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A POSTERIORI ERROR ESTIMATES WITH POST-PROCESSING FOR NONCONFORMING FINITE ELEMENTS

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Abstract. For a nonconforming finite element approximation of an elliptic model problem, we propose *a posteriori* error estimates in the energy norm which use as an additive term the "post-processing error" between the original nonconforming finite element solution and an easy computable conforming approximation of that solution. Thus, for the error analysis, the existing theory from the conforming case can be used together with some simple additional arguments. As an essential point, the property is exploited that the nonconforming finite element space contains as a subspace a conforming finite element space of first order. This property is fulfilled for many known nonconforming spaces. We prove local lower and global upper *a posteriori* error estimates for an enhanced error measure which is the discretization error in the discrete energy norm plus the error of the best representation of the exact solution by a function in the conforming space used for the post-processing. We demonstrate that the idea to use a computed conforming approximation of the nonconforming solution can be applied also to derive an *a posteriori* error estimate for a linear functional of the solution which represents some quantity of physical interest.

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Introduction

Nonconforming finite elements are attractive in the field of mixed finite element approximations or saddle point problems like for instance the Navier-Stokes equations as a typical problem in Computational Fluid Dynamics [6, 9, 12]. An advantage of many nonconforming finite elements compared to conforming ones is that each degree of freedom is associated with the interior of an element or the interior of a $(d-1)$-dimensional face of the d-dimensional elements. This implies that each degree of freedom belongs to at most two elements which simplifies the local communication for a parallelization of the method, particularly in the 3D case (see [21, 22]).

In order to get an efficient numerical method for solving a partial differential equation, it is important to use adaptive mesh-refinement. This requires an *a posteriori* error estimator for evaluating the quality of the numerical solution. Nowadays, there exists a lot of work in the literature concerning the construction and analysis of *a posteriori* error estimators for the case of conforming finite elements (see [1, 25] and the references cited therein). However, there exist not so much papers treating the nonconforming case. It turned out that in this case some extra terms have to be added to the well-known *a posteriori* error estimator used for the

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conforming case. In [8,10,11], these extra terms are the jumps across the element faces of the derivatives of the finite element solution in tangential direction with respect to the element faces. In [14], two other approaches for constructing an *a posteriori* error estimator are considered which are based on the solution of local subproblems or on a two-level splitting of the space of piecewise quadratic conforming finite elements. For deriving an L^2-error estimator, John [15,16] has used as additional term the jumps of the nonconforming finite element solution itself across the element edges.

In this paper, an alternative approach is presented which is based on the usage of a "smoothed" conforming finite element approximation $R_h u_h$ of the nonconforming solution u_h. Then, the computable quantities $\eta_K^{(p)}$, defined as the local norms of the "post-processing error" $u_h - R_h u_h$ on the elements K, are used as additional terms in the *a posteriori* error estimator. With this idea the existing theory from the conforming case can be used together with some simple additional arguments to prove *a posteriori* error estimates in the energy norm. This approach, which has been proposed also in [2] in an abstract setting, is based on the property that the nonconforming finite element space contains as a subspace a first order conforming finite element space. This property is fulfilled for many known nonconforming spaces and will be discussed at the end of Section 3.

In Section 2, we derive an *a posteriori* error estimator for the nonconforming case and prove a global upper estimate of the discretization error in the energy norm. In Section 3, we define an "enhanced error measure" of the discrete solution u_h with respect to the exact solution and the chosen conforming space V_h^c used for the post-processing. This error measure is the discretization error in the discrete energy norm plus the error of the best representation of the exact solution by a function in the conforming space V_h^c. We prove an upper *a posteriori* error estimate of the global enhanced error measure which shows that our nonconforming error estimator is reliable. Furthermore, we prove a lower *a posteriori* error estimate of the local enhanced error measure which justifies that the element-wise contributions of our nonconforming error estimator can be used as local error indicators for controlling the adaptive mesh refinement.

In many applications, one is not interested in all details of the solution but in the value of some physical quantity which can be regarded as a functional of the solution. Therefore, it is important to have an *a posteriori* error estimate for the error with respect to a functional applied to the solution. In contrast to the case of conforming finite element spaces, where a lot of work on this subject has been done by Rannacher and his coworkers (see [3,4,18,19]), there is not as much known for the nonconforming case. The main problem here is the loss of Galerkin orthogonality of the discretization error. In Section 4, we show that the idea of using a conforming approximation $R_h u_h$ of the nonconforming solution u_h can be applied again to overcome this problem. Together with some simple arguments we can use the existing theory for the conforming case. We derive residual based *a posteriori* error estimates using the discrete solution of a dual problem together with additional terms depending on the computable post-processing error $u_h - R_h u_h$. Furthermore, we derive local error indicators for controlling the adaptive procedure to create a suitable locally refined mesh.

Concerning the space dimension and the order of the finite element space, we treat the two- and three-dimensional case and the general case of a finite element space of order $r \geq 1$ in a unified fashion. However, concerning the treated partial differential equation, we restrict the presentation in this paper to the case of a simple Poisson problem in order to avoid technical difficulties and to make the underlying ideas as clear as possible. Generalizations to more general elliptic partial differential equations and also to mixed problems as for instance the Stokes problem can be done in a relatively straightforward way.

1. Preliminaries and notations

As a model problem we consider the Poisson equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega\,, \\ u &= 0 \text{ on } \partial\Omega\,, \end{aligned} \tag{1}$$

in a connected, bounded and polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, where $f \in L^2(\Omega)$ is a given function.

At first, let us introduce some notation and formulate necessary assumptions. We use the standard notation $|.|_{m,G}$ and $\|.\|_{m,G}$ for the semi-norm and norm in the Sobolev space $\mathrm{H}^m(G)$ and we denote by $|\cdot|_{1,h,\Omega}$ the "broken" H^1-semi-norm given as

$$|v|_{1,h,\Omega} := \left(\sum_{K\in\mathcal{T}_h} |v|^2_{1,K} \right)^{1/2}, \tag{2}$$

where $\mathcal{T}_h$ is the set of mesh cells K forming a regular partition of the domain Ω. The cells are assumed to be shape-regular and supposed to be of simplicial, quadrilateral (for $d = 2$) or hexahedral (for $d = 3$) shape. The diameter of element K is denoted by h_K and the global mesh-size is defined as $h := \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{E}$ denote the set of all $(d-1)$-dimensional faces of the elements $K \in \mathcal{T}_h$, $\mathcal{E}_0$ the set of all inner faces and $\langle\cdot,\cdot\rangle_E$, for $E \in \mathcal{E}$, the inner product in $\mathrm{L}^2(E)$. For $K \in \mathcal{T}_h$, let $F_K : \widehat{K} \to K$ be the one-to-one mapping between the reference element $\widehat{K}$ and K, where for instance $\widehat{K} := [-1,+1]^d$ for a quadrilateral ($d = 2$) or hexahedral ($d = 3$) element K. By $\widehat{\mathcal{P}}$ we denote the polynomial space of shape functions on the reference element $\widehat{K}$. Then, the local space of finite element functions on the original element K is defined by

$$\mathcal{P}_K := \left\{ p = \widehat{p} \circ F_K^{-1} :\ \widehat{p} \in \widehat{\mathcal{P}} \right\} \cdot \tag{3}$$

These local spaces form the discontinuous discrete space

$$V(\mathcal{T}_h) := \{ v \in \mathrm{L}^2(\Omega) :\ v\big|_K \in \mathcal{P}_K \quad \forall\, K \in \mathcal{T}_h \} \cdot \tag{4}$$

The finite element space V_h will be chosen as the subspace of those functions $v_h \in V(\mathcal{T}_h)$ where the jump $[v_h]_E$ on any face $E \in \mathcal{E}$ (defined by (15) in Sect. 2) satisfies a certain "smallness condition". As an example, it holds $[v_h]_E \equiv 0$ for a conforming space $V_h \subset \mathrm{H}^1_0(\Omega)$ or $\int_E [v_h]_E \, d\gamma = 0$ for the nonconforming finite element spaces proposed in [7, 9, 20].

Now, we want to describe in a uniform way the discretization of the model problem (1) for both the conforming and nonconforming case, respectively. We start with the weak formulation of (1) in the function space $V := \mathrm{H}^1_0(\Omega)$, which reads

$$u \in V : \qquad a(u,v) = (f,v) \qquad \forall\, v \in V\,, \tag{5}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathrm{L}^2(\Omega)$ and $a : V \times V \to \mathbb{R}$ a bilinear form defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \qquad \forall\, u,v \in V\,. \tag{6}$$

For the description of the discrete problem, we generalize the bilinear form $a(\cdot,\cdot)$ to a bilinear form $a_h : \mathrm{H}^1(\Omega,\mathcal{T}_h) \times \mathrm{H}^1(\Omega,\mathcal{T}_h) \to \mathbb{R}$ on the larger "broken H^1-space"

$$\mathrm{H}^1(\Omega,\mathcal{T}_h) := \{ v \in \mathrm{L}^2(\Omega) :\ v\big|_K \in \mathrm{H}^1(K) \quad \forall\, K \in \mathcal{T}_h \} \tag{7}$$

by means of

$$a_h(u,v) := \sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v \, \mathrm{d}x \qquad \forall\, u,v \in \mathrm{H}^1(\Omega,\mathcal{T}_h)\,. \tag{8}$$

Obviously, we have $V_h \subset V(\mathcal{T}_h) \subset \mathrm{H}^1(\Omega,\mathcal{T}_h)$ and the identities

$$a_h(u,v) := a(u,v) \qquad \forall\, u,v \in V\,, \tag{9}$$

$$|v|_{1,h,\Omega} = |v|_{1,\Omega} \qquad \forall\, v \in V. \tag{10}$$

Note that, due to the boundary conditions and the properties of V_h on inner element faces, the semi-norm $|\cdot|_{1,h,\Omega}$ is a norm on the space $V + V_h$. For a given finite element space $V_h \subset V(\mathcal{T}_h)$, the discrete problem reads

$$u_h \in V_h : \qquad a_h(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h\,. \tag{11}$$

In order to guarantee existence and uniqueness of the solution of problem (5) and (11) and for our error analysis we use the assumption that the bilinear form $a_h(\cdot,\cdot)$ is coercive and continuous, *i.e.* that there exist h-independent positive constants α and M such that

(A1) $\alpha\, |v|^2_{1,h,\Omega} \le a_h(v,v) \qquad \forall\, v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$,

(A2) $a_h(u,v) \le M \sum_{K\in\mathcal{T}_h} |u|_{1,K}|v|_{1,K} \le M|u|_{1,h,\Omega}|v|_{1,h,\Omega} \qquad \forall\, u,v \in \mathrm{H}^1(\Omega,\mathcal{T}_h)$.

For our particular bilinear form defined in (8), these assumptions are obviously fulfilled with $\alpha = M = 1$.

Finally, let us introduce some general notation. We denote by $\mathrm{card}(\mathcal{M})$ the number of elements of a finite set $\mathcal{M}$. Let X' be the dual space of a given space X. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively, and by $(\cdot,\cdot)_G$ the inner product in $\mathrm{L}^2(G)$. All constants in the estimates, which are denoted by $C_1, C_2, \ldots$, are independent of the mesh size, the solution u and the data f.

2. An *a posteriori* error estimator for nonconforming finite elements

At first, let us recall the arguments for the derivation of a residual based *a posteriori* error estimator in the conforming case (see [1,25]) and let us look at those points where the arguments fail in the nonconforming case, *i.e.* where $V_h \not\subset V$.

Step 1. Find a test function $v \in V = \mathrm{H}^1_0(\Omega)$ such that

$$\alpha\, |u - u_h|_{1,\Omega}\; |v|_{1,\Omega} \le a_h(u - u_h, v)\,. \tag{12}$$

From the coercivity of $a(\cdot,\cdot) = a_h(\cdot,\cdot)$ (see (A1)) and from the property $u_h \in V_h \subset V$ in the conforming case we see that $v = u - u_h \in V$ is such a function. However, in the nonconforming case, the error $u - u_h$ in general is not contained in V. The main idea in this paper to circumvent this problem is to analyze at first the error $u - u_h^c \in V$ where $u_h^c = R_h u_h \in V_h^c$ is a "smoothed" approximation of the nonconforming solution $u_h \in V_h$ in a second appropriate conforming finite element space $V_h^c \subset V$. The choice of V_h^c and the *post-processing operator* $R_h : V_h \to V_h^c$ should guarantee that the order of the discretization error $u - u_h$ is saved, *i.e.* that $|u - u_h|_{1,h,\Omega} = O(h^r)$ should imply $|u - R_h u_h|_{1,\Omega} = O(h^r)$. The properties of R_h, which we need to make our method work, will be specified later. So, in the nonconforming case, we get (by $v = u - R_h u_h \in V$) instead of (12) the estimate

$$\alpha\, |u - R_h u_h|_{1,\Omega}\; |v|_{1,\Omega} \le a_h(u - R_h u_h, v)\,. \tag{13}$$

Step 2. Represent the error functional $a_h(u - u_h, w)$ for arbitrary $w \in V$ by means of local contributions from each element. In the conforming case, we get for an arbitrary $w \in V$ (see [1], Eq. (3.7))

$$a_h(u - u_h, w) \;=\; \sum_{K\in\mathcal{T}_h} \left\{ (f + \triangle u_h, w)_K - \frac{1}{2} \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_0} \langle\, [\nabla u_h \cdot n_E]_E\,,\, w\,\rangle_E \right\}, \tag{14}$$

where $\mathcal{E}(K)$ denotes the set of the $(d-1)$-dimensional faces of element K, n_E a unit vector normal to E and $[v]_E$ the jump of the function v on the face E defined as follows. For an inner face $E \in \mathcal{E}_0$, there exist two uniquely defined elements $K^{\mathrm{out}}(E), K^{\mathrm{in}}(E) \in \mathcal{T}_h$ with $\bar{E} = \partial K^{\mathrm{out}}(E) \cap \partial K^{\mathrm{in}}(E)$ and the property that the normal vector

n_E points outward with respect to the element $K^{\text{out}}(E)$. Then, for an element-wise smooth function v and a point $x \in \text{int}(E)$, we define

$$[v]_E\,(x) := v\big|_{K^{\text{out}}(E)}(x) - v\big|_{K^{\text{in}}(E)}(x)\,. \tag{15}$$

The equation (14) remains true if the discrete solution u_h is only in the "broken H^1-space" $\mathrm{H}^1(\Omega,\mathcal{T}_h)$, *i.e.* (14) also holds in the nonconforming case.

Step 3. Use the *"Galerkin orthogonality"*. For the discretization error $u - u_h$, we can use the equation

$$a_h(u-u_h, v_h) = 0 \qquad \forall\, v_h \in V_h \cap V \tag{16}$$

(which is a consequence of (5, 9) and (11)) in order to get for the $v \in V$ from step 1

$$a_h(u-u_h, v) = a_h(u-u_h, v-v_h)\,. \tag{17}$$

Thus, we can apply the error equation (14) to $w = v - v_h$. In the conforming case, it holds $V_h = V_h \cap V$ and we can choose $v_h = \Pi_h v$ where $\Pi_h : V \to V_h$ is an H^1-stable interpolation operator as for example the operator of Scott and Zhang proposed in [24]. However, in the nonconforming case, the space $V_h \cap V$ is smaller than V_h. Therefore, the idea is to look for v_h in a conforming finite element space $V_h^{c,1}$ of first order which is a subspace of the nonconforming space V_h such that $V_h^{c,1} \subset V_h \cap V$. In many situations such a space $V_h^{c,1}$ exists (see Rem. 3.6 below). An example for $V_h^{c,1}$ would be the space of continuous piecewise linear functions if the nonconforming space V_h would be based on simplicial elements. Now, we can choose $v_h = \Pi_h^{c,1} v$ where $\Pi_h^{c,1} : V \to V_h^{c,1}$ is an interpolation operator with the approximation properties

$$|v - \Pi_h^{c,1} v|_{m,K} \le C_1\, h_K^{1-m} |v|_{1,\delta(K)} \qquad \forall\, K \in \mathcal{T}_h,\ m \in \{0,1\},\ v \in V \tag{18}$$

and

$$\|v - \Pi_h^{c,1} v\|_{\mathrm{L}^2(E)} \le C_2\, h_K^{1/2} |v|_{1,\delta(K)} \qquad \forall\, E \in \mathcal{E}(K),\ K \in \mathcal{T}_h,\ v \in V\,. \tag{19}$$

Here $\delta(K)$ denotes the vicinity of K consisting of all elements $\widetilde{K} \in \mathcal{T}_h$, which have a $V_h^{c,1}$-node in common with K. A possible choice for $\Pi_h^{c,1}$ satisfying (18) and (19) is the operator of Scott and Zhang [24] applied to the finite element space $V_h^{c,1}$. The choice $v_h = \Pi_h^{c,1} v$ can be used also in the conforming case if $V_h^{c,1} \subset V_h$ which is satisfied in many situations.

Step 4. Prove that there exists a constant C_3 such that

$$a_h(u-u_h, v) \ \le\ C_3\, \eta^{(c)}\, |v|_{1,\Omega} \tag{20}$$

where $\eta^{(c)}$ is the global error estimator in the conforming case which will be specified later. For the sake of a uniform representation, we assume that in both cases, the conforming and the nonconforming one, there exist a conforming subspace $V_h^{c,1} \subset V_h \cap V$ and an interpolation operator $\Pi_h^{c,1} : V \to V_h^{c,1}$ satisfying (18) and (19). Formally, we can choose $V_h^{c,1} = V_h$ and $\Pi_h^{c,1} = \Pi_h$ in the conforming case. To show (20) we use (17) for $v_h = \Pi_h^{c,1} v$ with the v from step 1 and (14) with $w = v - \Pi_h^{c,1} v \in V$ and obtain

$$\begin{aligned} a_h(u-u_h, v) &= a_h(u-u_h, v - \Pi_h^{c,1} v) \\ &= \sum_{K\in\mathcal{T}_h} \left\{ (f + \triangle u_h, v - \Pi_h^{c,1} v)_K - \frac{1}{2} \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_0} \Big\langle\, [\nabla u_h \cdot n_E]_E\,,\, v - \Pi_h^{c,1} v\, \Big\rangle_E \right\}. \end{aligned} \tag{21}$$

By means of the Cauchy-Schwarz inequality and the approximation properties (18) and (19), this implies

$$
\begin{aligned}
a_h(u-u_h,v) &\le (C_1^2 + C_4 C_2^2/2)^{1/2} \sum_{K\in\mathcal{T}_h} \eta_K^{(c)}\, |v|_{1,\delta(K)} \\
&\le (C_1^2 + C_4 C_2^2/2)^{1/2}\, \eta^{(c)} \left\{ \sum_{K\in\mathcal{T}_h} |v|^2_{1,\delta(K)} \right\}^{1/2} ,
\end{aligned} \tag{22}
$$

where $C_4 := \max_{K\in\mathcal{T}_h}(\mathrm{card}\{E :\ E\in\mathcal{E}(K)\cap\mathcal{E}_0\})$,

$$
\eta_K^{(c)} := \left\{ h_K^2 \|f + \triangle u_h\|^2_{0,K} \ + \ \frac{1}{2} \sum_{E\in\,\mathcal{E}(K)\cap\mathcal{E}_0} h_K \,\|\ [\nabla u_h\cdot n_E]_E\ \|^2_{\mathrm{L}^2(E)} \right\}^{1/2} \tag{23}
$$

and

$$
\eta^{(c)} := \left(\sum_{K\in\mathcal{T}_h} (\eta_K^{(c)})^2 \right)^{1/2} . \tag{24}
$$

Let C_5 be defined as $C_5 := \max_{\widetilde{K}\in\mathcal{T}_h}(\mathrm{card}\{K\in\mathcal{T}_h :\ \widetilde{K}\subset\delta(K)\})$. Since the grid $\mathcal{T}_h$ has been assumed to be shape-regular, the constants C_4 and C_5 are uniformly bounded with respect to the mesh-size. The last factor in (22) can be estimated by $C_5^{1/2}|v|_{1,\Omega}$. This proves (20) with $C_3 := (C_5(C_1^2 + C_4C_2^2/2))^{1/2}$. We see that the same arguments can be used also for the nonconforming case since in the above arguments we have not exploited the fact that $u_h\in V_h$ is a continuous function. We have only used that $w = v - \Pi_h^{c,1} v \in \mathrm{H}_0^1(\Omega)$ and that $u_h\in\mathrm{H}^1(\Omega,\mathcal{T}_h)$ is element-wise smooth.

Step 5. Derive an *a posteriori* error estimate. In the conforming case, we obtain as a consequence of (12) and (20)

$$
|u-u_h|_{1,\Omega} \le C_6\,\eta^{(c)} \tag{25}
$$

with $C_6 := \alpha^{-1}C_3$. In the nonconforming case, we can use the splitting

$$
a_h(u - R_h u_h, v) = a_h(u-u_h, v) \ + \ a_h(u_h - R_h u_h, v)
$$

and apply the estimate (20) for the first term on the right hand side. Together with (A2) this implies

$$
a_h(u - R_h u_h, v) \le \left\{ C_3\,\eta^{(c)} \ + \ M\,|u_h - R_h u_h|_{1,h,\Omega} \right\} |v|_{1,\Omega}\,. \tag{26}
$$

If we introduce the following computable "post-processing error"

$$
\eta^{(p)} := \left\{ \sum_{K\in\mathcal{T}_h} (\eta_K^{(p)})^2 \right\}^{1/2} \qquad \text{with} \qquad \eta_K^{(p)} := |u_h - R_h u_h|_{1,K} \tag{27}
$$

we obtain from (13) and (26) the following *a posteriori* error estimates for the error of the "smoothed" conforming approximation $R_h u_h\in V_h^c\subset V$ and the nonconforming discrete solution $u_h\in V_h$.

Theorem 2.1. *Assume that the grid $\mathcal{T}_h$ is shape regular. Furthermore, suppose that the bilinear form $a_h(\cdot,\cdot)$ and the nonconforming finite element space V_h fulfill (A1, A2) and the assumption*

(**A3**) *there exists a conforming finite element space of first order $V_h^{c,1} \subset V_h \cap V$ and an interpolation operator $\Pi_h^{c,1} : V \to V_h^{c,1}$ satisfying the approximation properties (18) and (19).*

Let $u \in V$ and $u_h \in V_h$ be the solutions of the continuous and discrete problem (5) and (11), respectively. Then, for the smoothed solution $R_h u_h \in V_h^c$ in a suitable conforming finite element space $V_h^c \subset V$, it holds the a posteriori error estimate

$$|u - R_h u_h|_{1,\Omega} \le C_6\, \eta^{(c)} + \alpha^{-1} M\, \eta^{(p)}\,, \tag{28}$$

where $\eta^{(c)}$ is the error estimator from the conforming case defined by (23, 24) and $\eta^{(p)}$ the post-processing error defined by (27). For the nonconforming finite element solution $u_h \in V_h$, it holds

$$|u - u_h|_{1,h,\Omega} \le C_6\, \eta^{(c)} + (\alpha^{-1} M + 1)\, \eta^{(p)}\,. \tag{29}$$

From the two parts $\eta^{(c)}$ and $\eta^{(p)}$ we can define the following *error estimator in the nonconforming case*

$$\eta^{(n)} := \left(\sum_{K\in\mathcal{T}_h} (\eta_K^{(n)})^2 \right)^{1/2} \qquad \text{with} \qquad \eta_K^{(n)} := \left\{ (\eta_K^{(c)})^2 + (\lambda\, \eta_K^{(p)})^2 \right\}^{1/2}, \tag{30}$$

where $\lambda > 0$ is a suitable h-independent scaling factor which can equilibrate the influence of the two parts $\eta^{(c)}$ and $\eta^{(p)}$. As a consequence of Theorem 2.1, we obtain the *a posteriori* error estimate

$$|u - u_h|_{1,h,\Omega} \le C_7\, \eta^{(n)} \tag{31}$$

with $C_7 := (C_6^2 + ((\alpha^{-1}M + 1)/\lambda)^2)^{1/2}$. The same *a posteriori* error bound also holds for the error $|u - R_h u_h|_{1,\Omega}$ of the "smoothed" discrete solution $R_h u_h \in V_h^c$.

3. Upper and lower *a posteriori* estimates of an enhanced error measure

Our error estimator in the nonconforming case $\eta^{(n)}$ does not only give a bound for the discretization error but also for the "representation error" of the exact solution by the conforming "target space" $V_h^c \subset V$ of the post-processing operator R_h. Therefore, we will define at first the following "enhanced error measure".

Definition 3.1. For a given subdomain $G \subset \Omega$, we define

$$\mathcal{T}_h(G) := \{K \in \mathcal{T}_h : K \subset G\}\cdot$$

We say that a subdomain $G \subset \Omega$ is composed of some elements $K \in \mathcal{T}_h$ if $\overline{G} = \bigcup_{K\in\mathcal{T}_h(G)} \overline{K}$. Let $u \in V$ and $u_h \in V_h$ be the exact and discrete solution, respectively, G a subdomain composed of some elements $K \in \mathcal{T}_h$ and $V_h^c \subset V$ a conforming finite element space based on the partition $\mathcal{T}_h$. Then the *enhanced error measure* of u_h with respect to u, G and V_h^c is defined as

$$\varepsilon(u_h, u, G, V_h^c) := |u - u_h|_{1,h,G} + \inf_{v_h^c \in V_h^c} |u - v_h^c|_{1,G}\,. \tag{32}$$

A small enhanced error measure $\varepsilon(u_h, u, G, V_h^c)$ means that on the one hand the local discretization error $|u - u_h|_{1,h,G}$ is small and on the other hand that the exact solution u on the subdomain G can be well approximated by a function u_h^c in a desired conforming finite element space V_h^c which is intended to be appropriate with respect to practical requirements. In particular, this means that in the region G the mesh-size and the polynomial order are chosen suitably.

From Theorem 2.1, (31) and the analogous estimate for $|u - R_h u_h|_{1,\Omega}$ we immediately get the following estimate of the enhanced error measure.

Theorem 3.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Then, it holds the global upper* a posteriori *estimate of the enhanced error measure*

$$\varepsilon(u_h, u, \Omega, V_h^c) \;\leq\; 2\, C_7\, \eta^{(n)}. \tag{33}$$

Now, we want to derive a local lower *a posteriori* estimate of the enhanced error measure $\varepsilon(u_h, u, \delta(K), V_h^c)$ where $\delta(K)$ denotes a local vicinity of the element $K \in \mathcal{T}_h$ which will be specified later. Let $K \in \mathcal{T}_h$ be a fixed element. Our aim is now to estimate each term of the local part $\eta_K^{(c)}$ of our estimator $\eta^{(n)}$ in terms of the discretization error and the representation error in the vicinity of K. At first, we want to estimate the term $h_K\|f + \triangle u_h\|_{0,K}$. Let the function f on K be approximated by a function f_K in a finite dimensional space of functions on K. The accuracy of f_K can be adjusted later on. We define a function $w_K \in V$ by $w_K(x) = 0$ for $x \in \Omega \setminus K$ and

$$w_K(x) := \big(f_K(x) + \triangle u_h(x)\big)\, b_K(x) \qquad \text{for } x \in K\,, \tag{34}$$

where $b_K \in \mathrm{H}_0^1(K)$ is a suitable non-negative bubble function. From Theorem 3.3 in [1] it follows that b_K can be chosen such that

$$C_8\|f_K + \triangle u_h\|_{0,K}\, \|w_K\|_{0,K} \leq (f_K + \triangle u_h, w_K)_K\,, \tag{35}$$

$$|w_K|_{1,K} \leq C_9\, h_K^{-1}\|w_K\|_{0,K}\,, \tag{36}$$

where C_8 and C_9 are positive constants independent of f_K, u_h, w_K and the mesh-size h_K. If we use the error equation (14) and the fact that $w_K = 0$ on $\Omega \setminus K$, we obtain

$$\begin{aligned}
(f_K + \triangle u_h, w_K)_K &= (f + \triangle u_h, w_K)_K \;+\; (f_K - f, w_K)_K \\
&= a_h(u - u_h, w_K) \;+\; (f_K - f, w_K)_K \\
&\leq \big\{ M C_9 h_K^{-1} |u - u_h|_{1,K} + \|f_K - f\|_{0,K} \big\} \|w_K\|_{0,K}\,.
\end{aligned}$$

Together with (35) this implies

$$h_K\, \|f + \triangle u_h\|_{0,K} \;\leq\; M C_9 C_8^{-1} |u - u_h|_{1,K} \;+\; (1 + C_8^{-1})\, h_K\, \|f_K - f\|_{0,K}\,. \tag{37}$$

The approximation f_K of f will be chosen such that, in comparison to the local discretization error $|u - u_h|_{1,K}$, the term $h_K\|f_K - f\|_{0,K}$ is of higher order small asymptotically for $h_K \to 0$.

Next we want to estimate the term $h_K^{1/2}\|\, [\nabla u_h \cdot n_E]_E \,\|_{\mathrm{L}^2(E)}$ of $\eta_K^{(c)}$ where $E \in \mathcal{E}(K) \cap \mathcal{E}_0$ is an inner $(d-1)$-dimensional face of the element $K \in \mathcal{T}_h$ and n_E and the jump $[\cdot]_E$ are defined as in Section 2 (see step 2, Eq. (15)). To E we assign the local region $\Omega_E := \mathrm{int}(\overline{K^{\mathrm{out}}(E)} \cup \overline{K^{\mathrm{in}}(E)})$ where $K^{\mathrm{out}}(E)$ and $K^{\mathrm{in}}(E)$ are (like in Sect. 2) the two uniquely determined elements with the common face E. Let us consider the function $v_E := [\nabla u_h \cdot n_E]_E$ which is a smooth function living in a finite dimensional space of functions on E. This function can be continuously extended to a smooth function $\tilde{v}_E : \Omega_E \to \mathbb{R}$. Now we define a function $w_E \in V$ by $w_E(x) = 0$ for $x \in \Omega \setminus \Omega_E$ and

$$w_E(x) := \tilde{v}_E(x)\, b_E(x) \qquad \text{for } x \in \Omega_E\,, \tag{38}$$

where $b_E \in \mathrm{H}_0^1(\Omega_E)$ is a suitable non-negative bubble function. From Theorem 3.5 in [1] it follows that b_E can be chosen such that

$$\frac{1}{2}\|v_E\|^2_{\mathrm{L}^2(E)} \le C_{10} \langle v_E, w_E \rangle_E \,, \tag{39}$$

$$\|w_E\|_{0,\Omega_E} + h_K |w_E|_{1,\Omega_E} \le C_{11}\, h_K^{1/2} \|v_E\|_{\mathrm{L}^2(E)} \,, \tag{40}$$

where C_{10} and C_{11} are positive constants independent of v_E and the mesh-size h_K. If we use the error equation (14), the fact that $w_E = 0$ on $\Omega \setminus \Omega_E$ and (40), we obtain

$$\begin{aligned}\langle v_E, w_E \rangle_E &= \sum_{\widetilde{K} \subset \Omega_E} (f + \triangle u_h, w_E)_{\widetilde{K}} - a_h(u - u_h, w_E) \\ &\le C_{11} \left\{ \left(\sum_{\widetilde{K} \subset \Omega_E} h_K^2 \|f + \triangle u_h\|^2_{0,\widetilde{K}} \right)^{1/2} + M |u - u_h|_{1,h,\Omega_E} \right\} h_K^{-1/2} \|v_E\|_{\mathrm{L}^2(E)} \,.\end{aligned}$$

For the fixed element $K \in \mathcal{T}_h$, let us define the local patch

$$\triangle(K) := \bigcup_{E \in \mathcal{E}(K) \cap \mathcal{E}_0} \; \bigcup_{\widetilde{K} \subset \Omega_E} \widetilde{K} \,, \tag{41}$$

consisting of element K and all of its face neighbours. Since the mesh $\mathcal{T}_h$ is assumed to be shape regular, there exists a positive constant C_{12} such that

$$C_{12} h_{\widetilde{K}} \le h_K \le C_{12}^{-1} h_{\widetilde{K}} \qquad \forall\, \widetilde{K} \subset \triangle(K) \,. \tag{42}$$

Applying (39) and (42) we get

$$\frac{1}{2} h_K^{1/2} \|v_E\|_{\mathrm{L}^2(E)} \le C_{10} C_{11} \left\{ C_{12} \left(\sum_{\widetilde{K} \subset \Omega_E} h_{\widetilde{K}}^2 \|f + \triangle u_h\|^2_{0,\widetilde{K}} \right)^{1/2} + M |u - u_h|_{1,h,\Omega_E} \right\} .$$

Using (37) for $K = \widetilde{K}$ implies

$$\frac{1}{2} \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}_0} h_K^{1/2} \|v_E\|_{\mathrm{L}^2(E)} \le C_{13}\, |u - u_h|_{1,h,\triangle(K)} + C_{14} \sum_{\widetilde{K} \subset \triangle(K)} h_{\widetilde{K}} \|f - f_{\widetilde{K}}\|_{0,\widetilde{K}} \tag{43}$$

with $C_{13} := C_{10} C_{11} M (C_{12} C_9 C_8^{-1} + 1)(1 + C_4)^{1/2}$ and $C_{14} := C_{10} C_{11} C_{12} (1 + C_8^{-1})(1 + C_4)$. Finally, the definition (23) of $\eta_K^{(c)}$ and (37) yield the lower estimate

$$\eta_K^{(c)} \le C_{15}\, |u - u_h|_{1,h,\triangle(K)} + C_{16} \sum_{\widetilde{K} \subset \triangle(K)} h_{\widetilde{K}} \|f - f_{\widetilde{K}}\|_{0,\widetilde{K}} \,, \tag{44}$$

where $C_{15} := C_{13} + M C_9 C_8^{-1}$ and $C_{16} := C_{14} + 1 + C_8^{-1}$.

Now we want to derive a lower estimate for the part $\eta_K^{(p)}$ of our nonconforming error estimator $\eta_K^{(n)}$. For the "post-processing operator" R_h, we need the following assumptions. Let R_h be a linear operator $R_h : V_h + V_h^c \to V_h^c$ satisfying

(**A4**) $R_h v_h = v_h \qquad \forall\, v_h \in V_h^c$, (**A5**) *for each $K \in \mathcal{T}_h$, there exists a local vicinity $\delta(K)$ consisting of elements near to K such that with a constant C_S independent of K it holds*

$$|R_h v_h|_{1,K} \le C_S\, |v_h|_{1,h,\delta(K)} \qquad \forall\, v_h \in V_h + V_h^c$$

and $\triangle(K) \subset \delta(K)$ with $\triangle(K)$ defined by (41).

An example of the operator R_h would be the interpolation operator of Scott and Zhang [24] if the space V_h would be a subspace of $\mathrm{H}_0^1(\Omega)$. However, this is not the case here. Therefore, in [23], we have proposed an operator R_h satisfying (A4) and (A5) for a general case of finite element spaces V_h and V_h^c. The only requirement that we need in [23] to show (A5) is the following "weak continuity" of the functions $v_h \in V_h$ along element faces. We assume that

$$(\mathbf{A6}) \qquad \langle\, [v_h]_E \,, 1\, \rangle_E \;=\; 0 \qquad \forall\, E \in \mathcal{E},\;\; v_h \in V_h\,,$$

where $[v_h]_E$ denotes the jump of v_h on E which is defined for inner faces $E \in \mathcal{E}_0$ by (15) and for boundary faces $E \subset \partial\Omega$ by $[v_h]_E := v_h\big|_{K(E)}$ where $K(E)$ is the uniquely determined element that contains the face E. For a boundary face $E \in \mathcal{E} \setminus \mathcal{E}_0$, the assumption (A6) represents a weak homogeneous Dirichlet boundary condition for the functions $v_h \in V_h$. With such an operator R_h, which is also cheap with respect to the computational costs, we get the following estimate for the post-processing error $\eta_K^{(p)}$.

Lemma 3.3. *Let the assumptions (A4, A5) and (A6) for the operator $R_h : V_h + V_h^c \to V_h^c$ be satisfied where $V_h^c \subset V$ is a conforming finite element space. Then, for each $K \in \mathcal{T}_h$, it holds the estimate*

$$\eta_K^{(p)} = |u_h - R_h u_h|_{1,K} \;\le\; C_{17}\; \varepsilon(u_h, u, \delta(K), V_h^c) \tag{45}$$

where $\varepsilon(u_h, u, \delta(K), V_h^c)$ is the enhanced error measure from Definition 3.1.

Proof. We choose an arbitrary function $v_h^c \in V_h^c$. Then, by the triangle inequality and (A4) we have

$$\eta_K^{(p)} \;\le\; |u_h - u|_{1,K} + |u - v_h^c|_{1,K} + |R_h(v_h^c - u_h)|_{1,K}\,.$$

From (A5) and the triangle inequality we get

$$|R_h(v_h^c - u_h)|_{1,K} \;\le\; C_S \big\{\, |v_h^c - u|_{1,h,\delta(K)} \;+\; |u_h - u|_{1,h,\delta(K)} \,\big\}.$$

Thus, using the definition of $\varepsilon(u_h, u, \delta(K), V_h^c)$, we obtain (45) with $C_{17} = 1 + C_S$. ☐

If we combine the estimate (44) and (45) we get the following local and global lower estimate for our nonconforming *a posteriori* error estimator $\eta_K^{(n)}$.

Theorem 3.4. *Let the grid $\mathcal{T}_h$ be shape regular. Furthermore, suppose that the bilinear form $a_h(\cdot,\cdot)$ and the nonconforming finite element space V_h fulfill the assumptions (A1–A3) and (A6). Let $V_h^c \subset V$ be a conforming finite element space and $R_h : V_h + V_h^c \to V_h^c$ a post-processing operator satisfying the assumptions (A4) and (A5) with the local vicinities $\triangle(K)$ and $\delta(K)$ of an element $K \in \mathcal{T}_h$. Then, it holds the lower local a posteriori estimate of the enhanced error measure*

$$\eta_K^{(n)} \;\le\; C_{18}\, \varepsilon(u_h, u, \delta(K), V_h^c) \;+\; C_{16} \sum_{\widetilde{K} \subset \triangle(K)} h_{\widetilde{K}} \|f - f_{\widetilde{K}}\|_{0,\widetilde{K}} \tag{46}$$

and the lower global estimate

$$\eta^{(n)} \;\le\; C_{19}\, \varepsilon(u_h, u, \Omega, V_h^c) \;+\; C_{20} \left\{ \sum_{K \in \mathcal{T}_h} h_K^2 \|f - f_K\|_{0,K}^2 \right\}^{1/2}. \tag{47}$$

Proof. From (44, 45) and $\triangle(K) \subset \delta(K)$ we conclude

$$\eta_K^{(n)} \le \eta_K^{(c)} + \lambda\,\eta_K^{(p)} \le (C_{15} + \lambda C_{17})\,\varepsilon(u_h, u, \delta(K), V_h^c) + C_{16} \sum_{\widetilde{K}\subset\triangle(K)} h_{\widetilde{K}} \|f - f_{\widetilde{K}}\|_{0,\widetilde{K}}\,,$$

which proves (46) with $C_{18} = C_{15} + \lambda C_{17}$. From the definition (32) of the enhanced error measure, using again the constant $C_5 = \max_{\widetilde{K}\in\mathcal{T}_h}(\mathrm{card}\{K \in \mathcal{T}_h : \widetilde{K} \subset \delta(K)\})$, we get

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \big(\varepsilon(u_h, u, \delta(K), V_h^c)\big)^2 &\le 2 \sum_{K\in\mathcal{T}_h} |u - u_h|^2_{1,h,\delta(K)} + 2 \sum_{K\in\mathcal{T}_h} \left(\inf_{v_h^c\in V_h^c} |u - v_h^c|_{1,\delta(K)}\right)^2 \\ &\le 2\,C_5 |u - u_h|^2_{1,h,\Omega} + 2\,C_5 \inf_{v_h^c\in V_h^c} |u - v_h^c|^2_{1,\Omega}\,. \end{aligned} \tag{48}$$

The local estimate (46) and the fact that $\max_{K\in\mathcal{T}_h}(\mathrm{card}\{\widetilde{K} \in \mathcal{T}_h : \widetilde{K} \subset \triangle(K)\}) = 1 + C_4$ with the above defined constant $C_4 = \max_{K\in\mathcal{T}_h}(\mathrm{card}\{E : E \in \mathcal{E}(K) \cap \mathcal{E}_0\})$, yield

$$\big(\eta_K^{(n)}\big)^2 \le 2\,C_{18}^2 \,\big(\varepsilon(u_h, u, \delta(K), V_h^c)\big)^2 + 2\,C_{16}^2(1 + C_4) \sum_{\widetilde{K}\subset\triangle(K)} h_{\widetilde{K}}^2 \|f - f_{\widetilde{K}}\|^2_{0,\widetilde{K}}\,.$$

If we take the sum over all elements $K \in \mathcal{T}_h$ and use (48) and $\max_{\widetilde{K}\in\mathcal{T}_h}(\mathrm{card}\{K \in \mathcal{T}_h : \widetilde{K} \subset \triangle(K)\}) = 1 + C_4$, we obtain

$$\big(\eta^{(n)}\big)^2 \le 4\,C_{18}^2\,C_5 \big(\varepsilon(u_h, u, \Omega, V_h^c)\big)^2 + 2\,C_{16}^2(1 + C_4)^2 \sum_{K\in\mathcal{T}_h} h_K^2 \|f - f_K\|^2_{0,K}\,,$$

which proves (47) with $C_{19} = 2\,C_{18} C_5^{1/2}$ and $C_{20} = \sqrt{2} C_{16}(1 + C_4)$. $\square$

Remark 3.5. The second terms on the right hand side of (46) and (47), respectively, are of higher order small since we can use sufficiently good approximations f_K for the data function f on element K. Therefore, the estimates (46) and (47) also hold without these second terms but with little larger constants C_{18} and C_{19} if the mesh size is sufficiently small. Similarly, the numerical integration for evaluating the term $\|f + \triangle u_h\|_{0,K}$ in $\eta_K^{(c)}$ can be done such that this causes again only additional terms in the *a posteriori* estimates which are of higher order small (see [1]).

Remark 3.6. The essential assumptions that we need to make our nonconforming error estimator work are (A3, A6) for the space V_h and (A4, A5) for the post-processing operator $R_h : V_h + V_h^c \to V_h^c$. Assumption (A6) is a minimal requirement to ensure consistency for a finite element discretization, *i.e.* it is a non-restrictive condition which is satisfied for nearly every nonconforming space V_h. Examples of nonconforming finite element spaces V_h, that satisfy assumption (A3), are, in the case of simplicial elements, the elements of Crouzeix and Raviart [9] and, in the case of quadrilateral elements, the modified Rannacher-Turek element [17]. The modification is that the nonconforming "bubble function" associated with the $\widehat{K}$-polynomial $\widehat{x}_1\widehat{x}_2$ is added to the original space $\widehat{\mathcal{P}}$ of shape functions for V_h on the reference element $\widehat{K}$. In practical computations, these bubble functions can be removed by static condensation. For the higher order two-dimensional elements of Hennart, Jaffre and Roberts [13], assumption (A3) is satisfied automatically since here we have $\mathbb{Q}_1(\widehat{K}) \subset \mathbb{P}_2(\widehat{K}) \subset \widehat{\mathcal{P}}$ which implies that the space of the conforming $\mathbb{Q}_1$-elements is contained in V_h. However, in the three-dimensional case, this argument works only for elements of order larger than two. Therefore, the elements of lower order also have to be modified by adding to $\widehat{\mathcal{P}}$ appropriate nonconforming "bubble functions" such that $\mathbb{Q}_1(\widehat{K}) \subset \widehat{\mathcal{P}}$.

Examples for the post-processing operator $R_h : V_h + V_h^c \to V_h^c$ are the regularization operator of Bernardi and Girault [5] in the two-dimensional case and the general transfer operator proposed in [23] in the two- and three-dimensional case.

4. *A POSTERIORI* ERROR ESTIMATES FOR FUNCTIONALS

In the previous sections, we have constructed and analyzed *a posteriori* error estimates for the global and local energy norm of the discretization error. However, in applications one often wants to know with a certain accuracy approximate values for some quantities of physical interest. Therefore, it is important to have also an *a posteriori* estimate for the error with respect to such a quantity. For the case of conforming finite element spaces, a lot of work on this subject has been done by Rannacher and his coworkers (see [3, 4, 18, 19]). An extension of these techniques to the case of two-dimensional nonconforming first order elements has been done in [17].

In this section, we want to present an alternative approach for nonconforming finite elements which uses again a conforming approximation $R_h u_h \in V_h^c$ of the nonconforming solution $u_h \in V_h$. Let us assume that the quantity of interest q can be computed by applying a linear functional $J \in V'$ to the solution $u \in V$ of the continuous problem, *i.e.* let $q = J(u)$. Furthermore, we assume that the numerical approximation q_h of the quantity q is computed from the discrete solution u_h by $q_h = J(u_h)$ and that $J \in V_h'$ where $\|J\|_{V_h'} \le M_J$ with an h-independent constant M_J.

In the following, we show how the "smoothed" conforming approximation $R_h u_h \in V_h^c$ of the solution $u_h \in V_h$ can also be used for the estimation of the error with respect to our quantity, *i.e.* for estimating $q - q_h = J(u) - J(u_h)$.

For a given functional $J \in V'$, let $z \in V$ be the solution of the continuous dual problem

$$a(\phi, z) = J(\phi) \qquad \forall\, \phi \in V. \tag{49}$$

Then, we can prove the following error representation.

Lemma 4.1. *Let the assumptions of Theorem* 2.1 *be satisfied. Furthermore, let $R_h u_h \in V_h^c$ be the smoothed conforming approximation of the discrete solution u_h and $\Pi_h^{c,1} z \in V_h^{c,1}$ a first order conforming interpolation of the solution $z \in V$ of the dual problem (49). Then, for the error corresponding to the functional $J(\cdot)$, we have the representation*

$$J(u) - J(u_h) = \sum_{K \in \mathcal{T}_h} \eta_K^{(c)}(u_h, z) \quad + \quad \eta^{(p)}(u_h, z) \tag{50}$$

where

$$\eta_K^{(c)}(u_h, z) := (f + \triangle u_h, z - \Pi_h^{c,1} z)_K - \frac{1}{2} \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}_0} \Big\langle [\nabla u_h \cdot n_E]_E \,,\, z - \Pi_h^{c,1} z \Big\rangle_E \,, \tag{51}$$

$$\eta^{(p)}(u_h, z) := a_h(u_h - R_h u_h, z) \,-\, J(u_h - R_h u_h). \tag{52}$$

Proof. Since $V_h^c \subset V$, we have that $e_h^{(c)} := u - R_h u_h \in V$. Taking $\varphi = e_h^{(c)}$ in (49) and using the Galerkin orthogonality (16) for $v_h = \Pi_h^{c,1} z \in V_h^{c,1} \subset V \cap V_h$, we obtain

$$\begin{aligned} J(e_h^{(c)}) &= a\left(e_h^{(c)}, z\right) \;=\; a_h\left(e_h^{(c)}, z\right) \\ &= a_h(u - u_h, z - \Pi_h^{c,1} z) \;+\; a_h(u_h - R_h u_h, z) \\ &= \sum_{K \in \mathcal{T}_h} \left(\nabla(u - u_h), \nabla(z - \Pi_h^{c,1} z)\right)_K \quad + \quad a_h(u_h - R_h u_h, z)\,. \end{aligned} \tag{53}$$

Partial integration and the fact that $z - \Pi_h^{c,1} z$ is zero on the boundary $\partial\Omega$, imply that the sum in (53) is equal to the sum of all $\eta_K^{(c)}(u_h, z)$. Thus, we get

$$J(u) - J(u_h) = J\left(e_h^{(c)}\right) - J(u_h - R_h u_h) = \sum_{K \in \mathcal{T}_h} \eta_K^{(c)}(u_h, z) + a_h(u_h - R_h u_h, z) - J(u_h - R_h u_h),$$

which proves (50). □

Note that the terms of $\eta_K^{(c)}(u_h, z)$ in (51) are the same as for the error representation in the conforming case (see [4, 19]). The only difference is the additional term $\eta^{(p)}(u_h, z)$, which depends on the computable post-processing error $u_h - R_h u_h$. It can be regarded as a consistency error in the dual problem (49) for the nonconforming test function $\phi = u_h - R_h u_h \in V_h + V$. In the conforming case, where $V_h \subset V$, we would have $u_h - R_h u_h \in V$ such that the consistency error $\eta^{(p)}(u_h, z)$ would be zero.

The problem with the error representation (50) is that the solution $z \in V$ of the dual problem (49) is not known exactly. However, this problem is not a new phenomenon of the nonconforming case, it also occurs in the conforming situation. A way to overcome this problem in both cases, the conforming and the nonconforming one, is to use an approximation $\widetilde{z}_{h'} \in \widetilde{V}_{h'}$ of z in an appropriate finite element space $\widetilde{V}_{h'} \approx V$. The notation $\widetilde{V}_{h'}$ indicates that the space $\widetilde{V}_{h'}$ can be associated with another order than V_h or a finer grid $\mathcal{T}_{h'}$ instead of $\mathcal{T}_h$. Some choices for the space $\widetilde{V}_{h'}$ and the approximation $\widetilde{z}_{h'} \approx z$ have been discussed in [4,18,19]. One possibility is to compute $\widetilde{z}_{h'} \in \widetilde{V}_{h'}$ as the solution of the *discrete dual problem*

$$a(\phi_{h'}, \widetilde{z}_{h'}) = J(\phi_{h'}) \qquad \forall\, \phi_{h'} \in \widetilde{V}_{h'}\,. \tag{54}$$

In our context we could take for $\widetilde{V}_{h'}$ also a nonconforming finite element space. Once the approximation $\widetilde{z}_{h'}$ of z has been calculated, an estimate for the error with respect to our quantity of interest is

$$J(u) - J(u_h) \approx \eta(u_h, \widetilde{z}_{h'}) \qquad \text{with} \qquad \eta(u_h, \widetilde{z}_{h'}) := \sum_{K \in \mathcal{T}_h} \eta_K^{(c)}(u_h, \widetilde{z}_{h'}) + \eta^{(p)}(u_h, \widetilde{z}_{h'})\,. \tag{55}$$

The difference to the conforming case is the computable term $\eta^{(p)}(u_h, \widetilde{z}_{h'})$ defined by (52). In [4,18] it has been noted that the term $\eta(u_h, \widetilde{z}_{h'})$, calculated by direct numerical evaluation of the error representation (50), gives the best estimate for $J(u) - J(u_h)$. If we would apply the triangle inequality to (50) in the following way

$$|J(u) - J(u_h)| \lessapprox \overline{\eta}(u_h, \widetilde{z}_{h'}) := \sum_{K \in \mathcal{T}_h} \overline{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'}) + |\eta^{(p)}(u_h, \widetilde{z}_{h'})|\,, \tag{56}$$

where $\overline{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'})$ is a computable upper bound of $|\eta_K^{(c)}(u_h, \widetilde{z}_{h'})|$, we would loose in general the asymptotic sharpness of the global error estimate. However, the upper bounds $\overline{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'})$ are needed in order to compute element-wise error indicators η_K for getting mesh refinement criteria. For details we will refer to [3, 4, 18]. Here we will only mention the differences to the conforming case. The idea is to split the term $|\eta^{(p)}(u_h, \widetilde{z}_{h'})|$ in (56) into element-wise contributions. Therefore, we decompose the global functional $J(\cdot)$ into a sum of local functionals $J_K(\cdot)$, *i.e.*

$$J(\phi) = \sum_{K \in \mathcal{T}_h} J_K(\phi) \qquad \forall\, \phi \in V + V_h\,. \tag{57}$$

The choice of the local functionals will be discussed below. Based on the splitting

$$\eta^{(p)}(u_h, \widetilde{z}_{h'}) = \sum_{K \in \mathcal{T}_h} \left\{ a_{h,K}(u_h - R_h u_h, \widetilde{z}_{h'}) - J_K(u_h - R_h u_h) \right\},$$

where $a_{h,K}(v,w) := (\nabla v, \nabla w)_K$ in the case of our model problem (1), we can use the computable term

$$\overline{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'}) \;:=\; |\, a_{h,K}(u_h - R_h u_h, \widetilde{z}_{h'}) \;-\; J_K(u_h - R_h u_h)\,| \tag{58}$$

as a part of the error indicator η_K defined by

$$\eta_K := \overline{\eta}_K^{(c)}(u_h, \widetilde{z}_{h'}) \,+\, \overline{\eta}_K^{(p)}(u_h, \widetilde{z}_{h'})\,. \tag{59}$$

Based on the computed error indicators $\{\eta_K : K \in \mathcal{T}_h\}$, there are different strategies for controlling the mesh refinement. In [4, 18] and [19], an *error balancing strategy*, a *fixed fraction strategy* and a *mesh optimization strategy* are discussed for the case of a conforming finite element discretization. The same strategies can be applied also to the case of nonconforming finite elements. The only difference concerns the choice of the local error indicators η_K as mentioned above.

Finally, let us consider an example for the splitting of the functional $J(\cdot)$. Assume that $\Omega \subset \mathbb{R}^2$ and

$$J(\phi) \;:=\; \int_\Gamma \phi \,\mathrm{d}s \qquad \forall\, \phi \in V + V_h\,,$$

where $\Gamma \subset \overline{\Omega}$ is a given contour line. Then, we can define a disjoint partition of the contour $\Gamma = \bigcup_{K \in \mathcal{T}_h} \Gamma_K$ with $\Gamma_K \subset \Gamma \cap \overline{K}$ and $\Gamma_K \cap \Gamma_{K'} = \emptyset$ for all pairs of different elements $K, K' \in \mathcal{T}_h$. Based on this partition we define the local functional for an element $K \in \mathcal{T}_h$ and an arbitrary function $\phi \in V + V_h$ as

$$J_K(\phi) \;:=\; \int_{\Gamma_K} \phi \,\mathrm{d}s\,,$$

i.e. an element K yields a non-zero contribution only if $\overline{K}$ contains a line segment of Γ with positive length.

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