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THE BLOCKING OF AN INHOMOGENEOUS BINGHAM FLUID. APPLICATIONS TO LANDSLIDES

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Abstract. This work is concerned with the flow of a viscous plastic fluid. We choose a model of Bingham type taking into account inhomogeneous yield limit of the fluid, which is well-adapted in the description of landslides. After setting the general threedimensional problem, the blocking property is introduced. We then focus on necessary and sufficient conditions such that blocking of the fluid occurs. The anti-plane flow in twodimensional and onedimensional cases is considered. A variational formulation in terms of stresses is deduced. More fine properties dealing with local stagnant regions as well as local regions where the fluid behaves like a rigid body are obtained in dimension one.

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1. Introduction

Due the importance of evaluation of landslide risk, great efforts have been devoted to analyzing, modeling, and predicting such phenomena in the last decades. A stability analysis, which treats the geologic material as a rigid viscoplastic body, may provide information on the safety factor of stable mass of soil. One of the simplest and convenient viscoplastic constitutive relation is the one modeling a Bingham fluid [1], exhibiting viscosity and yield stress.

Although the Bingham model deals with fluids, it was also seen as a solid, called the "Bingham solid" (see for instance [20]) and investigated to describe the deformation and displacement of many solid bodies. Therefore this model was often used in metal-forming processes; it was first introduced for wire drawing in [3] and intensively used thereafter [4,13]. More recently, the inhomogeneous (or density-dependent) Bingham fluid was considered in landslides modeling [2, 5, 6]. In this work, the inhomogeneous yield limit is a key point in describing a natural slope. Indeed, due to their own weight, the geomaterials are compacted (*i.e.*, their density increase with depth), so that the mechanical properties also vary with depth. Therefore the choice of a Bingham model in which the yield limit g and the viscosity coefficient η vary with density is motivated.

Keywords and phrases. Viscoplastic fluid, inhomogeneous Bingham model, landslides, blocking property, nondifferentiable variational inequalities, local qualitative properties.

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A particularity of the Bingham model lies in the presence of rigid zones located in the interior of the flow of the Bingham solid/fluid. As the yield limit g increases, these rigid zones become larger and may completely block the flow. This property is called the *blocking property*. When considering oil transport in pipelines, in the process of oil drilling or in the case of metal forming, the blocking of the solid/fluid is a catastrophic event to be avoided. In a completely opposite context, when modeling landslides, the solid is blocked in its natural configuration and the beginning of a flow can be seen as a disaster.

This paper deals with some boundary-value problems describing the flow of an inhomogeneous Bingham fluid through a bounded domain in $\mathbb{R}^3$. We focus on the blocking phenomenon, the description of the rigid zones and also the stagnant regions (*i.e.*, the zones near the boundary of the domain where the fluid does not move). More precisely, we study the link between the yield limit distribution and the external forces distribution (or the mass density distribution) for which the flow of the Bingham fluid is blocked or exhibits rigid zones. In opposition to the previous works dealing only with homogeneous Bingham fluids [8, 9, 11, 12, 16–18], we are interested in a fluid whose yield limit is inhomogeneous.

An outline of the paper is as follows. The equations modeling the flow of a Bingham fluid are introduced in Section 2 and the corresponding variational formulation is recalled. Section 3 is concerned with the blocking property in the threedimensional context. There we give a necessary and sufficient condition which characterizes the blocking property in the inhomogeneous case. The stationary anti-plane problem (twodimensional) is considered in Section 4. We obtain a variational formulation in terms of stresses which is useful in the description of the rigid zones.

The onedimensional problem describing the flow between two infinite planes is studied in Section 5. Several necessary and sufficient conditions for blocking and also local conditions of blocking (stagnant zones) and rigid body behavior are obtained in this case. Finally, we examine in Section 6 a simple onedimensional problem (flow between an infinite plane and a rigid roof) in which the exact solution can be determined analytically.

2. Statement of the 3D-problem

We consider here the evolution equations in the time interval $(0,T)$, $T>0$ describing the flow of an inhomogeneous Bingham fluid in a domain $\mathcal{D}\subset\mathbb{R}^3$ with a smooth boundary $\partial\mathcal{D}$. The notation $\boldsymbol{u}$ stands for the velocity field, $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor field, $p=-\operatorname{trace}(\boldsymbol{\sigma})/3$ represents the pressure and $\boldsymbol{\sigma}'=\boldsymbol{\sigma}+pI$ is the deviatoric part of the stress tensor. The momentum balance law in the Eulerian coordinates reads

$$\rho\Big(\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big)-\mathbf{div}\,\boldsymbol{\sigma}'+\nabla p=\rho\boldsymbol{f}\quad\text{ in }\mathcal{D}\times(0,T),\tag{1}$$

where $\rho=\rho(t,x)\geq\underline{\rho}>0$ is the mass density distribution and $\boldsymbol{f}$ denotes the body forces. Since we deal with an incompressible fluid, we get

$$\operatorname{div}\,\boldsymbol{u}=0\quad\text{ in }\mathcal{D}\times(0,T).\tag{2}$$

The conservation of mass becomes

$$\frac{\partial\rho}{\partial t}+\boldsymbol{u}\cdot\nabla\rho=0\quad\text{ in }\mathcal{D}\times(0,T).\tag{3}$$

We notice from the above equation that, excepting some special cases (see Sects. 4–6), the flow of an incompressible fluid with inhomogeneous mass density is not stationary.

If we denote by $D(\boldsymbol{u})=(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}^T\boldsymbol{u})/2$ the rate deformation tensor, the constitutive equation of the Bingham fluid can be written as follows:

$$\boldsymbol{\sigma}'=2\eta D(\boldsymbol{u})+g\frac{D(\boldsymbol{u})}{|D(\boldsymbol{u})|}\qquad\text{if }|D(\boldsymbol{u})|\neq0,\tag{4}$$

$$|\boldsymbol{\sigma}'|\leq g\qquad\text{if }|D(\boldsymbol{u})|=0,\tag{5}$$

where $\eta \geq \eta_0 > 0$ is the viscosity distribution and $g \geq 0$ is a nonnegative continuous function which stands for the yield limit distribution in $\mathcal{D}$. The type of behavior described by equations (4–5) can be observed in the case of some oils or sediments used in the process of oil drilling. The Bingham model, also denominated "Bingham solid" (see for instance [20]) was considered in order to describe the deformation of many solid bodies. In metal-forming processes, it was introduced for wire drawing in [3] and intensively studied in [4, 13]. Recently, the inhomogeneous (or density-dependent) Bingham fluid was chosen in landslides modeling [2, 5, 6].

When considering a density-dependent model, the viscosity coefficient η and the yield limit g depend on the density ρ through two constitutive functions, *i.e.*,

$$\eta = \eta(\rho(t,x)), \quad g = g(\rho(t,x)). \tag{6}$$

In order to complete equations (1–6) with some boundary conditions we assume that $\partial\mathcal{D}$ is divided into two disjoint parts so that $\partial\mathcal{D} = \partial_0\mathcal{D} \cup \partial_1\mathcal{D}$ and

$$\boldsymbol{u} = 0 \quad \text{on} \quad \partial_0\mathcal{D} \times (0,T), \quad \boldsymbol{\sigma n} = 0 \quad \text{on} \quad \partial_1\mathcal{D} \times (0,T), \tag{7}$$

where $\boldsymbol{n}$ stands for the outward unit normal on $\partial\mathcal{D}$. Finally the initial conditions are given by

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \rho|_{t=0} = \rho_0. \tag{8}$$

Setting

$$\mathcal{V} = \left\{\boldsymbol{v} \in H^1(\mathcal{D})^3,\ \text{div}\ \boldsymbol{v} = 0\ \text{ in } \mathcal{D}, \boldsymbol{v} = 0\ \text{ on } \partial_0\mathcal{D}\right\},$$

we give the variational formulation of (1), (2), (4), (5) and (7) for the velocity field (see [8])

$$\begin{cases} \forall t \in (0,T), & \boldsymbol{u}(t,\cdot) \in \mathcal{V}, \\ \forall \boldsymbol{v} \in \mathcal{V}, & \displaystyle\int_{\mathcal{D}} \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right)\cdot(\boldsymbol{v}-\boldsymbol{u}) \\ & \displaystyle + \int_{\mathcal{D}} 2\eta D(\boldsymbol{u}) : (D(\boldsymbol{v}) - D(\boldsymbol{u})) \\ & \displaystyle + \int_{\mathcal{D}} g|D(\boldsymbol{v})| - \int_{\mathcal{D}} g|D(\boldsymbol{u})| \geq \int_{\mathcal{D}} \rho \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u}). \end{cases} \tag{9}$$

Finally the problem of the flow of a inhomogeneous Bingham fluid becomes:

Find the velocity field $\boldsymbol{u}$ and the mass density field ρ such that conditions (3), (6), (8) and (9) hold.

As far as we know there does not exist any uniqueness result for this problem. For the Navier-Stokes model (*i.e.*, when $g = 0$) existence results can be found in [7, 15, 19].

3. The blocking property for the 3-D problem

When considering a viscoplastic model of Bingham type, one can observe rigid zones (*i.e.*, zones where $D(\boldsymbol{u}) = 0$) in the interior of the flow of the solid/fluid. When g increases, the rigid zones are growing and if g becomes sufficiently large, the fluid stops flowing (see [10]). Commonly called the *blocking property*, such a behavior can lead to unfortunate consequences in oil transport in pipelines, in the process of oil drilling or in the case of metal forming. On the contrary, in landslides modeling, it is precisely the blocking phenomenon which ensures stability of the slope.

We suppose in what follows that the volume forces are independent of time, *i.e.* $\boldsymbol{f} = \boldsymbol{f}(x)$. We say that the Bingham fluid is blocked if $\boldsymbol{u} \equiv 0$ satisfies equations and conditions (3), (6), (8), (9). One can easily check that the fluid is blocked if and only if the density has no time evolution (*i.e.*, $\rho(t, x) = \rho_0(x)$) and fulfills:

$$\int_{\mathcal{D}} g(\rho_0(x))|D(\boldsymbol{v}(x))| \, \mathrm{d}x \geq \int_{\mathcal{D}} \rho_0(x)\boldsymbol{f}(x) \cdot \boldsymbol{v}(x) \, \mathrm{d}x, \quad \forall \boldsymbol{v} \in \mathcal{V}.$$

Hence the study of the blocking property consists in finding the link between ρ_0 and $\boldsymbol{f}$ such that the above inequality holds. Since in landslides modeling the yield limit $g = g(\rho)$ depends also on some other parameters (as water concentration), another formulation of the blocking property is more adequate. Indeed if we define

$$\boldsymbol{b}(x) = \rho_0(x)\boldsymbol{f}(x), \quad g(x) = g(\rho_0(x)),$$

then the blocking of the Bingham fluid can be characterized by:

$$\int_{\mathcal{D}} g(x)|D(\boldsymbol{v}(x))| \, \mathrm{d}x \geq \int_{\mathcal{D}} \boldsymbol{b}(x) \cdot \boldsymbol{v}(x) \, \mathrm{d}x, \quad \forall \boldsymbol{v} \in \mathcal{V}. \tag{10}$$

Now the main problem consists in finding properties on $\boldsymbol{b}$ and g such that inequality (10) holds.

We suppose in what follows that

$$\boldsymbol{b} \in L^\infty(\mathcal{D})^3, \quad \text{and} \quad \int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{r} = 0, \quad \forall \boldsymbol{r} \in \mathcal{R} \cap \mathcal{V}, \tag{11}$$

where $\mathcal{R} = \ker D = \{\boldsymbol{r} : \mathcal{D} \to \mathbb{R}^3 \,; \boldsymbol{r}(x) = m + n \wedge x\}$ is the set of rigid motions. The first condition in (11) is a natural assumption for the body forces. The second one, which is implied by (10), is always satisfied if $\partial_0\mathcal{D} \neq \emptyset$. These two conditions ensure the existence of a blocking state for large enough yield limit. More precisely we have:

Proposition 3.1. *If (11) holds then*

$$g^*_{\text{hom}} := \sup_{\boldsymbol{v} \in \mathcal{V} \setminus \mathcal{R}} \frac{\displaystyle\int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{v}}{\displaystyle\int_{\mathcal{D}} |D(\boldsymbol{v})|} < +\infty$$

and if $g(x) \geq g^*_{\text{hom}}$, a.e. $x \in \mathcal{D}$ then the blocking occurs, i.e. *(10) holds.*

Proof. The space $LD(\mathcal{D}) = \{\boldsymbol{v} \in L^1(\mathcal{D})^3; D(\boldsymbol{v}) \in L^1(\mathcal{D})^{3\times 3}\}$ is a closed subspace of the space $BD(\mathcal{D})$ introduced in [21, 24] and includes (strictly) the space $W^{1,1}(\mathcal{D})^3$ (see [23]). We define $l \in \mathcal{V}'$ by:

$$l(\boldsymbol{v}) = \int_{\mathcal{D}} \boldsymbol{b}(x) \cdot \boldsymbol{v}(x) \, \mathrm{d}x, \qquad \forall \boldsymbol{v} \in \mathcal{V},$$

and let $\mathcal{W}$ be the subspace of $LD(\mathcal{D})$:

$$\mathcal{W} = \left\{\boldsymbol{v} \in LD(\mathcal{D}); \ \mathrm{div}\ \boldsymbol{v} = 0 \ \text{ in } \mathcal{D}, \boldsymbol{v} = 0 \ \text{ on } \partial_0\mathcal{D}\right\}\cdot$$

Since $\boldsymbol{b} \in L^\infty(\mathcal{D})^3$ we deduce that $l \in \mathcal{W}'$, hence there exists $C_1 > 0$ such that $|l(\boldsymbol{v})| \leq C_1||\boldsymbol{v}||_{LD(\mathcal{D})}$ for all $\boldsymbol{v} \in LD(\mathcal{D})$. On the other hand, from the Korn inequality in $LD(\mathcal{D})$ (see [23]), we deduce that there exists $C_2 > 0$ such that, for all $\boldsymbol{v} \in \mathcal{W}$ there exists $\boldsymbol{r}_v \in \mathcal{R} \cap \mathcal{W}$ which satisfies $\|\boldsymbol{v} - \boldsymbol{r}_v\|_{LD(\mathcal{D})} \leq C_2 \int_{\mathcal{D}} |D(\boldsymbol{v})|$. Using the last two inequalities we get $|l(\boldsymbol{v})| = |l(\boldsymbol{v} - \boldsymbol{r}_v)| \leq C_1 C_2 \int_{\mathcal{D}} |D(\boldsymbol{v})|$ for all $\boldsymbol{v} \in \mathcal{V}$, thus $g^*_{\text{hom}} \leq C_1 C_2$. If $g(x) \geq g^*_{\text{hom}}$ a.e. $x \in \mathcal{D}$ then

$$\int_{\mathcal{D}} g|D(\boldsymbol{v})| \geq g^*_{\text{hom}} \int_{\mathcal{D}} |D(\boldsymbol{v})| \geq \int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in \mathcal{V} \setminus \mathcal{R}$$

and (10) holds, taking (11) into account. □

In the homogeneous case, it is easy to check that the condition $g \geq g^*_{\text{hom}}$ is a complete characterization of the blocking property. In the inhomogeneous case it is only a rough sufficient condition. Indeed the following statement gives a more accurate condition for (10).

We define

$$\mathcal{H} = \{\boldsymbol{\tau} \in L^2(\mathcal{D})^{3\times 3}; \quad \tau_{ij} = \tau_{ji}, \quad \text{trace}(\boldsymbol{\tau}) = 0 \text{ in } \mathcal{D}\}$$

which stands for the deviatoric subspace of $L^2(\mathcal{D})^{3\times 3}_S$, and we consider

$$\mathcal{A}_{\boldsymbol{b}} = \left\{\boldsymbol{\tau} \in \mathcal{H}\,;\ \exists p \in L^2(\mathcal{D}), \quad \textbf{div}\,\boldsymbol{\tau} - \nabla p = -\boldsymbol{b} \text{ in } \mathcal{D}, \quad (\boldsymbol{\tau} - pI)\boldsymbol{n} = 0 \text{ on } \partial_1 \mathcal{D}\right\},$$

where $(\boldsymbol{\tau} - pI)\boldsymbol{n} = 0$ lies in $H^{-1/2}(\partial\mathcal{D})^3$. Using the characterization of the gradient of a distribution (see for instance [22], p. 14) we obtain another characterization of the set $\mathcal{A}_{\boldsymbol{b}}$:

$$\mathcal{A}_{\boldsymbol{b}} = \left\{\boldsymbol{\tau} \in \mathcal{H}\,; \quad \int_{\mathcal{D}} \boldsymbol{\tau} : D(\boldsymbol{v}) = \int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{v},\ \forall \boldsymbol{v} \in \mathcal{V}\right\}.$$

Theorem 3.1. *The Bingham fluid is blocked,* i.e., *(10) holds, if and only if there exists a function $\boldsymbol{\sigma} \in \mathcal{A}_{\boldsymbol{b}}$ such that $g(x) \geq |\boldsymbol{\sigma}(x)|$, a.e. $x \in \mathcal{D}$.*

Proof. Let $\boldsymbol{\sigma} \in \mathcal{A}_{\boldsymbol{b}}$ such that $g(x) \geq |\boldsymbol{\sigma}(x)|$, a.e. $x \in \mathcal{D}$. Then for all $\boldsymbol{v} \in \mathcal{V}$ we have

$$\int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{v} = \int_{\mathcal{D}} \boldsymbol{\sigma} : D(\boldsymbol{v}) \leq \int_{\mathcal{D}} |\boldsymbol{\sigma}|\ |D(\boldsymbol{v})| \leq \int_{\mathcal{D}} g|D(\boldsymbol{v})|$$

and (10) follows.

Suppose now that (10) holds and we consider $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ given by

$$\mathcal{J}(\boldsymbol{\tau}) = \frac{1}{2}\int_{\mathcal{D}} [|\boldsymbol{\tau}| - g]^2_+,$$

where $[\ ]_+$ denotes the positive part. From standard arguments of convex analysis we deduce that there exists (at least) a $\boldsymbol{\sigma} \in \mathcal{A}_{\boldsymbol{b}}$ solution of the minimization problem $\mathcal{J}(\boldsymbol{\sigma}) \leq \mathcal{J}(\boldsymbol{\tau})$ for all $\boldsymbol{\tau} \in \mathcal{A}_{\boldsymbol{b}}$. Indeed $\mathcal{J}$ is a convex continuous functional and from $[|\boldsymbol{\tau}| - g]^2_+ \geq |\boldsymbol{\tau}|^2 - 2g|\boldsymbol{\tau}|$ we get that $\mathcal{J}$ is coercive on $\mathcal{H}$. Euler's equation for $\mathcal{J}$ reads

$$\int_{\mathcal{D}} \frac{[|\boldsymbol{\sigma}| - g]_+}{|\boldsymbol{\sigma}|}\boldsymbol{\sigma} : \boldsymbol{\tau} = 0, \qquad \forall\, \boldsymbol{\tau} \in \mathcal{A}_0,$$

where $\mathcal{A}_0$ is $\mathcal{A}_{\boldsymbol{b}}$ for $\boldsymbol{b} = 0$, the tangent space to $\mathcal{A}_{\boldsymbol{b}}$. (Here $[|\boldsymbol{\sigma}| - g]_+\boldsymbol{\sigma}/|\boldsymbol{\sigma}|$ stands for 0 if $\boldsymbol{\sigma} = 0$.)

Since the orthogonal subspace of $\mathcal{A}_0$ in $\mathcal{H}$ is $D(\mathcal{V})$ we deduce that there exists $\boldsymbol{w} \in \mathcal{V}$ such that $\dfrac{[|\boldsymbol{\sigma}| - g]_+}{|\boldsymbol{\sigma}|}\boldsymbol{\sigma} = D(\boldsymbol{w})$. We put $\boldsymbol{v} = \boldsymbol{w}$ in (10). Then we get

$$\int_{\mathcal{D}} g|D(\boldsymbol{w})| \geq \int_{\mathcal{D}} \boldsymbol{b} \cdot \boldsymbol{w} = \int_{\mathcal{D}} \boldsymbol{\sigma} : D(\boldsymbol{w}) = \int_{\mathcal{D}} \frac{[|\boldsymbol{\sigma}| - g]_+}{|\boldsymbol{\sigma}|}|\boldsymbol{\sigma}|^2.$$

Bearing in mind that $|D(\boldsymbol{w})| = [|\boldsymbol{\sigma}| - g]_+$ we obtain $\int_{\mathcal{D}}[|\boldsymbol{\sigma}| - g]_+(g - |\boldsymbol{\sigma}|) \geq 0$ which implies that $|\boldsymbol{\sigma}(x)| \leq g(x)$ a.e. $x \in \mathcal{D}$. □

4. The stationary anti-plane flow

We consider in this section the particular case of the stationary anti-plane flow. Therefore, $\mathcal{D} = \Omega \times \mathbb{R}$ where Ω is a bounded domain in $\mathbb{R}^2$. The boundary of Ω, denoted by Γ, is divided into two parts $\Gamma = \Gamma_0 \cup \Gamma_1$, such that $\partial_0\mathcal{D} = \Gamma_0 \times \mathbb{R}$, $\partial_1\mathcal{D} = \Gamma_1 \times \mathbb{R}$. We are looking for a flow in the Ox_3 direction, *i.e.* $\boldsymbol{u} = (0, 0, u)$, which does not depend on x_3 and t so that $\rho = \rho(x_1, x_2)$ and $u = u(x_1, x_2)$. Note that under these assumptions the equations (2–3) are satisfied, hence the density ρ represents now a parameter of the inhomogeneous problem and we cannot talk about a density dependent model anymore. Indeed the density is implied only in the spatial distribution of inhomogeneous parameters g, η and the body forces f are defined as follows

$$\eta(x_1, x_2) = \eta(\rho(x_1, x_2)), \quad g(x_1, x_2) = g(\rho(x_1, x_2)), \quad f(x_1, x_2) = \rho(x_1, x_2) f_3(x_1, x_2),$$

where f_3 denotes the component of the forces in the Ox_3 direction. We suppose in the following that

$$f, g, \eta \in L^\infty(\Omega), \quad g \geq 0, \quad \eta(x) \geq \eta_0 > 0, \text{ a.e. } x \in \Omega.$$

If we define

$$V = \{v \in H^1(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma_0\}$$

then the variational formulation (9) for the anti-plane flow becomes

$$u \in V, \qquad \forall v \in V, \qquad \int_\Omega \eta(x) \nabla u(x) \cdot \nabla(v(x) - u(x))\,\mathrm{d}x$$
$$+ \int_\Omega g(x)|\nabla v(x)|\,\mathrm{d}x - \int_\Omega g(x)|\nabla u(x)|\,\mathrm{d}x \geq \int_\Omega f(x)(v(x) - u(x))\,\mathrm{d}x. \quad (12)$$

The above problem is a standard variational inequality. If $\mathrm{meas}(\Gamma_0) > 0$ then it has a unique solution u. If $\Gamma_0 = \emptyset$ and $\int_\Omega f(x)\,\mathrm{d}x = 0$ then a solution exists and it is unique up to an additive constant. In the following we will always assume that one of these cases holds.

In order to give the variational formulation in terms of stresses for (12) we define $H = (L^2(\Omega))^2$ and

$$A_f = \{\boldsymbol{\tau} \in H; \quad \mathrm{div}\,\boldsymbol{\tau} = -f \quad \text{in} \quad \Omega, \quad \boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \quad \text{on}\, \Gamma_1\}, \tag{13}$$

where $\boldsymbol{\tau} \cdot \boldsymbol{n}$ is considered in $H^{-\frac{1}{2}}(\Gamma)$. Let $J : H \to \mathbb{R}$ be the following functional

$$J(\boldsymbol{\tau}) = \int_\Omega \frac{1}{2\eta(x)}[|\boldsymbol{\tau}(x)| - g(x)]_+^2\,\mathrm{d}x. \tag{14}$$

Proposition 4.1. *There exists at least a $\boldsymbol{\sigma} \in A_f$ minimizing J on A_f,* i.e. *$J(\boldsymbol{\sigma}) \leq J(\boldsymbol{\tau})$, for all $\tau \in A_f$, which is characterized by*

$$\boldsymbol{\sigma} \in A_f \quad \textit{and} \quad \int_\Omega \frac{[|\boldsymbol{\sigma}(x)| - g(x)]_+}{\eta(x)|\boldsymbol{\sigma}(x)|} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\tau}(x)\,\mathrm{d}x = 0, \quad \forall\, \boldsymbol{\tau} \in A_0 \tag{15}$$

where A_0 is A_f with $f = 0$.

Proof. From $[|\boldsymbol{\tau}| - g]_+^2 \geq |\boldsymbol{\tau}|^2 - 2g|\boldsymbol{\tau}|$ we deduce that J is coercive on H and since J is a convex and continuous functional we get the existence of $\boldsymbol{\sigma}$. The variational equation (15) is Euler's equation associated with the minimization problem. □

The following theorem gives the connection between (12) and (15).

Theorem 4.1. *Let u be the solution of (12) and let $\boldsymbol{\sigma}$ be a solution of (15). Then we have*

$$\nabla u(x) = \frac{[|\boldsymbol{\sigma}(x)| - g(x)]_+}{\eta(x)|\boldsymbol{\sigma}(x)|}\boldsymbol{\sigma}(x), \quad a.e. \quad x \in \Omega. \tag{16}$$

Proof. Let $\boldsymbol{\sigma}$ be a solution of (15). This implies that $\bar{\boldsymbol{\sigma}} = \frac{1}{\eta}[1 - \frac{g}{|\boldsymbol{\sigma}|}]_+ \boldsymbol{\sigma}$ belongs to the orthogonal subspace of A_0 in H. So there exists $w \in V$ such that $\nabla w = \bar{\boldsymbol{\sigma}}$. We now prove that for all $r \in \mathbb{R}^2$ and a.e. $x \in \Omega$ we have

$$\boldsymbol{\sigma}(x) \cdot (r - \bar{\boldsymbol{\sigma}}(x)) \leq \eta(x)\bar{\boldsymbol{\sigma}}(x) \cdot (r - \bar{\boldsymbol{\sigma}}(x)) + g(x)|r| - g(x)|\bar{\boldsymbol{\sigma}}(x)|. \tag{17}$$

Indeed if $|\boldsymbol{\sigma}(x)| > g(x)$ then $\boldsymbol{\sigma}(x) = \left(\eta(x) + \frac{g(x)}{|\bar{\boldsymbol{\sigma}}(x)|}\right)\bar{\boldsymbol{\sigma}}(x)$ and (17) follows. If $|\boldsymbol{\sigma}(x)| \leq g(x)$ then $\bar{\boldsymbol{\sigma}}(x) = 0$ which implies (17).

Since $\boldsymbol{\sigma} \in A_f$ we have

$$\int_\Omega f(x)(v(x) - w(x))\,\mathrm{d}x = \int_\Omega \boldsymbol{\sigma}(x) \cdot (\nabla v(x) - \nabla w(x))\,\mathrm{d}x.$$

If we put $r = \nabla v(x)$ in (17) and then integrate over Ω we deduce that w is a solution of (12). Since the solution of (12) is unique we obtain $u = w, \nabla u = \bar{\boldsymbol{\sigma}}$, and (16) follows. □

The above theorem gives the opportunity to describe the rigid zones Ω_r and the shearing zones Ω_s defined by

$$\Omega_r = \{x \in \Omega; \quad |\nabla u(x)| = 0\}, \quad \Omega_s = \{x \in \Omega; \quad |\nabla u(x)| > 0\}\cdot$$

Indeed, from (16) we have the following result.

Corollary 4.1. *The solution $\boldsymbol{\sigma}$ of (15) is unique in Ω_s,* i.e., *if $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ are two solutions of (15) then $\boldsymbol{\sigma}_1(x) = \boldsymbol{\sigma}_2(x)$ a.e. $x \in \Omega_s$. For any $\boldsymbol{\sigma}$ solution of (15) we have*

$$\Omega_r = \{x \in \Omega; \quad |\boldsymbol{\sigma}(x)| \leq g(x)\}, \quad \Omega_s = \{x \in \Omega; \quad |\boldsymbol{\sigma}(x)| > g(x)\}\cdot \tag{18}$$

Proof. If $x \in \Omega_s$ then $|\nabla u(x)| > 0$ and from (16) we have

$$\boldsymbol{\sigma}(x) = \left(\eta(x) + \frac{g(x)}{|\nabla u(x)|}\right)\nabla u(x).$$

The uniqueness follows. □

The previous description of the rigid zones can be used to study the blocking property, *i.e.*, when the whole Ω is a rigid zone ($\Omega = \Omega_r$). In this case $u \equiv 0$ is the solution of (12) characterized by the following problem: *Find the link between f and g such that*

$$\int_\Omega g(x)|\nabla v(x)|\,\mathrm{d}x \geq \int_\Omega f(x)v(x)\,\mathrm{d}x, \qquad \forall v \in V. \tag{19}$$

As in the threedimensional case, the blocking always occurs for large enough yield distribution. Indeed, there exists an homogeneous yield limit $g^*_{\mathrm{hom}} > 0$ given by

$$g^*_{\mathrm{hom}} = \sup_{v \in V,\ v \neq \mathrm{const}} \frac{\displaystyle\int_\Omega f(x)v(x)\,\mathrm{d}x}{\displaystyle\int_\Omega |\nabla v(x)|\,\mathrm{d}x}$$

such that if $g(x) \geq g^*_{\mathrm{hom}}$, a.e. $x \in \Omega$ then the blocking occurs, *i.e.* (19) holds. Moreover we have the the following complete characterization of the blocking property.

Proposition 4.2. *The Bingham fluid is blocked if and only if there exists $\boldsymbol{\sigma} \in A_f$ such that $|\boldsymbol{\sigma}(x)| \leq g(x)$ a.e. $x \in \Omega$.*

Proof. If $|\boldsymbol{\sigma}(x)| \leq g(x)$ a.e. $x \in \Omega$ then $J(\boldsymbol{\sigma}) = 0 \leq J(\boldsymbol{\tau})$, for all $\boldsymbol{\tau} \in A_f$ and from (16) we deduce $\nabla u \equiv 0$. $\square$

5. Flow between two infinite planes

We shall consider here the anti-plane flow in one dimension, *i.e.* $\mathcal{D} = \Omega \times \mathbb{R}^2$, with $\Omega = (0, \ell) \subset \mathbb{R}$. The choice of $\partial_0 \mathcal{D} = \Gamma_0 \times \mathbb{R}^2$ with $\Gamma_0 = \{0, \ell\}$ corresponds to the flow between two infinite planes $x = 0$ and $x = \ell$. In this case $V = H^1_0(\Omega)$. Let u be the solution of (12), satisfying:

$$u \in V, \qquad \forall v \in V, \quad \int_\Omega \eta(x)\ u'(x)\ (v'(x) - u'(x))\ \mathrm{d}x + \int_\Omega g(x)(|v'(x)| - |u'(x)|)\ \mathrm{d}x \geq \int_\Omega f(x)(v(x) - u(x))\ \mathrm{d}x. \tag{20}$$

We denote by F the antiderivative of f such that

$$F(x) := \int_0^x f(s)\ \mathrm{d}s.$$

Note that in the case of the flow between two infinite planes, A_f becomes an affine set of dimension one. Indeed from (13) we have

$$A_f = -F + \mathbb{R} = \{-F - C; \quad C \in \mathbb{R}\},$$

and the functional J defined in (14) can be reduced to the one-variable functional $j : \mathbb{R} \to \mathbb{R}$ given by

$$j(C) = \int_0^\ell \frac{1}{2\eta(x)} \left[|F(x) + C| - g(x)\right]^2_+\ \mathrm{d}x.$$

In the general twodimensional case the minimizer of J is not unique. But here, if blocking does not occur, then the uniqueness of the minimizer of j can be proved. More precisely we have:

Theorem 5.1. *Either the flow is blocked,* i.e. $u \equiv 0$ *in Ω, or the minimizer C_0 of j over $\mathbb{R}$, is unique. In the latter case, C_0 is the solution of the scalar equation*

$$\int_\Omega \frac{\big[|F(x) + C_0| - g(x)\big]_+}{\eta(x)} \operatorname{sign}(F(x) + C_0)\ \mathrm{d}x = 0, \tag{21}$$

and u is given by

$$u(x) = -\int_0^x \frac{\big[|F(s) + C_0| - g(s)\big]_+}{\eta(s)} \operatorname{sign}(F(s) + C_0)\ \mathrm{d}s. \tag{22}$$

Proof. We suppose that there exist two minimizers $C_1 \neq C_2$ of j over $\mathbb{R}$. We have $j'(C_1) = j'(C_2) = 0$. This implies that $(j'(C_1) - j'(C_2))(C_1 - C_2) = 0$. Define h on $\mathbb{R} \times \Omega$ by $h(t, x) = t - g(x)$ if $t > g(x)$, $h(t, x) = 0$ if $|t| \leq g(x)$ and $h(t, x) = t + g(x)$ if $t < -g(x)$. Then $j'(C) = \int_\Omega h(F(x) + C, x)\ \mathrm{d}x$ and we have

$\int_\Omega (h(F(x)+C_1,x)-h(F(x)+C_2,x))(C_1-C_2)\,\mathrm{d}x=0$. Since $t\to h(t,x)$ is nondecreasing for all $x\in\Omega$ we have $h(F(x)+C_1,x)=h(F(x)+C_2,x)$ in Ω. Bearing in mind that $t\to h(t,x)$ is increasing on $\mathbb{R}\setminus[-g(x),g(x)]$ we deduce that $|F(x)+C_i|\le g(x)$ for all $x\in\Omega$, $i=1,2$. We can use now Proposition 4.2 to deduce that the Bingham fluid is blocked, a contradiction. The expression of u follows from (16), keeping in mind that $\boldsymbol{\sigma}=-F-C_0$. □

Since the minimizer C_0 cannot be obtained directly from the equation (21), we complete here the picture of the solution u with some qualitative results. Of particular interest in applications are the *rigid zones* and the *blocking zones*, that is, intervals where $u'\equiv 0$ and where $u\equiv 0$ respectively.

The following theorem characterizes the intervals of monotony for the solution u. It makes use of a preliminary lemma.

Lemma 5.1. *Let $I=[a,b]\subset\overline{\Omega}$; The following statements are equivalent:*

$$\max_{x\in I}\,[-g(x)-F(x)]\le\min_{x\in I}\,[g(x)-F(x)]\,, \tag{23}$$

$$\forall(x,y)\in I\times I,\qquad g(x)+g(y)\ge|F(y)-F(x)|\,, \tag{24}$$

$$\exists C>0,\ \forall x\in I,\qquad |C+F(x)|\le g(x). \tag{25}$$

Theorem 5.2. *Let $I\subset\overline{\Omega}$ satisfying (23). Then u is monotone in I.*

Remark 1. The solution u is monotone in any interval I such that

$$\int_I|f|\le 2\min_I g. \tag{26}$$

For instance, it is sufficient to have for the length of I:

$$|I|\le 2\frac{\min_\Omega g}{\max_\Omega|f|}\,\cdot \tag{27}$$

Indeed if (26) is satisfied, then (24) obviously holds in I.

Proof of Lemma 5.1. We will prove that (25) ⇒ (23) ⇒ (24) ⇒ (25).

Proof of (25) ⇒ (23). We have from (25) $(F(x)+C)^2\le g(x)^2$ for all $x\in I$, hence $(C+F(x)-g(x))(C+F(x)+g(x))\le 0$. That implies $-g(x)-F(x)\le C\le g(x)-F(x)$. Since this must be true for all $x\in I$, and C is a constant, we deduce (23).

Proof of (23) ⇒ (24). From (23), we have, for any $(x,y)\in I^2$, $-g(x)-F(x)\le g(y)-F(y)$, or equivalently $g(x)+g(y)\ge F(y)-F(x)$. Exchanging x and y, we have also $g(x)+g(y)\ge F(x)-F(y)$, hence (24).

Proof of (24) ⇒ (25). Since $g+F$ is a continuous function on I, it attains a global minimum at some $x_0\in I$, which gives $g(x)+F(x)\ge g(x_0)+F(x_0)$ for all $x\in I$. On the other hand, we have from (24) $g(x_0)+g(x)\ge F(x)-F(x_0)$. Defining $C:=-g(x_0)-F(x_0)$, this yields $g(x)\ge -F(x)-C$ and $g(x)\ge F(x)+C$. This proves (25). □

Proof of Theorem 5.2. Assume that there exist $\alpha<\beta$ in I such that $u(\alpha)=u(\beta)$. We define $J:=[\alpha,\beta]$. For any $w\in L^2(J)$ satisfying $\int_J w=0$, define $W(t):=0$ for $t<\alpha$, and $W(t)=\int_\alpha^t w$ otherwise. Then $W\in H_0^1(J)$, $v:=u+W\in H_0^1(\Omega)$, so from (20) we get

$$\forall w\in L^2(J),\ \int_J w=0,\qquad \int_J\eta u'w+\int_J g(|u'+w|-|u'|)\ge-\int_J Fw. \tag{28}$$

Since $\int_J u' = u(\beta) - u(\alpha) = 0$, we can apply this inequality to $w = u'$ and to $w = -u'$:

$$\int_J \eta(u')^2 + \int_J g\,|u'| \geq -\int_J Fu'$$
$$-\int_J \eta(u')^2 - \int_J g\,|u'| \geq \int_J Fu'.$$

This yields

$$\int_J \eta(u')^2 + \int_J g\,|u'| = -\int_J Fu'.$$

From the Lemma 5.1, we know that (23) implies (25), that is, there exists $C \in \mathbb{R}$ such that $-F - C \leq g$. Hence using again $\int_J u' = 0$:

$$\int_J \eta(u')^2 + \int_J g\,|u'| = -\int_J Fu' = \int_J (-C - F)u' \leq \int_J g\,|u'|\,.$$

We conclude that $\int_J \eta(u')^2 = 0$, that is $u' \equiv 0$ almost everywhere in $J = [\alpha, \beta]$. Since this holds for any α, β such that $u(\alpha) = u(\beta)$, u is monotone in I. □

Now we give some necessary conditions for the existence of the rigid zones.

Proposition 5.1. *Let $I \subset \overline{\Omega}$ be a rigid zone ($u' \equiv 0$ in I). Then any of (23-25) hold.*

Proof. Since $u' \equiv 0$ in I, we deduce from (20) that for any $v \in H_0^1(I)$, one has $\int_I g\,|v'| \geq \int_I fv = -\int_I Fv'$. Considering the same property for $-v$, and defining $w = v'$, we deduce that

$$\forall w \in L^2(I),\ \int_I w = 0, \qquad \int_I g\,|w| \geq \left|\int_I Fw\right|.$$

Let φ_1 be a continuous function on $\mathbb{R}$ with compact support in $(-1, 1)$. We define $\varphi_n(t) := n\varphi_1(nt)$ so that (φ_n) converges to the Dirac mass at 0 as $n \to \infty$. Let x, y be given in the interior of I, and $w_n(t) := \varphi_n(t-x) - \varphi_n(t-y)$. For n large enough, we have $\int_I w_n = 0$, so $\int_I g\,|w_n| \geq \left|\int_I Fw_n\right|$. In the limit, this yields $g(x) + g(y) \geq |F(x) - F(y)|$. We deduce (24) for interior points of I, and then for any $(x, y) \in I^2$ by continuity. □

The following corollary gives sufficient conditions for the existence of the rigid zones and the blocking property.

Property 5.1. *Assume that $I = [a, b] \subset \overline{\Omega}$ satisfies any of (23–25). Then*

1. *If $u(a) = u(b)$, then $u \equiv u(a)$ in I (that is, I is a rigid zone).*
2. *If there exists $\sigma \in \{-1, +1\}$ such that the following three conditions hold:*
   (a) *$F(b) - F(a) = \sigma(g(a) + g(b))$,*
   (b) *either $a = 0$ or $F + \sigma g$ is increasing in a left-neighborhood of a,*
   (c) *either $b = \ell$ or $F - \sigma g$ is increasing in a right-neighborhood of b,*
   then $I = [a, b]$ is a maximal rigid zone (no interval strictly containing I is a rigid zone).
3. *If $I = \overline{\Omega}$ then the fluid is blocked ($u \equiv 0$).*

Proof. From the Theorem 5.2 we know that u is monotone in I; if $u(a) = u(b)$ we deduce that u is constant in I.

If $\int_a^b f = \pm(g(a) + g(b))$ we have equality in (23). For instance, if $\int_a^b f = g(a) + g(b)$, we obtain:

$$\max_I(-g - F) \geq -g(a) - F(a) = g(b) - F(b) \geq \min_I(g - F).$$

If $a > 0$, the additional assumption on a implies that for any α in a left-neighborhood of a, we have $\max_{[\alpha,a]}(-g-F) > -g(a) - F(a)$. This implies that the interval $[\alpha, b]$ does not satisfy (23) and therefore is not a rigid zone from Proposition 5.1. A similar argument near b gives the maximality of I. □

The monotonicity property proved in Theorem 5.2 implies that u does not attain a strict extremum in Ω. The following property gives a more precise estimate:

Property 5.2. *Assume that $g > 0$ in Ω. Then all local extremums of u in Ω are intervals $[\alpha, \beta]$ of rigid zones such that:*

$$\beta - \alpha \geq 2\frac{\min_{[\alpha,\beta]} g}{\max_{[\alpha,\beta]} |f|} \geq 2\frac{\min_\Omega g}{\max_\Omega |f|}\,\cdot \tag{29}$$

In particular, the solution u does not have any local strict extremum in Ω.

Proof. Assume that the interval $I = [\alpha, \beta] \subset \Omega$ (with $\beta \geq \alpha$) is a local maximizer of u. That is, there exists $\omega \subset \Omega$, an open neighborhood of $[\alpha, \beta]$ such that $x \in \omega \setminus I$ implies $u(x) < m := u(y)$, $\forall y \in I$. For $\varepsilon > 0$ small enough, $u^{-1}([m-\varepsilon, m]) \cap \omega = [\alpha_\varepsilon, \beta_\varepsilon] =: I_\varepsilon$, and $\lim_{\varepsilon\to 0} \alpha_\varepsilon = \alpha$, $\lim_{\varepsilon\to 0} \beta_\varepsilon = \beta$.

Since $u(\alpha_\varepsilon) = u(\beta_\varepsilon)$ but u is not constant in I_ε, we deduce from Proposition 5.1 that (23) is not satisfied in I_ε, that is, there exists $x_\varepsilon, y_\varepsilon \in I_\varepsilon$ such that

$$-g(x_\varepsilon) - F(x_\varepsilon) > g(y_\varepsilon) - F(y_\varepsilon). \tag{30}$$

On the other hand, $u' \equiv 0$ in I, hence from Proposition 5.1, $x_\varepsilon \notin I$ or $y_\varepsilon \notin I$. We assume for instance that $x_\varepsilon \notin I$.

Hence $\lim_{\varepsilon\to 0} x_\varepsilon = \alpha$, and extracting subsequences, we may assume $y_\varepsilon \to \gamma \in I$. Should they have the same limit ($\gamma = \alpha$), (30) would give $-g(\alpha) \geq g(\alpha)$, in contradiction to the assumption $g > 0$. Hence $\alpha < \beta$ and $\gamma \in\,]\alpha, \beta]$; we get from (30) in the limit

$$g(\alpha) + g(\gamma) \leq \left| \int_\alpha^\gamma f(t)\,\mathrm{d}t \right| \leq (\gamma - \alpha) \max_{[\alpha,\gamma]} |f| \leq (\beta - \alpha) \max_{[\alpha,\beta]} |f|$$

and (29) follows. □

In the physical model the sign of f is constant. We consider now the case $f \geq 0$; similar properties hold for $f \leq 0$ (changing u with $-u$).

Theorem 5.3. *Assume that the fluid is not blocked ($u \not\equiv 0$) and that $f \geq 0$ in Ω. Then there exists $x_0 \in \Omega$ such that u is nondecreasing in $[0, x_0]$ and nonincreasing in $[x_0, \ell]$.*

Additionally if g is homogeneous ($g(x) = g_{\mathrm{hom}} = \mathrm{const.}$) then there exists a unique rigid (non blocking) zone for u in the interior of Ω.

Proof. Let C_0 be the constant number given in Theorem 5.1. Since $f \geq 0$, $\widehat{F}(x) := F(x) + C_0$ is nondecreasing. Hence there exists $x_0 \in [0, \ell]$ such that $\widehat{F}(x) \leq 0$ in $[0, x_0[$ and $\widehat{F}(x) \geq 0$ in $]x_0, \ell]$. From (21),

$$\int_0^\ell \left[|\widehat{F}(x)| - g(x) \right]_+ \mathrm{sign}(\widehat{F}(x))\, \frac{\mathrm{d}x}{\eta(x)} = 0 \tag{31}$$

so either the integrand is zero in Ω (and then $u \equiv 0$ contrary to the assumptions of the Theorem) or the integrand shows positive and negative values. In the latter case, $\widehat{F}$ also shows positive and negative values, and x_0 is an interior point of Ω.

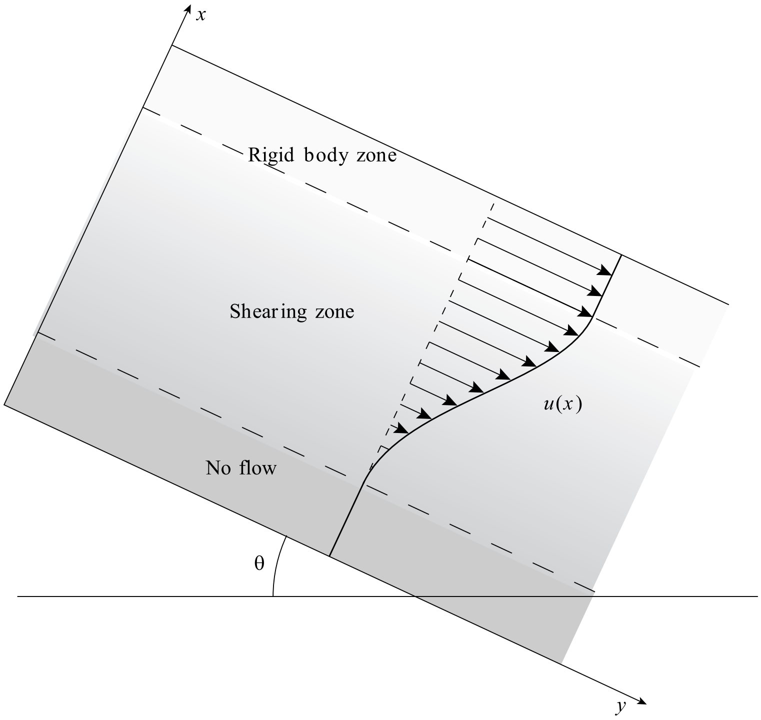

FIGURE 1. The geometry of a landslide on a natural slope.

Using (22) we get

$$-u'(x)\eta(x) = \left[|\widehat{F}(x)| - g(x)\right]_+ \operatorname{sign}(\widehat{F}(x)).$$

Therefore u is nondecreasing in $[0, x_0]$ and nonincreasing in $[x_0, \ell]$.

If we assume that g is constant, then the rigid zones are given by $\Omega_r = \{x \in \Omega\,;\, |\widehat{F}(x)| \le g_{\mathrm{hom}}\}$, which is an interval $[a_r, b_r]$ since $\widehat{F}$ is nondecreasing. If $a_r = 0$ then the integrand in (31) is everywhere nonnegative, a contradiction. Hence $a_r > 0$ and similarly $b_r < \ell$. This concludes the proof of the Theorem. □

6. Flow between an infinite plane and a rigid roof

We suppose that $\Gamma_0 = \{0\}$ and $\Gamma_1 = \{\ell\}$, *i.e.* $V = \{v \in H^1(\Omega);\ v(0) = 0\}$. Such a boundary condition corresponds to the flow on the plane $x = 0$ with a rigid roof at $x = \ell$.

Remark 2. The problem described here occurs in modeling the landslides on a natural slope (see [5]). If we consider the geometry plotted in Figure 1 and if we denote by γ the vertical gravitational acceleration then the body forces are given by $f(x) = \gamma\rho(x)\sin\theta$, where θ is the angle of the slope.

If we consider (13) in our case we come to the conclusion that the set A_f is reduced to a single function σ and $|\sigma|$ defines the *inhomogeneous critical yield limit* g^*, that is

$$\sigma(x) = \int_x^\ell f(s)\,\mathrm{d}s, \quad A_f = \{\sigma\}, \quad g^*(x) = \left|\int_x^\ell f(s)\,\mathrm{d}s\right|.$$

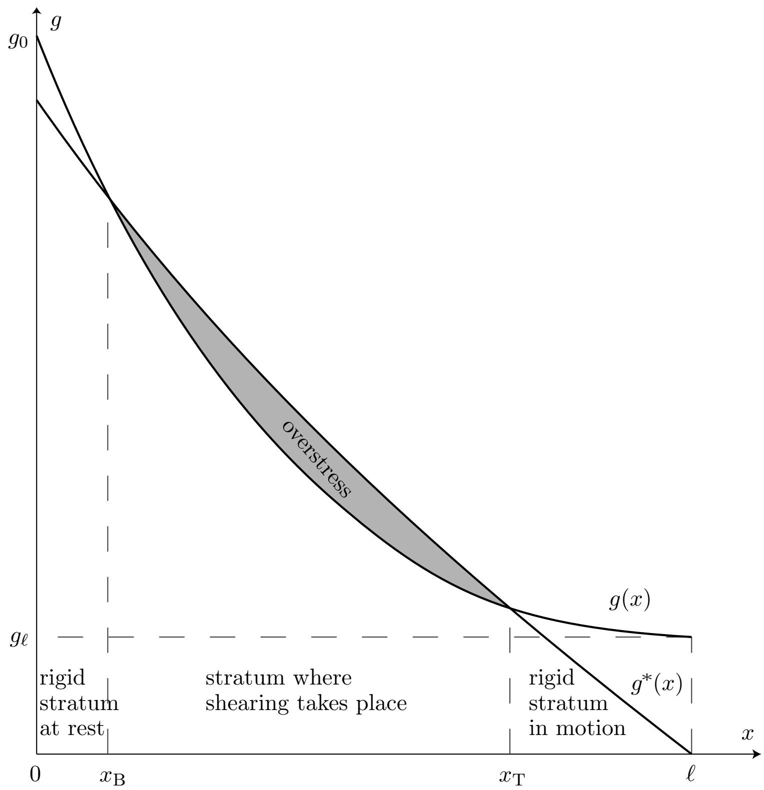

FIGURE 2. The spatial distribution of the critical yield limit $g^*(x)$ compared with the yield limit $g(x)$.

We point out that in this particular case the inhomogeneous critical yield limit g^* characterizes the rigid and the shearing zones. Indeed, from Corollary 4.1 we have

$$\Omega_r = \{x \in \Omega; \quad g(x) \geq g^*(x)\}, \qquad \Omega_s = \{x \in \Omega; \quad g(x) < g^*(x)\} \cdot \tag{32}$$

From (16) we get the analytical expression of the solution

$$u(x) = \int_0^x \frac{1}{\eta(s)} \left[g^*(s) - g(s)\right]_+ \operatorname{sign} \sigma(s) \, \mathrm{d}s.$$

In order to illustrate the previous simple result we consider, as in [5], the case of a natural slope involving a *linear variation with depth* of the density

$$\rho(x) = \rho_0 - \frac{\rho_0 - \rho_\ell}{\ell} x \tag{33}$$

where ρ_0 is the density at the bottom and $0 < \rho_\ell < \rho_0$ is the density at the top. In this case the critical yield stress is

$$g^*(x) = \gamma \sin\theta \ (\ell - x) \left[\rho_0 - \frac{\rho_0 - \rho_\ell}{2\ell}(\ell + x)\right]. \tag{34}$$

We remark that the variation with depth of this critical yield stress is quadratic. If the yield stress g has a non-linear variation with depth of the type:

$$g(x) = (g_0 - g_\ell)\left(1 - \frac{x}{\ell}\right)^m + g_\ell$$

with $m > 1$, where g_0 is the yield limit on the bottom and $g_\ell < g_0$ the yield limit on the top (see [5]). In this case the rigid and shearing zones can be easily deduced (see Fig. 2) from (32). The intersection between the graphs of g and g^* represents the separation boundary between of the rigid and shearing zones. Note that the rigid zone on the bottom $x = 0$ is a blocking (or non-flow zone).

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References

[1] E.C. Bingham, *Fluidity and plasticity.* Mc Graw-Hill, New-York (1922).

[2] O. Cazacu and N. Cristescu, Constitutive model and analysis of creep flow of natural slopes. *Ital. Geotech. J.* **34** (2000) 44–54.

[3] N. Cristescu, Plastical flow through conical converging dies, using viscoplastic constitutive equations. *Int. J. Mech. Sci.* **17** (1975) 425–433.

[4] N. Cristescu, On the optimal die angle in fast wire drawing. *J. Mech. Work. Technol.* **3** (1980) 275–287.

[5] N. Cristescu, *A model of stability of slopes* in Slope Stability 2000. Proceedings of Sessions of Geo-Denver 2000, D.V. Griffiths, G.A. Fenton and T.R. Martin (Eds.). Geotechnical special publication **101** (2000) 86–98.

[6] N. Cristescu, O. Cazacu and C. Cristescu, A model for slow motion of natural slopes. *Can. Geotech. J.* (to appear).

[7] R.J. DiPerna and P.-L. Lions, Ordinary differential equations, Sobolev spaces and transport theory. *Invent. Math.* **98** (1989) 511–547.

[8] G. Duvaut and J.-L. Lions, *Les inéquations en mécanique et en physique.* Dunod, Paris (1972).

[9] R. Glowinski, *Lectures on numerical methods for nonlinear variational problems.* Notes by M.G. Vijayasundaram and M. Adimurthi. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 65. Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York (1980).

[10] R. Glowinski, J.-L. Lions and R. Trémolières, *Analyse numérique des inéquations variationnelles. Tome 1 : Théorie générale et premières applications. Tome 2 : Applications aux phénomènes stationnaires et d'évolution.* Méthodes Mathématiques de l'Informatique, 5. Dunod, Paris (1976).

[11] I. Ionescu and M. Sofonea, The blocking property in the study of the Bingham fluid. *Int. J. Engng. Sci.* **24** (1986) 289–297.

[12] I. Ionescu and M. Sofonea, *Functional and numerical methods in viscoplasticity.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1993).

[13] I. Ionescu and B. Vernescu, A numerical method for a viscoplastic problem. An application to the wire drawing. *Internat. J. Engrg. Sci.* **26** (1988) 627–633.

[14] J.-L. Lions and G. Stampacchia, Variational inequalities. *Comm. Pure. Appl. Math.* **XX** (1967) 493–519.

[15] P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vol 1: Incompressible models.* Oxford University Press (1996).

[16] P.P. Mosolov and V.P. Miasnikov, Variational methods in the theory of the fluidity of a viscous-plastic medium. *PPM, J. Mech. and Appl. Math.* **29** (1965) 545–577.

[17] P.P. Mosolov and V.P. Miasnikov, On stagnant flow regions of a viscous-plastic medium in pipes. *PPM, J. Mech. and Appl. Math.* **30** (1966) 841–854.

[18] P.P. Mosolov and V.P. Miasnikov, On qualitative singularities of the flow of a viscoplastic medium in pipes. *PPM, J. Mech and Appl. Math.* **31** (1967) 609–613.

[19] A. Nouri and F. Poupaud, An existence theorem for the multifluid Navier-Stokes problem. *J. Differential Equations* **122** (1995) 71–88.

[20] J.G. Oldroyd, A rational formulation of the equations of plastic flow for a Bingham solid. *Proc. Camb. Philos. Soc.* **43** (1947) 100–105.

[21] P. Suquet, Un espace fonctionnel pour les équations de la plasticité. *Ann. Fac. Sci. Toulouse Math. (6)* **1** (1979) 77–87.

[22] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis.* North-Holland, Amsterdam (1979).

[23] R. Temam, *Problèmes mathématiques en plasticité.* Gauthiers-Villars, Paris (1983).

[24] R. Temam and G. Strang, Functions of bounded deformation. *Arch. Rational Mech. Anal.* **75** (1980) 7–21.