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ON MONOTONE AND SCHWARZ ALTERNATING METHODS FOR NONLINEAR ELLIPTIC PDES *

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Abstract. The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains. In this paper, proofs of convergence of some Schwarz alternating methods for nonlinear elliptic problems which are known to have solutions by the monotone method (also known as the method of subsolutions and supersolutions) are given. In particular, an additive Schwarz method for scalar as well as some coupled nonlinear PDEs are shown to converge for finitely many subdomains. These results are applicable to several models in population biology.

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1. Introduction

The Schwarz alternating method was devised by H.A. Schwarz more than one hundred years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as an efficient algorithm for parallel computers. See the fundamental work of Lions in [10] and [11]. The literature on this method for the linear boundary value problem is huge, see the recent reviews of Chan and Mathew [4], Le Tallec [23], and Xu and Zou [25], and the books of Smith, Bjorstad and Gropp [19] and Quarteroni and Valli [17]. The literature for nonlinear problems is rather sparse. Besides Lions' works, see also Badea [2], Zou and Huang [26], Cai and Dryja [3], Tai [20], Pao [16], Xu [24], Dryja and Hackbusch [5], Tai and Espedal [21], Tai and Xu [22], Lui [13], Lui [14], Lui [12] and references therein. The effectiveness of Schwarz methods for nonlinear problems (especially those in fluid mechanics) has been demonstrated in many papers. See proceedings of the annual domain decomposition conferences beginning with [7].

This paper is a continuation of previous works by this author attempting to survey various classes of nonlinear elliptic PDEs for which Schwarz methods are applicable. We consider elliptic PDEs amenable to analysis by the monotone method (also known as the method of subsolutions and supersolutions). The paper of Keller and Cohen [9] was one of the first to employ such method to solve boundary value problems. Subsequent works by these two authors as well as by Sattinger [18], Amann [1], and many others have made this method into one of the important tools in nonlinear analysis. See Pao [15] for a very complete reference with many applications

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as well as a good bibliography. Lions [11] shows the convergence of a multiplicative Schwarz method for the Poisson's equation using the monotone method. Here, we prove convergence for an additive Schwarz method on finitely many subdomains for scalar as well as coupled systems of nonlinear elliptic PDEs. Our results on coupled systems can be applied to the three types of Lotka-Volterra models in population biology: competition, cooperation and predator-prey.

In Section 2, we prove convergence of two Schwarz methods for a class of scalar nonlinear elliptic PDEs. In Section 3, we treat the so-called quasi-monotone non-increasing case of a coupled system of PDEs, giving a proof of convergence of an additive Schwarz method on finitely many subdomains. The other two cases (quasi-monotone non-decreasing and mixed quasi-monotone) will be discussed in Section 4. In the final section, we conclude and suggest some open problems. In the remaining part of this introduction, we set some notations.

Let Ω be a bounded, connected domain in $\mathbb{R}^N$ with a smooth boundary. Suppose Ω is composed of $m \geq 2$ subdomains, that is, $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$. The boundary of each subdomain is also assumed to be smooth. Let $X = C^\alpha(\overline{\Omega}) \cap C^2(\Omega)$ for some $0 < \alpha \leq 1$. We shall look for solutions of PDEs lying in this space. Let λ_1 be the principal eigenvalue of $-\triangle$ on Ω with homogeneous Dirichlet boundary condition, and ϕ_1 be its corresponding (positive) eigenfunction.

2. Scalar equations

Consider the PDE

$$-\triangle u = f(x,u) \text{ on } \Omega, \qquad u = h \text{ on } \partial\Omega. \tag{2.1}$$

We shall be concerned with positive solutions only, that is $u > 0$ on Ω. A smooth function $\underline{u} \in X$ is a *subsolution* of the above PDE if

$$-\triangle \underline{u} - f(x,\underline{u}) \leq 0 \text{ on } \Omega \quad \text{and} \quad \underline{u} \leq h \text{ on } \partial\Omega.$$

Similarly, a *supersolution* satisfies the above with both inequalities reversed.

Let us now record the assumptions for the above PDE. Suppose that it has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ which satisfy $0 < \underline{u} \leq \overline{u}$ on Ω. Define the sector of smooth functions

$$\mathcal{A} \equiv \{u \in X,\ \underline{u} \leq u \leq \overline{u} \text{ on } \overline{\Omega}\}.$$

Assume f is a smooth (Holder continuous) function defined on $\overline{\Omega} \times \mathcal{A}$ and h is a smooth non-negative function defined on the boundary. In addition, suppose there exists some bounded non-negative function c defined on Ω so that

$$-c(x)(u-v) \leq f(x,u) - f(x,v), \qquad x \in \Omega, \quad v \leq u \in \mathcal{A}. \tag{2.2}$$

Finally, assume that

$$F(x,u) - F(x,v) < 0, \qquad x \in \Omega, \quad v < u \in \mathcal{A} \tag{2.3}$$

where

$$F(x,u) \equiv \frac{f(x,u)}{u}.$$

With these assumptions, it is known (Sect. 3.3 in [15]) that the PDE has a unique solution in the sector $\mathcal{A}$.

With additional assumptions on the nonlinearity, we can specify a subsolution and a supersolution. If $h \equiv 0$, then if we assume some positive number M so that $f(x,M) \leq 0$ and that $F(x,0) > \lambda_1$ for every $x \in \Omega$, then it is easy to see that candidates for a positive subsolution and supersolution are $\underline{u} = \epsilon\phi_1$, where ϵ is a sufficiently small positive real number, and $\overline{u} = M$. In case $h \geq 0$ and not identically zero, then assume $F(x,0) > 0$ for every $x \in \Omega$. Let

$$-\triangle \underline{u} = 0 \text{ on } \Omega, \qquad \underline{u} = \epsilon h \text{ on } \partial\Omega,$$

where ϵ is a sufficiently small positive number. It is not difficult to show that $\underline{u}$ is a positive subsolution and $\overline{u} = M$ is a supersolution where $M \geq \sup_{\partial\Omega} h$ and $f(x, M) \leq 0$. As an example, consider

$$-\triangle u = u(a-u) \text{ on } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where $a > \lambda_1$. Then this PDE has a unique positive solution. For a non-negative, non-zero boundary condition, the PDE has a positive solution for all positive a. Such PDEs have applications in population biology, chemical kinetics, etc.

We begin with a comparison lemma.

Lemma 2.1. *Suppose $-\triangle u \geq f(x,u)$ and $-\triangle v \leq f(x,v)$ on Ω with both $u, v \in X$ and positive in Ω and $u \geq v \geq 0$ on $\partial\Omega$. Then $u \geq v$ on Ω.*

Proof. Let $S = \{x \in \Omega,\ u(x) < v(x)\}$. Assume that S is nonempty. By continuity of the functions, S has positive measure. Now on $\partial S,\ u = v$ and

$$\frac{\partial u}{\partial n} \geq \frac{\partial v}{\partial n} \ \text{ almost everywhere } \ \text{(a.e.)},$$

where n denotes the outward unit normal on ∂S. Multiply the inequality $-\triangle u \geq f(x,u)$ by v and multiply the inequality $-\triangle v \leq f(x,v)$ by u and then subtract the two resulting inequalities followed by an integration over S to obtain

$$\begin{array}{rcl} \displaystyle\int_{\partial S} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) & \leq & \displaystyle\int_S f(x,v)u - f(x,u)v \\ 0 \ \leq \ \displaystyle\int_{\partial S} \left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} \right) u & \leq & \displaystyle\int_S \big(F(x,v) - F(x,u)\big)uv \ < 0 \end{array}$$

which is a contradiction. Hence $S = \emptyset$ or $u \geq v$ on Ω. This completes the proof. ☐

We now show convergence of a (multiplicative) Schwarz sequence for the PDE (2.1) for the two subdomain case. Let u denote the unique solution in the sector $\mathcal{A}$. For convenience, we suppress the dependence of f on $x \in \Omega$.

Theorem 2.1. *Let $u^{(0)} = \underline{u}$ on Ω with $\underline{u} = h$ on $\partial\Omega$. Define the Schwarz sequence by ($n \geq 0$)*

$$\begin{array}{rclcrcl} -\triangle u^{(n+\frac{1}{2})} & = & f(u^{(n+\frac{1}{2})}) \ \textit{on } \Omega_1, & \qquad & u^{(n+\frac{1}{2})} & = & u^{(n)} \ \textit{on } \partial\Omega_1, \\ -\triangle u^{(n+1)} & = & f(u^{(n+1)}) \ \textit{on } \Omega_2, & \qquad & u^{(n+1)} & = & u^{(n+\frac{1}{2})} \ \textit{on } \partial\Omega_2. \end{array}$$

Then $u^{(n+\frac{i}{2})} \to u$ in $C^2(\overline{\Omega}_i),\ i = 1, 2$.

Proof. The proof can be divided into four steps. We first show that each element of the Schwarz sequence is well defined in $\mathcal{A}$. Second, we demonstrate that the sequences obey the monotone properties

$$u^{(n-\frac{1}{2})} \leq u^{(n+\frac{1}{2})} \text{ on } \Omega_1 \ \text{ and } \ u^{(n)} \leq u^{(n+1)} \text{ on } \Omega_2.$$

Since the sequences are bounded above, the following limits are well defined

$$\lim_{n\to\infty} u^{(n+\frac{1}{2})} = u_1 \text{ on } \Omega_1, \qquad \lim_{n\to\infty} u^{(n)} = u_2 \text{ on } \Omega_2.$$

In the third step, we prove that the function u_i satisfies the PDE on Ω_i. Finally, we prove that $u_i = u$.

The details of each step are now given. First, for the problem

$$-\triangle u^{(\frac{1}{2})} = f(u^{(\frac{1}{2})}) \text{ on } \Omega_1, \qquad u^{(\frac{1}{2})} = u^{(0)} \text{ on } \partial\Omega_1,$$

note that $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution (on Ω_1) and thus $u^{(\frac{1}{2})}$ exists and is the unique solution in $\mathcal{A}$. The existence and uniqueness of $u^{(n+\frac{1}{2})}$ follows by the induction hypothesis ($\underline{u} \le u^{(n)} \le \overline{u}$ on $\partial\Omega_1$) and a similar argument.

The second step (proving monotonicity) is also shown by induction. The base step is trivial if we define $u^{(-\frac{1}{2})} = u^{(0)}$. In the induction step, we prove $u^{(n+\frac{1}{2})} \ge u^{(n-\frac{1}{2})}$ on Ω_1 (and simultaneously $u^{(n+1)} \ge u^{(n)}$ on Ω_2) using the comparison lemma, noting that $u^{(n+\frac{1}{2})} = u^{(n)} \ge u^{(n-1)} = u^{(n-\frac{1}{2})}$ on $\partial\Omega_1$ by the induction hypothesis.

Next, we show that the limit u_i satisfies the PDE on Ω_i using the usual elliptic regularity argument. It is sufficient to carry this out for $i = 1$. Since $f(u^{(n+\frac{1}{2})}) \to f(u_1)$ (pointwise convergence), $f(u^{(n+\frac{1}{2})})$ is uniformly bounded in $L^p(\Omega_1)$, $p \ge 1$. Since $-\triangle u^{(n+\frac{1}{2})} = f(u^{(n+\frac{1}{2})})$ on Ω_1, the uniform boundedness of $u^{(n+\frac{1}{2})}$ in $W^{2,p}(\Omega_1)$ follows. Taking $p > N$, $\alpha \equiv 1 - N/p > 0$, we obtain the uniform boundedness of $u^{(n+\frac{1}{2})}$ in $C^{1+\alpha}(\overline{\Omega}_1)$ by embedding theory. This implies that $f(u^{(n+\frac{1}{2})})$ is uniformly bounded in $C^{\alpha}(\overline{\Omega}_1)$. From the defining PDE for $u^{(n+\frac{1}{2})}$, we conclude that $u^{(n+\frac{1}{2})}$ is uniformly bounded in $C^{2+\alpha}(\overline{\Omega}_1)$. By the Arzela-Ascoli theorem, a subsequence of $u^{(n+\frac{1}{2})}$ converges in $C^2(\overline{\Omega}_1)$. Since $\{u^{(n+\frac{1}{2})}\}$ is monotone converging pointwise to u_1, we must have $u^{(n+\frac{1}{2})} \to u_1$ in $C^2(\overline{\Omega}_1)$.

Finally, we show that $u_1 = u_2$ on $\Omega_1 \cap \Omega_2$ so that $u_1 = u_2$ on Ω and both are equal to the unique solution of the PDE. On $\Omega_1 \cap \Omega_2$, $-\triangle u_i = f(u_i)$, $i = 1, 2$. From the boundary conditions, $u_1 = u_2$ on $\partial\Omega_1 \cap \Omega$ and $u_2 = u_1$ on $\partial\Omega_2 \cap \Omega$ and thus u_i satisfy the same PDE and boundary conditions on $\Omega_1 \cap \Omega_2$ meaning that $u_1 = u_2$ there. This completes the proof. ☐

The above Schwarz iteration is a direct generalization of the classical Schwarz iteration for the Poisson's equation. The next Schwarz method is called an additive Schwarz method. It generalizes the additive method for linear PDEs first introduced by Dryja and Widlund [6]. It is sometimes preferable to the (multiplicative) Schwarz method above because the subdomain PDEs are independent and hence can be solved in parallel. We consider the general m-subdomain case.

Theorem 2.2. *Let $u^{(0)} = \underline{u}$ on Ω with $\underline{u} = h$ on $\partial\Omega$. Define the additive Schwarz sequence by ($n \ge 1$)*

$$-\triangle u_i^{(n)} = f(u_i^{(n)}) \text{ on } \Omega_i, \qquad u_i^{(n)} = u^{(n-1)} \text{ on } \partial\Omega_i, \quad i = 1, \cdots, m.$$

Here, $u_i^{(n)}$ is defined as $u^{(n-1)}$ on $\overline{\Omega} \setminus \overline{\Omega}_i$ and

$$u^{(n)} = (1 - m\omega)u^{(n-1)} + \omega \sum_{i=1}^{m} u_i^{(n)} \text{ on } \overline{\Omega}.$$

The relaxation parameter ω satisfies $0 < \omega < 1/m$. Then $u_i^{(n)} \to u$ in $C^2(\overline{\Omega}_i)$, $i = 1, \cdots, m$.

Proof. It is sufficient to just outline the proof as the details are quite similar to the last proof. The existence and monotone properties for $u_i^{(n)}$ and $u^{(n)}$ in $\mathcal{A}$

$$u_i^{(n)} \le u_i^{(n+1)} \text{ on } \Omega_i, \qquad u^{(n)} \le u^{(n+1)} \text{ on } \Omega$$

are shown by induction. Note that $u_i^{(n)} \le u_i^{(n+1)}$ on Ω_i can be shown using the comparison lemma as before and on Ω,

$$u^{(n+1)} = (1-m\omega)u^{(n)} + \omega \sum_{i=1}^{m} u_i^{(n+1)} \ge (1-m\omega)u^{(n-1)} + \omega \sum_{i=1}^{m} u_i^{(n)} = u^{(n)}.$$

Also,

$$u^{(n+1)} \le (1-m\omega)\overline{u} + \omega \sum_{i=1}^{m} \overline{u} = \overline{u}.$$

Next, we define, for $i = 1, \cdots, m$,

$$\lim_{n\to\infty} u_i^{(n)} = u_i \text{ on } \Omega_i, \qquad \lim_{n\to\infty} u^{(n)} = u_0 \text{ on } \Omega$$

and show using elliptic regularity theory that the limit u_i satisfies the same PDE on $\Omega_i,\ i = 1, \cdots, m$. Finally, we prove that $u_i = u$ on Ω_i in detail below.

By induction and the comparison lemma, it is not difficult to show that $u_i^{(n)} \le u$ on $\Omega_i,\ i = 1, \cdots, m$. Taking the limit, we obtain $u_i \le u$ on Ω_i. Define

$$w = \min_{1\le i\le m} u_i \text{ on } \overline{\Omega}.$$

We have

$$w \le u_i \le u \text{ on } \Omega_i. \tag{2.4}$$

Since every u_i is continuous on Ω, so is w.

Now Ω can be partitioned into a number of regions R_i where $-\triangle w = f(w)$ inside each region but w may not be smooth across the regions. Denote the common boundary of these regions by $\Gamma_{ij} = \partial R_i \cap \partial R_j$ and say $w = u_{k(i)}$ in R_i, for some $1 \le k(i) \le m$. Let n_i denote the unit outward normal on ∂R_i. Note that along $\Gamma_{ij},\ n_i = -n_j$ and

$$\frac{\partial u_{k(i)}}{\partial n_i} \ge \frac{\partial u_{k(j)}}{\partial n_i} \quad \text{a.e.}$$

We now show that

$$-\triangle w - f(w) \ge 0 \text{ on } \Omega \tag{2.5}$$

in the weak sense. That is, for any non-negative test function ϕ with support in Ω,

$$\int_\Omega \nabla w \cdot \nabla \phi \ge \int_\Omega f(w)\phi$$

where w is considered as an $H^1(\Omega)$ function. Now

$$\begin{aligned}
\int_\Omega \nabla w \cdot \nabla \phi &= \sum_i \int_{R_i} \nabla u_{k(i)} \cdot \nabla \phi \\
&= -\sum_i \int_{R_i} \triangle u_{k(i)}\ \phi + \sum_i \int_{\partial R_i} \frac{\partial u_{k(i)}}{\partial n_i}\, \phi \\
&= \sum_i \int_{R_i} f(u_{k(i)})\ \phi + \sum_{i \neq j} \int_{\Gamma_{ij}} \left(\frac{\partial u_{k(i)}}{\partial n_i} + \frac{\partial u_{k(j)}}{\partial n_j} \right) \phi \\
&= \int_\Omega f(w)\ \phi + \sum_{i \neq j} \int_{\Gamma_{ij}} \left(\frac{\partial u_{k(i)}}{\partial n_i} - \frac{\partial u_{k(j)}}{\partial n_i} \right) \phi \\
&\geq \int_\Omega f(w)\ \phi.
\end{aligned}$$

Since the comparison lemma still holds for continuous piecewise smooth supersolution (to be shown below), we have $w \geq u$ on Ω (noting that $w = u$ on $\partial\Omega$). In light of (2.4), $u = u_i$ on Ω_i.

Finally, we prove an extension of the comparison lemma for the piecewise smooth supersolution w. Recall that $w = u$ on $\partial\Omega$ and we try to show that $w \geq u$ on Ω. Let $S = \{x \in \Omega,\ w(x) < u(x)\}$. Assume S is nonempty. By continuity, S has positive measure. On ∂S, $u = w$ and

$$\frac{\partial w}{\partial n} \geq \frac{\partial u}{\partial n} \quad \text{a.e.},$$

where n is the unit outward normal on ∂S. Since $w \in H^1(\Omega)$, we have from (2.5),

$$-\int_{\partial S} \frac{\partial w}{\partial n} u + \int_S \nabla w \cdot \nabla u \geq \int_S f(w) u.$$

Since $-\triangle u = f(u)$, we have

$$-\int_{\partial S} \frac{\partial u}{\partial n} w + \int_S \nabla w \cdot \nabla u = \int_S f(u) w.$$

Subtract these two results to obtain

$$0 \geq \int_{\partial S} \left(\frac{\partial u}{\partial n} - \frac{\partial w}{\partial n} \right) u \geq \int_S \big(F(w) - F(u)\big) uw > 0$$

which is a contradiction. Hence $S = \emptyset$ or $w \geq u$ on Ω. This completes the proof. □

We remark that in practice, the subdomains can be colored so that subdomains with a nonempty intersection are assigned different colors. All subdomains having the same color can be relabeled as one new subdomain. Suppose k is the minimum number of colors needed to color the subdomains. For m large, k can be much smaller than m. This way, the constraint on ω becomes $0 < \omega < 1/k$ which may be much less restrictive.

In the next two sections, we consider some coupled systems of nonlinear elliptic PDEs and their solution by an additive Schwarz method.

3. Quasi-monotone non-increasing coupled systems

Consider the system

$$-\triangle u = f(u,v), \qquad -\triangle v = g(u,v) \qquad \text{on } \Omega, \tag{3.1}$$

$$u = r, \qquad v = s \qquad \text{on } \partial\Omega.$$

The pairs of smooth functions $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$-\triangle\underline{u} - f(\underline{u},\overline{v}) \le 0 \le -\triangle\overline{u} - f(\overline{u},\underline{v}) \text{ on } \Omega,\ -\triangle\underline{v} - g(\overline{u},\underline{v}) \le 0 \le -\triangle\overline{v} - g(\underline{u},\overline{v}) \text{ on } \Omega,$$

and

$$\underline{u} \le r \le \overline{u}, \qquad \underline{v} \le s \le \overline{v} \quad \text{on } \partial\Omega.$$

Furthermore, they are said to be *ordered* if

$$\underline{u} \le \overline{u}, \qquad \underline{v} \le \overline{v} \quad \text{on } \overline{\Omega}.$$

We seek positive solutions, that is, both $u, v > 0$ on Ω and thus r, s are required to be smooth non-negative functions defined on the boundary.

Define the sector

$$\mathcal{A} \equiv \left\{ \begin{bmatrix} u \\ v \end{bmatrix},\ u,v \in X,\ \underline{u} \le u \le \overline{u},\ \underline{v} \le v \le \overline{v} \text{ on } \overline{\Omega} \right\}.$$

Suppose $f, g \in C^1(\mathcal{A})$. Our system of PDEs is called *quasi-monotone non-increasing* if

$$\frac{\partial F}{\partial v},\ \frac{\partial G}{\partial u} \le 0 \text{ on } \mathcal{A}, \qquad \text{where } F \equiv \frac{f}{u},\ G \equiv \frac{g}{v}.$$

Note that the definition of subsolution and supersolution depends on the assumptions on the nonlinearities. Later on, this definition changes for a different set of assumptions.

Suppose our system of PDEs is quasi-monotone non-increasing. Then it can be shown (Sect. 8.4 in [15]) that it has a solution in $\mathcal{A}$. Without further assumptions, it may have more than one solution. Despite this, the following additive Schwarz sequence converges for an appropriately chosen initial guess. Note that the subdomain problems at each iteration are independent and are decoupled.

Theorem 3.1. *Suppose the system (3.1) is quasi-monotone non-increasing and let $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ be positive ordered subsolution and supersolution pairs. Suppose*

$$\frac{\partial F}{\partial u},\ \frac{\partial G}{\partial v} < 0 \text{ on } \mathcal{A}.$$

Let

$$u^{(0)} = \underline{u} \text{ and } v^{(0)} = \overline{v} \text{ on } \Omega \text{ with } \underline{u} = r \text{ and } \overline{v} = s \text{ on } \partial\Omega. \tag{3.2}$$

Define the Schwarz sequence for $i = 1, \cdots, m$ and $n \ge 1$

$$\begin{aligned} -\triangle u_i^{(n)} &= f(u_i^{(n)}, v^{(n-1)}) \text{ on } \Omega_i, & u_i^{(n)} &= u^{(n-1)} \text{ on } \partial\Omega_i \\ -\triangle v_i^{(n)} &= g(u^{(n-1)}, v_i^{(n)}) \text{ on } \Omega_i, & v_i^{(n)} &= v^{(n-1)} \text{ on } \partial\Omega_i. \end{aligned}$$

Here, $u_i^{(n)}$ is defined as $u^{(n-1)}$ on $\overline{\Omega} \setminus \overline{\Omega}_i$ and

$$u^{(n)} = (1 - m\omega)u^{(n-1)} + \omega \sum_{i=1}^{m} u_i^{(n)} \text{ on } \overline{\Omega}$$

and $v_i^{(n)}$ and $v^{(n)}$ are similarly defined. The relaxation parameter ω satisfies $0 < \omega < 1/m$. Then $u_i^{(n)} \to \underline{u}_0$ and $v_i^{(n)} \to \overline{v}_0$ in $C^2(\overline{\Omega}_i)$, $i = 1, \cdots, m$, where $(\underline{u}_0, \overline{v}_0)$ is a positive solution of (3.1) in $\mathcal{A}$. If (u, v) is any solution in the sector $\mathcal{A}$, then $\underline{u}_0 \le u$ and $v \le \overline{v}_0$.

If $u^{(0)} = \overline{u}$ and $v^{(0)} = \underline{v}$ on Ω with $\overline{u} = r$ and $\underline{v} = s$ on $\partial\Omega$ replace the assumption (3.2), then the above Schwarz sequence satisfies $u_i^{(n)} \to \overline{u}_0$ and $v_i^{(n)} \to \underline{v}_0$ in $C^2(\overline{\Omega}_i)$, $i = 1, \cdots, m$, where $(\overline{u}_0, \underline{v}_0)$ is also a positive solution of (3.1) in $\mathcal{A}$. If (u, v) is any solution in the sector $\mathcal{A}$, then $u \le \overline{u}_0$ and $v \ge \underline{v}_0$.

Proof. We only consider the case where $u^{(0)} = \underline{u}$ and $v^{(0)} = \overline{v}$. The proof can be divided into four steps. We first show that each element of the Schwarz sequence is well defined in $\mathcal{A}$ and the sequences obey the monotone properties

$$u_i^{(n)} \le u_i^{(n+1)}, \qquad v_i^{(n+1)} \le v_i^{(n)} \quad \text{on } \Omega_i, \quad i = 1, \cdots, m$$

and

$$\underline{u} \le u^{(n)} \le u^{(n+1)} \le \overline{u}, \qquad \underline{v} \le v^{(n+1)} \le v^{(n)} \le \overline{v} \quad \text{on } \Omega.$$

Since the sequences are bounded, the following limits are well defined

$$\lim_{n\to\infty} u_i^{(n)} = \underline{u}_i, \qquad \lim_{n\to\infty} v_i^{(n)} = \overline{v}_i \qquad \text{on } \Omega_i, \quad i = 1, \cdots, m$$

and

$$\lim_{n\to\infty} u^{(n)} = \underline{u}_0, \qquad \lim_{n\to\infty} v^{(n)} = \overline{v}_0 \quad \text{on } \Omega.$$

In the second step, we prove that the limit functions satisfy the following PDEs on Ω_i:

$$-\triangle \underline{u}_i = f(\underline{u}_i, \overline{v}_0), \qquad -\triangle \overline{v}_i = g(\underline{u}_0, \overline{v}_i), \quad i = 1, \cdots, m. \tag{3.3}$$

Third, we show that $\underline{u}_i = \underline{u}_0$ and $\overline{v}_i = \overline{v}_0$ on Ω_i, $i = 1, \cdots, m$ so that $(\underline{u}_0, \overline{v}_0)$ is a solution of (3.1). Finally, we prove that any solution (u, v) of (3.1) in $\mathcal{A}$ must satisfy

$$\underline{u}_i \le u \text{ and } v \le \overline{v}_i \text{ on } \Omega_i. \tag{3.4}$$

The details of each step are now given. First, the existence and monotone properties of the sequences are shown together by induction. Note that for the problem

$$-\triangle u_i^{(1)} = f(u_i^{(1)}, v^{(0)}) \text{ on } \Omega_i, \qquad u_i^{(1)} = u^{(0)} \text{ on } \partial\Omega_i, \quad i = 1, \cdots, m$$

$u^{(0)}$ is a subsolution while $\overline{u}$ is a supersolution since

$$-\triangle \overline{u} - f(\overline{u}, v^{(0)}) \ge -\triangle \overline{u} - f(\overline{u}, \underline{v}) \ge 0 \text{ on } \Omega_i$$

and of course $u^{(0)} = \underline{u} \le u^{(0)} \le \overline{u}$ on $\partial\Omega_i$. According to the theory for the scalar equation (2.1), $u_i^{(1)}$ is the unique solution in $\mathcal{A}$. (Recall that $f \in C^1(\mathcal{A})$ and thus (2.2) holds.)

For the problem

$$-\triangle u_i^{(n+1)} = f(u_i^{(n+1)}, v^{(n)}) \text{ on } \Omega_i, \qquad u_i^{(n+1)} = u^{(n)} \text{ on } \partial\Omega_i, \tag{3.5}$$

note

$$-\triangle \underline{u} - f(\underline{u}, v^{(n)}) \le -\triangle \underline{u} - f(\underline{u}, \overline{v}) \le 0 \text{ on } \Omega_i$$

and

$$-\triangle \overline{u} - f(\overline{u}, v^{(n)}) \ge -\triangle \overline{u} - f(\overline{u}, \underline{v}) \ge 0 \text{ on } \Omega_i.$$

By the induction hypothesis, $\underline{u} \le u^{(n)} \le \overline{u}$ on $\partial\Omega_i$ and so $\underline{u}$ is a subsolution while $\overline{u}$ is a supersolution for the problem. Consequently, $u_i^{(n+1)}$ exists and is the unique solution in $\mathcal{A}$.

Now we prove $u_i^{(n)} \le u_i^{(n+1)}$ and $v_i^{(n+1)} \le v_i^{(n)}$ on Ω_i. By the induction hypothesis, $v^{(n-1)} \ge v^{(n)}$ on Ω. Thus

$$-\triangle u_i^{(n)} = f(u_i^{(n)}, v^{(n-1)}) \le f(u_i^{(n)}, v^{(n)}) \text{ on } \Omega_i.$$

Noting that $u_i^{(n)} = u^{(n-1)} \le u^{(n)} = u_i^{(n+1)}$ on $\partial\Omega_i$ and comparing with (3.5), we have $u_i^{(n)} \le u_i^{(n+1)}$ on Ω_i. The inequality $v_i^{(n+1)} \le v_i^{(n)}$ on Ω_i can be shown similarly.

Now we prove $u^{(n)} \le u^{(n+1)} \le \overline{u}$ on Ω:

$$\begin{aligned} u^{(n+1)} &= (1 - m\omega)u^{(n)} + \omega \sum_{i=1}^{m} u_i^{(n+1)} \\ &\ge (1 - m\omega)u^{(n-1)} + \omega \sum_{i=1}^{m} u_i^{(n)} \;=\; u^{(n)} \end{aligned}$$

and

$$u^{(n+1)} \le (1 - m\omega)\overline{u} + \omega \sum_{i=1}^{m} \overline{u} = \overline{u}$$

by the induction hypothesis. Similarly, we prove $\underline{v} \le v^{(n+1)} \le v^{(n)}$ on Ω.

Next, we show (3.3). Using the usual elliptic regularity argument as before, we show that $u_i^{(n)}$ and $v_i^{(n)}$ are bounded uniformly in $C^{1+\alpha}(\overline{\Omega}_i)$. Now, we claim that $v^{(n)}$ is bounded uniformly in $C^{1+\alpha}(\overline{\Omega})$. If not, then from the definition of $v^{(n+1)}$,

$$\begin{aligned} \frac{\|v^{(n+1)}\|}{\|v^{(n)}\|} &\le 1 - m\omega + \omega \frac{\sum_{i=1}^{m} \|v_i^{(n+1)}\|}{\|v^{(n)}\|} \\ &\le 1 - m\omega + o(1) \;<\; 1 \end{aligned}$$

for all n sufficiently large, contradicting the assumption that $\|v^{(n)}\|$ is unbounded. In the above, $\|\cdot\|$ denotes the $C^{1+\alpha}(\overline{\Omega})$ norm. Since $v^{(n)}$ is bounded uniformly in $C^{1+\alpha}(\overline{\Omega})$, it follows that $f(u_i^{(n)}, v^{(n-1)})$ is uniformly bounded in $C^{\alpha}(\overline{\Omega}_i)$. From the PDE defining $u_i^{(n)}$, we obtain the uniform boundedness of $u_i^{(n)}$ in $C^{2+\alpha}(\overline{\Omega}_i)$. An application of the theorem of Arzela-Ascoli yields the convergence $u_i^{(n)} \to \underline{u}_i$ in $C^2(\overline{\Omega}_i)$. In a similar manner, we obtain $v_i^{(n)} \to \overline{v}_i$ in $C^2(\overline{\Omega}_i)$. These two facts together prove (3.3).

Next, we show $\underline{u}_i = \underline{u}_0$ and $\overline{v}_i = \overline{v}_0$ on $\Omega_i,\ i = 1, \cdots, m$. Let

$$w = \min_{1 \le i \le m} \underline{u}_i \quad \text{and} \quad z = \max_{1 \le i \le m} \underline{u}_i \quad \text{on } \overline{\Omega}. \tag{3.6}$$

We have

$$w \le \underline{u}_i \le z \text{ on } \Omega_i. \tag{3.7}$$

As before, we show that w is a piecewise smooth supersolution, $-\triangle w \ge f(w, \overline{v}_0)$ on Ω while z is a piecewise smooth subsolution, $-\triangle z \le f(z, \overline{v}_0)$. Since $w = z$ on $\partial\Omega$, comparison implies that $z \le w$ on $\overline{\Omega}$. In light of (3.7), we obtain

$$z = w = \underline{u}_i = \underline{u}_0 \text{ on } \Omega_i. \tag{3.8}$$

In a similar manner, we can show $\overline{v}_i = \overline{v}_0$ on Ω_i.

Finally, we show that any solution (u, v) of (3.1) in $\mathcal{A}$ must satisfy (3.4). This can be accomplished by essentially repeating the arguments in step one of this proof. Alternatively, observe that $(\underline{u}, v)$ and $(u, \overline{v})$ form subsolution and supersolution pairs. Apply the result of step one for these two pairs to establish (3.4). This completes the proof. □

Three generalizations are immediate. The Laplacian operator can be replaced by a self-adjoint, positive definite, linear second-order differential operator. Smoothness requirement of the nonlinearities can be relaxed. Also f and g can depend on $x \in \Omega$ as well. In the latter case, all properties of the nonlinearities mentioned above must hold for all $x \in \Omega$.

One example where a quasi-monotone non-increasing system occurs is the Lotka-Volterra competition model

$$-\triangle u = u(a_1 - b_1 u - c_1 v), \qquad -\triangle v = v(a_2 - b_2 u - c_2 v).$$

Here u, v stand for the population of two species competing for the same food sources and/or territories and all other parameters are positive constants. See [8].

4. Other coupled systems

In this section, we also consider positive solutions of the system (3.1) with two other sets of assumptions on the nonlinearities. For the first of these, the pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$-\triangle \underline{u} - f(\underline{u}, \underline{v}) \le 0 \le -\triangle \overline{u} - f(\overline{u}, \overline{v}) \text{ on } \Omega,$$

$$-\triangle \underline{v} - g(\underline{u}, \underline{v}) \le 0 \le -\triangle \overline{v} - g(\overline{u}, \overline{v}) \text{ on } \Omega,$$

and

$$\underline{u} \le r \le \overline{u}, \qquad \underline{v} \le s \le \overline{v} \quad \text{on } \partial\Omega.$$

Assuming that the subsolution-supersolution pairs are ordered, our system of PDEs is called *quasi-monotone non-decreasing* if

$$\frac{\partial F}{\partial v},\ \frac{\partial G}{\partial u} \ge 0 \text{ on } \mathcal{A},$$

where $\mathcal{A}$ is defined as above.

Suppose our system of PDEs is quasi-monotone non-decreasing. Then it can be shown (Sect. 8.4 in [15]) that it has a solution in $\mathcal{A}$. Without further assumptions, it may have more than one solution.

Theorem 4.1. *Suppose the system (3.1) is quasi-monotone non-decreasing and let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be positive ordered subsolution and supersolution pairs. Suppose*

$$\frac{\partial F}{\partial u}, \ \frac{\partial G}{\partial v} < 0 \ \text{on } \mathcal{A}.$$

Let

$$u^{(0)} = \underline{u} \text{ and } v^{(0)} = \underline{v} \text{ on } \Omega \text{ with } \underline{u} = r \text{ and } \underline{v} = s \text{ on } \partial\Omega. \tag{4.1}$$

Define the Schwarz sequence, for $i = 1, \cdots, m$ and $n \geq 1$

$$\begin{aligned} -\triangle u_i^{(n)} &= f(u_i^{(n)}, v^{(n-1)}) \text{ on } \Omega_i, & u_i^{(n)} &= u^{(n-1)} \text{ on } \partial\Omega_i \\ -\triangle v_i^{(n)} &= g(u^{(n-1)}, v_i^{(n)}) \text{ on } \Omega_i, & v_i^{(n)} &= v^{(n-1)} \text{ on } \partial\Omega_i. \end{aligned}$$

Here, $u_i^{(n)}$ is defined as $u^{(n-1)}$ on $\overline{\Omega} \setminus \overline{\Omega}_i$ and

$$u^{(n)} = (1 - m\omega)u^{(n-1)} + \omega \sum_{i=1}^{m} u_i^{(n)} \text{ on } \overline{\Omega}$$

and $v_i^{(n)}$ and $v^{(n)}$ are similarly defined. The relaxation parameter ω satisfies $0 < \omega < 1/m$. Then $u_i^{(n)} \to \underline{u}_0$ and $v_i^{(n)} \to \underline{v}_0$ in $C^2(\overline{\Omega}_i)$, $i = 1, \cdots, m$, where $(\underline{u}_0, \underline{v}_0)$ is a positive solution of (3.1) in $\mathcal{A}$. If (u, v) is any solution in the sector $\mathcal{A}$, then $\underline{u}_0 \leq u$ and $\underline{v}_0 \leq v$.

If $u^{(0)} = \overline{u}$ and $v^{(0)} = \overline{v}$ on Ω with $\overline{u} = r$ and $\overline{v} = s$ on $\partial\Omega$ replace the assumption (4.1), then the above Schwarz sequence satisfies $u_i^{(n)} \to \overline{u}_0$ and $v_i^{(n)} \to \overline{v}_0$ in $C^2(\overline{\Omega}_i)$, $i = 1, \cdots, m$, where $(\overline{u}_0, \overline{v}_0)$ is also a positive solution of (3.1) in $\mathcal{A}$. If (u, v) is any solution in the sector $\mathcal{A}$, then $u \leq \overline{u}_0$ and $v \leq \overline{v}_0$.

Proof. The proof is similar to the previous one and thus we only give a sketch. Suppose $u^{(0)} = \underline{u}$ and $v^{(0)} = \underline{v}$. The first step is to show that each element of the Schwarz sequence is well defined in $\mathcal{A}$ and that the sequences obey the monotone properties

$$u_i^{(n)} \leq u_i^{(n+1)}, \qquad v_i^{(n)} \leq v_i^{(n+1)} \quad \text{on } \Omega_i, \quad i = 1, \cdots, m$$

and

$$\underline{u} \leq u^{(n)} \leq u^{(n+1)} \leq \overline{u}, \qquad \underline{v} \leq v^{(n)} \leq v^{(n+1)} \leq \overline{v} \quad \text{on } \Omega.$$

Since the sequences are bounded above, the following limits are well defined

$$\lim_{n\to\infty} u_i^{(n)} = \underline{u}_i, \qquad \lim_{n\to\infty} v_i^{(n)} = \underline{v}_i \quad \text{on } \Omega_i, \qquad i = 1, \cdots, m$$

and

$$\lim_{n\to\infty} u^{(n)} = \underline{u}_0, \qquad \lim_{n\to\infty} v^{(n)} = \underline{v}_0 \quad \text{on } \Omega.$$

In the second step, we prove using elliptic regularity theory, that the limit functions satisfy the following PDEs on Ω_i:

$$-\triangle \underline{u}_i = f(\underline{u}_i, \underline{v}_0), \qquad -\triangle \underline{v}_i = g(\underline{u}_0, \underline{v}_i), \quad i = 1, \cdots, m.$$

In the third step, we prove that $\underline{u}_i = \underline{u}_0$ and $\underline{v}_i = \underline{v}_0$ on Ω_i by following the chain of arguments similar to the one starting at (3.6). This demonstrates that $(\underline{u}_0, \underline{v}_0)$ is a solution of (3.1). Finally, we show that any solution (u, v) of (3.1) must satisfy $\underline{u}_0 \leq u$ and $\underline{v}_0 \leq v$ on Ω. This completes the sketch of the proof. $\square$

One example where a quasi-monotone non-decreasing system occurs is the Lotka-Volterra cooperating model

$$-\triangle u = u(a_1 - b_1 u + c_1 v), \qquad -\triangle v = v(a_2 + b_2 u - c_2 v).$$

Here u, v stand for the population of two species which have a symbiotic relationship and all other parameters are positive constants.

Finally, we consider positive solutions for the third class of coupled systems. The pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$\begin{aligned} -\triangle \underline{u} - f(\underline{u}, \overline{v}) &\leq 0 \leq -\triangle \overline{u} - f(\overline{u}, \underline{v}) \text{ on } \Omega, \\ -\triangle \underline{v} - g(\underline{u}, \underline{v}) &\leq 0 \leq -\triangle \overline{v} - g(\overline{u}, \overline{v}) \text{ on } \Omega, \end{aligned}$$

and

$$\underline{u} \leq r \leq \overline{u}, \qquad \underline{v} \leq s \leq \overline{v} \quad \text{ on } \partial\Omega.$$

In case the subsolution-supersolution pairs are ordered, our system of PDEs is called *mixed quasi-monotone* if

$$\frac{\partial F}{\partial v} \leq 0 \text{ and } \frac{\partial G}{\partial u} \geq 0 \text{ on } \mathcal{A}.$$

Suppose our system of PDEs is mixed quasi-monotone and either

$$\frac{\partial f}{\partial u} > \lambda_1 \text{ or } \frac{\partial g}{\partial v} > \lambda_1 \text{ on } \mathcal{A} \tag{4.2}$$

holds. Then it can be shown (Sect. 8.5 in [15]) that it has a solution in $\mathcal{A}$. Without further assumptions, it may have more than one solution.

Theorem 4.2. *Suppose the system (3.1) is mixed quasi-monotone and let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be positive ordered subsolution and supersolution pairs. Suppose*

$$\frac{\partial F}{\partial u}, \frac{\partial G}{\partial v} < 0 \textit{ on } \mathcal{A}$$

and (4.2) holds. Let $\underline{u}^{(0)} = \underline{u}$, $\overline{u}^{(0)} = \overline{u}$, $\underline{v}^{(0)} = \underline{v}$, and $\overline{v}^{(0)} = \overline{v}$ on Ω with $\underline{u} = \overline{u} = r$ and $\underline{v} = \overline{v} = s$ on $\partial\Omega$. Define the Schwarz sequences, for $i = 1, \cdots, m$ and $n \geq 1$

$$\begin{aligned} -\triangle \underline{u}_i^{(n)} &= f(\underline{u}_i^{(n)}, \overline{v}^{(n-1)}) \textit{ on } \Omega_i, \qquad \underline{u}_i^{(n)} = \underline{u}^{(n-1)} \textit{ on } \partial\Omega_i, \\ -\triangle \overline{u}_i^{(n)} &= f(\overline{u}_i^{(n)}, \underline{v}^{(n-1)}) \textit{ on } \Omega_i, \qquad \overline{u}_i^{(n)} = \overline{u}^{(n-1)} \textit{ on } \partial\Omega_i, \\ -\triangle \underline{v}_i^{(n)} &= g(\underline{u}^{(n-1)}, \underline{v}_i^{(n)}) \textit{ on } \Omega_i, \qquad \underline{v}_i^{(n)} = \underline{v}^{(n-1)} \textit{ on } \partial\Omega_i, \\ -\triangle \overline{v}_i^{(n)} &= g(\overline{u}^{(n-1)}, \overline{v}_i^{(n)}) \textit{ on } \Omega_i, \qquad \overline{v}_i^{(n)} = \overline{v}^{(n-1)} \textit{ on } \partial\Omega_i. \end{aligned}$$

Here, $\underline{u}_i^{(n)}$ is defined as $\underline{u}^{(n-1)}$ on $\overline{\Omega}\setminus\overline{\Omega}_i$ and

$$\underline{u}^{(n)} = (1-m\omega)\underline{u}^{(n-1)} + \omega\sum_{i=1}^{m}\underline{u}_i^{(n)} \ \text{ on } \overline{\Omega}$$

and $\overline{u}_i^{(n)}$, etc. are similarly defined. The relaxation parameter ω satisfies $0<\omega<1/m$. Then $\underline{u}_i^{(n)}\to\underline{u}_0$, $\overline{u}_i^{(n)}\to\overline{u}_0$, $\underline{v}_i^{(n)}\to\underline{v}_0$, and $\overline{v}_i^{(n)}\to\overline{v}_0$ in $C^2(\overline{\Omega}_i)$, $i=1,\cdots,m$, where $(\underline{u}_0,\underline{v}_0)$ and $(\overline{u}_0,\overline{v}_0)$ are positive solutions of (3.1) in $\mathcal{A}$.

Furthermore, if (u,v) is any solution in the sector $\mathcal{A}$, then $\underline{u}_0\le u\le\overline{u}_0$ and $\underline{v}_0\le v\le\overline{v}_0$.

Proof. The current case is slightly more complicated than the previous two cases because the four pairs of Schwarz iterates are not completely independent. However, the ideas and techniques of the proof are essentially the same. Hence we only give a sketch of the proof.

We first show that each element of the Schwarz sequences is well defined in $\mathcal{A}$ and that the sequences obey the monotone properties

$$\underline{u}_i^{(n)}\le\underline{u}_i^{(n+1)}\le\overline{u}_i^{(n+1)}\le\overline{u}_i^{(n)},\qquad \underline{v}_i^{(n)}\le\underline{v}_i^{(n+1)}\le\overline{v}_i^{(n+1)}\le\overline{v}_i^{(n)}$$

on $\Omega_i,\ i=1,\cdots,m$ and on Ω,

$$\underline{u}^{(n)}\le\underline{u}^{(n+1)}\le\overline{u}^{(n+1)}\le\overline{u}^{(n)},\qquad \underline{v}^{(n)}\le\underline{v}^{(n+1)}\le\overline{v}^{(n+1)}\le\overline{v}^{(n)}.$$

Since the sequences are bounded, the following limits are well defined on $\Omega_i,\ i=1,\cdots,m$

$$\lim_{n\to\infty}\underline{u}_i^{(n)}=\underline{u}_i,\quad \lim_{n\to\infty}\overline{u}_i^{(n)}=\overline{u}_i,\quad \lim_{n\to\infty}\underline{v}_i^{(n)}=\underline{v}_i,\quad \lim_{n\to\infty}\overline{v}_i^{(n)}=\overline{v}_i$$

and on Ω,

$$\lim_{n\to\infty}\underline{u}^{(n)}=\underline{u}_0,\quad \lim_{n\to\infty}\overline{u}^{(n)}=\overline{u}_0,\quad \lim_{n\to\infty}\underline{v}^{(n)}=\underline{v}_0,\quad \lim_{n\to\infty}\overline{v}^{(n)}=\overline{v}_0.$$

In the second step, we prove using elliptic regularity theory that the limit functions satisfy the following PDEs on $\Omega_i,\ i=1,\cdots,m$:

$$-\triangle\underline{u}_i=f(\underline{u}_i,\overline{v}_0),\qquad -\triangle\overline{u}_i=f(\overline{u}_i,\underline{v}_0),$$

and

$$-\triangle\underline{v}_i=g(\underline{u}_0,\underline{v}_i),\qquad -\triangle\overline{v}_i=g(\overline{u}_0,\overline{v}_i).$$

After this, we prove that $\underline{u}_i=\underline{u}_0,\ \overline{u}_i=\overline{u}_0,\ \underline{v}_i=\underline{v}_0$ and $\overline{v}_i=\overline{v}_0$ on $\Omega_i,\ i=1,\cdots,m$ by following the chain of arguments similar to the one starting at (3.6). Then we demonstrate that $(\underline{u}_0,\underline{v}_0)$ and $(\overline{u}_0,\overline{v}_0)$ are solutions of (3.1). Finally, we show that any solution (u,v) in $\mathcal{A}$ of (3.1) must satisfy $\underline{u}_0\le u\le\overline{u}_0$ and $\underline{v}_0\le v\le\overline{v}_0$ on Ω.

We provide the details for the penultimate step for the case $f_u>\lambda_1$ on $\mathcal{A}$. Multiply each of the equations

$$-\triangle\overline{u}_0=f(\overline{u}_0,\underline{v}_0),\qquad -\triangle\underline{u}_0=f(\underline{u}_0,\overline{v}_0)$$

by ϕ_1 and then subtract them. This is followed by an integration over Ω to obtain

$$\int_{\partial\Omega} \frac{\partial \phi_1}{\partial n}(\overline{u}_0 - \underline{u}_0) + \lambda_1 \int_\Omega \phi_1(\overline{u}_0 - \underline{u}_0) = \int_\Omega \left[\frac{\partial f(*)}{\partial u}(\overline{u}_0 - \underline{u}_0) + \frac{\partial f(*)}{\partial v}(\underline{v}_0 - \overline{v}_0)\right]\phi_1$$

or

$$0 \geq \int_\Omega \left(\frac{\partial f(*)}{\partial u} - \lambda_1\right)(\overline{u}_0 - \underline{u}_0)\phi_1$$

which is possible only if $\underline{u}_0 \equiv \overline{u}_0$ on Ω. (Recall by monotonicity that $\overline{u}_0 \geq \underline{u}_0$ on $\overline{\Omega}$.) Here, $*$ denotes an argument between $(\overline{u}_0, \underline{v}_0)$ and $(\underline{u}_0, \overline{v}_0)$ given by Taylor's theorem. Hence on Ω,

$$-\triangle \underline{u}_0 = f(\underline{u}_0, \underline{v}_0), \qquad -\triangle \underline{v}_0 = g(\underline{u}_0, \underline{v}_0),$$

$$-\triangle \underline{u}_0 = f(\underline{u}_0, \overline{v}_0), \qquad -\triangle \overline{v}_0 = g(\underline{u}_0, \overline{v}_0),$$

which means that $(\underline{u}_0, \underline{v}_0)$ and $(\underline{u}_0, \overline{v}_0)$ are solutions to (3.1). This completes the sketch of the proof. □

Note that a sufficient condition for a unique solution to the mixed quasi-monotone system (3.1) in $\mathcal{A}$ is that both conditions in (4.2) hold.

One example where a mixed quasi-monotone system occurs is the Lotka-Volterra predator-prey model

$$-\triangle u = u(a_1 - b_1 u - c_1 v), \qquad -\triangle v = v(a_2 + b_2 u - c_2 v).$$

Here u stands for the population of a prey while v denotes the population of a predator and all other parameters are positive constants.

5. Conclusion

In this paper, we have shown convergence of some Schwarz methods for nonlinear PDEs whose solutions can be demonstrated by the monotone method. Our results include a parallel additive Schwarz method for a domain which is decomposed into finitely many subdomains. Both scalar and coupled systems can be handled. For the latter, subdomain problems are both independent and decoupled in each iteration. Even in the presence of multiple solutions, the Schwarz iteration converges to a specified solution.

Some future work include obtaining a rate of convergence of the Schwarz sequence. For the linear equation, it is known that the Schwarz sequence converges geometrically in L^∞. It is unclear if geometric convergence holds for the nonlinear elliptic PDEs considered here. Another research direction is to consider Schwarz methods for elliptic PDEs (even fully nonlinear ones) which are amenable to analysis by the viscosity solution method of Crandall and Lions. This method generalizes the monotone method in that solutions only need to be uniformly continuous and are not required to be smooth.

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Note added in proof. Recently we have successfully analyzed improved versions of these Schwartz methods, where the subdomain problems are now linear. As a result, monotonicity assumptions such as (2.3) are no longer necessary and thus we have enlarged the class of PDEs for which we can apply these Schwarz methods. These new results will appear in a forthcoming paper.

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