}$ if and only if*

$$h \in \mathcal{E} \ \textit{ and } \ \int_\Omega h \,\text{grad}\, u_a \,\text{grad}\, v = 0 \ \textit{ for all } \ v \in V \ \textit{ implies } \ h = 0.$$

As a consequence we observe that if $a \in C$ is identifiable in $\mathcal{E}$ and $h \neq 0$, $h \in \mathcal{E}$, then necessarily $h \,\text{grad}\, u_a \neq 0$.

**Lemma 2.2.** *For $a \in C$ and $h \in \mathcal{E}$ define $u = u_a$ and $v = \frac{hu}{a}$. Then $v \in V$ and*

$$\int_\Omega h \,\text{grad}\, u \cdot \,\text{grad}\, v = \frac{1}{2}\int_\Omega \frac{h^2}{a} |\,\text{grad}\, u|^2 + \frac{1}{2}\int_\Omega \frac{h^2}{a^2}\, u\, f + \frac{1}{2}\int_{\Gamma_N} \frac{h^2}{a^2}\, u\, g + \sum_{i=1}^{N} \frac{h_i^2}{a_i^2}\, u_i Q_i.$$

*Proof.* Since $C \subset \mathcal{E} \subset C^{0,1}(\bar{\Omega})$ one has $v = \frac{hu}{a} \in H^1(\Omega)$. Moreover $v$ satisfies the boundary conditions defining V and hence $v \in V$. It follows that

$$\int_\Omega h \,\text{grad}\, u \cdot \,\text{grad}\, v = \int_\Omega \frac{h^2}{a} |\,\text{grad}\, u|^2 + \frac{1}{2}\int_\Omega \frac{u}{a} \,\text{grad}\, u \cdot \,\text{grad}\, h^2 - \int_\Omega \frac{h^2 u}{a^2} \,\text{grad}\, a \cdot \,\text{grad}\, u.$$

Integrating by parts the second term on the right hand side implies

$$\begin{aligned} \int_\Omega h \,\text{grad}\, u \cdot \,\text{grad}\, v =& \frac{1}{2}\int_\Omega \frac{h^2}{a} |\,\text{grad}\, u|^2 - \frac{1}{2}\int_\Omega h^2 \left(\frac{u}{a}\, \Delta u + \frac{u}{a^2} \,\text{grad}\, a \cdot \,\text{grad}\, u\right) \\ & + \frac{1}{2}\int_{\Gamma_N} u\, g \frac{h^2}{a^2} + \frac{1}{2} \sum_{i=1}^{N} u_i Q_i\, \frac{h_i^2}{a_i^2}, \end{aligned}$$

which, utilizing $-a\Delta u -$ grad $a \cdot$ grad $u = f$ gives the desired result. ☐

**Theorem 2.1.** *Let the data of the problem satisfy*

- $f = g = 0$;
- $Q_i, i = 1, \cdots, N$ *are not all zero;*
- $0 < a_m \leq a_M$;
- $|\Gamma_i|, i = 1, \cdots, N$, *are sufficiently small.*

*Then for all* $n$, *all* $a \in C_n$ *are identifiable in* $\mathcal{E}_n$ *and (H) holds.*

*Proof.* Let $n \in \mathbb{N}$ be given. Let $a \in C_n$ and $u_a$ denote the solution to (2.4). We argue that grad $u(a)$ cannot vanish on a set $I$ of positive measure. Let $\gamma$ denote a curve in $\Omega$ connecting the inner boundaries $\Gamma_i$ to $\Gamma_D \cup \Gamma_N$ such that $\Omega \backslash \gamma$ is simply connected and meas $\gamma = 0$.

Then $I_\gamma = (\Omega \backslash \gamma) \cap I$ satisfies meas $I_\gamma > 0$. From [1], Theorem 2.1 and Remark it follows that either $u_a$ is constant on $\Omega \backslash \gamma$ and hence on $\Omega$ or $u_a$ has only isolated critical points, *i.e.* points $z$ satisfying $\nabla u(z)$. But $u_a$ cannot equal a constant over $\Omega$ since this violates the boundary conditions at the wells $\Gamma_i$ at which $Q_i \neq 0$. On the other hand the number of isolated critical points in $I_\gamma$ can be at most countable, and hence meas $I_\gamma = 0$. Consequently meas $\{x : \nabla u_a(x) = 0\} = 0$.

Suppose that $\Gamma_i$ surrounds for each $i = 1, \cdots, N$, a fixed source/sink location $x_i$. If $|\Gamma_i| \to 0$, for all $i = 1, \cdots, N$, the solution $u_a$ converges towards the weak solution associated to a right-hand side with Dirac source term $\sum_{i=1}^{N} Q_i \delta(x - x_i)$, which is singular at $x_i$. Hence $u_a|_{\Gamma_i} = u_{a,i} \to \infty$ if $Q_i > 0$ and $u_a|\Gamma_i \to -\infty$ if $Q_i < 0$. Since $C_n$ is compact and $a \to u_{a,i}$ is continuous, we conclude that for $|\Gamma_i|$ sufficiently small the solution $u_a$ satisfies

$$u_{a,i}\, Q_i \geq 0 \ \text{ for } \ i = 1, \cdots, N, \ \text{ and all } \ a \in C_n.$$

Henceforth it is assumed that $|\Gamma_i|, i = 1, \cdots, N$, is sufficiently small. For each $a \in C_n$, Lemma 2.2 implies that

$$\begin{cases} \text{for each } \ h \in \mathcal{E}_n \ \text{ and } \ v = \dfrac{h u_a}{a} \\ \displaystyle\int_\Omega h \text{ grad } u_a \cdot \text{ grad } v \geq \frac{1}{2} \int_\Omega \frac{h^2}{a} |\text{grad } u_a|^2. \end{cases}$$

Since $|\text{grad } u_a(x)| > 0$ a.e. on $\Omega$,

$$\int_\Omega h \ \text{ grad } u_a \cdot \ \text{ grad } w = 0 \ \text{ for all } \ w \in V$$

implies, by choosing $w = \frac{h u_a}{a}$, that $h^2 = 0$ a.e. on $\Omega$ and hence $a$ is identifiable in $\mathcal{E}_n$ by Lemma 2.1.

## 3. Decomposition of $\mathbb{L}^2(\Omega)$

It will be convenient to denote by $(\cdot, \cdot)$ the scalar products in $L^2(\Omega)$ and $\mathbb{L}^2(\Omega)$. Then for every $a, b \in C$ we obtain from the variational formulation (Q) defining $u_a$ and $u_b$ that

$$((a - b) \text{ grad } u_a, \text{ grad } v) = (b( \text{ grad } u_b - \text{ grad } u_a), \text{ grad } v), \tag{3.1}$$

for all $v \in V$. This suggests to associate to $V$ an equivalence relation $\sim$ of vectorfields in $\mathbb{L}^2(\Omega)$ according to

$$\vec{q} \sim \vec{q'} \ \text{ if } \ \vec{q} \cdot \text{grad } v = \vec{q'} \cdot \text{grad } v \ \text{ for all } \ v \in V, \tag{3.2}$$

to denote by

$$\begin{cases} G = L^2(\Omega)/_\sim & \text{the quotient space} \\ G^\perp & \text{the orthogonal complement} \end{cases} \tag{3.3}$$

and by

$$P \text{ and } P^\perp \text{ the orthogonal projection in } \mathbb{L}^2(\Omega) \text{ onto } G \text{ and } G^\perp. \tag{3.4}$$

The decomposition

$$\mathbb{L}^2(\Omega) = G \oplus G^\perp$$

is, by construction, adapted to the elliptic problem (Q). We further introduce

$$W = \{\psi \in H^1(\Omega) \colon \psi|_{\Gamma_N} = 0\},$$

where the condition $\int_\Omega \psi = 0$ is added to the definition of $W$ if $\Gamma_N = \phi$. For $\varphi \in H^1(\Omega)$ and $\vec{\psi} \in H^1(\Omega) \times H^1(\Omega)$ we define

$$\vec{\mathrm{rot}}\,\varphi = \begin{pmatrix} \dfrac{\partial \varphi}{\partial x_2} \\ \\ -\dfrac{\partial \varphi}{\partial x_1} \end{pmatrix} \text{ and } \mathrm{rot}\,\vec{\psi} = \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2}\cdot$$

The following representation for $G$ and $G^\perp$ can be obtained.

**Proposition 3.1.**

$$G = \{\mathrm{grad}\,\varphi : \ \varphi \in V\}$$
$$G^\perp = \{\mathrm{rot}\,\psi : \ \psi \in W\}\cdot$$

*Moreover, for every $\vec{q} \in \mathbb{L}^2(\Omega)$ one has*

$$P\vec{q} = \mathrm{grad}\,\varphi,\ P^\perp \vec{q} = \vec{\mathrm{rot}}\,\psi,$$

*where $\varphi \in V$ and $\psi \in W$ are given by*

$$\begin{array}{lll} (\mathrm{grad}\,\varphi,\ \mathrm{grad}\,v) & = (\vec{q},\ \mathrm{grad}\,v) & \textit{for all } v \in V, \\ (\vec{\mathrm{rot}}\,\psi,\ \vec{\mathrm{rot}}\,v) & = (\vec{q},\ \vec{\mathrm{rot}}\,v) & \textit{for all } v \in W. \end{array}$$

Except for the atypical boundary conditions this decomposition is rather standard. For convenience an outline of the proof is given in the Appendix. The identifiability condition can now be reformulated as

**Proposition 3.2.** *A parameter $a \in C$ (respectively $C_n$) is identifiable in $\mathcal{E}$ (resp. $\mathcal{E}_n$) if and only if:*

$$h \neq 0 \textit{ and } h \in \mathcal{E} \textit{ (resp. } \mathcal{E}_n\textit{) imply } P(h\,\mathrm{grad}\,u_a) \neq 0.$$

The proposition follows directly from Proposition 3.1 and Lemma 2.1.

## 4. From which direction can an identifiable parameter be recovered in a stable way?

Let $a \in C$ be a given reference parameter and let $b \in C$ be a possibly different parameter. We investigate in this section conditions on $b$ for which the stability estimate (S) holds.

From (3.1) we have

$$(a-b) \text{ grad } u_a \sim b(\text{ grad } u_b - \text{ grad } u_a), \tag{4.1}$$

so that

$$\|(a-b) \text{ grad } u_a\|_G = \|b(\text{ grad } u_b - \text{ grad } u_a)\|_G \leq |b(\text{ grad } u_b - \text{ grad } u_a|_{\mathbb{L}^2}. \tag{4.2}$$

We decompose $(a-b)$ grad $u_a$ on $G \oplus G^\perp$:

$$(a-b) \text{ grad } u_a = \text{ grad } \varphi + \vec{\text{rot}}\ \psi, \tag{4.3}$$

where $\varphi \in V$ and $\psi \in W$ are given according to Proposition 3.1 by

$$(\text{grad } \varphi,\ \text{grad } v) = ((a-b) \text{ grad } u_a,\ \text{grad } v) \ \text{ for all } \ v \in V, \tag{4.4}$$

$$(\vec{\text{rot}}\ \psi,\ \vec{\text{rot}}\ v) = ((a-b) \text{ grad } u_a,\ \vec{\text{rot}}\ v) \ \text{ for all } \ v \in W. \tag{4.5}$$

Clearly one has

$$\text{grad } \varphi = P((a-b) \text{ grad } u_a),\ \vec{\text{rot}}\ \psi = P^\perp((a-b) \text{ grad } u_a). \tag{4.6}$$

Moreover

$$\|(a-b) \text{ grad } u_a\|_G = |P((a-b) \text{ grad } u_a)|_{\mathbb{L}^2}$$

which together with (4.2) implies

$$|P((a-b) \text{ grad } u_a|_{\mathbb{L}^2} \leq |b(\text{grad} u_b - \text{ grad } u_a)|_{\mathbb{L}^2}. \tag{4.7}$$

Defining for $M > 0$ the set

$$S_M(a) = \{b \in C : |P^\perp((b-a) \text{ grad } u_a)|_{\mathbb{L}^2} \leq M|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}\}, \tag{4.8}$$

we conclude that

$$|(b-a) \text{ grad } u_a|_{\mathbb{L}^2} \leq (1+M^2)^{1/2}|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}, \tag{4.9}$$

for all $b \in S_M(a)$. Combining (4.7) and (4.9) implies

**Proposition 4.1.** *Let $M > 0$ and $a \in C$ be given. Then for all $b \in S_M(a)$ the stability estimate (S) holds with $k = (1+M^2)^{1/2}$:*

$$|(b-a) \text{ grad } u_a|_{\mathbb{L}^2} \leq (1+M^2)^{1/2}|b(\text{ grad } u_b - \text{ grad } u_a)|_{\mathbb{L}^2}. \tag{4.10}$$

Hence the directions $b - a$, with $b \in S_M(a)$ are those from which the parameter $a$ can be recovered within $C$ with a stability constant $(1+M^2)^{1/2}$. We investigate now the shape of $S_M(a)$. The set $S_M(a)$ is the intersection of $C$ with a wedge having its vertex at $a$. Clearly $a \in S_M(a)$. If $a$ is in the $\mathcal{E}$–interior of $C$, $S_M(a)$ also contains parameters of the form $b = a + t\gamma(u_a)$ for $t$ small enough, where $\gamma$ is any regular function. In fact, in this case the gradients of $b - a$ and $u_a$ are collinear so that $P^\perp((b - a) \text{ grad } u_a) = 0$ (see (4.14) below), and hence (S) holds with $M = 0$ and $k = 1$.

**Proposition 4.2.** *Let $a \in C$ be identifiable in $\mathcal{E}$. Then*

$$\bigcup_{M>0} S_M(a) = C. \tag{4.11}$$

*Proof.* Let $b \in C$ with $b \neq a$. As $a$ is identifiable it follows from Proposition 3.2 that $P((b - a) \text{ grad } u_a) \neq 0$. Hence $b \in S_M(a)$ for $M$ sufficiently large. □

We next interpret the quantities which enter the definition of $S_M(a)$.

**Lemma 4.1.** *For every $a \in C$ and $h \in \mathcal{E}$ we have*

$$\| \text{ div } (h \text{ grad } u_a)\|_{H^{-1}} \leq |P(h \text{ grad } u_a)|_{\mathbb{L}^2} \leq |h \text{ grad } u_a|_{\mathbb{L}^2}, \tag{4.12}$$

$$\begin{aligned} \|\vec{\text{rot}}\, h \text{ grad } u_a\|_{W^*} &= |P^\perp(h \text{ grad } u_a)|_{\mathbb{L}^2} \\ &\leq \min\{C_W|\vec{\text{rot}}\, h \cdot \text{ grad } u_a|, |h \text{ grad } u_a|\}, \end{aligned} \tag{4.13}$$

*where $C_W$ is the Poincare constant in $W$.*

*Proof.* From Proposition 3.1 we have

$$|P(h \text{ grad } u_a)|_{\mathbb{L}^2} = | \text{ grad } \varphi|_{\mathbb{L}^2} = \sup\{( \text{ grad } \varphi, \vec{q}) \,:\, \vec{q} \in \mathbb{L}^2, |\vec{q}|_{\mathbb{L}^2} = 1\}\cdot$$

But $\vec{q} = \text{grad } v + \vec{\text{rot}}\, w$ with $v \in V$ and $w \in W$, so that

$$\begin{aligned} |P(h \text{ grad } u_a)|_{\mathbb{L}^2} &= \sup \{(\text{grad}\varphi,\ \text{grad } v + \vec{\text{rot}}\, w) \colon \\ &\qquad v \in V, w \in W, | \text{ grad } v|^2 + |\vec{\text{rot}}\, w|^2 = 1\} \\ &= \sup \{(\text{grad}\varphi,\ \text{grad } v) \colon v \in V, | \text{ grad } v| = 1\} \end{aligned}$$

and

$$|P(h \text{ grad } u_a)|_{\mathbb{L}^2} = \sup \left\{ \int_\Omega h \text{ grad } u_a \cdot \text{grad} v \,:\, v \in V, | \text{ grad } v| = 1 \right\}\cdot$$

These estimates imply by the Cauchy–Schwartz the second inequality in (4.12). The first follows from $H_0^1(\Omega) \subset V$. From Proposition 3.1 we obtain as well that

$$|P^\perp(h \text{ grad } u_a)|_{\mathbb{L}^2} = \sup \left\{ \int_\Omega h \text{ grad } u_a \cdot \vec{\text{rot}}\, w : w \in W,\ |\text{grad } w| = 1 \right\}\cdot \tag{4.14}$$

By Green's formula we find:

$$\int_\Omega h \text{ grad } u_a \cdot \vec{\text{rot}}\, w = \int_\Omega \text{ rot } (h \text{ grad } u_a)\, w - \int_{\partial\Omega} h \frac{\partial u_a}{\partial \tau}\, w. \tag{4.15}$$

Since $u_a = 0$ on $\Gamma_D$ and $u_a =$ const on each $\Gamma_i$, we have $\frac{\partial u_a}{\partial \tau} = 0$ on $\Gamma_D$ and on $\Gamma_i, i = 1, \cdots, N$, and $w = 0$ on $\Gamma_N$. Thus all boundary terms vanish. From (4.14, 4.15) and the fact that $\text{rot}(h \text{ grad } u_a) = \vec{\text{rot}}\, h \cdot \text{ grad } u_a$ we find

$$|P^\perp(h \text{ grad } u_a)|_{\mathbb{L}^2} = \sup_{w \in W, |\text{ grad } w|=1} \int_\Omega (\vec{\text{rot}}\, h \cdot \text{ grad } u_a) w,$$

which, by Poincaré inequality in $W$ shows that

$$|P^\perp(h \text{ grad } u_a)|_{\mathbb{L}^2} = |\vec{\text{rot}}\, h \cdot \text{ grad } u_a|_{W^*} \le C_W |\vec{\text{rot}}\, h \cdot \text{ grad } u_a|_{L^2}.$$

Combining (4.14) and the last estimate implies (4.13) and the lemma is proved. □

Lemma 4.1 implies that $S_M(a)$ contains all $b \in C$ such that $h = b - a$ satisfies

$$C_W |\vec{\text{rot}}\, h \text{ grad } u_a| \le M |\text{ div } (h \text{ grad } u_a)|_{H^{-1}}.$$

Hence we see that the perturbations from which $a$ can be stably recovered are, speaking loosely those whose gradient tends to be mostly oriented along the flow lines of $u_a$ at each point $x \in \Omega$. In particular, $a$ cannot be recovered stably from perturbations $h$ such that $\text{div}(h \text{ grad } u_a) = 0$. When $u_a$ is harmonic (*e.g.* if $f = 0$ and $a =$ const), these unstable perturbations are such that grad $h\cdot$ grad $u_a = 0$. This corresponds to the intuition that the observation of the pressure field grad $u_a$ gives little information on the diffusion parameter $a$ orthogonal to flow lines.

We next show that if $a$ can be recovered stably from $b$ it can also be recovered stably, but with a larger constant, from all $c \in C$ in some $L^2-$ neighborhood of $b$:

**Theorem 4.1.** *Let $a \in C$ be identifiable in $\mathcal{E}$, $b \in C$, $b \ne a$ be given, and define:*

$$0 \le M = \frac{|P^\perp((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}}{|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}} < \infty.$$

*Then for every $M' > M$ there exists $\epsilon > 0$ such that*

$$S_{M'}(a) \supset C \cap \{c \in L^2(\Omega) \colon |c - b|_{L^2} \le \epsilon\} \cdot \tag{4.16}$$

*Proof.* The mappings $\Lambda(h) = P(h \text{ grad } u_a)$ and $\Lambda^\perp(h) = P^\perp(h \nabla u_a)$ are continuous from $\mathbb{L}^2(\Omega)$ into itself. Hence we get for $|c - b|_{L^2} \le \epsilon$

$$|P^\perp((c-a) \text{ grad } u_a)| \le |P^\perp((b-a) \text{ grad } u_a)| + \|\Lambda^\perp\| \epsilon,$$

$$|P((c-a) \text{ grad } u_a)| \ge |P((b-a) \text{ grad } u_a)| - \|\Lambda\| \epsilon.$$

Since $a$ is identifiable we have $P((b-a) \text{ grad } u_a) \ne 0$. Hence for $M' > M$ there exists $\epsilon > 0$ such that $|P^\perp((c-a) \text{ grad } u_a)|\, |P((c-a) \text{ grad } u_a|^{-1} \le M'$ as soon as $|c - b| \le \epsilon$. This implies (4.16). Of course $c = a$ cannot belong to this neighborhood of $b$! □

At this point the question arises whether it is possible for the stability estimate (S), or (4.10) to hold uniformly for some $k = (1 + M^2)^{1/2}$ for all $a, b \in C$. In other terms we search for $C$ satisfying (2.6, 2.7) and

$$a, b \in C \text{ implies } b \in S_M(a). \tag{4.17}$$

We give a negative answer in the sense that there is no infinite dimensional $C$ with nonempty $\mathcal{E}$-interior satisfying the specified properties.

Suppose that such a $C$ exists, and let $a$ be an element of the $\mathcal{E}$-interior of C. Further let $B$ denote a ball with center $a$ and radius $\epsilon$ contained in $C$. Then (4.17) would imply that $B \subset S_M(a)$ and hence as $S_M(a)$ is the intersection of $C$ with $a$ wedge, that

$$|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2} \leq M|P(h \text{ grad } u_a)|_{\mathbb{L}^2} \text{ for all } h \in \mathcal{E}. \tag{4.18}$$

We show on a simple example that (4.18) cannot be true in general. For this purpose consider (Q) with $\Omega$ the unit square in $\mathbb{R}^2$, $f = 0$, $g = 0$ on top and bottom, $g = -1$, on the left and $g = 1$ on the right lateral boundary, and no internal sources and sinks. The solution corresponding to $a = 1$ is given by $u_a(x, y) = x - \frac{1}{2}$.

- We check that $a = 1$ is identifiable in $\mathcal{E}$. For every $h \in \mathcal{E}$ we have from Lemma 2.2 that for $v = \frac{hu_a}{a} \in V$

  $$\int_\Omega h \text{ grad } u_a \text{ grad } v = \frac{1}{2}\int_\Omega h^2 + \frac{1}{2}\int_{\Gamma_N} h^2 u\, g.$$

  Since $u\, g \geq 0$ on $\partial\Omega$ we see that

  $$\int_\Omega h \text{ grad } u_a \text{ grad } v = 0 \text{ for all } v \in V \text{ implies } h = 0,$$

  and thus by Lemma 2.1 $a$ is identifiable in $\mathcal{E}$.
- Next we consider parameter perturbations which are orthogonal to the flow lines. This results in choosing ($x, y$ denote the coordinates in $\mathbb{R}^2$)

  $$h(x, y) = h(y).$$

We shall require that

$$h \in C^{0,1}(0,1), \int_0^1 h = 0 \text{ and } h(0) = h(1) = 0. \tag{4.19}$$

Under these conditions we estimate lower and upper bounds to $|P^{\perp}(h \text{ grad } u_a)|$ and $|P(h \text{ grad } u_a)|$.

(i) From Lemma 4.1 we have

$$|P^{\perp}(h \text{ grad } u_a)| = \| \text{ rot } h \text{ grad } u_a\|_{W^*} = \|h'\|_{W^*},$$

and by (4.19)

$$\|h'\|_{W^*} = \sup\left\{\int_\Omega h\frac{\partial w}{\partial y} : w \in W, |\text{ grad } w| = 1\right\}. \tag{4.20}$$

Let $H$ be the primitive of $h$:

$$H(y) = \int_0^y h(\tau)\mathrm{d}\tau$$

and note that $H(0) = H(1) = 0$. We define

$$\tilde{w}(x, y) = x(1 - x)H(y),$$

so that $\widetilde{w} = 0$ on $\partial\Omega = \Gamma_N$, and hence $\widetilde{w} \in W$, with

$$|\text{ grad } \widetilde{w}|^2 = \tfrac{1}{3}(|H|^2 + \tfrac{1}{10}|h|^2).$$

Choosing $w = \frac{\widetilde{w}}{|\text{ grad } \widetilde{w}|}$ in (4.20) gives

$$\|h'\|_{W^*} \geq \frac{\sqrt{3}|h|^2}{6\sqrt{|H|^2 + \frac{1}{10}|h|^2}}.$$

Since $|H|^2 \leq \frac{1}{2}|h|^2$ we obtain

$$|P^{\perp}(h \text{ grad } u_a)| \geq \tfrac{2}{3}|h|. \tag{4.21}$$

(ii) From the proof of Lemma 4.1 we get

$$|P(h \text{ grad } u_a)| = \sup \left\{ \int_\Omega h \frac{\partial v}{\partial x} \;:\; v \in V, |\text{ grad } v| = 1 \right\},$$

and, integrating by parts with respect to $x$,

$$|P(h \text{ grad } u_a)| = \sup \left\{ \int_0^1 h(y)(v(1,y) - v(0,y))dy \;:\; v \in V, |\text{grad } v| = 1 \right\}.$$

Let $\gamma = \partial\Omega \cup \{x = 1\}$, the right lateral boundary of $\Omega$, and denote by $c$ the continuity constant from $V$ to $H^{1/2}(\gamma)$. Then $|\text{ grad } v| = 1$ implies $\|v|_\gamma\|_{H^{1/2}(\gamma)} \leq c$, so that by symmetry

$$|P(h \text{ grad } u_a)| \leq 2 \sup \left\{ \int_0^1 h\,\xi \colon \xi \in H^{1/2}(\gamma), \|\xi\|_{H^{1/2}(\gamma)} \leq c \right\}.$$

Associating to $h$ a function $H \in H^{1/2}(\gamma)$ satisfying

$$((H, \xi))_{H^{1/2}(\gamma)} = \int_0^1 h\,\xi, \quad \text{for all } \; \xi \in H^{1/2}(\gamma), \tag{4.22}$$

we have

$$|P(h \text{ grad } u_a)| \leq 2 \sup_{\|\xi\|_{H^{1/2}(\gamma)} \leq c} ((H, \xi))_{H^{1/2}} = 2c\|H\|_{H^{1/2}}. \tag{4.23}$$

From (4.21, 4.23) we see that (4.18) would imply that

$$|h| \leq 3Mc\,\|H\|_{H^{1/2}(\gamma)}, \tag{4.24}$$

for all $h \in \mathcal{E}$ satisfying (4.19). But we can choose a sequence of functions $h_n$ satisfying

$$h_n \in C^{0,1}(0,1), \int_0^1 h_n = 0, h_n(0) = h_n(1) = 0,$$

and

$$|h_n|_{L^2} = \text{ const }, |h_n|_{H^{-1/2}(\gamma)} \to 0 \;\text{ for }\; n \to \infty.$$

From (4.22) we see that $\|H_n\|_{H^{1/2}(\gamma)} \to 0$ for $n \to \infty$. This contradicts (4.24) and consequently (4.18) as well. $\square$

It is hence impossible to find an infinite dimensional set $C$ with nonempty interior on which the stability estimate holds uniformly. This motivates the reduction to finite dimensional parameter sets in the remaining sections.

## 5. Finite dimensional stability estimates

We turn to the finite dimensional setting of Section 2 with $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}$ satisfying (2.11). We recall the definition $C_n = C \cap \mathcal{E}_n$ and suppose throughout this section that (H) holds. Identifiability of $a$ in $\mathcal{E}_n$ implies by Proposition 3.2 that for all $a, b \in C_n$ with $b \neq a$ we have

$$P((b-a) \text{ grad } u_a) \neq 0. \tag{5.1}$$

Hence we can define for each $n \in \mathbb{N}$

$$M_n = \sup_{\substack{a,b \in C_n \\ a \neq b}} \frac{|P^{\perp}((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}}{|P((b-a) \text{ grad } u_a)|_{\mathbb{L}^2}} \, . \tag{5.2}$$

From Proposition 3.2 we know that (5.1) holds with $b - a$ replaced by any $h \in \mathcal{E}_n$, $h \neq 0$, and hence

$$M_n \leq \sup_{a \in C_n} \sup_{h \in \mathcal{E}_n, \|h\|_{\mathcal{E}}=1} \frac{|P^{\perp}(h \text{ grad } u_a)|_{\mathbb{L}^2}}{|P(h \text{ grad } u_a)|_{\mathbb{L}^2}} \, . \tag{5.3}$$

As the fraction in (5.3) is a continuous function of $h$ and $a$, and the suprema are taken on compact sets it follows that $M_n$ is finite for each $n$. Since $\mathcal{E}_0$ consists of constant functions $M_0 = 0$ by Lemma 4.1. Moreover since $\mathcal{E}_n \subset \mathcal{E}_{n+1}$, for all $n$, we have

$$0 = M_0 \leq M_1 \leq \cdots \leq M_n \cdots < \infty. \tag{5.4}$$

From the example at the end of the previous section we can generally expect that $\lim_{n \to \infty} M_n = \infty$.

Proposition 4.1 implies the following stability estimates:

**Theorem 5.1.** *Let (2.5–2.7, 2.10, 2.11) and (H) hold. Then for every $n \in \mathbb{N}$*

$$|(b-a) \text{ grad } u_a|_{\mathbb{L}^2} \leq (1 + M_n^2)^{1/2} \, |b(\text{grad } u_b - \text{ grad } u_a)|_{\mathbb{L}^2},$$

*for all $a, b \in C_n$, where $M_n$ is defined in (5.2).*

This theorem gives a rigorous justification to the numerical observation [12, 13, 16] that the sensitivity of the inversion of the $a \to u_a$ mapping is a decreasing function of the scale at which the parameter is estimated. This observation together with the analysis of the nonlinearity of the mapping $a \to u_a$, to be given in the next section, has motivated the introduction of successful multiscale approaches to parameter estimation [9, 16].

We close the section with an estimate of $M_n$ in the case where (H) is satisfied by virtue of Theorem 2.1.

**Theorem 5.2.** *Let (2.5–2.7, 2.10, 2.11) hold, and suppose that the hypotheses of Theorem 2.1 are satisfied. Then*

$$M_n \leq \sup_{a \in C_n} \sup_{h \in \mathcal{E}_n, h \neq 0} M_n(a, h),$$

*where, for $a \in C_n$ and $h \in \mathcal{E}_n$, $h \neq 0$:*

$$M_n(a,h) = 2 \left( \frac{a_M}{a_m} \right)^{1/2} \frac{|\text{ grad } (\frac{h}{a}) u_a|}{|\frac{h}{a} \text{ grad } u_a|} \min \left\{ 1, C_W \frac{|\text{ } \vec{\text{rot}} \text{ } h \text{ grad } u_a|}{|h \text{ grad } u_a|} \right\} . \tag{5.5}$$

*Proof.* Choosing $v = \frac{hu_a}{a}|\text{ grad }(\frac{hu_a}{a})|_{\mathbb{L}^2}^{-1}$ on the right hand side of the equality above (4.14) we obtain by Theorem 2.1 and Lemma 2.2

$$|P(h\text{ grad }u_a)|_{\mathbb{L}^2} \geq \frac{1}{2}\,\frac{|\frac{h}{a^{1/2}}\text{ grad }u_a|_{\mathbb{L}^2}^2}{|\text{grad }(\frac{hu_a}{a})|_{\mathbb{L}^2}}$$

and hence

$$|P(h\text{ grad }u_a)|_{\mathbb{L}^2} \geq \frac{a_m^{1/2}}{2a_M^{1/2}}\,\frac{|\frac{h}{a}\text{ grad }u_a|\,|h\text{ grad }u_a|}{|\text{grad }(\frac{h}{a}u_a)}.$$

Combining this estimate with (4.13) of Lemma 4.1 and (5.3) implies the theorem. □

For $a, b \in C_n$ the stability constant $M_n(a, b-a)$ of (5.5) allows the following interpretation:

- the first factor is related to the size of $C_n$;
- the second factor tends to infinity when the dimension of $\mathcal{E}_n$ goes to infinity. It does not depend on the relative orientations of grad $h$ and grad $u_a$;
- the third factor is bounded by 1 and tends to zero as grad $h$ becomes collinear with grad $u_a$.

## 6. Finite dimensional sensitivity, deflection and curvature estimates

In this section we analyze the geometric quantities associated to the parameter-to-solution mapping $a \to u_a$ defined by (Q). Knowledge of these quantities will be required in the next section to prove that the output set $D_n$ is strictly quasiconvex, and hence the OLS-identifiability of $a$.

For $a_0, a_1 \in C_n$ and $t \in [0,1]$ we set

$$h = a_1 - a_0, \tag{6.1}$$

$$a = (1-t)a_0 + t\,a_1. \tag{6.2}$$

The geometric quantities are related to the curve $t \in [0,1] \to \nabla u_a \in \mathbb{L}^2(\Omega)$ in the range of the mapping $a \to \nabla u_a$. We denote by $\eta$ the velocity and $\xi$ the acceleration, *i.e.* the first and second derivatives of $u_a$ with respect to $t$. The equations characterizing $\eta \in V$ and $\xi \in V$ are found to be:

$$\int_\Omega a\text{ grad }\eta\cdot\text{ grad }v = -\int_\Omega h\text{ grad }u_a\text{ grad }v,\quad\text{for all }\ v \in V, \tag{6.3}$$

$$\int_\Omega a\text{ grad }\xi\cdot\text{ grad }v = -2\int_\Omega h\text{ grad }\eta\text{ grad }v,\quad\text{for all }\ v \in V. \tag{6.4}$$

For given $n \in \mathbb{N}$ the objective is to find $0 < \alpha_m \leq \alpha_M$, $\Theta \geq 0$ and $R > 0$ such that the following inequalities hold uniformly for all $a_0, a_1 \in C_n$ and $t \in [0,1]$:

$$\alpha_m|h|_{L^2} \leq |\text{grad }\eta|_{\mathbb{L}^2} \leq \alpha_M|h|_{L^2}, \tag{6.5}$$

$$|\text{grad }\xi|_{\mathbb{L}^2} \leq \Theta|\text{grad }\eta|_{\mathbb{L}^2}, \tag{6.6}$$

$$|\text{grad }\xi|_{\mathbb{L}^2} \leq \frac{1}{R}|\text{grad }\eta|_{\mathbb{L}^2}^2. \tag{6.7}$$

These inequalities can be interpreted as follows:

- $\alpha_m$ and $\alpha_M$ are lower and upper bounds to the first derivative of $a \to \nabla u_a$. They are referred to as minimal and maximal sensitivity;

- $\Theta$ is an upper bound to the deflection along the curve $t \to \nabla u_a$ (*i.e.* to the angle between the tangents at two points of the curve);
- $\frac{1}{R}$ is an upper bound to the curvature along the curve.

**Theorem 6.1.** *Let (2.5–2.7, 2.10, 2.11) and (H) hold. Then, for every $n \in \mathbb{N}$, $a_0, a_1 \in C_n$ and $t \in [0,1]$, (6.5–6.7) hold with*

$$\alpha_m = \frac{\gamma_{m,n}}{a_M(1+M_n^2)^{1/2}}, \quad \alpha_M = \frac{\gamma_M}{a_m}, \tag{6.8}$$

$$\Theta = 2\,\frac{\operatorname{diam}_\infty(C_n)}{a_m}, \tag{6.9}$$

$$\frac{1}{R} = 2\,K_n\,\frac{(1+M_n^2)^{1/2}}{\gamma_{m,n}}\,\frac{a_M}{a_m}, \tag{6.10}$$

*where $\gamma_M$ is defined in (2.8), and*

$$\gamma_{m,n} = \inf_{a\in C_n}\ \inf_{h\in\mathcal{E}_n,\ |h|_{L^2}=1} |h \text{ grad } u_a|_{\mathbb{L}^2} > 0, \tag{6.11}$$

$$K_n = \sup_{h\in\mathcal{E}_n, |h|_{L^2}=1} |h|_{L^\infty}. \tag{6.12}$$

*Proof.* Let $t + \mathrm{d}t \in [0,1]$ and denote by $a_t$ and $a_{t+\mathrm{d}t}$ the corresponding values of $a$ given by (6.2). Choosing $a = a_t$ and $b = a_{t+\mathrm{d}t}$ in Theorem 5.1 and letting $\mathrm{d}t$ tend to zero implies that

$$|h \text{ grad } u_a|_{\mathbb{L}^2} \le (1+M_n^2)^{1/2}\ |a \text{ grad } \eta|_{\mathbb{L}^2}. \tag{6.13}$$

Hence the left inequality of (6.5) is satisfied with $\alpha_m$ defined by (6.8, 6.11). The strict positivity of $\gamma_{m,n}$ follows from (H) which ensures that the argument of the inf is strictly positive, and hence the inf itself is strictly positive as it is taken over a compact set. The right inequality of (6.5) is obtained by choosing $v = \eta$ in (6.3) and using (2.9). Setting $v = \xi$ in (6.4) gives

$$|a^{1/2} \text{ grad } \xi|_{\mathbb{L}^2} \le 2\ |\frac{h}{\sqrt{a}}|_{L^\infty}\ |\text{grad } \eta|,$$

which implies (6.9), and also (6.10) using (6.11–6.13). □

Notice first that $\gamma_{m,n}$ can be understood as a lower bound to the local mean value of $|\text{grad} u_a|$ at scale $n$. In general $u_a$ will have stationary points where grad $u_a(x) = 0$, in which case one expects that $\gamma_{m,n} \to 0$ for $n \to \infty$. In special cases, as for instance the example of Section 4, see also [15,17] for further examples, it can happen that $|\text{grad } u_a(x)| \ge \gamma_m > 0$ for all $a \in C$ and $x \in \Omega$, in which case $\gamma_{m,n} \ge \gamma_m > 0$ for all $n$.

Let us now discuss the behavior of the constants $\alpha_m$, $\alpha_M$, $\Theta$ and $R$ as $n$ tends to infinity. In case (H) is satisfied through the assumptions of Theorem 2.1 an upper bound to $M_n$ can obtained by Theorem 5.2:

$$M_n \le 2\left(\frac{a_M}{a_m}\right)^{1/2}\left(1 + \frac{u_M}{\gamma_{m,n}}\ \sup_{a\in C_n}\ \sup_{h\in\mathcal{E}_n, |\frac{h}{a}|_{L^2}=1}\left|\text{grad } \frac{h}{a}\right|\right). \tag{6.14}$$

In case the finite dimensional spaces $\mathcal{E}_n$ are obtained from a regular family of triangulations of $\Omega$ by elements of size $\Delta x$, the right hand side of (6.14) is of the order $\frac{1}{\Delta x}$ for $n \to \infty$. In this situation the continuity constant $K_n$ of the $L^2 \to L^\infty$ injection (for $h \in \mathcal{E}_n$) is also of the order $\frac{1}{\Delta x}$ as $n \to \infty$. These considerations imply the following

**Corollary 6.1.** *Under the conditions of Theorem 6.1, as the scale parameter $n \to \infty$ we have*

- *the minimal sensitivity $\alpha_m \to 0$;*
- *the maximal sensitivity $\alpha_M$ remains bounded;*
- *the deflection $\Theta$ remains bounded;*
- *the curvature $\frac{1}{R} \to \infty$.*

*In case (H) is satisfied through the assumption of Theorem 2.1 and $\mathcal{E}_n$ is constructed based on a regular triangulation of $\Omega$ by elements characterized by meshsize $\Delta x$, one has*

- $\alpha_m \geq$ const $\Delta x\, \gamma_{m,n}$*;*
- $\frac{1}{R} \leq$ const$/(\Delta x^2\, \gamma_{m,n})$.

## 7. Output least squares identifiability of $a$

We study in this section the structure of the nonlinear least squares problem $(P)$. We have seen in Section 4 that there is no hope for $(P)$ itself to be quadratically wellposed, or equivalently for $a$ to be OLS – identifiable on $C$. Therefore we choose finite dimensional subspaces $\mathcal{E}_n$ satisfying (2.10, 2.11) and consider for each $n \in \mathbb{N}$ the finite dimensional estimation problems

$$(\mathrm{P}_n) \qquad \text{minimize } \frac{1}{2}\, |\text{grad } u_a - z|^2_{\mathbb{L}^2} \text{ over } C_n.$$

Using the results of [3,4] and the geometric estimates of Section 6, one sees that the output sets $D_n = \{\text{grad } u_a \in \mathbb{L}^2(\Omega)\colon a \in C_n\}$ are strictly quasiconvex as soon as $\Theta \leq \frac{\pi}{2}$. They are also closed because of the finite dimension of the closed and bounded sets $C_n$, and the properties of closed and strictly quasiconvex sets imply that the projection on $D_n$ is Q-wellposed in the sense of Definition 2.1.

Hence the following theorem holds:

**Theorem 7.1.** *Let (2.5–2.7, 2.10, 2.11) and (H) hold and suppose that $C$ satisfies*

$$\Theta = \frac{2}{a_m}\, \text{diam}_\infty C \leq \frac{\pi}{2}\,\cdot \tag{7.1}$$

*Then $(P_n)$ is quadratically wellposed on*

$$\mathcal{V}_n = \left\{ \vec{q} \in \mathbb{L}^2(\Omega)\colon \text{ dist } (\vec{q}, D_n) < R_n = \frac{\gamma_{m,n}}{2K_n(1+M_n^2)^{1/2}} \right\} \tag{7.2}$$

*for all $n \in \mathbb{N}$, that is*

1. *for every $z \in \mathcal{V}_n$, $(P_n)$ has a unique solution $\hat{a}_n$;*
2. *for every $z \in \mathcal{V}_n$, the function $a \to \frac{1}{2}\,|\,\text{grad}\, u_a - z|^2_{\mathbb{L}^2}$ has no local minima;*
3. *the mapping $z \to \hat{a}_n$ is Lipschitz continuous: for all $z_0, z_1 \in \mathcal{V}_n$ satisfying $|z_0 - z_1|_{\mathbb{L}^2} + \max_{j=0,1} d(z_j, D_n) \leq d_n < R_n$,*

$$\alpha_m |\hat{a}_{n,0} - \hat{a}_{n,1}|_{L^2} \leq L \leq \left(1 - \frac{d_n}{R_n}\right)^{-1} |z_0 - z_1|_{\mathbb{L}^2},$$

   *where $L$ is the arc length in $\mathbb{L}^2(\Omega)$ of the curve $t \in [0,1] \to \text{grad} u_a \in \mathbb{L}^2(\Omega)$ with $a = (1-t)\hat{a}_{n,0} + t\hat{a}_{n,1}$;*
4. *every minimizing sequence of $(P_n)$ converges to $\hat{a}_n$.*

The above results are to be compared with the numerical observations made in [16] about the behavior of the gradient algorithms for the solution of $(P_n)$ when a multiscale optimization is performed: at coarse scales, the gradient algorithms were found to converge quickly toward a global minimum; this correspond to the case where, in the above theorem, the data $z$ belongs at these coarse scales to the neighborhood $\mathcal{V}_n$ on which $(P_n)$ is wellposed - in particular, the objective function has no local minima other than the global one, so the gradient algorithm, if it converges, has to converge to the global minimum. When the scale at which the parameter is searched for is refined, the size of the neighborhood $\mathcal{V}_n$ shrinks, until the data $z$ ends up outside of $\mathcal{V}_n$. The gradient algorithm will then in general converge to a local minimum only. But it is reasonable to expect that, because of the good initial guess obtained at coarser scales, this local minimum is not too far from the global one, as it was observed in [16] – but this still needs to be proved.

**Remark.** In the 1–D case [5], the mapping $b = \frac{1}{a} \to u_b$ is "quasilinear". So the question arises of using $b = \frac{1}{a}$ in place of $a$ as optimization variable. Additional motivations are given by the facts that $b \to u_b$ is affine when $b$ is constant and that $a \to \frac{1}{a}$, which is in some sense contained in $a \to u_a$, is not twice differentiable on $L^2(\Omega)$.

In order to see whether $b \to u_b$ is less non-linear and its inverse is less illposed than $a \to u_a$, one can carry the analysis of the previous sections, with the appropriate modifications, which leads to new sensitivity, deflection and curvature constants $\tilde{\alpha}_m, \widetilde{\Theta}, 1/\widetilde{R}$ (the details can be found in the preprint [8]). The results are as follows:

- the two parameterizations have the same behavior with respect to sensitivity:
  $\tilde{\alpha}_m$ and $\alpha_m \to 0$ at similar rates when $n \to \infty$, $\tilde{\alpha}_M$ and $\alpha_M$ are both bounded with respect to $n$;
- at coarse scales, the $b$–parameterization is advantageous for deflection and curvature:
  $\widetilde{\Theta}$ and $1/\widetilde{R} \to 0$ for $n \to 0$.
  This reflects the fact that the problem becomes "less nonlinear" for the $b$-parameterization as the scale gets coarser.
- At fine scales the $a$–parameterization is advantageous as
  $\Theta$ remains bounded, whereas $\widetilde{\Theta}$ goes to infinity with $n$ (for fixed value of $\mathrm{diam}_\infty C$),
  $1/R$ does not go to infinity as fast as $1/\widetilde{R}$.

The fact that $\widetilde{\Theta} \to \infty$ with $n$ is a big drawback as this will require to reduce the size of $\widetilde{C}_n$ when $n \to \infty$, if one wants to ensure the $Q$-well posedness over $\widetilde{C}_n$ for the $b$-parameterization. Of course, $\widetilde{\Theta}$ is only an upper bound to the maximum deflection. We do not know if the maximum deflection over $\widetilde{C}_n$ actually tends to infinity with the scale $n$.

## Appendix

*Proof of Proposition 3.1.*

**Step 1.** Following [11], Chapter 1 we define

$$H = \left\{ \vec{q} \in \mathbb{L}^2(\Omega) \colon \ \mathrm{div}\ \vec{q} = 0, \vec{q} \cdot n|\Gamma_N = 0, \int_{\Gamma_i} \vec{q} \cdot n = 0 \right\},$$

where div is understood in the variational sense. As in [11] one argues that $H$ is a closed subspace of $\mathbb{L}^2(\Omega)$, and hence we have the decomposition

$$\mathbb{L}^2(\Omega) = H \oplus H^\perp. \tag{A.1}$$

**Step 2.** We argue that $H^\perp = G$. Since $G$ is closed in $\mathbb{L}^2(\Omega)$ it suffices to show that $H = G^\perp$. For this purpose choose and fix $\vec{q} \in H$ arbitrarily. Then for every $\varphi \in V$ we have $(\vec{q}, \nabla\varphi) = \sum_{i=1}^N \ \varphi_i \int_{\Gamma_i} \vec{q} \cdot n = 0$ and hence $\vec{q} \in G^\perp$. Conversely if $\vec{q} \in G^\perp$, then $(\vec{q}, \nabla\varphi) = 0$ for all $\varphi \in V$, in particular for all $\varphi \in \mathcal{D}(\Omega)$ and hence div $\vec{q} = 0$. Choosing $\varphi \in V$ implies $\vec{q} \cdot n|\Gamma_N = 0$ and $\displaystyle\int_{\Gamma_i} \vec{q} \cdot n = 0$, for $i = 1, \cdots, N$. Hence $\vec{q} \in H$ and $H^\perp = G$.

**Step 3.** We show that $H = \{\vec{\text{rot}}\ \psi : \psi \in W\}$. For $\psi \in W$ we have div $\vec{\text{rot}}\ \psi$ in the variational sense and $\int_{\Gamma_i} \vec{\text{rot}}\ \psi \cdot n = \int_{\Gamma_i} \nabla\psi \cdot \tau = 0$, where $\tau$ denotes the tangent to $\Gamma_i$. Moreover $\vec{\text{rot}}\ \psi \cdot n = \nabla\psi \cdot \tau = 0$ on $\Gamma_N$ and hence $\vec{\text{rot}}\ \psi \in H$. Conversely, if $\vec{q} \in H$, then by the arguments in [11], (p. 36) there exists $\psi \in H^1(\Omega)$ such that $\vec{\text{rot}}\ \psi = \vec{q}$. Since $\vec{q} \cdot n = \vec{\text{rot}}\ \psi \cdot n = \nabla\psi \cdot \tau = 0$ on $\Gamma_N$ it follows that $\psi =$ const a.e. on $\Gamma_N$ and without loss of generality we may take this constant to be 0. Hence $\vec{\text{rot}}\ \psi = \vec{q}$ with $\psi \in W$.

**Step 4.** Let $\vec{q} \in \mathbb{L}^2(\Omega)$. Then the elliptic problem

$$(\text{grad } \varphi,\ \text{grad } v) = (\vec{q},\ \text{grad } v) \quad \text{for all } v \in V \tag{A.2}$$

has a unique solution in $V$. Consider $\vec{q} - \text{grad } \varphi \in \mathbb{L}^2(\Omega)$, and note that div $(\vec{q} - \text{grad } \varphi) = 0$. Moreover $(\vec{q} - \text{grad } \varphi) \cdot n|\Gamma_N = 0$ and $\int_{\Gamma_i} (\vec{q} - \text{grad } \varphi) \cdot n = 0$, for $i = 1, \cdots, M$. It follows that $\vec{q} - \text{grad } \varphi \in H$ and hence there exists $\psi \in W$ such that $\vec{\text{rot}}\ \psi = \vec{q} - \text{grad } \varphi$ in $H$. Consequently

$$(\vec{\text{rot}}\ \psi,\ \vec{\text{rot}}\ v) = (\vec{q} - \text{grad } \varphi,\ \vec{\text{rot}}\ v) \quad \text{for all } v \in W,$$

and utilizing div $\varphi = 0$ and boundary conditions for $v$ and $\varphi$

$$(\vec{\text{rot}}\ \psi,\ \vec{\text{rot}}\ v) = (\vec{q},\ \vec{\text{rot}}\ v), \quad \text{for all } v \in W. \tag{A.3}$$

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