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Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type

$\mathcal{N}$UMDAM

GLOBAL EXISTENCE AND ONE-DIMENSIONAL NONLINEAR STABILITY OF SHEARING MOTIONS OF VISCOELASTIC FLUIDS OF OLDROYD TYPE (*)

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Résumé. — Dans cet article nous étudions l'écoulement de cisaillement et l'écoulement de Poiseuille d'un fluide de type Oldroyd (ou Johnson-Segalman) avec temps de retard. Nous montrons que le mouvement existe pour tout temps, et pour des données initiales quelconques. Nous examinons la stabilité (au sens de Lyapunov) de l'écoulement stationnaire perturbé par des perturbations d'amplitude finie.

Abstract. — In this paper we discuss shearing motions and Poiseuille flows of Oldroyd (Johnson-Segalman) fluids with retardation time. We show that the motion exists for arbitrary time and arbitrary initial data. We investigate the (Lyapunov) stability of the basic steady flow to one-dimensional finite amplitude perturbations.

1. INTRODUCTION

This paper is concerned with one dimensional motions of a class of viscoelastic fluids of Oldroyd type [1], [9], [11], i.e. satisfying the constitutive law

$$\underset{=}{\tau} + \lambda_1 \frac{\mathscr{D}_a \underset{=}{\tau}}{\mathscr{D}t} = 2\,\eta\left(\underset{=}{D}[\underset{\sim}{u}] + \mu\,\frac{\mathscr{D}_a\,\underset{=}{D}[\underset{\sim}{u}]}{\mathscr{D}t}\right) \qquad (1.1)$$

where $\underline{\tau}$ is the extra-stress tensor, $\underline{D}[\underline{u}]$ is the symmetric part of the velocity gradient, η is the elastic viscosity, λ_1 is the relaxation time, and μ is the retardation time, $0 \leqslant \mu < \lambda_1$. $\dfrac{\mathscr{D}_a}{\mathscr{D}t}$ is an invariant (frame indifferent) time derivative

$$\frac{\mathscr{D}_a \underline{\underline{\tau}}}{\mathscr{D}t} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + (\underline{u} \cdot \nabla) \underline{\underline{\tau}} + \underline{\underline{\tau}}\underline{\underline{W}} - \underline{\underline{W}}\underline{\underline{\tau}} - a(\underline{\underline{D}}\underline{\underline{\tau}} + \underline{\underline{\tau}}\underline{\underline{D}}) , \qquad (1.2)$$

where $\underline{\underline{W}}$ is the skew-symmetric part of the velocity gradient and $-1 \leqslant a \leqslant 1$. The case $a = \pm 1$, 0 corresponds to the Maxwell models (with retardation time), also called Jeffreys models.

Equation (1.1) is coupled with the following equations, given by the balance of momentum and the incompressibility,

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \right) + \nabla p = \nabla \cdot \underline{\underline{\tau}} + \underline{f} , \qquad (1.3)$$

$$\mathrm{div}\, \underline{u} = 0 ; \qquad (1.4)$$

$\underline{f}$ is some given body force.

We shall assume that the retardation time μ is different from zero and decompose $\underline{\underline{\tau}}$ into a viscous stress plus an elastic stress $\underline{\underline{\tau}} = \underline{\underline{\tau}}_s + \underline{\underline{\tau}}_p$, where

$$\underline{\underline{\tau}}_s = 2\, \eta\, \frac{\mu}{\lambda_1}\, \underline{\underline{D}} \underset{\mathrm{def}}{=} 2\, \eta_s\, \underline{\underline{D}} ,$$

$$\underline{\underline{\tau}}_p + \lambda_1 \frac{\mathscr{D}_a}{\mathscr{D}t} \underline{\underline{\tau}}_p = 2\, \eta\, (1 - \mu/\lambda_1)\, \underline{\underline{D}} \underset{\mathrm{def}}{=} 2\, \eta_p\, \underline{\underline{D}} .$$

Let us denote $\underline{\underline{\tau}}_p$ by $\underline{\underline{\tau}}$. Then equations (1.1), (1.3), (1.4) reduce to

$$\begin{cases} \rho \left(\dfrac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \right) + \nabla p = \eta_s\, \Delta \underline{u} + \nabla \cdot \underline{\underline{\tau}} + \underline{f} , \\[2mm] \mathrm{div}\, \underline{u} = 0 , \\[2mm] \underline{\underline{\tau}} + \lambda_1 \dfrac{\mathscr{D}_a \underline{\underline{\tau}}}{\mathscr{D}t} = 2\, \eta_p\, \underline{\underline{D}} . \end{cases} \qquad (1.5)$$

Another reduction is obtained by using dimensionless variables, and introducing the Weissenberg number $\mathrm{We} = \lambda_1 U/L$ and the Reynolds number $\mathrm{Re} = \rho UL/\eta$. ($U$ and L represent a typical velocity and a typical length of the flow.) Namely we set

$$\underline{x} = \frac{x^*}{L} , \underline{u} = \frac{u^*}{U} , t = t^* \frac{U}{L} , \underline{\underline{\tau}} = \frac{\underline{\underline{\tau}}^* L}{\eta U} , p = \frac{p^* L}{\eta U},$$

where stars are attached to dimensional variables. The dimensionless constitutive and momentum equations are

$$\begin{cases} \underline{\underline{\tau}} + \text{We} \, \dfrac{\mathscr{D}_a \, \underline{\underline{\tau}}}{\mathscr{D}t} = 2 \, \omega \underline{\underline{D}} \, , \\[2mm] \text{Re} \left(\dfrac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \, \underline{u} \right) + \nabla p = (1 - \omega) \, \Delta \underline{u} + \nabla \cdot \underline{\underline{\tau}} + \underline{f} \, , \\[2mm] \text{div} \, \underline{u} = 0 \, , \end{cases} \qquad (1.6)$$

where the retardation parameter ω is defined by

$$\omega = \eta_p/\eta = 1 - \mu/\lambda_1 \, , \quad 0 < \omega < 1 \, . \qquad (1.7)$$

The " total " stress is thus given by $\underline{\underline{\tau}} + 2 (1 - \omega) \, \underline{\underline{D}}$.

We shall now restrict our study to specific one-dimensional motions. For plane shear flow (plane Couette flow) between two parallel planes, we take

$$\underline{u}(\underline{x}, t) = (0, v(x, t)) \, , \quad \underline{\underline{\tau}}(\underline{x}, t) = \begin{pmatrix} \sigma(x, t) & \tau(x, t) \\ \tau(x, t) & \gamma(x, t) \end{pmatrix} \, ,$$

where $\underline{x} = (x, y)$ denotes a point of $\mathbb{R}^2$.

The Weissenberg and Reynolds numbers are defined *via* the velocity U of the upper plane and the distance L between the planes.

We easily deduce from (1.6) the equations of motion

$$\begin{cases} \text{Re} \, v_t = (1 - \omega) \, v_{xx} + \tau_x \\[2mm] \sigma_t + \dfrac{\sigma}{\text{We}} = (1 + a) \, \tau v_x \\[2mm] \gamma_t + \dfrac{\gamma}{\text{We}} = - (1 - a) \, \tau v_x \\[2mm] \tau_t + \dfrac{\tau}{\text{We}} = \dfrac{\omega}{\text{We}} \, v_x + \left(\dfrac{1 + a}{2} \, \gamma - \dfrac{1 - a}{2} \, \sigma \right) v_x \, . \end{cases} \qquad (1.8)$$

System (1.8) holds for $x \in (0, 1) = I$, $t \in \mathbb{R}_+$ and is supplemented by the boundary conditions

$$v(t, 0) = 0 \, , \quad v(t, 1) = 1 \, , \quad \forall t \in \mathbb{R}_+ \, , \qquad (1.9)$$

and the initial conditions

$$v(0, x) = v_0(x) \, , \quad \underline{\underline{\tau}}(0, x) = \begin{pmatrix} \sigma_0(x) & \tau_0(x) \\ \tau_0(x) & \gamma_0(x) \end{pmatrix} \, . \qquad (1.10)$$

System (1.8) can be further reduced to a system of only three equations.

This is obvious for $a = \pm 1$, where the equations for σ and γ can be solved at once. When $a \neq \pm 1$, we define the following combinations of σ and γ

$$\alpha = \frac{1-a}{2}\sigma - \frac{1+a}{2}\gamma , \quad \beta = (1-a)\sigma + (1+a)\gamma ; \qquad (1.11)$$

and system (1.8) takes the form

$$
\begin{cases}
\text{Re } v_t - (1-\omega) v_{xx} = \tau_x \\[2mm]
\beta_t + \dfrac{\beta}{\text{We}} = 0 \\[2mm]
\alpha_t + \dfrac{\alpha}{\text{We}} = (1-a^2)\tau v_x \\[2mm]
\tau_t + \dfrac{\tau}{\text{We}} = \left(\dfrac{\omega}{\text{We}} - \alpha \right) v_x .
\end{cases}
\qquad (1.12)
$$

Equation (1.12) is decoupled and trivial, so we are left with a system of three equations for the three unknowns v, α, τ.

For plane Poiseuille flow we obtain in a similar fashion the system ($t \geqslant 0$ and $x \in (-1, 1) = I$),

$$
\begin{cases}
\text{Re } v_t - (1-\omega) v_{xx} = \tau_x - f , \\[2mm]
\alpha_t + \dfrac{\alpha}{\text{We}} = (1-a^2)\tau v_x , \\[2mm]
\tau_t + \dfrac{\tau}{\text{We}} = \left(\dfrac{\omega}{\text{We}} - \alpha \right) v_x ,
\end{cases}
\qquad (1.13)
$$

where f is the (constant) pressure gradient in the flow direction. The boundary conditions are

$$v(t, -1) = v(t, 1) = 0 , \quad t \geqslant 0 . \qquad (1.14)$$

In what follows, we shall consider only the genuinely nonlinear case where $|a| < 1$.

Let us now describe the content of the paper. In section 2 we review some facts about steady motions. In particular we show that for a suitable range of the parameters, the basic steady Poiseuille flow fails to be C^1 at two points. In section 3, we prove global existence for solutions of the systems (1.12), (1.9), (1.10) and the systems (1.13), (1.14), (1.10), for arbitrary time and arbitrary data. Moreover we show that the stresses are uniformly bounded in space and time. (This is also true for the corresponding nonlinear hyperbolic systems obtained with a zero retardation time.) In section 4 we use energy methods to study the nonlinear (Lyapunov) stability of the basic steady solution of the plane Couette problem, for one dimensional

perturbations. In section 5 we relate these results to linear stability analysis by studying the spectral properties of the linearized operator. In the last section we briefly discuss extensions of some of our results to models with several relaxation times and to time dependent shearings.

Results related to those of section 2 have been obtained by Kolkka *et al.* [7], [8]. We thank Professor B. Plohr who kindly brought these works to our attention (*cf.* also J. Yerushalmi *et al.* [14]).

We would like to thank J. M. Ghidaglia and C. Jouron for fruitful discussions.

2. STEADY SOLUTIONS

We review here some facts about steady solutions for the aforementioned flows. In this section, ε denotes the constant $1 - \omega$, $0 < \varepsilon < 1$.

2.1. Steady solutions for the plane Couette flow

The basic steady solution for the plane Couette flow is given by

$$\begin{cases} v(x) = x\,, & \forall x \in (0, 1)\,, \\ \tau = \text{Cst.}, & \alpha = \text{Cst.} \end{cases} \tag{2.1}$$

which, by (1.13), implies that the total shear stress has the value

$$\tilde{\tau} = \tau + \varepsilon v_x = \frac{1 - \varepsilon}{1 + \text{We}^2(1 - a^2)} + \varepsilon\,,$$

or

$$\tilde{\tau} = \frac{1 + \varepsilon \, \text{We}^2(1 - a^2)}{1 + \text{We}^2(1 - a^2)}\,, \tag{2.2}_1$$

while

$$\alpha = \frac{1 - \varepsilon - \tau}{\text{We}} = \frac{(1 - \varepsilon) \, \text{We}\,(1 - a^2)}{1 + \text{We}^2(1 - a^2)}\,. \tag{2.2}_2$$

We make explicit the dependence of the dimensional shear stress on the shear rate $\gamma = U/L$. Setting $k = \lambda_1 / (1 - a^2)^{1/2}$, one obtains

$$\tilde{\tau}^*(\gamma) = \eta \gamma \, \frac{1 + \varepsilon k^2 \gamma^2}{1 + k^2 \gamma^2}\,. \tag{2.3}$$

It is readily seen from (2.3) that, if $1/9 < \varepsilon < 1$, then $\tilde{\tau}^*$ is strictly increasing in γ, while if $0 < \varepsilon < 1/9$ then $\{(\gamma, \tilde{\tau}^*(\gamma)), \gamma > 0\}$ is a S-shaped curve, shown in figure 2.1.
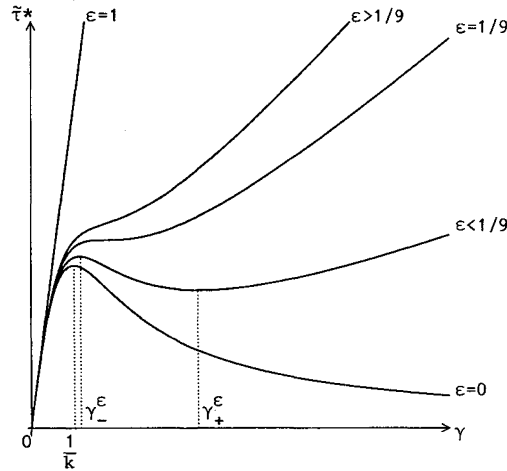
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Figure 2.1. — Curve $\{(\gamma, \tilde{\tau}^*(\gamma)), \gamma > 0\}$ for different values of ε, $0 \leqslant \varepsilon \leqslant 1$.

The real numbers γ_-^ε and γ_+^ε are the two positive roots of the equation

$$\varepsilon k^4 \gamma^4 - (1 - 3\varepsilon) k^2 \gamma^2 + 1 = 0 \; ; \tag{2.4}$$

thus

$$\gamma_\pm^\varepsilon = \frac{1}{k} \left(\frac{1 - 3\varepsilon \pm (1 - \varepsilon)^{1/2}(1 - 9\varepsilon)^{1/2}}{2\varepsilon} \right)^{1/2} . \tag{2.5}$$

Notice that $\gamma_\pm^\varepsilon > 1/k$, for all ε's, $0 < \varepsilon < 1/9$.

2.2. Steady solutions for plane Poiseuille flow

Here f denotes a (positive) constant pressure gradient driving the flow. From now on we set $k^2 = \mathrm{We}^2(1 - a^2)$. Steady solutions for the plane Poiseuille flow are given by

$$\tau = \frac{(1 - \varepsilon) v_x}{1 + k^2 v_x^2}, \quad \alpha = \frac{1 - \varepsilon}{\mathrm{We}} \frac{k v_x^2}{1 + k^2 v_x^2}, \tag{2.6}$$

where v_x is a solution of an algebraic equation of degree 3,

$$(1 + \varepsilon k^2 v_x^2) v_x = f x (1 + k^2 v_x^2) . \tag{2.7}$$

It is instructive to study first the limiting case $\varepsilon = 0$: there exists a critical $f_c = 1/(2k)$ such that, if $f > f_c$, equation (2.7) is not solvable if

$|x| < 1$ is large enough, while for $f \leqslant f_c$, equation (2.7) admits a unique solution in $\mathscr{C}^\infty(-1, 1)$ given by

$$v_x = \frac{2\,fx}{1 + \sqrt{1 - 4\,f^2\,k^2\,x^2}}\,, \qquad -1 \leqslant x \leqslant 1 . \tag{2.8}$$

The profiles of the steady Poiseuille flow obtained for $\varepsilon = 0$ and for $\varepsilon = 1$ are shown in figure 2.2.
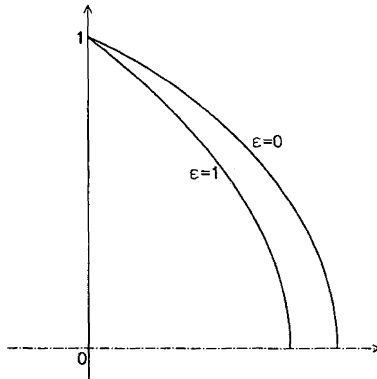

Figure 2.2. — Profiles on $(0, 1)$ of the velocity for steady Poiseuille flows for $f \leqslant f_c$:

(a) Case $\varepsilon = 1$, $v(x) = \dfrac{f}{2}\,(x^2 - 1)$, $x \in (0, 1)$;

(b) Case $\varepsilon = 0$, $v(x) = \displaystyle\int_1^x \frac{2\,fs}{1 + (1 - 4\,f^2\,k^2\,s^2)^{1/2}}\,ds$, $x \in (0, 1)$.

For $\varepsilon > 0$, equation (2.7) coincides with equation (2.3), with the following change of notation :

$$\gamma \to v_x\,, \quad \eta \to 1/f\,, \quad \tilde{\tau}^*(\gamma) \to x\,.$$

Clearly, for $\varepsilon \geqslant 1/9$ equation (2.7) admits a unique solution v_x continuous on $(-1, 1)$, while for $0 < \varepsilon < 1/9$ equation (2.7) admits one or three solutions depending on the magnitude of $f\,|x|$, $x \in (-1, 1)$.

Stationary solutions of plane Poiseuille flow are described by the following result.

PROPOSITION 2.1 :

(a) Let ε be in $[1/9, 1]$. There exists a unique $\mathscr{C}^1$ stationary Poiseuille flow $v = v^\varepsilon(x)$, $x \in (-1, 1)$. This solution is $\mathscr{C}^\infty$ on $(-1, 1)$, except possibly for $\varepsilon = 1/9$.

(b) Let ε be in $(0, 1/9)$. There exists some critical $f_c^\varepsilon > 0$ such that :

(i) *if* $f \leqslant f_c^\varepsilon$, *then there exists a unique* $\mathscr{C}^1$ *stationary Poiseuille flow (which is* $\mathscr{C}^\infty$ *in* $(-1, 1)$ *)* ;

(ii) *if* $f > f_c^\varepsilon$, *there does not exist a* $\mathscr{C}^1$ *steady flow, but there exists a continuum of* $\mathscr{C}^0$ *stationary flows which are* $\mathscr{C}^\infty$ *except at two points* x^ε *and* $-x^\varepsilon$ *in* $(-1, 1)$.

Proof : Equation (2.7) is solved by taking the inverse of the function shown in figure 2.1 (after having made the aforementioned change of notation).

(*a*) For $\varepsilon = 1$, (2.7) gives the Newtonian parabolic steady Poiseuille flow. For $1 > \varepsilon \geqslant 1/9$, equation (2.7) admits a unique continuous solution $v_x^\varepsilon(x)$, $x \in (-1, 1)$; the (unique) steady Poiseuille flow is then obtained by integration of v_x, and is $\mathscr{C}^\infty$ on $(-1, 1)$, except possibly for $\varepsilon = 1/9$, where v_{xx} can be infinite at two points of $(-1, 1)$.

(*b*) For $0 < \varepsilon < 1/9$, the curve in figure 2.1 is S-shaped. Let f_c^ε be the maximal value of $f > 0$ for which the function $f \rightarrow v_x(1)$, which is a solution of equation (2.7), is monotonic increasing :

$$f_c^\varepsilon = \gamma_-^\varepsilon \frac{1 + \varepsilon k^2 (\gamma_-^\varepsilon)^2}{1 + k^2 (\gamma_-^\varepsilon)^2} \, , \tag{2.9}$$

where γ_-^ε is given by (2.5).

(i) If $f \leqslant f_c^\varepsilon$, then the situation is similar to the case where $\varepsilon = 0$, $f \leqslant f_c$, and to case (*a*) : there exists a unique $\mathscr{C}^1$ Poiseuille flow which is $\mathscr{C}^\infty$ on $(-1, 1)$.

(ii) Let f be large enough, namely $f > f_c^\varepsilon$. Because the solution $v_x = v_x(x, f)$ of equation (2.7) is not single-valued, there cannot be any $\mathscr{C}^1$ steady Poiseuille flow. Necessarily, such a v_x has to have a jump at the points x^ε and $-x^\varepsilon$, where $x_+^\varepsilon \leqslant x^\varepsilon \leqslant x_-^\varepsilon$, and

$$x_\pm^\varepsilon = \frac{\gamma_\pm^\varepsilon}{f} \frac{1 + \varepsilon (k\gamma_\pm^\varepsilon)^2}{1 + (k\gamma_\pm^\varepsilon)^2} \, . \tag{2.10}$$

Thus there exists a continuum of steady solutions which are $\mathscr{C}^\infty$ on $(-1, 1)$ except at the two points x^ε and $-x^\varepsilon$ (*cf. fig.* 2.3 and 2.4).
This proves the proposition.

Remark 2.2.

(*a*) When $\varepsilon \rightarrow 0$, then $(x_-^\varepsilon, \gamma_-^\varepsilon)$ goes to $\left(\dfrac{1}{2\,kf}, \dfrac{1}{k} \right)$, while $(x_+^\varepsilon, \gamma_+^\varepsilon)$ goes to $(0, +\infty)$.
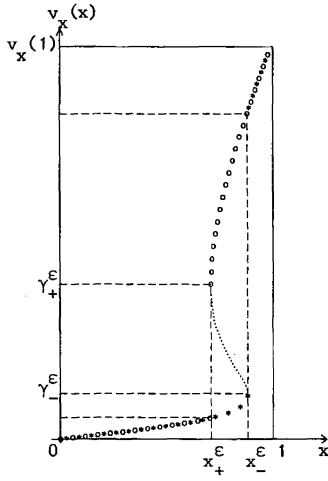
Figure 2.3. — Case $0 < \varepsilon < 1/9$, and $f > f_c^\varepsilon$.
Profiles on $(0, 1)$ of the two " extremal " discontinuous solutions v_x of equation (2.7) :

o o o o v_x is discontinuous at $x = x_-^\varepsilon$;

* * * * v_x is discontinuous at $x = x_+^\varepsilon$.

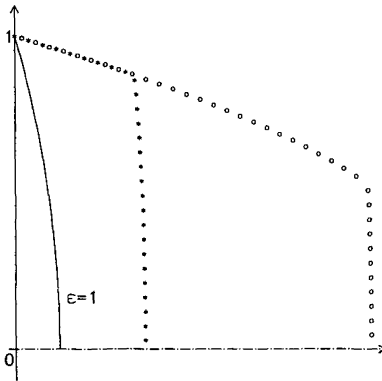Figure 2.4. — Case $0 < \varepsilon < 1/9$, and $f > f_c^\varepsilon$.
Profiles on $(0, 1)$ of the two " extremal " velocity fields solution :

o o o o v is $\mathscr{C}^\infty$ except at x_-^ε ;

* * * * v is $\mathscr{C}^\infty$ except at x_+^ε.

(b) Let us choose f such that $f > f_c$. Then, $f > f_c^\varepsilon$ if ε is small enough, so there exists a continuum of steady Poiseuille flows which are regular except at the points x^ε and $-x^\varepsilon$. As $\varepsilon \to 0$, the solution v_+^ε, which is singular at x_+^ε, converges to a function that is infinite at $x = 0$, which therefore has not been considered in the case $\varepsilon = 0$. In the other hand, as $\varepsilon \to 0$, the solution v_-^ε, which is singular at x_-^ε, converges on the interval $\left(-\dfrac{1}{2\,kf}, \dfrac{1}{2\,kf} \right)$ to the solution obtained for $\varepsilon = 0$. Moreover, on the intervals $\left\{ |x| > \dfrac{1}{2\,kf} \right\}$, the profile of the velocity v_-^ε becomes more and more parallel to the walls at $x = \pm 1$. This phenomenon can be viewed as if those viscoelastic fluids having a weak Newtonian part nearly slip along the walls (cf. W. R. Schowalter [13]).

(c) Proposition 2.1(b) (ii) gives solutions which are $\mathscr{C}^1$ except at two points. Similarly equation (2.7) can be used to produce solutions which are $\mathscr{C}^1$ except at finitely many points.

(d) This phenomenon of steady solutions which are not $\mathscr{C}^1$ appears here in a model with Newtonian viscosity, contrary to the viscoelastic model studied by J. K. Hunter and M. Slemrod [15]. We do not have a criterion for selecting the physically admissible solutions among this infinity of solutions.

3. GLOBAL EXISTENCE

In this section, we shall prove a global existence theorem for unsteady plane Poiseuille and plane Couette flows, valid for arbitrary time and data.

Before stating our results, we recall some standard notation.

If J is an interval of $\mathbb{R}_+$, $L^2(J)$ will stand for the space of measurable square integrable functions, equipped with the norm

$$|u| = \left(\int_J |u(x)|^2 \, dx \right)^{1/2} .$$

The Sobolev space $H^k(J)$ is the space of L^2 functions on J having weak derivatives up to order k in L^2, equipped with the norm

$$\|u\|_k = \left(\sum_{j=0}^{k} \int_J \left| \frac{\partial^j u}{\partial x_j}(x) \right|^2 dx \right)^{1/2} .$$

For $1 \leq p \leq +\infty$, $L^p(\mathbb{R}_+ ; H^k)$ is the space of functions u of t and x such that

$$\|u\|_{p,k} \underset{\text{def}}{=} \left(\int_{\mathbb{R}_+} \|u(.,t)\|_k^p \, dt \right)^{1/p} < \infty ,$$

(with the usual modification when $p = +\infty$).

As seen in section 1, the one-dimensional flow problems that we consider here can be reduced to a system in (v, x, τ) which takes the form

$$
\begin{cases}
\operatorname{Re} v_t + \varepsilon v_{xx} = \tau_x - f \,, \\[2mm]
\alpha_t + \dfrac{\alpha}{\mathrm{We}} = (1 - a^2)\, \tau v_x \,, \\[2mm]
\tau_t + \dfrac{\tau}{\mathrm{We}} = \left(\dfrac{1 - \varepsilon}{\mathrm{We}} - \alpha \right) v_x \,, \quad \text{a.e. in } I \times \mathbb{R} \,,
\end{cases}
\tag{3.1}
$$

$$
v(x, t) = g(x) \,, \quad \text{a.e. } x \in \partial I \,, \quad t \in \mathbb{R}_+ \,,
\tag{3.2}
$$

where f and g are given and independent of t. For plane Couette flow, we set

$$
f = 0 \,, \quad I = (0, 1) \,, \quad g(0) = 0 \,, \quad g(1) = 1 \,;
$$

for plane Poiseuille flow, we set

$$
f \in \mathbb{R}_+ \,, \quad I = (-1, 1) \,, \quad g(-1) = g(1) = 0 \,.
$$

Moreover, the plane Couette flow problem can be reduced to a system in (u, α, τ) involving homogeneous boundary conditions for u by setting $u(x, .) = v(x, .) - x$, $x \in (0, 1)$. This has the effect of introducing affine terms in the α and τ equations.

We recall the following local existence result, which is a particular case of the general results of Theorem 1 in [2].

PROPOSITION 3.1 : *Let* $0 < \varepsilon < 1$ *and* $v_0 \in H^2(I) \cap H_0^1(I)$, $\tau_0 \in H^2(I)$, $\alpha_0 \in H^2(I)$, $f \in H^1(I)$. *Then there exists a unique solution* (v, τ, α) *of* (3.1), (3.2) *such that*

$$
v \in L^2(0, T^* ; H^3(I)) \cap \mathscr{C}([0, T^*] ; H^2(I)), \quad \alpha, \tau \in \mathscr{C}([0, T^*] ; H^2(I))
$$

where $T^* > 0$ *depends only on the data.*

We now state a global existence result for the aforementioned system, together with a uniform bound on the solution. Here f is a fixed real number, and ε is strictly positive.

THEOREM 3.2 : *Let* $0 < \varepsilon < 1$.

(i) Uniqueness. *There exists at most one solution* (v, α, τ) *of* (3.1)-(3.2) *in the space* $L^\infty(\mathbb{R}_+ ; L^2) \cap L_{\mathrm{loc}}^2(\mathbb{R}_+ ; H^1) \times L^\infty(I \times \mathbb{R}_+)^2$.

(ii) Existence. *Let* $v_0 \in H_0^1(I)$, α_0, $\tau_0 \in H^1(I)$. *Then, for all* $T > 0$, *there exists a unique solution* (v, α, τ) *of* (3.1)-(3.2) *in the space* $\mathscr{C}(0, T ; H^1) \cap L^2(0, T ; H^2) \times \mathscr{C}(0, T ; H^1)^2$.

(iii) Uniform bound. *Let (v, α, τ) be the solution given in* (ii). *Then* $v \in \mathscr{C}_b(\mathbb{R}_+ \; ; L^2)$ [1]. *Moreover* $\alpha, \tau \in \mathscr{C}_b(I \times \mathbb{R}_+)$, *uniformly in* $\varepsilon \in [0, 1]$. *More precisely*

$$\left(\alpha(x, t) - \frac{1 - \varepsilon}{\mathrm{We}} \right)^2 + (1 - a^2) \tau^2(x, t) \leqslant$$

$$\leqslant \left(\left(\alpha_0(x) - \frac{1 - \varepsilon}{\mathrm{We}} \right)^2 + (1 - a^2) \tau_0^2(x) \right) e^{-t/\mathrm{We}} +$$

$$+ \left(\frac{1 - \varepsilon}{\mathrm{We}} \right)^2 (1 - e^{-t/\mathrm{We}}), \quad (x, t) \in I \times \mathbb{R}_+ . \quad (3.3)$$

Proof: We first prove the uniqueness result, then the existence result together with the uniform bound.

(i) *Uniqueness*. Let (v_1, α_1, τ_1) and (v_2, α_2, τ_2) be two solutions in the aforementioned class ; the functions $v = v_1 - v_2$, $\tau = \tau_1 - \tau_2$, $\alpha = \alpha_1 - \alpha_2$ satisfy the (linear) system,

$$\begin{cases} \mathrm{Re} \, v_t - \varepsilon v_{xx} = \tau_x \, , \\[2mm] \alpha_t + \dfrac{\alpha}{\mathrm{We}} = (1 - a^2)(\tau v_{1x} + \tau_2 v_x), \\[2mm] \tau_t + \dfrac{\tau}{\mathrm{We}} = \dfrac{1 - \varepsilon}{\mathrm{We}} v_x - (\alpha v_{1x} + \alpha_2 v_x) \, , \end{cases} \quad (3.4)$$

a.e. in $I \times (0, T)$, together with

$$\begin{cases} v(., 0) = \alpha(., 0) = \tau(., 0) = 0 \text{ a.e. in } I \, , \\ v(x, t) = 0 \text{ a.e. } x \in \partial I \, , \quad t \in (0, T) \, . \end{cases} \quad (3.5)$$

We multiply $(3.4)_1$ by $\dfrac{1 - \varepsilon}{\mathrm{We}} v$, $(3.3)_2$ by α, $(3.4)_3$ by $(1 - a^2) \tau$, integrate over I, and add the resulting relations to get :

$$\frac{1}{2} \frac{d}{dt} \left(\frac{\mathrm{Re}}{\mathrm{We}} (1 - \varepsilon) |v|^2 + \frac{1}{1 - a^2} |\alpha|^2 + |\tau|^2 \right)$$

$$+ \frac{1}{\mathrm{We}} \left(\varepsilon (1 - \varepsilon) |v_x|^2 + \frac{1}{1 - a^2} |\alpha|^2 + |\tau|^2 \right) \quad (3.6)$$

$$= \int_I (\tau_2 \alpha - \alpha_2 \tau) v_x \, dx \, .$$

Using the Cauchy-Schwarz inequality, and the fact that α_2 and τ_2 are in $L^\infty(I \times \mathbb{R}_+)$, we deduce from (3.6) that there exists a constant c such that

---

[1] $\mathscr{C}_b(\mathbb{R}_+ \; ; X)$ denotes the space of bounded continuous X-valued functions on $\mathbb{R}_+$.

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\mathrm{Re}}{\mathrm{We}}(1-\varepsilon)|v|^2 + \frac{1}{1-a^2}|\alpha|^2 + |\tau|^2\right)$$

$$+ \frac{1}{\mathrm{We}}\left(\frac{\varepsilon(1-\varepsilon)}{2}|v_x|^2 + \frac{1}{1-a^2}|\alpha|^2 + |\tau|^2\right) \tag{3.7}$$

$$\leq c(|\alpha|^2 + |\tau|^2),$$

which gives $v \equiv \alpha \equiv \tau \equiv 0$ by Gronwall's lemma.

(ii) *Existence and uniform bounds.* We shall first establish some a priori estimates satisfied by a smooth solution (v, α, τ). Recall that $\omega = 1 - \varepsilon$.

(a) To start with, we prove that the elastic stress components α and τ are bounded in the space $L^\infty(I \times \mathbb{R}_+)$, uniformly for $\varepsilon \in [0, 1]$. We multiply equation $(3.2)_2$ by $\alpha - \frac{\omega}{\mathrm{We}}$, equation $(3.1)_3$ by $(1 - a^2)\tau$, and add the two resulting equations. We obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left(\alpha - \frac{\omega}{\mathrm{We}}\right)^2 + (1-a^2)\tau^2\right) + \frac{1}{\mathrm{We}}\left(\left(\alpha - \frac{\omega}{\mathrm{We}}\right)^2 + (1-a^2)\tau^2\right)$$

$$= -\frac{\omega}{\mathrm{We}^2}\left(\alpha - \frac{\omega}{\mathrm{We}}\right), \quad \text{in } I \times \mathbb{R}_+. \tag{3.8}$$

Setting $\varphi(x, t) = \left(\alpha(x, t) - \frac{\omega}{\mathrm{We}}\right)^2 + (1-a^2)\tau^2(x, t)$, for fixed $x \in I$, $t \in \mathbb{R}_+$, and denoting $\varphi' = \frac{\partial \varphi}{\partial t}$, we deduce, from (3.7) and from the Cauchy-Schwarz inequality,

$$\varphi' + \frac{1}{\mathrm{We}}\varphi \leq \frac{\omega^2}{\mathrm{We}^3} \quad \text{in } \mathbb{R}_+, \tag{3.9}$$

which, by integration over $(0, t)$, gives inequality (3.3).

(b) Next we derive an L^∞ bound on v. Let us define the function u by $u(x, t) = v(x, t)$ for the plane Poiseuille flow, and by $u(x, t) = v(x, t) - x$ for the plane Couette flow. We multiply $(3.1)_1$ by u and integrate over I. This gives

$$\frac{1}{2}\frac{d}{dt}(\mathrm{Re}|u|^2) + \varepsilon|u_x|^2 = -\int_I \tau u_x \, dx - \int_I f u \, dx. \tag{3.10}$$

Using the Cauchy-Schwarz inequality, and the Poincaré inequality (that is $|u_x| \geq \frac{\pi}{|I|}|u|$, $\forall u \in H_0^1(I)$, where $|I|$ denotes the length of the interval I), we obtain

$$\frac{d}{dt}(\mathrm{Re}|u|^2) + \varepsilon\frac{\pi^2}{|I|^2}|u|^2 \leq \frac{2}{\varepsilon}\left(|\tau|^2 + \frac{|I|^2}{\pi^2}|f|^2\right). \tag{3.11}$$

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Taking (3.3) into account, we deduce from (3.11), that the function $|u(.,t)|$ is bounded on $\mathbb{R}_+$, and therefore also the function $|v(.,t)|$. More precisely there exists an increasing positive function C_1, such that

$$\sup_{t \in \mathbb{R}_+} |v(t)| \leq C_1 \left(|v_0|, |\alpha_0|, |\tau_0|, (1-a^2)^{1/2}, \frac{1-\varepsilon}{\text{We}}, \frac{1}{\varepsilon} \right) . \quad (3.12)$$

Combining this with (3.10) yields

$$\varepsilon \left(\int_0^T |v_x|^2 \, dt \right) \leq C_2 \, T, \quad \text{for every } T > 0 . \quad (3.13)$$

(c) The next step is to derive some a priori estimates for (v, α, τ) in $H^2(I) \times H^1(I) \times H^1(I)$. We multiply $(3.1)_1$ by $-\dfrac{1-\varepsilon}{\text{We}} v_{xx}$ (which equals $-\dfrac{1-\varepsilon}{\text{We}} u_{xx}$) and integrate over I. We differentiate $(3.1)_2$ and $(3.1)_3$ once with respect to x and take the scalar product in L^2 with $\dfrac{\alpha_x}{1-a^2}$ and τ_x respectively. Adding the three resulting equations, we get

$$\frac{1}{2} \frac{d}{dt} \left(\frac{\text{Re}}{\text{We}} (1-\varepsilon) |u_x|^2 + \frac{1}{1-a^2} |\alpha_x|^2 + |\tau_x|^2 \right)$$

$$+ \frac{1}{\text{We}} \left((1-\varepsilon) \varepsilon |v_{xx}|^2 + \frac{1}{1-a^2} |\alpha_x|^2 + |\tau_x|^2 \right)$$

$$= \frac{1-\varepsilon}{\text{We}} \left(\int_I f v_{xx} \, dx \right) + \int_I v_{xx} (\tau \alpha_x - \alpha \tau_x) \, dx .$$

Using the uniform bound (3.3) for α and τ, and the Cauchy-Schwarz inequality, we obtain

$$\frac{d}{dt} \left(\frac{\text{Re}}{\text{We}} (1-\varepsilon) |u_x|^2 + \frac{1}{1-a^2} |\alpha_x|^2 + |\tau_x|^2 \right)$$

$$+ \frac{1}{\text{We}} \left((1-\varepsilon) \varepsilon |v_{xx}|^2 + \frac{1}{1-a^2} |\alpha_x|^2 + |\tau_x|^2 \right) \quad (3.14)$$

$$\leq \frac{C_3}{\varepsilon} \left(\frac{1}{1-a^2} |\alpha_x|^2 + |\tau_x|^2 \right) + \frac{1-\varepsilon}{\varepsilon \text{We}} |f|^2 ,$$

where $C_3 = C_3 \left(\|\alpha_0\|_\infty, \|\tau_0\|_\infty, (1-a^2)^{1/2}, \dfrac{1-\varepsilon}{\text{We}} \right)$. A straightforward use of Gronwall's lemma implies

$$\begin{cases} \|v\|_{L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2)} \leq C_4 , \\ \|\tau\|_{L^\infty(0,T;H^1)}, \quad \|\alpha\|_{L^\infty(0,T;H^1)} \leq C_4 , \end{cases} \quad (3.15)$$

where C_4 depends only on $\|v_0\|_1$, $\|\alpha_0\|_1$, $\|\tau_0\|_1$, $(1 - a^2)^{1/2}$, $\dfrac{1 - \varepsilon}{\mathrm{We}}$, $\dfrac{1}{\varepsilon}$ and T.

(d) We are now in a position to prove the existence part of Theorem 3.2. Let η be a positive number. We choose a smooth sequence $\{v_0^\eta, \alpha_0^\eta, \tau_0^\eta\}_\eta$, converging to (v_0, α_0, τ_0) as η goes to zero. Let $(v^\eta, \alpha^\eta, \tau^\eta)$ be the corresponding local solution given by Proposition 3.1, defined on some interval $[0, T^*(\eta))$. Actually this solution is smooth enough to derive rigorously the estimates (3.15), which are therefore valid on $[0, T^*(\eta))$. Since the constant C_4 is bounded for bounded values of $T^*(\eta)$, it follows that $T^*(\eta) = +\infty$, and that (3.15) is valid for $(v^\eta, \alpha^\eta, \tau^\eta)$, for any $T > 0$.

Using a standard compactness argument we see that $(v^\eta, \alpha^\eta, \tau^\eta)$ converges to a solution (v, α, τ) of (3.1)-(3.2) which satisfies the requirement of Theorem 3.2, parts (ii) and (iii). This proves the theorem. □

One can easily prove a regularity property of the solution (v, α, τ) of (3.1)-(3.2).

THEOREM 3.3 : *If* $v_0 \in H^2(I)$, $\alpha_0, \tau_0 \in H^2(I)$, *then the solution* (v, α, τ) *of* (3.1)-(3.2) *satisfies*

$$v \in L^\infty(0, T ; H^2) \cap L^2(0, T ; H^3) \,,$$

$$\alpha, \tau \in L^\infty(0, T ; H^2) \,, \quad \text{for any } T > 0 \,.$$

Proof: Again it suffices to derive the corresponding a priori estimates for the local smooth solution given by Proposition 3.1. They are obtained in a fashion similar to the proof of part (ii) in Theorem 3.2. We leave the details to the reader.

Remark 3.4 : Inequality (3.3) provides a uniform bound of the stress components of a purely elastic fluid ($\varepsilon = 0$), that is

$$\left(\alpha(x, t) - \frac{1}{\mathrm{We}} \right)^2 + (1 - a^2)\, \tau^2(x, t)$$

$$\leqslant \left(\left(\alpha_0(x) - \frac{1}{\mathrm{We}} \right)^2 + (1 - a^2)\, \tau_0^2(x) \right) e^{-t/\mathrm{We}} \qquad (3.16)$$

$$+ \frac{1}{\mathrm{We}^2} \left(1 - e^{-t/\mathrm{We}} \right) \,,$$

for every (x, t) in $I \times \mathbb{R}_+$. However, as proved in [11], there exists, in this case ($\varepsilon = 0$), smooth solutions whose spatial gradient develops singularities in finite time.

4. ENERGY METHODS AND LYAPUNOV STABILITY FOR THE BASIC STEADY COUETTE FLOW

In this section we investigate the Lyapunov stability for arbitrary perturbations (v, α, τ) of the steady solution of the Couette flow. This basic flow is given, for $x \in (0, 1)$, by

$$v_s(x) = x, \ \alpha_s = \frac{1 - \varepsilon}{\text{We}} \frac{k^2}{1 + k^2}, \ \tau_s = \frac{1 - \varepsilon}{1 + k^2},$$

where $k^2 = \text{We}^2 (1 - a^2) > 0$, and $0 < \varepsilon < 1$.

Expressing that $(v + v_s, \ \alpha + \alpha_s, \ \tau + \tau_s)$ is solution of (1.12) yields the following system for (v, α, τ):

$$\begin{cases} \text{Re } v_t - \varepsilon v_{xx} = \tau_x, \\ \alpha_t + \dfrac{\alpha}{\text{We}} = (1 - a^2)(\tau_s v_x + \tau) + (1 - a^2)\tau v_x, \\ \tau_t + \dfrac{\tau}{\text{We}} = \dfrac{\tau_s}{\text{We}} v_x - \alpha - \alpha v_x, \quad \text{in} \quad (0, 1) \times \mathbb{R}_+, \end{cases} \qquad (4.1)$$

$$v(0, .) = v(1, .) = 0, \quad \text{on} \quad \mathbb{R}_+, \qquad (4.2)$$

$$v(., 0) = v_0, \alpha(., 0) = \alpha_0, \tau(., 0) = \tau_0 \quad \text{on} \quad (0, 1). \qquad (4.3)$$

We first state a Lyapunov stability result which is unconditional in $L^2(0, 1)$.

THEOREM 4.1 : *Let* $\varepsilon \in (0, 1)$ *and* $k^2 = \text{We}^2 (1 - a^2)$. *If* $\varepsilon < 1/5$, *we assume that*

$$k < k_0(\varepsilon) \underset{\text{def}}{=} 2 \left(\frac{\varepsilon}{1 - 5\,\varepsilon} \right)^{1/2}. \qquad (4.4)$$

(*If* $\varepsilon \geqslant 1/5$ *we put no restriction on* k.) *Then the steady Couette flow* (v_s, α_s, τ_s) *is unconditionally Lyapunov stable in* $L^2(0, 1)$, *and conditionally stable in* $H^2(0, 1)$.

Proof : We proceed as in the proof of part (ii) of Theorem 3.2 and derive energy estimates in $L^2(0, 1)$ and in $H^1(0, 1)$. The proof of the conditional stability in $H^2(0, 1)$ is similar, and is left to the reader.

(i) Let us first examine the Lyapunov stability in $L^2(0, 1)$. To derive an energy equation in L^2, we multiply (4.1)$_1$ by $\dfrac{\tau_s}{\text{We}} v$, (4.1)$_2$ by $\dfrac{\alpha}{1 - a^2}$,

$(4.1)_3$ by τ, and integrate over $(0, 1)$. Adding the resulting equations, we obtain

$$
\begin{cases}
\dfrac{1}{2} \dfrac{d}{dt} \left(\dfrac{\mathrm{Re}\, \tau_s}{\mathrm{We}} |v|^2 + \dfrac{1}{1 - a^2} |\alpha|^2 + |\tau|^2 \right) \\[2mm]
\quad + \dfrac{1}{\mathrm{We}} \left(\varepsilon \tau_s |v_x|^2 + \dfrac{1}{1 - a^2} |\alpha|^2 + |\tau|^2 \right) \\[2mm]
\quad = \tau_s \left(\displaystyle\int_0^1 v_x \,\alpha\, dx \right) .
\end{cases} \tag{4.5}
$$

For (x, t) fixed, let us define the quadratic form in $\mathbb{R}^3$,

$$
Q(X, Y, Z) = AX^2 + BY^2 + CZ^2 + 2\, DXY ,
$$

where $X = v_x(x, t)$, $Y = \alpha(x, t)$, $Z = \tau(x, t)$, and

$$
A = \varepsilon \tau_s, \quad B = \frac{1}{1 - a^2}, \quad C = 1, \quad D = \frac{\mathrm{We}\, \tau_s}{2} .
$$

The quadratic form Q is positive definite if and only if

$$
AB - D^2 > 0 ; \tag{4.6}
$$

in this case we have

$$
Q(X, Y, Z) \geqslant \lambda_0 (X^2 + Y^2 + Z^2) , \quad \forall X, Y, Z \in \mathbb{R}^3 , \tag{4.7}
$$

where λ_0 is the smallest eigenvalue of Q,

$$
\lambda_0 = \min \left(1, \frac{A + B - ((A - B)^2 + 4\, D^2)^{1/2}}{2} \right) .
$$

Here condition (4.6) reads

$$
4\, \varepsilon > k^2 \tau_s ,
$$

which is equivalent to condition (4.4) if $\varepsilon < 1/5$, and which imposes no restriction on k if $\varepsilon \geqslant 1/5$.

Using inequality (4.7) and Poincaré's inequality we deduce from (4.5) that the function defined by

$$
\varphi(t) = \frac{\mathrm{Re}\, \tau_s}{\mathrm{We}} |v|^2 + \frac{1}{1 - a^2} |\alpha|^2 + |\tau|^2
$$

satisfies an inequality having the form

$$
\varphi'(t) + \lambda \varphi(t) \leqslant 0 , \quad t \geqslant 0 , \tag{4.8}
$$

for some $\lambda < 0$, provided that condition (4.4) holds if $\varepsilon < 1/5$. Inequality (4.8) implies the L^2-unconditional Lyapunov stability of the steady solution.

(ii) Proceeding as in part (ii) of the proof of Theorem 3.2, we obtain an energy equation for the derivative (v_x, α_x, τ_x) of the solution of (4.1)-(4.3), i.e.

$$
\left\{
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\frac{\mathrm{Re}\,\tau_s}{\mathrm{We}} \, |v_x|^2 + \frac{1}{1-a^2} \, |\alpha_x|^2 + |\tau_x|^2 \right) \\
& + \frac{1}{\mathrm{We}} \left(\varepsilon \tau_s |v_{xx}|^2 + \frac{1}{1-a^2} \, |\alpha_x|^2 + |\tau_x|^2 \right) \\
& = \tau_s \left(\int_0^1 v_{xx}\, \alpha_x\, dx \right) + \int_0^1 (\tau \alpha_x - \alpha \tau_x)\, v_{xx}\, dx \,.
\end{aligned}
\right.
\tag{4.9}
$$

Under condition (4.4) if $\varepsilon < 1/5$, the quadratic form $Q(v_{xx}, \alpha_x, \tau_x)$ defined in part (i) is positive definite, so that

$$
\left\{
\begin{aligned}
& \frac{d}{dt} \left(\frac{\mathrm{Re}\,\tau_s}{\mathrm{We}} \, |v_x|^2 + \frac{1}{1-a^2} \, |\alpha_x|^2 + |\tau_x|^2 \right) \\
& + \tilde{\lambda}(|v_{xx}|^2 + |\alpha_x|^2 + |\tau_x|^2) \\
& = \left| \int_0^1 (\tau \alpha_x - \alpha \tau_x)\, v_{xx}\, ds \right|
\end{aligned}
\right.
\tag{4.10}
$$

for some $\tilde{\lambda} > 0$.

We estimate the last term of (4.9) as follows,

$$
\begin{aligned}
\left| \int_0^1 (\tau \alpha_x - \alpha \tau_x)\, v_{xx}\, dx \right| & \leq (\|\tau\|_\infty \, |\alpha_x| + \|\alpha\|_\infty \, |\tau_x|)\, |v_{xx}| \\
& \leq c_1 \|\tau\|_1 \, \|\alpha\|_1 \, |v_{xx}| \,.
\end{aligned}
\tag{4.11}
$$

for some positive constant c_1. Using the Cauchy-Schwarz inequality and (4.11), we deduce from (4.10) and (4.8) that there exists a δ, $0 < \delta < \tilde{\lambda}$, such that

$$
\left\{
\begin{aligned}
& \frac{d}{dt} \left(\frac{\mathrm{Re}\,\tau_s}{\mathrm{We}} \, \|v\|_1^2 + \frac{1}{1-a^2} \, \|\alpha\|_1^2 + \|\tau\|_1^2 \right) \\
& + \delta (\|v_x\|_1^2 + \|\alpha\|_1^2 + \|\tau\|_1^2) \\
& \leq c_2 (\|\tau\|_1^2 + \|\alpha\|_1^2)^2 \,,
\end{aligned}
\right.
\tag{4.12}
$$

for some positive constant c_2. We use (4.8) and the inequality

$$
|v_{xx}| \geq \pi |v_x| \,, \quad \forall v \in H_0^1(0, 1) \cap H^2(0, 1) \,,
$$

so that (4.12) becomes the Ricatti inequality,

$$\varphi' + \lambda\varphi \leqslant c_3\,\varphi^2\,, \quad \text{for some} \quad \lambda > 0, c_3 > 0\,, \tag{4.13}$$

where

$$\varphi(t) = \frac{\text{Re }\tau_s}{\text{We}}\,\|v(t)\|_1^2 + \frac{1}{1-a^2}\,\|\alpha(t)\|_1^2 + \|\tau(t)\|_1^2\,.$$

We thus obtain the uniform bound for φ,

$$\varphi(t) \leqslant e^{-\lambda t}\,\frac{\varphi_0}{1 - c_3\,\varphi_0(1 - e^{-\lambda t})/\lambda}\,, \quad \text{for all} \quad t \geqslant 0\,,$$

provided $\varphi(0) \leqslant \lambda/c_3$. This estimate shows the conditional Lyapunov stability of the steady solution in $H^1(0, 1)$. $\square$

Remark 4.2 : The function $k_0 = k_0(\varepsilon)$ obtained in (4.4) is an increasing function of ε, such that $k_0(0) = 0$, $k_0(1/9) = 1$, $k_0(1/5) = +\infty$. $\square$

For small values of k, we also have a conditional Lyapunov stability result in $H^1(0, 1)$ (and consequently in $H^2(0, 1)$), but the computations are somewhat tedious).

THEOREM 4.3 : *Let $0 < \varepsilon < 1$ and $k^2 = \text{We}^2\,(1 - a^2)$. Then the steady Couette solution is conditionally Lyapunov stable in $H^1(0, 1)$, for all $k^2 \leqslant 1$.*

Proof : We change the unknown function by setting

$$A = \frac{\text{We}^2}{k^2}\,\alpha - \text{We }\tau\,,$$

so that the differential equation satisfied by A does not contain any linear term in v_x. The functions v, A and τ satisfy the following equations

$$\begin{cases} \text{Re }v_t - \varepsilon v_{xx} = \tau_x\,, \\[2mm] A_t + \dfrac{1-k^2}{\text{We}}\,A = (1 + k^2)\,\tau + (1 + k^2)\,\tau v_x + \dfrac{k^2}{\text{We}}\,Av_x\,, \\[2mm] \tau_t + \dfrac{1+k^2}{\text{We}}\,\tau = \dfrac{\tau_s\,v_x}{\text{We}} - (1 - a^2)\,A - (1 - a^2)\,Av_x - \dfrac{k^2}{\text{We}}\,\tau v_x\,. \end{cases} \tag{4.14}$$

We now multiply $(4.14)_1$ by $\dfrac{\tau_s\,v}{\text{We}}$, $(4.14)_2$ by $\dfrac{(1 - a^2)\,A}{1 + k^2}$, $(4.14)_3$ by τ, and obtain

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\mathrm{Re}\,\tau_s}{\mathrm{We}}\,|v|^2 + \frac{1-a^2}{1+k^2}\,|A|^2 + |\tau|^2\right)$$

$$+ \frac{1}{\mathrm{We}}\left(\varepsilon\tau_s\,|v_x|^2 + \frac{(1-a^2)(1-k^2)}{1+k^2}\,|A|^2 + (1+k^2)|\tau|^2\right)$$

$$= \frac{k^2}{\mathrm{We}}\left(\int_0^1\left(\frac{1-a^2}{1+k^2}A^2 - \tau^2\right)v_x\,dx\right). \tag{4.15}$$

We take the x-derivative of (4.14) and multiply the resulting system by $\dfrac{\tau_s\,v_x}{\mathrm{We}}$, $\dfrac{1-a^2}{1+k^2}A_x$, τ_x respectively. We obtain

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\mathrm{Re}\,\tau_s}{\mathrm{We}}\,|v_x|^2 + \frac{1-a^2}{1+k^2}\,|A_x|^2 + |\tau_x|^2\right)$$

$$+ \frac{1}{\mathrm{We}}\left(\varepsilon\tau_s|v_{xx}|^2 + \frac{(1-a^2)(1-k^2)}{1+k^2}\,|A_x|^2 + (1+k^2)|\tau_x|^2\right)$$

$$= \frac{k^2}{\mathrm{We}}\frac{(1-a^2)}{1+k^2}\int (Av_x)_x\,A_x\,dx + (1-a^2)\int (\tau v_x)_x\,A_x\,dx$$

$$- (1-a^2)\int (Av_x)_x\,\tau_x\,dx - \frac{k^2}{\mathrm{We}}\int (\tau v_x)_x\,\tau_x\,dx. \tag{4.16}$$

The absolute value of the right hand side is easily majorized by $\eta\|v\|_2^2 + C(\eta)(\|A\|_1^4 + \|\tau\|_1^4)$ for every $\eta > 0$. This, together with (4.15) shows that the function φ, defined by

$$\varphi(t) = \frac{\mathrm{Re}\,\tau_s}{\mathrm{We}}\,\|v(t)\|_1^2 + \frac{1-a^2}{1+k^2}\,\|A(t)\|_1^2 + \|\tau(t)\|_1^2,$$

satisfies the Ricatti inequality (4.13), with different constants. This shows the conditional Lyapunov stability in $H^1(0,1)$ of the steady solution for $k^2 < 1$. This result for $k^2 = 1$ will be obtained while proving Theorem 4.4.□

A systematic derivation of energy estimates in $H^1(0,1)$ made by using some appropriate linear combinations of α and τ gives a conditional Lyapunov stability result valid for all k's small enough, the bound on k being larger than the one obtained in Theorem 4.1. This result can be stated as follows.

THEOREM 4.4 : *Let* $k^2 = \mathrm{We}^2\,(1-a^2)$.

(i) *If* ε *is in* $(1/9, 1)$, *then the steady Couette flow is conditionally Lyapunov stable in* $H^1(0,1)$ *for all values of* k.

(ii) *Let* $0 < \varepsilon \leqslant 1/9$. *Then there exists a function* $k_1 = k_1(\varepsilon) > k_0(\varepsilon)$ *such that the steady Couette flow is conditionally Lyapunov stable in* $H^1(0, 1)$ *for all* $k < k_1(\varepsilon)$. (k_0 *is given in Theorem* 4.1).

Proof: We define $A = \alpha + \dfrac{\xi}{\text{We}} \tau$, ξ being a real number. Using the function A instead of α in equations (4.1) implies that v, A and τ satisfy the following

$$\begin{cases} \text{Re } v_t - \varepsilon v_{xx} = \tau_x \, , \\[2mm] A_t + \dfrac{1 + \xi}{\text{We}} A = \dfrac{\tau_s}{\text{We}^2} (k^2 + \xi) + \dfrac{k^2 + \xi^2}{\text{We}^2} \tau(1 + v_x) - \dfrac{\xi}{\text{We}} A v_x \, , \quad (4.17) \\[2mm] \tau_t + \dfrac{1 - \xi}{\text{We}} \tau = \dfrac{\tau_s}{\text{We}} v_x - (1 + v_x) A + \dfrac{\xi}{\text{We}} \tau v_x \, . \end{cases}$$

We now derive an energy estimate in $L^2(0, 1)$ in the same way as above : it reads

$$\begin{cases} \dfrac{1}{2} \dfrac{d}{dt} \left(\dfrac{\text{Re } \tau_s}{\text{We}} |v|^2 + \dfrac{\text{We}^2}{k^2 + \xi^2} |A|^2 + |\tau|^2 \right) \\[3mm] + \dfrac{1}{\text{We}} \left(\varepsilon \tau_s |v_x|^2 + \text{We}^2 \dfrac{1 + \xi}{k^2 + \xi^2} |A|^2 + (1 - \xi)|\tau|^2 \right) \qquad (4.18) \\[3mm] = \tau_s \dfrac{k^2 + \xi}{k^2 + \xi^2} \left(\int_0^1 v_x A \, dx \right) - \dfrac{\xi}{\text{We}} \left(\int_0^1 \left(\dfrac{\text{We}^2}{k^2 + \xi^2} A^2 - \tau^2 \right) v_x \, dx \right) \, . \end{cases}$$

Similarly an energy estimate in $H^1(0, 1)$ is derived (the coefficients of the quadratic terms are the same as those in (4.18)). The cubic terms are majorized by $\eta \|v\|_1^2 + C(\eta)(\|A\|_1^4 + \|\tau\|_1^4)$, $\forall \eta > 0$. We shall therefore obtain a Ricatti inequality for

$$\varphi(t) = \dfrac{\text{Re } \tau_s}{\text{We}} \|v\|_1^2 + \dfrac{\text{We}^2}{k^2 + \xi^2} \|A\|_1^2 + \|\tau\|_1^2$$

(and thus the conditional stability in H^1), provided that a certain quadratic form (*cf.* below) is positive definite.

In relation (4.18), we have to choose

$$-1 < \xi < 1 \, . \qquad (4.19)$$

Notice that the special case where ξ is equal to $-k^2$ is compatible with (4.19) only if $k^2 < 1$: this is the case studied in Theorem 4.3.

Let us assume $k^2 \geqslant 1$ and define the quadratic form Q by

$$Q(X, Y, Z) = \varepsilon \tau_s X^2 + \text{We}^2 \dfrac{1 + \xi}{k^2 + \xi^2} Y^2 + (1 - \xi) Z^2 - \tau_s \text{We} \dfrac{k^2 + \xi}{k^2 + \xi^2} XY \, .$$

Condition (4.6) reads

$$\varepsilon \tau_s \frac{1+\xi}{k^2+\xi^2} - \left(\frac{\tau_s}{2} \text{We} \frac{k^2+\xi}{k^2+\xi^2} \right)^2 > 0 \,,$$

which is equivalent to

$$\frac{1-\varepsilon}{\varepsilon} < 4(1+k^2) \frac{(k^2+\xi^2)(1+\xi)}{(k^2+\xi)^2} \,. \qquad (4.20)$$

Set

$$\ell_k(\xi) = \frac{(k^2+\xi^2)(1+\xi)}{(k^2+\xi)^2} \,, \quad \text{for} \quad -1 < \xi < 1 \,.$$

(i) Let us assume that $k^2 = 1$. Then $\ell_1(\xi)$ is unbounded for $\xi = -1$. Therefore, for every $\varepsilon > 0$, there exists ξ, $-1 < \xi < 0$, such that condition (4.20) holds. This proves the case $k^2 = 1$ in Theorem 4.3.

(ii) Let $\varepsilon > 1/9$. Since $\max\limits_{-1 < \xi < 1} \ell_k(\xi) \geqslant \ell_k(1) = \dfrac{2}{1+k^2}$, $\forall k > 1$, there exists ξ, $0 < \xi < 1$, such that condition (4.20) holds for every $k > 1$. This proves the case $\varepsilon > 1/9$ of the current theorem.

(iii) Let $0 < \varepsilon \leqslant 1/9$. In order to define the function $k_1 = k_1(\varepsilon)$ introduced in the theorem, we first notice that there exists $\bar{k}$, $1 < \bar{k} < 2$, such that

$$\max_{-1 < \xi < 1} \ell_{\bar{k}}(\xi) = \ell_{\bar{k}}(\bar{\xi}) = \ell_{\bar{k}}(1) \,,$$

where $\bar{\xi} = \xi_{\bar{k}}$ is in $(-1, 0)$. The real number $\bar{k}$ is characterized by the following property :

— if $1 < k < \bar{k}$, then $\max\limits_{-1 < \xi < 1} \ell_k(\xi) = \ell_k(\xi_k)$, $\xi_k \in (-1, 0)$;

— if $k > \bar{k}$, then $\max\limits_{-1 < \xi < 1} \ell_k(\xi) = \ell_k(1)$.

Actually, an easy calculation shows that $\bar{k}^2$ satisfies $\bar{k}^4 = 1 + \bar{k}^2$, and that $\bar{\xi} = -\dfrac{1}{\bar{k}^2}$. Therefore $\bar{k}^2 = (1 + \sqrt{5})/2$.

Moreover, the function $k \to \ell_k(\xi_k)$ decreases from $(1, \bar{k})$ onto $\left(\dfrac{2}{1+k^2}, \infty \right)$. We define the function ε_1 on $[1, \bar{k}]$, by

$$\varepsilon_1(k) = (1 + 4(1 + k^2) \ell_k(\xi_k))^{-1} \,.$$

The function $k_1 = k_1(\varepsilon)$ is then the inverse function of ε_1, and therefore is an increasing function from $[0, 1/9]$ onto $[1, \bar{k}]$. We clearly have $k_1(\varepsilon) > k_0(\varepsilon)$, $\forall \varepsilon < 1/9$, where k_0 has been introduced in (4.4).

Fix now ε in $(0, 1/9)$, and $k < k_1(\varepsilon)$. Then condition (4.20) holds for $\xi = \xi_k$, where ξ_k (as defined above) is the real number in $(-1, 0)$ where ℓ_k attains its maximum. This completes the proof of the theorem.

5. LINEAR STABILITY

It is well known in the case of the Navier-Stokes equations (at least for flows in a bounded domain) that the nonlinear stability can be determined by the analysis of the eigenvalues of the linearized stationary operator [4], [5], [10], [12]. No such result is known in the context of flows of viscoelastic fluids. It is even not clear whether the asymptotic behavior of the linearized equation is governed by the spectrum of the linearized stationary operator.

In this section we shall give a partial answer to these questions in the case of one dimensional shearing motions. A discussion of 2 or 3 dimensional perturbations is postponed to a forthcoming paper.

First we investigate the linear operator in (4.1). Thus we define an unbounded linear $\mathscr{L}$ operator in $H = L^2(I)^3$ by

$$\mathscr{L}\begin{pmatrix} v \\ \alpha \\ \tau \end{pmatrix} = \begin{pmatrix} -\dfrac{\varepsilon}{\mathrm{Re}}\,\partial_x^2 & 0 & -\dfrac{1}{\mathrm{Re}}\,\partial_x \\[2ex] -(1-a^2)\,\tau_s\,\partial_x & \dfrac{1}{\mathrm{We}} & -(1-a^2) \\[2ex] -\dfrac{\tau_s}{\mathrm{We}}\,\partial_x & 1 & \dfrac{1}{\mathrm{We}} \end{pmatrix}\begin{pmatrix} v \\ \alpha \\ \tau \end{pmatrix}. \qquad (5.1)$$

The domain $D(\mathscr{L})$ is defined as the set of $(v, \alpha, \tau) \in H_0^1(I) \times L^2(I)^2$ such that $-\varepsilon v_{xx} - \tau_x \in L^2(I)$.

The following lemma states the main properties of $\mathscr{L}$ (see T. Kato [6] for the definitions).

LEMMA 5.1 :

(i) $\mathscr{L}$ *is a closed operator in* H.

(ii) $\mathscr{L}$ *is m-sectorial with vertex* $-\Lambda$ *for some* $\Lambda > 0$ *and semi-angle* $\dfrac{\pi}{4}$.

(iii) *If* $\varepsilon > \dfrac{1}{9}$, *or if* $0 < \varepsilon < \dfrac{1}{9}$ *and* $k^2 = \mathrm{We}^2(1 - a^2)$ *is different from*

$\kappa_\pm$, *where* $\kappa_\pm = \dfrac{1 - 3\,\varepsilon \pm (9\,\varepsilon^2 - 10\,\varepsilon + 1)^{1/2}}{2\,\varepsilon}$, *the spectrum* $\sigma(\mathscr{L})$ *of* $\mathscr{L}$ *consists only of a countable set of eigenvalues of finite multiplicity.*

If $0 < \varepsilon \leqslant \dfrac{1}{9}$, *and* $k^2 = \kappa_\pm$, *then* $\sigma(\mathscr{L})$ *contains in addition* 0 *as an eigenvalues of infinite multiplicity.*

Proof : (i) This is obvious.

(ii) Let us prove first that the numerical range of $\mathscr{L}$ is included in a complex sector of vertex $-\Lambda$ and semi-angle $\dfrac{\pi}{4}$. Setting $U = \begin{pmatrix} v \\ \alpha \\ \tau \end{pmatrix}$, we obtain

$$(\mathscr{L}U, U) = \frac{\varepsilon}{\mathrm{Re}}\,|v_x|^2 + \frac{1}{\mathrm{We}}\,|\alpha|^2 + \frac{1}{\mathrm{We}}\,|\tau|^2 + \frac{1}{\mathrm{Re}}\int_I \tau \bar{v}_x\,dx$$

$$- \frac{\tau_s}{\mathrm{We}}\int_I v_x\,\bar{\tau}\,dx - (1 - a^2)\,\tau_s \int_I v_x\,\bar{\alpha}\,dx - (1 - a^2)\int_I \tau\bar{\alpha}\,dx + \int_I \alpha\bar{\tau}\,dx\,.$$

Therefore,

$$\mathscr{R}\mathrm{e}\,(\mathscr{L}U, U) \geqslant \frac{\varepsilon}{\mathrm{Re}}\,|v_x|^2 + \frac{1}{\mathrm{We}}\,|\tau|^2 + \frac{1}{\mathrm{We}}\,|\alpha|^2 - \left(\frac{\tau_s}{\mathrm{We}} + \frac{1}{\mathrm{Re}}\right)|\tau|\,|v_x|$$

$$- (1 - a^2)\,\tau_s\,|v_x|\,|\alpha| - (2 - a^2)|\alpha|\,|\tau|\,,$$

while

$$\mathscr{I}\mathrm{m}\,(\mathscr{L}U, U) \leqslant$$

$$\leqslant \left(\frac{\tau_s}{\mathrm{We}} + \frac{1}{\mathrm{Re}}\right)|\tau|\,|v_x| + 2(1 - a^2)\,\tau_s|v_x|\,|\alpha| + 2(2 - a^2)|\tau|\,|\alpha|\,,$$

and

$$\mathscr{R}\mathrm{e}\,(\mathscr{L}U, U) + 2\left(\frac{\tau_s}{\mathrm{We}} + \frac{1}{\mathrm{Re}}\right)|\tau|\,|v_x| + 2(1 - a^2)\,\tau_s|v_x|\,|\alpha|$$

$$+ 2(2 - a^2)|\tau|\,|\alpha|$$

$$\geqslant \frac{\varepsilon}{\mathrm{Re}}\,|v_x|^2 + \frac{1}{\mathrm{We}}\,|\tau|^2 + \frac{1}{\mathrm{We}}\,|\alpha|^2 + \left(\frac{\tau_s}{\mathrm{We}} + \frac{1}{\mathrm{Re}}\right)|\tau|\,|v_x|$$

$$+ (1 - a^2)\,\tau_s|v_x|\,|\alpha| + (2 - a^2)|\alpha|\,|\tau|$$

$$\geqslant \mathscr{I}\mathrm{m}\,(\mathscr{L}U, U)\,.$$

Finally,

$$\mathscr{I}\mathrm{m}\,(\mathscr{L}U, U) \leqslant \mathscr{R}\mathrm{e}\,(\mathscr{L}U, U) + \left(\frac{1}{\mathrm{Re}} + \frac{\tau_s}{\mathrm{We}} + 2 - a^2\right)|\alpha|^2 +$$

$$+ ((1 - a^2)\,\tau_s + 2 - a^2)|\alpha|^2 + \left(\frac{1}{\mathrm{Re}} + \frac{\tau_s}{\mathrm{We}} + (1 - a^2)\,\tau_s\right)|v_x|$$

$$\leqslant \mathscr{R}\mathrm{e}\,(\mathscr{L}U, U) + \Lambda(|\alpha|^2 + |\tau|^2 + |v_x|^2)\,, \qquad (5.2)$$

where

$$\Lambda = \mathrm{Max}\left(\frac{1}{\mathrm{Re}} + \frac{\tau_s}{\mathrm{We}} + 2 - a^2, (1 - a^2)\,\tau_s + 2 - a^2,\right.$$

$$\left.\frac{1}{\mathrm{Re}} + \frac{\tau_s}{\mathrm{We}} + (1 - a^2)\,\tau_s\right)$$

which shows that

$$\mathscr{I}\mathrm{m}\,(\mathscr{L}U, U) \leqslant \mathscr{R}\mathrm{e}\,((\mathscr{L} + \Lambda I)\,U, U)\,.$$

The numerical range of $\mathscr{L}$ is therefore included in the complex sector of vertex $-\Lambda$ and semi-angle $\frac{\pi}{4}$.

Let us show that $\mathscr{L}$ is quasi-m-accretive ; the first part of the proof has just shown that $\mathscr{L}$ is quasi-accretive. It remains to prove that there exists μ such that

$$(\mathscr{L} + (\mu + \xi)\,I)^{-1} \in \mathscr{L}(H)\ \text{(the space of bounded linear operators on } H)$$

and

$$\|(\mathscr{L} + (\mu + \xi)\,I)^{-1}\|_{\mathscr{L}(H)} \leqslant \frac{1}{\mathscr{R}\mathrm{e}\,\xi}\,, \quad \mathscr{R}\mathrm{e}\,\xi > 0\,.$$

To do this, we consider the system, for $(f, g, h) \in L^2(I)^3$,

$$\begin{cases} -\dfrac{\varepsilon}{\mathrm{Re}}\,v_{xx} - \dfrac{1}{\mathrm{Re}}\,\tau_x + (\mu + \xi)\,v = f\,, \\[2mm] \dfrac{\alpha}{\mathrm{We}} - (1 - a)^2\,\tau_s\,v_x - (1 - a^2)\,\tau + (\mu + \xi)\,\alpha = g\,, \\[2mm] \dfrac{\tau}{\mathrm{We}} - \dfrac{\tau_s}{\mathrm{We}}\,v_x + \alpha + (\mu + \xi)\,\tau = h\,, \\[2mm] v(0) = v(1) = 0\,. \end{cases} \qquad (5.3)$$

We readily obtain the following estimate from (5.3) :

$$\frac{\varepsilon}{\text{Re}} |v_x|^2 + \frac{1}{\text{We}} (|\alpha|^2 + |\tau|^2) + (\mu + \text{Re } \xi)(|v|^2 + |\alpha|^2 + |\tau|^2)$$

$$\leqslant (1 - a^2) \tau_s |v_x| |\alpha| + \left(\frac{1}{\text{Re}} + \frac{\tau_s}{\text{We}} \right) |v_x| |\tau| + |\alpha| |\tau| \qquad (5.4)$$

$$+ |f| |v| + |g| |\alpha| + |h| |\tau| .$$

From Young's inequality and (5.4), with $\mu > 0$ large enough, we obtain an inequality of the type

$$|v|^2 + |\alpha|^2 + |\tau|^2 \leqslant c(|f|^2 + |g|^2 + |h|^2) .$$

This shows that $(\mathscr{L} + (\mu + \xi) I)^{-1}$ is a bounded operator in H.

Now, let $V \in H$ and U be such that $(\mathscr{L} + (\mu + \xi) I) U = V$. Then

$$((\mathscr{L} + \mu I) U, U) + (\xi U, U) = (V, U) ,$$

which gives

$$\mathscr{R}\text{e } \xi |U|^2 + ((\mathscr{L} + \mu I) U, U) \leqslant |V| |U| .$$

Since $((\mathscr{L} + \mu I) U, U) \geqslant 0$ for μ large enough, we get

$$|U| \leqslant \frac{1}{\text{Re } \xi} |V| .$$

(iii) Let λ be an eigenvalue of $\mathscr{L}$. Then there exists such that

$$\begin{cases} - \varepsilon v_{xx} - \tau_x = \lambda \text{ Re } v , \\ - (1 - a^2) \tau_s v_x + \frac{1}{\text{We}} \alpha - (1 - a^2) \tau = \lambda \alpha , \\ - \frac{\tau_s}{\text{We}} v_x + \alpha + \frac{1}{\text{We}} \tau = \lambda \tau , \\ v(0) = v(1) = 0 . \end{cases} \qquad (5.5)$$

We eliminate α in $(5.5)_2$, $(5.5)_3$ and use the expression for τ_s. Thus,

$$\tau_x = - v_{xx} \frac{1 - \varepsilon}{1 + k^2} \frac{k^2 + \text{We } \lambda - 1}{k^2 + (1 - \lambda \text{ We})^2} \underset{\text{def}}{=} - v_{xx} F(\varepsilon, \text{We}, a, \lambda)$$

where $k^2 = \text{We}^2(1 - a^2)$. Finally, (5.5) implies

$$[F(\varepsilon, \text{We}, a, \lambda) - \varepsilon] v_{xx} = \lambda \text{ Re } v . \qquad (5.6)$$

We assume first that $F(\varepsilon, \text{We}, a, \lambda) \neq \varepsilon$. Then, (5.6) with the boundary conditions $(5.5)_4$ yields

$$v(x) = A_n \sin n\pi x , \quad n = 1, 2, \ldots, \tag{5.7}$$

$$\frac{\lambda_n \, \mathrm{Re}}{\varepsilon - F(\varepsilon, \mathrm{We}, a, \lambda_n)} = n^2 \pi^2 , \quad n = 1, 2, \ldots$$

More explicitly, we end up with a polynomial of degree 3 in λ_n,

$$\mathrm{Re} \, \mathrm{We}^2 (1 + k^2) \lambda_n^3 - \mathrm{We} \, (1 + k^2)[2 \, \mathrm{Re} + n^2 \pi^2 \, \mathrm{We} \, \varepsilon] \lambda_n^2$$
$$+ \lambda_n \{ \mathrm{Re} \, (1 + k^2)^2 + n^2 \pi^2 \, \mathrm{We} \, [1 + \varepsilon + 2 \, \varepsilon k^2] \} \tag{5.8}$$
$$- n^2 \pi^2 \{ \varepsilon (1 + k^2)^2 + (1 - \varepsilon)(1 - k^2) \} = 0 .$$

Once v is chosen by (5.7), α and τ are uniquely determined by solving (5.5)$_2$, (5.5)$_3$. This gives eigenvalues only of finite multiplicity. Indeed, let $\lambda_n^{(i)}$, $i = 1, 2, 3$, be the roots of (5.8). The eigenvalue λ is not simple precisely when $\lambda = \lambda_n^{(i)} = \lambda_m^{(j)}$ for $n \neq m$ and $i, j \in \{1, 2, 3\}$. This is only possible for at most a finite number of couples (n, m). (Look at the resultant of the corresponding polynomials in (5.8).)

Now we examine the case $F(\varepsilon, \mathrm{We}, a, \lambda) = \varepsilon$. This implies $\lambda = 0$ and $F(\varepsilon, \mathrm{We}, a, 0) = \varepsilon$, i.e. $(1 - \varepsilon)(k^2 - 1) = \varepsilon (1 + k^2)^2$. Solving this equation in $\kappa = k^2$ yields

$$\varepsilon \kappa^2 + \kappa (3 \varepsilon - 1) + 1 = 0 .$$

There are two positive roots $\kappa_\pm$ if and only if $9 \varepsilon^2 - 10 \varepsilon + 1 > 0$, that is if and only if $0 < \varepsilon \leqslant \frac{1}{9}$. In this case

$$k^2 = \mathrm{We}^4 \, (1 - a^2)^2 = \frac{1 - 3 \varepsilon \pm \sqrt{9 \varepsilon^2 - 10 \varepsilon + 1}}{2 \varepsilon} = \kappa_\pm . \tag{5.9}$$

Thus, for $0 < \varepsilon \leqslant \frac{1}{9}$ and $k^2 = \kappa_\pm$, 0 is an eigenvalue of $\mathscr{L}$ with *infinite multiplicity*, since every $v \in H_0^1(I) \cap H^2(I)$ is then a solution of (5.6). Note that this is the only case where 0 is an eigenvalue of $\mathscr{L}$.

We shall prove now that the spectrum of $\mathscr{L}$ consists only of eigenvalues, i.e. if $\lambda \in \mathbb{C} \backslash \Sigma$, where $\Sigma = \{0\} \cup \{\lambda_n, n \in \mathbb{N}$, solution of (5.8)$\}$, then λ belongs to the resolvent set of $\mathscr{L}$. Let f, g, h be arbitrary elements in $L^2(I)$. It is sufficient to show that the following system has a unique solution in $D(\mathscr{L})$

$$\begin{cases} - \varepsilon v_{xx} - \tau_x + \lambda \, \mathrm{Re} \, v = \mathrm{Re} \, f , \\[2mm] - (1 - a^2) \tau_s \, v_x + \left(\dfrac{1}{\mathrm{We}} + \lambda \right) \alpha - (1 - a^2) \tau = g , \\[2mm] - \dfrac{\tau_s}{\mathrm{We}} v_x + \alpha + \left(\dfrac{1}{\mathrm{We}} + \lambda \right) \alpha - (1 - a^2) \tau = h , \\[2mm] v(0) = v(1) = 0 , \end{cases} \tag{5.10}$$

and that $(\mathscr{L} + \lambda I)^{-1} \in \mathscr{L}(H, H)$. From $(5.10)_2$, $(5.10)_3$, we get

$$\tau_x = -v_{xx} F(\varepsilon, \text{We}, a, -\lambda)$$
$$-\frac{1}{k^2 + (1 + \lambda \text{We})^2} [-\text{We}^2 g_x + \text{We} (1 + \lambda \text{We}) h_x].$$

Since $\lambda \in \mathbb{C} \backslash \Sigma$ one can solve (5.10) uniquely in v and obtain the estimate

$$|v| \le C(|f| + |g| + |h|).$$

We can then solve the linear system in α, τ, and get

$$|\alpha| + |\tau| \le C(|f| + |g| + |h|).$$

This completes the proof of the theorem. $\square$

The solution (v_s, τ_s, α_s) is called *linearly stable* if the spectrum of $\mathscr{L}$ lies in $\{\mathscr{R}e\, z > 0\}$. Then obviously, Inf $\mathscr{R}e\, \sigma(\mathscr{L}) > 0$. Since $\mathscr{L}$ is the infinitesimal generator of an analytic semi-group, this implies that the solution of the linearized version of (4.1) tends to 0 exponentially as $t \to +\infty$.

The next lemma states a necessary and sufficient condition for linear stability. This result is essentially contained in [14] but we give a direct proof.

LEMMA 5.2 :

(i) If $\frac{1}{9} < \varepsilon < 1$, then (v_s, τ_s, α_s) is linearly stable.

(ii) Let $0 < \varepsilon \le \frac{1}{9}$. Then (v_s, τ_s, α_s) is linearly stable if and only if $0 \le k < (\kappa_-)^{1/2}$ or $k > (\kappa_+)^{1/2}$, where $k = \text{We}^2(1 - a^2)$, and $\kappa_\pm$ are given by (5.9).

Proof: We use the Routh-Hurwitz criterion (*cf.* [3]) to locate the solutions of (5.8) : the roots of (5.8) belong to the half plane $\{\mathscr{R}e\, z > 0\}$ if and only if

$$\varepsilon(1 + k^2)^2 + (1 - k^2)(1 - \varepsilon) > 0, \tag{5.11}$$

$$\alpha_1 \alpha_2 - \alpha_0 \alpha_3 > 0, \tag{5.12}$$

where

$$\alpha_1 = \text{Re}\,(1 + k^2)^2 + n^2 \pi^2 \text{We}\,(1 + \varepsilon + 2\varepsilon k^2),$$
$$\alpha_2 = \text{We}\,(1 + k^2)(2\,\text{Re} + n^2 \pi^2 \text{We}\,\varepsilon),$$
$$\alpha_3 = \text{Re}\,\text{We}^2(1 + k^2),$$
$$\alpha_0 = n^2 \pi^2 [\varepsilon(1 + k^2)^2 + (1 - \varepsilon)(1 - k^2)].$$

Condition (5.12) can be expressed as

$$2 \operatorname{Re}^2 \kappa^2 + \kappa [2 (\operatorname{Re} + \varepsilon n^2 \pi^2)(2 \operatorname{Re} + n^2 \pi^2 \operatorname{We} \varepsilon) + (1 - 3 \varepsilon) \operatorname{Re} \operatorname{We} n^2 \pi^2]$$

$$+ (\operatorname{Re} + n^2 \pi^2 \operatorname{We} \varepsilon)(2 \operatorname{Re} + n^2 \pi^2 \operatorname{We} (1 + \varepsilon)) > 0, \quad \forall \kappa \geqslant 0 \quad (5.13)$$

where $\kappa = k^2$.

We have already encountered condition (5.11). It holds for any k provided that $\frac{1}{9} < \varepsilon \leqslant 1$. On the other hand, if $0 < \varepsilon < \frac{1}{9}$, it holds if $k^2 < \kappa_-$ or $k^2 > \kappa_+$.

We note that (5.12) is always satisfied. Indeed it clearly suffices to show that the roots of (5.13) are negative. Their product is positive. Furthermore the sign of their sum is the sign of

$$- 2 (\operatorname{Re} + \varepsilon X)(2 \operatorname{Re} + \operatorname{We} \varepsilon X) + (3 \varepsilon - 1) \operatorname{Re} \operatorname{We} X$$

where $X = n^2 \pi^2$. Since this expression equals

$$- 2 \varepsilon^2 \operatorname{We} X^2 - \operatorname{Re} X [4 \varepsilon + (1 - \varepsilon) \operatorname{We}] - 4 \operatorname{Re}^2 < 0,$$

it shows that the roots of (5.13) are negative.

The lemma is proved.

Remark 5.3 :

1. The intervals of linear stability are precisely those where the curve in figure 2.1 is increasing.

2. We do not know how to show that linear stability implies nonlinear stability (e.g. in L^2). However, we proved in section 4 that the solution (v_s, τ_s, α_s) is stable for all k's if $\varepsilon > \frac{1}{9}$, and for $k \in [0, k_1(\varepsilon))$ if $0 < \varepsilon \leqslant \frac{1}{9}$. This $k_1(\varepsilon)$ is smaller than κ_-.

3. If $0 < \varepsilon \leqslant \frac{1}{9}$, the linear stability of (v_s, τ_s, α_s) is lost when the spectrum of $\mathscr{L}$ crosses the imaginary axis by 0, eigenvalue of infinite multiplicity.

6. RELATED PROBLEMS

In this section we investigate some generalizations of our previous results which are of physical interest.

6.1. Several relaxation times

The models under study in the previous sections involve a single relaxation time (λ_1). For an integral model, this corresponds to a relaxation

kernel of exponential type. More realistic models possess a more general kernel. Taking a sum of exponentials leads, for the differential form of the model, to several relaxation times. Namely, the extrastress tensor $\underline{\underline{\tau}}$ is decomposed as

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}_s + \underline{\underline{\tau}}_p \, , \tag{6.1}$$

where

$$\underline{\underline{\tau}}_p = \sum_{i=1}^{N} \underline{\underline{\tau}}_i \, ,$$

and

$$\underline{\underline{\tau}}_i + \lambda_i \frac{\mathscr{D}a}{\mathscr{D}t} \underline{\underline{\tau}}_i = 2 \, \eta_i \, \underline{\underline{D}} \, , \quad \eta_i, \lambda_i > 0 \, , \quad i = 1, ..., N \, , \tag{6.2}$$

$$\underline{\underline{\tau}}_s = 2 \, \eta_s \, \underline{\underline{D}} \, , \quad \eta_s \geqslant 0 \, .$$

We shall restrict ourselves to the case $\eta_s > 0$ (non-zero Newtonian contribution). The full equations of motion become (cf. (1.5)),

$$\begin{cases} \rho \left(\dfrac{\partial u}{\partial t} + (u \cdot \nabla) u \right) + \nabla p - \eta_s \, \Delta u = \displaystyle\sum_{i=1}^{N} \nabla \cdot \underline{\underline{\tau}}_i + f \, , \\[2mm] \operatorname{div} u = 0 \, , \\[2mm] \underline{\underline{\tau}}_i + \lambda_i \dfrac{\mathscr{D}_a \, \underline{\underline{\tau}}_i}{\mathscr{D}t} = 2 \, \eta_i \, \underline{\underline{D}} \, , \end{cases} \tag{6.3}$$

or, in nondimensional form

$$\begin{cases} \operatorname{Re} \left(\dfrac{\partial u}{\partial t} + (u \cdot \nabla) u \right) + \nabla p - (1 - \omega) \, \Delta u = \nabla \cdot \underline{\underline{\tau}}_i + f \, , \\[2mm] \operatorname{div} u = 0 \, , \\[2mm] \underline{\underline{\tau}}_i + W_i \dfrac{\mathscr{D}_a \, \underline{\underline{\tau}}_i}{\mathscr{D}t} = 2 \, \omega_i \, \underline{\underline{D}} \, , \end{cases} \tag{6.4}$$

where here

$$\omega = \frac{\eta_p}{\eta} \, , \quad \omega_i = \frac{\eta_i}{\eta} \, , \quad \eta = \eta_p + \eta_s \, , \quad \eta_p = \sum_{i=1}^{N} \eta_i \, , \quad W_i = \lambda_i \frac{U}{L} \, ,$$

and $\operatorname{Re} = \dfrac{\rho UL}{\eta}$.

Setting $\underline{\underline{\tau}}_i = \begin{pmatrix} \sigma_i & \tau_i \\ \tau_i & \gamma_i \end{pmatrix}$, we reduce the system (5.4), for the Poiseuille or the plane Couette flow, to

$$
\begin{cases}
\mathrm{Re}\, v_t - (1 - \omega)\, v_{xx} = \sum_{i=1}^{N} \tau_{ix} - f\,, \\[2mm]
\alpha_{it} + \dfrac{\alpha_i}{W_i} = (1 - a^2)\, \tau_i\, v_x\,, \\[2mm]
\tau_{it} + \dfrac{\tau_i}{W_i} = \left(\dfrac{\omega_i}{W_i} - \alpha_i \right) v_x\,,
\end{cases}
\tag{6.5}
$$

where

$$
\alpha_i = \frac{1-a}{2}\, \sigma_i - \frac{1+a}{2}\, \gamma_i\,,
$$

$$
\beta_i = (1 - a)\, \sigma_i + (1 + a)\, \gamma_i\,.
$$

As in (1.13) the equations for β_i are linear and uncoupled. The system (6.5) is satisfied for $t \geq 0$, $x \in I$ ($I = (0, 1)$ for the shear flow, $I = (-1, 1)$ for the plane Poiseuille flow).

The boundary conditions are

$$
\begin{aligned}
v|_{x=0} = 0\,, \quad & v|_{x=1} = 1 \text{ for the shear flow,} \\
v|_{x=-1} = 0\,, \quad & v|_{x=1} = 0 \text{ for the Poiseuille flow.}
\end{aligned}
\tag{6.6}
$$

The force f is zero in the first case, and equals a positive number in the second case.

Most of the results of the previous sections carry over to the system (6.5), (6.6). For instance, Theorem 3.2 holds *mutatis mutandis*.

The basic steady solution for the plane Couette flow is

$$
v(x) = x\,, \quad \tilde{\tau} = \tau + \varepsilon v_x = \sum_{i=1}^{N} \frac{\omega_i}{1 + W_i^2 (1 - a^2)} + 1 - \omega
$$

($\tilde{\tau}$ is the total shear stress),

$$
\alpha = (1 - a^2) \sum_{i=1}^{N} \frac{\omega_i\, W_i}{1 + W_i^2 (1 - a^2)}\,.
$$

Steady solutions for the plane Poiseuille flow are given by

$$
\tau = v_x \sum_{i=1}^{N} \frac{\omega_i}{1 + W_i^2 (1 - a^2)\, v_x^2}\,,
\tag{6.7}
$$

$$
\alpha = (1 - a^2)\, v_x^2 \sum_{i=1}^{N} \frac{w_i\, W_i}{1 + W_i^2 (1 - a^2)\, v_x^2}\,,
\tag{6.8}
$$

where v_x is a solution of the algebraic equation of degree $2N + 1$,

$$fx = (1 - \omega)\, v_x + v_x \sum_{i=1}^{N} \left(\frac{\omega_i}{1 + (1 - a^2)\, W_i\, v_x^2} \right). \qquad (6.9)$$

6.2. Non uniform shearing motions

We consider the shearing motions between two infinite parallel walls, the lower wall being fixed and the upper wall moving with the time dependent velocity $K(t)$. For instance, $K(t)$ could be a periodic function $K(t) = U \sin bt$.

If for instance $U = \underset{t}{\mathrm{Sup}}\ K(t)$ is used to define the Reynolds number and

the Weissenberg number, the analog of (1.13) is

$$\begin{cases} \mathrm{Re}\, v_t - (1 - \omega)\, v_{xx} = \tau_x - \mathrm{Re}\, K'(t)\, x\,, \\[2mm] \alpha_t + \dfrac{\alpha}{\mathrm{We}} = (1 - a^2)\, \tau v_x + (1 - a^2)\, K(t)\,, \\[2mm] \tau_t + \dfrac{\tau}{\mathrm{We}} = \left(\dfrac{\omega}{\mathrm{We}} - \alpha \right) v_x + \left(\dfrac{\omega}{\mathrm{We}} - \alpha \right) K(t)\,, \end{cases} \qquad (6.10)$$

where $v(x, t)$ satisfies the homogeneous boundary conditions

$$v(0, t) = v(1, t) = 0\,. \qquad (6.11)$$

When $K(t)$ is a smooth function (with K and K' bounded), Theorem 3.2 can be easily extended to the situation described by (6.10), (6.11) : we get global existence, uniqueness and uniform bounds on the solution.

Note added in proof. After this paper was accepted for publication, the work [16] was brought to our attention by J. Nohel.

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