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INCOMPRESSIBLE LIMIT BEHAVIOUR OF SLIGHTLY COMPRESSIBLE NONLINEAR ELASTIC MATERIALS (*)

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Abstract. — *This paper deals with the convergence of the deformations and stresses of slightly compressible elastic bodies to the incompressible ones as the compressibility goes to zero. The result concerning deformations is a very general convergence result stated in the context of J. Ball's poly-convex stored energy functions. The convergence of the stresses in obtained first formally through an assumption of existence of an asymptotic expansion for the deformation. This asymptotic expansion is then proved to exist in a special case.*

Résumé. — *On s'intéresse à la convergence quand la compressibilité tend vers zéro des défor-mations et contraintes d'un corps élastique peu compressible vers les quantités correspondantes incompressibles. On montre dans le cadre général des densités d'énergie polyconvexes introduites par J. Ball, la convergence des déformations. Pour les contraintes, on donne d'abord un résultat formel reposant sur une hypothèse d'existence d'un développement asymptotique pour la défor-mation. On montre enfin l'existence de ce développement dans un cas particulier.*

0. INTRODUCTION

The materials of the physical world are all more or less compressible. Nevertheless, for certain materials which are usually almost incompressible, such as rubbers, incompressible elasticity is commonly used with good nume-rical results, see Le Tallec [9]. The situation is somewhat surprising, since the equations of compressible elasticity and the equations of incompressible elasticity appear to be very different at first sight. Therefore, the problem of justifying incompressible elasticity by some limit process is posed, cf. Truesdell, Noll [15], p. 122. In the case of linear elasticity, the problem is solved by Lions [10], see also Geymonat, Sanchez Palencia [7], Pelissier [13]. In the nonlinear case, we can mention Ebin [6] for the motion of a slightly compressible fluid.

For a related approximation of the incompressible Navier-Stokes equations, see Témam [15].

We shall be concerned with the behaviour of equilibrium deformations of slightly compressible bodies as the compressibility goes to zero, and the corresponding behaviour of the stresses. The paper is divided in three parts. Part A is devoted to a very general convergence result of compressible solutions to incompressible ones. Part B deals with the formal behaviour of the stress *via* the assumption of the existence of an asymptotic expansion for the solution. In Part C, we partly justify this assumption by showing the existence of such an asymptotic expansion in a special case.

Our analysis in Part A rests mainly on J. Ball's existence and weak continuity results [2]. We penalize a polyconvex coercive incompressible stored energy function in a way first introduced by Ogden [12]. That is, with standard notations :

$$W_\varepsilon(F) = W(F) + \frac{1}{\varepsilon} h(\det F).$$

Then the coercivity and the weak continuity of null Lagrangians, see Ball-Currie-Olver [3], allow us to prove the weak convergence in a Sobolev space $W^{1,\gamma}(\Omega)^3$ of compressible energy minimizers to incompressible energy minimizers.

In the case of Ogden's materials, we show that the convergence is actually strong in $W^{1,\gamma}(\Omega)$. For related results and an extensive numerical study, see Pouyot [14]. For a general strictly polyconvex material, strong convergence also holds. We only indicate how to prove this by using the same ideas as in Ball-Marsden [4].

In part B, we assume that the solutions ϕ_ε belong to $W^{2,p}(\Omega)^3$ for some $p > 3$ and that they satisfy strongly the equilibrium equations. The formal assumption is that there exists an asymptotic expansion :

$$\phi_\varepsilon = \phi_0 + \varepsilon\phi_1 + o(\varepsilon) \quad \text{in} \quad W^{2,p}(\Omega)^3.$$

Then, we show that ϕ_0 is necessarily an incompressible deformation satisfying the equilibrium equation of the limit incompressible problem. Moreover, the hydrostatic pressure p_0 thus associated to ϕ_0, see Le Dret [8], is equal to one third of the limit of the trace of the Cauchy stress tensor corresponding to ϕ_ε. Therefore, the good behaviour of the incompressible model is established for the stresses, at least formally.

Finally, in part C, we show the existence of such an asymptotic expansion in the special case of the pure displacement problem, with small enough body forces, under suitable ellipticity assumptions on the incompressible Piola-

Kirchhoff stress at $\phi = \text{Id}$. We essentially use the results of [8] concerning the local invertibility of the incompressible nonlinear elasticity operator. The existence of the asymptotic expansion follows from the implicit function theorem applied to the equations of equilibrium written in a appropriate form.

A. CONVERGENCE OF THE DEFORMATIONS

I. Notations, preliminary remarks

1) *Notations and hypotheses*

Let Ω be a bounded open connex subset of $\mathbb{R}^3$ to be considered as a fixed reference configuration for all our bodies. Throughout the paper, we shall use the following notations :

ϕ : a deformation, that is a mapping from Ω to $\mathbb{R}^3$ we take in a Sobolev space $W^{1,\gamma}(\Omega)^3$, for some $1 < \gamma < +\infty$.

$F = \nabla\phi$: the gradient of the deformation.

$\text{Adj } F$: the transpose of the cofactor matrix of F.

We consider a one parameter family of compressible hyperelastic materials the stored-energy function of which has the following form (for brevity we shall call the corresponding material the ε-material) :

$$W_\varepsilon(F) = W(F) + \frac{1}{\varepsilon} h(\det F), \quad \varepsilon > 0. \tag{1}$$

where W is an incompressible stored-energy function which is polyconvex, i.e. : there exists a convex function $\overline{W} : \mathbb{R}^9 \times \mathbb{R}^9 \to \mathbb{R}$ such that :

$$W(F) = \overline{W}(F, \text{Adj } F) \quad \text{for all} \quad F \in \mathbb{R}^9.$$

Moreover $\overline{W}$ satisfies the following hypotheses, Ball [2] :

H1 : $\overline{W}$ *is continuous.*

H2 : *There exist constants* $K_1 > 0$, $\gamma > \dfrac{3}{2}$, $\mu > 1$ *and* $\dfrac{1}{\gamma} + \dfrac{1}{\mu} < \dfrac{4}{3}$, *with* :

$$\overline{W}(F, H) \geq K_1(|F|^\gamma + |H|^\mu) \quad (coercivity).$$

We suppose that the function h satisfies :

H3 : $h : \mathbb{R}^+ \to \mathbb{R}$ *is convex.*

H4 : $\lim\limits_{\delta \to 0^+} h(\delta) = \lim\limits_{\delta \to +\infty} h(\delta) = +\infty$, *and* $h(\delta) = 0$ *if and only if* $\delta = 1$.

H5 : *There exists a number r > 1 such that* :

$$h(\delta) \geqq c \mid \delta - 1 \mid^r .$$

We see at once that under the hypotheses H1 to H5, the stored-energy function W_ε is polyconvex and satisfies the coercivity assumptions that allow us to apply J. Ball's existence theorems [2]. Let us consider a measurable partition of the boundary of Ω : $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$ with $\partial\Omega_1 \cap \partial\Omega_2 = \varnothing$. Let there be given an imposed deformation $\overline{\phi}$ on $\partial\Omega_1$ such that $\overline{\phi}$ belongs to $W^{1-1/\gamma,\gamma}(\partial\Omega_1)^3$, an imposed traction t on $\partial\Omega_2$ such that t belongs to $L^\sigma(\partial\Omega_2)^3$, with σ as in theorem 7.6 of J. Ball [2] (depending on the value of γ). Finally, let there be given a body force b in $L^{\gamma'}(\Omega)^3$ $\left(\dfrac{1}{\gamma} + \dfrac{1}{\gamma'} = 1\right)$. Then, the dead loading mixed displacement traction boundary value problem for the body made of the ε-material filling Ω consists in minimizing the following energy integral :

$$I_\varepsilon(\phi) = \int_\Omega W_\varepsilon(\nabla\phi) - \left[\int_\Omega b.\phi + \int_{\partial\Omega_2} t.\phi \right], \qquad (2)$$

over the set of admissible deformations $\mathscr{A}$:

$$\mathscr{A} = \{ \phi \in W^{1,\gamma}(\Omega)^3, \phi_{\mid \Omega_1} = \overline{\phi}, \text{Adj } \nabla\phi \in L^\mu(\Omega)^9,$$

$$\det \nabla\phi \in L^r(\Omega) \text{ and } \det \nabla\phi > 0 \text{ almost everywhere} \} .$$

For the incompressible body whose stored-energy function is W, the same problem reduces to minimizing the energy :

$$I_0(\phi) = \int_\Omega W(\nabla\phi) - \left[\int_\Omega b.\phi + \int_{\partial\Omega_2} t.\phi \right], \qquad (3)$$

over the set $\mathscr{A}_0$:

$$\mathscr{A}_0 = \{ \phi \in W^{1,\gamma}(\Omega)^3, \phi_{\mid\partial\Omega_1} = \phi, \text{Adj } \nabla\phi \in L^\mu(\Omega)^9,$$

$$\det \nabla\phi \equiv 1 \text{ almost everywhere} \} .$$

For details, see J. Ball [2].

2) *Interpretation*

We wish to investigate the behaviour of equilibrium solutions for slightly compressible bodies as the « compressibility » goes to zero, and see whether there is a connection with incompressible elasticity. This latter point is not

so obvious, for the corresponding Euler-Lagrange equations, the equations of equilibrium in mechanics, are very different in nature. The compressible ones are a nonlinear system of second order partial differential equations of standard form. On the other hand, the solution of the incompressible equations must satisfy a highly nonlinear constraint, $\det \nabla \phi = 1$, and there appears a kind of Lagrange multiplier, the hydrostatic pressure p, which is not a function of the deformation gradient $\nabla \phi$, see [16], [8]. It should be interesting to study in general the relationship between this Lagrange multiplier p, and the trace of the stress tensor corresponding to the compressible solution in the limit. However, the results that we shall obtain in the subsequent analysis, are too weak for this purpose. We thus limit ourselves in Part A to the study of the deformations alone.

   The first main problem is to decide what is to be meant by « slightly compressible ». Indeed, such a statement should rather be thought of as the outcome of an experiment, than as an a priori requirement. We adopt here Ogden's point of view in the case of compressible rubberlike solids [12]. A detailed discussion of this kind of stored-energy function may be found there. Let us simply remark that relation (1), while keeping the polyconvexity and coercivity needed by J. Ball's existence theory, is the simplest one and very natural for our purposes. Indeed, the number ($\det F - 1$) is the local measure of compressibility. Adding the term $\frac{1}{\varepsilon} h(\det F)$ only means that an energy supply is needed in order to perform a local change of volume. The penalization factor $\frac{1}{\varepsilon}$ stands here to account for the word « slightly » : as ε goes to zero, the energy needed to change the volume tends to infinity, therefore the material is less and less compressible.

II. Limit behaviour of the solutions as $\varepsilon \to 0$

   Let us make one last hypothesis :

   H6 : *There exists a deformation* $\overline{\phi}$ *in* $\mathscr{A}_0$ *such that* :

$$I_0(\overline{\phi}) < + \infty .$$

As an immediate consequence of J. Ball's theorem 7.6 [2], we obtain :

   PROPOSITION II.1 : *For all* $\varepsilon \geqq 0$, *the mixed displacement traction boundary value problem* :

$$\underset{\phi \in \mathscr{A}}{\text{Inf}} \ I_\varepsilon(\phi), \qquad (\phi \in \mathscr{A}_0 \quad for \quad \varepsilon = 0),$$

has a solution ϕ_ε *in* $\mathscr{A}$ (ϕ_0 *in* $\mathscr{A}_0$).

Proof : We just remark that $\mathscr{A}_0 \subset \mathscr{A}$. Therefore, since $h(1) = 0$,

$$\overline{\phi} \in \mathscr{A} \quad \text{and} \quad I_\varepsilon(\overline{\phi}) = I_0(\overline{\phi}) < + \infty \, .$$

Then J. Ball's theorem gives the result.　□

THEOREM II.2 : *For each sequence* $\varepsilon_n \to 0$, *there exists a subsequence* $\varepsilon_{n'}$ *and a solution* ϕ_0' *of the incompressible problem such that :*

$$\phi_{\varepsilon_n'} \rightharpoonup \phi_0' \quad \text{weakly in} \quad W^{1,\gamma}(\Omega)^3 \, ,$$

$$\text{Adj } F_{\varepsilon_n'} \rightharpoonup \text{Adj } F_0' \quad \text{weakly in} \quad L^\mu(\Omega)^9 \, ,$$

$$\det F_\varepsilon \to 1 \quad \text{strongly in} \quad L^r(\Omega) \, .$$

Proof : For brevity, we shall omit the index n and not explicitly extract subsequences, while keeping this is mind.

Since $\mathscr{A}_0 \subset \mathscr{A}$, we have :

$$I_\varepsilon(\phi_\varepsilon) = \inf_{\phi \in \mathscr{A}} I_\varepsilon(\phi) \leqq \inf_{\phi \in \mathscr{A}_0} I_\varepsilon(\phi) = I_0(\phi_0) \, .$$

Then, by the coercivity hypotheses H2 and H5, one sees, as in Ball [2], that :

$$\| F_\varepsilon \|_\gamma < C \, ,$$

$$\| \text{Adj } F_\varepsilon \|_\mu < C \, ,$$

$$\| \det F_\varepsilon \| < C \, ,$$

for some constant C independant of ε.

Therefore, there exists (ϕ_0', H, δ) in $W^{1,\gamma}(\Omega)^3 \times L^\mu(\Omega)^9 \times L^r(\Omega)$ such that (the symbol $\rightharpoonup$ denotes weak convergence) :

$$\phi_\varepsilon \rightharpoonup \phi_0' \quad \text{in} \quad W^{1,\gamma}(\Omega)^3 \, ,$$

$$\text{Adj } F_\varepsilon \rightharpoonup H \quad \text{in} \quad L^\mu(\Omega)^9 \, ,$$

$$\det F_\varepsilon \rightharpoonup \delta \quad \text{in} \quad L^r(\Omega) \, .$$

By the weak continuity of the adjoint and of the determinant, we infer that :

$$H = \text{Adj } F_0' \, , \quad \delta = \det F_0' \quad \text{and} \quad \phi_0' \in \mathscr{A} \, .$$

Since W is polyconvex, the incompressible energy I_0 is weakly lower semi continuous, and so :

$$I_0(\phi_0') \leqq \underline{\lim} \, I_0(\phi_\varepsilon) \, .$$

Now :

$$\varepsilon I_\varepsilon(\phi_\varepsilon) = \varepsilon I_0(\phi_\varepsilon) + \int_\Omega h(\det F_\varepsilon),$$

and :

$$\varepsilon I_0(\phi_\varepsilon) \leqq \varepsilon I_\varepsilon(\phi_\varepsilon) \leqq \varepsilon I_0(\phi_0).$$

Since $I_0(\phi_\varepsilon)$ is bounded below, we infer that $\varepsilon I_\varepsilon(\phi_\varepsilon) \to 0$ as $\varepsilon \to 0$.
   As $\varepsilon I_0(\phi_\varepsilon) \to 0$, we get :

$$\int_\Omega h(\det F_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Using hypothesis H5, this yields :

$$\det F_\varepsilon \to 1 \quad \text{in} \quad L'(\Omega),$$

and therefore :

$$\det F_0' = \delta = 1, \quad \text{or} \quad \phi_0' \in \mathcal{A}_0.$$

Now :

$$\underset{\phi \in \mathcal{A}_0}{\text{Inf}} \, I_0(\phi) = I_0(\phi_0) \geqq \underline{\lim} \, I_\varepsilon(\phi_\varepsilon) \geqq \underline{\lim} \, I_0(\phi_\varepsilon) + \underline{\lim} \frac{1}{\varepsilon} \int_\Omega h(\det F_\varepsilon).$$

The mapping I_0 is weakly lower semi continuous, then :

$$I_0(\phi_0) \geqq I_0(\phi_0') + \underline{\lim} \frac{1}{\varepsilon} \int_\Omega h(\det F_\varepsilon) \geqq \underset{\phi \in \mathcal{A}_0}{\text{Inf}} \, I_0(\phi) + \underline{\lim} \frac{1}{\varepsilon} \int_\Omega h(\det F_\varepsilon).$$

Since h is positive, this inequality implies that :

$$\begin{cases} \underline{\lim} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega h(\det F_\varepsilon) = 0 \\[2ex] \phi_0' \text{ is a solution of the incompressible problem.} \quad \square \end{cases}$$

We obtain a little more information about the convergence of $\det F$ in the next proposition :

PROPOSITION II.3 : *We have (up to a subsequence)* :

$$\| \det F_\varepsilon - 1 \|_r = o(\varepsilon^{1/r}).$$

Proof : From the proof of proposition II.2, we see that :

$$\underline{\lim} \, I_\varepsilon(\phi_\varepsilon) = I_0(\phi_0').$$

We have :

$$I_0(\phi_\varepsilon) = I_\varepsilon(\phi_\varepsilon) - \frac{1}{\varepsilon} \int_\Omega h(\det F_\varepsilon) \leqq I_0(\phi_0).$$

Therefore passing to the inferior limit, we obtain :

$$I_0(\phi_0') \leqq \varliminf \left(I_\varepsilon(\phi_\varepsilon) - \frac{1}{\varepsilon} \int_\Omega h(\det F_\varepsilon) \right) \leqq I_0(\phi_0).$$

Since $I_0(\phi_0') = I_0(\phi_0) = \underset{\phi \in \mathscr{A}_0}{\text{Inf}}\ I_0(\phi),$

$$\varliminf \left(I_\varepsilon(\phi_\varepsilon) - \frac{1}{\varepsilon} \int_\Omega h(\det F_\varepsilon) \right) = I_0(\phi_0').$$

We extract a subsequence ε' such that :

$$I_{\varepsilon'}(\phi_{\varepsilon'}) \to I_0(\phi_0').$$

Then :

$$\varliminf \left(I_{\varepsilon'}(\phi_{\varepsilon'}) - \frac{1}{\varepsilon'} \int_\Omega h(\det F_{\varepsilon'}) \right) = I_0(\phi_0') - \varlimsup \frac{1}{\varepsilon'} \int_\Omega h(\det F_{\varepsilon'}).$$

Therefore :

$$\varlimsup \frac{1}{\varepsilon'} \int_\Omega h(\det F_{\varepsilon'}) = 0.$$

From the proof of proposition II.2, we obtain :

$$\frac{1}{\varepsilon'} \int_\Omega h(\det F_{\varepsilon'}) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By hypothesis H5, this means that :

$$\| \det F_{\varepsilon'} - 1 \|_r = o(\varepsilon'^{1/r}). \quad \square$$

As was pointed out to us by Prof. Le Tallec, this latter fact implies in the case of Ogden's materials, that all the convergences in theorem II.2 are actually strong.

PROPOSITION II.4 : *Let the body be made of an Ogden's material, i.e. :*

$$W(F) = a \operatorname{tr}(C^{\gamma/2}) + b \operatorname{tr}((\operatorname{Adj} C)^{\mu/2}) + \frac{1}{\varepsilon} h(\det F),$$

where : $C = F^T F$, $a > 0$, $b > 0$, *then* :

$$\phi_{\varepsilon'} \to \phi'_0 \quad \text{strongly in} \quad W^{1,\gamma}(\Omega)^3,$$

$$\text{Adj } F_{\varepsilon'} \to \text{Adj } F'_0 \quad \text{strongly in} \quad L^\mu(\Omega)^9.$$

Proof : From proposition II.34, we deduce at once that :

$$I_\varepsilon(\phi_\varepsilon) \to I_0(\phi'_0).$$

In the case of such an Ogden's material, the energy $I_\varepsilon(\phi_\varepsilon)$ involves the norms of F_ε in $L^\gamma(\Omega)^9$ and Adj F_ε in $L^\mu(\Omega)^9$:

$$I_\varepsilon(\phi_\varepsilon) = a \parallel F_\varepsilon \parallel_\gamma^\gamma + b \parallel \text{Adj } F_\varepsilon \parallel_\mu^\mu + \frac{1}{\varepsilon} h(\det F_\varepsilon) - \int_\Omega b.\phi_\varepsilon - \int_{\partial\Omega_2} t.\phi_\varepsilon.$$

Now :

$$\frac{1}{\varepsilon'} h(\det F_{\varepsilon'}) \to 0,$$

$$\int_\Omega b.\phi_{\varepsilon'} + \int_{\partial\Omega_2} t.\phi_{\varepsilon'} \to \int_\Omega b.\phi'_0 + \int_{\partial\Omega_2} t.\phi'_0,$$

Therefore :

$$a \parallel F_{\varepsilon'} \parallel_\gamma^\gamma + b \parallel \text{Adj } F_{\varepsilon'} \parallel_\mu^\mu \to a \parallel F'_0 \parallel_\gamma^\gamma + b \parallel \text{Adj } F'_0 \parallel_\mu^\mu.$$

Since the functions $\parallel . \parallel_\gamma^\gamma$ and $\parallel \text{Adj} . \parallel_\mu^\mu$ are weakly lower semi continuous, and a, b are strictly positive, this implies that :

$$\parallel F_{\varepsilon'} \parallel_\gamma \to \parallel F'_0 \parallel_\gamma \quad \text{as} \quad \varepsilon' \to 0,$$

$$\parallel \text{Adj } F_{\varepsilon'} \parallel_\mu \to \parallel \text{Adj } F'_0 \parallel_\mu \quad \text{as} \quad \varepsilon' \to 0.$$

Now, we use the fact that L^p spaces are uniformly convex for $1 < p < \infty$, and that in uniformly convex spaces strong convergence is equivalent to weak convergence together with the convergence of norms, to conclude that :

$$F_{\varepsilon'} \to F'_0 \quad \text{in} \quad L^\gamma(\Omega)^9 \quad \text{strongly},$$

$$\text{Adj } F_{\varepsilon'} \to \text{Adj } F'_0 \quad \text{in} \quad L^\mu(\Omega)^9 \quad \text{strongly}. \quad \square$$

For a related result in the discrete approximation case, see Le Tallec [9]. Proposition II.4 can be extended to more general Ogden's materials such as those described below.

We can actually state an almost general result, following ideas of Ball-Marsden [4], as follows :

THEOREM II.5 : *Let the function $\overline{W}$ be strictly convex. Then the same result as in proposition II.4 holds.*

Proof : We only sketch it, since it follows exactly the lines of Ball-Marsden [4] theorem 4.9. The point is to use the convergence of the energy and the strict convexity of $\overline{W}$ to extract a subsequence such that F_ε and Adj F_ε converge almost everywhere. Then we use the coercivity assumption H2 to control the norms $\| F_\varepsilon - F_0 \|_\gamma$ and $\| \text{Adj } F_\varepsilon - \text{Adj } F_0 \|_\mu$ and apply Fatou's lemma, which yields the result. $\square$

III. Remarks and conclusions

We make a few more comments about the choice of the stored-energy function (1). This choice is of course rather arbitrary. Indeed, not any actual slightly compressible elastic material can be imbedded in such a family : but the same results hold if we directly penalize any polyconvex stored energy function W in the same fashion, granted that we take the restriction of W to the manifold det $F = 1$ as incompressible energy function. There are other possible penalizations, however this particular one is simple and natural as explained in I.2. Ogden [12], used it to fit experimental data with calculations, and his justifications relie partly upon experimental evidence. Therefore, relation (1) surely defines a good model for an actual physically slightly compressible material. Let us also mention that, given any pair λ, μ of Lamé coefficients, it is possible to adjust an Ogden's material, i.e. a material whose stored-energy function is :

$$W(F) = \sum_{i=1}^{M} a_i \operatorname{tr}(C^{\alpha_i/2}) + \sum_{j=1}^{N} b_j \operatorname{tr}((\text{Adj } C)^{\beta_j/2}) + \Gamma(\det F),$$

(where $a_i > 0$, $b_j > 0$, $\alpha_i \geqq 1$, $\beta_j \geqq 1$, $C = F^T F$ and Γ is convex), so that the linearization at $F = \text{Id}$ of the corresponding stress tensor exactly is :

$$T(F) = \lambda \operatorname{tr} E + 2 \mu E, \quad \text{for} \quad E = \frac{1}{2}(F + F^T),$$

while in the same time, J. Ball's existence theorems work for the nonlinear problem, Ciarlet-Geymonat [5]. In the context of slightly compressible fluids, Ebin [6], uses exactly the same kind of penalized constitutive law in order to prove the convergence of compressible motions to incompressible ones, *via* differential geometric methods.

We have thus proved the convergence as ε tends to 0 of the compressible energy minimizers ϕ_ε to an incompressible energy minimizer ϕ_0. The convergence is strong in $W^{1,\gamma}(\Omega)$.

B. FORMAL BEHAVIOUR OF THE STRESS IN THE INCOMPRESSIBLE LIMIT

I. Hypotheses

In this part, we suppose that the compressible problems have a solution ϕ_ε in the space $W^{2,p}(\Omega)^3$, $p > 3$, for $\varepsilon_0 > \varepsilon > 0$, the deformation ϕ_ε satisfying strongly the equilibrium equations :

$$\left.\begin{array}{lll} \mathrm{div}\, T_R^\varepsilon(F_\varepsilon) + b = 0 & \mathrm{in}\quad \Omega,\, b \in L^p(\Omega)^3 \,, \\[2mm] T_R^\varepsilon(F_\varepsilon)\, n = t & \mathrm{on}\quad \partial\Omega_2 \,,\quad t \in W^{1-1/p,p}(\partial\Omega_2)^3 \,, \\[2mm] \phi_\varepsilon = \overline{\phi} & \mathrm{on}\quad \partial\Omega_1 \,,\quad \overline{\phi} \in W^{2,p}(\Omega)^3 \,, \end{array}\right\} \qquad (4_\varepsilon)$$

where T_R^ε denotes the first Piola-Kirchhoff stress tensor corresponding to the ε-material defined by relation (1), namely :

$$T_R^\varepsilon(F) = \frac{\partial W_\varepsilon}{\partial F}(F)\,, \quad \mathrm{for}\quad \varepsilon > 0\,.$$

The explicit expression of T_R^ε is therefore the following :

$$T_R^\varepsilon(F) = \frac{\partial W}{\partial F}(F) + \frac{1}{\varepsilon}\, h'(\det F)\, \mathrm{Adj}\, F^T\,.$$

More generally, we shall consider non necessarily hyperelastic materials the constitutive law of which we shall write in the following way.

$$T_R^\varepsilon(F) = T^D(F)\, \mathrm{Adj}\, F^T + \left(t_1(F) + \frac{1}{\varepsilon}\, h'(\det F)\right) \mathrm{Adj}\, F^T\,. \qquad (5)$$

Here, the tensor $T^D(F)$ is nothing but the deviatoric part of the Cauchy stress tensor $T(F)$. We shall assume that :

H5' : $h'(\delta) = 0$ *if and only if* $\delta = 1$.
H6 : $T^D(.) \in C^2(\mathbb{R}^9, \mathbb{R}^9)$, $t_1(.) \in C^2(\mathbb{R}^9, \mathbb{R})$ *and* $h(.) \in C^3(\mathbb{R})$.

Under these hypotheses, it follows from the results of Valent [17], that the operator :

$$W^{2,p}(\Omega)^3 \rightarrow W^{1,p}(\Omega)^9\,,$$

$$\phi \rightarrow T_R^\varepsilon(F)\,,$$

is of class C^1 and is locally bounded together with its first derivative. Therefore we shall be allowed to use Taylor-Young expansions up to the first order.

Hypothesis I.1 : *We assume that the solution* ϕ_ε *admits an asymptotic expansion of the form* :

$$\phi_\varepsilon = \phi_0 + \varepsilon\phi_1 + o(\varepsilon) \quad in \quad W^{2,p}(\Omega)^3, \quad where \quad \phi_0, \phi_1 \quad belong \ to \quad W^{2,p}(\Omega)^3$$

$$\phi_0 = \overline{\phi} \quad on \quad \partial\Omega_1, \quad \varepsilon\phi_1 + o(\varepsilon) = 0 \quad on \quad \partial\Omega_1 \ .$$

Hypothesis I.2 : *If* $\partial\Omega_2 \neq \varnothing$ *(mixed problem), we assume that* ϕ_0 *is a global diffeomorphism of* $\overline{\Omega}$ *on to* $\phi_0(\overline{\Omega})$.

Hypothesis I.3 : *If* $\partial\Omega_2 = \varnothing$ *(pure displacement problem), we assume that* $\overline{\phi}$ *is a global diffeomorphism of* $\overline{\Omega}$ *onto* $\overline{\phi}(\overline{\Omega})$ *such that* meas $(\phi(\Omega)) = $ meas Ω. *Moreover, we assume that* Ω *is diffeomorphic to the unit ball of* $\mathbb{R}^3$.

Remarks I.4 : Hypothesis I.1 might be too strong in general when $\partial\Omega_2 \neq \varnothing$ (apart from the fact that no general existence result is known for system (4_ε)). The solution should rather be expected to belong to $W^{1,p}(\Omega)^3$. However, in the pure displacement case if the body force b is small enough, we shall prove, in part C, the existence of such an asymptotic expansion.

II. LIMIT BEHAVIOUR AS ε GOES TO 0

Let us first recall some results about incompressible bodies. The constitutive law of an incompressible elastic body is defined by the tensor $T^D(F)$ only. We shall say that a deformation ϕ satisfies the restricted principle of virtual works if the following equalities hold, see [8] :

For all ψ in $H^1(\Omega)^3$ such that $\psi_{|\partial\Omega_1} = 0$ and Adj $\nabla\phi^T : \nabla\psi = 0$ in Ω,

$$-\int_\Omega T^D(F) \ \mathrm{Adj} \ F^T : \nabla\psi + \int_\Omega b.\psi + \int_{\partial\Omega_2} t.\psi = 0 \ , \tag{6}$$

where $F = \nabla\phi$, the dot denotes the scalar product of $\mathbb{R}^3$ and the colon denotes the scalar product of 3×3 matrices, i.e. :

$$A : B = \mathrm{tr} \ A^T B \ , \quad for \quad A, B \ in \ M_3 \ (the \ space \ of \ 3 \times 3 \ matrices) \ .$$

Then, the following result holds.

PROPOSITION II.1 [8] : *Let there be given a deformation* ϕ *satisfying the restricted principle of virtual works* (6). *Then, there exists a unique pressure field* p *in* $L^2(\Omega)$ *(* p *in* $L^2(\Omega)/\mathbb{R}$ *in the pure displacement case) such that* ϕ *satisfies the « full » principle of virtual works : for all* ψ *in* $H^1(\Omega)^3$, $\psi_{|\partial\Omega_1} = 0$,

$$-\int_\Omega (T^D + p \ \mathrm{Id}) \ \mathrm{Adj} \ F^T : \nabla\psi + \int_\Omega b.\psi + \int_{\partial\Omega_2} t.\psi = 0 \ . \tag{7}$$

Remarks : Equations (7) are the weak form of the equilibrium equations for the incompressible material defined by the tensor T^D. Proposition II.1 is thus a justification of the generally accepted constitutive law for incompressible bodies, [16] :

$$T(F) = H(F) + p \, \text{Id} \,,$$

where p is a so-called indeterminate hydrostatic pressure. Let us state the main result of this section :

THEOREM II.2 : *Under the hypotheses* I.1, I.2 *and* I.3, *we have*

i) $\det \nabla \phi_0 = 1$ *in* Ω,

ii) ϕ_0 *is a solution of the equilibrium equations of the incompressible body defined by the tensor* T^D,

iii) *the hydrostatic pressure* p_0 *associated to* ϕ_0 *by proposition* II.1 *satisfies* :

$$p_0 = \lim_{\varepsilon \to 0} \frac{1}{3} \, \text{tr} \, T^\varepsilon = t_1(F_0) + h''(1) \, (\text{Adj} \, F_0^T : F_1) \,,$$

where T^ε is the Cauchy stress tensor : $T^\varepsilon(F_\varepsilon) = \dfrac{1}{\det F_\varepsilon} \, T_R^\varepsilon(F_\varepsilon) \, F_\varepsilon^T$.

Remark : The hydrostatic pressure appears here (formally) as the strong limit of the trace of the Cauchy stress as the compressibility goes to zero.

Let us start with some lemmas.

LEMMA II.3 : *The matrix* $\text{Adj} \, F_\varepsilon$ *admits an asymptotic expansion* :

$$\text{Adj} \, F_\varepsilon = \text{Adj} \, F_0 + \varepsilon D_{F_0} \, \text{Adj}.F_1 + o(\varepsilon) \quad in \quad W^{1,p}(\Omega)^9 \,.$$

Proof : As a function from M_3 to M_3, the mapping Adj is a homogeneous polynomial of degree 2. We immediately infer that : for all A, B in M_3,

$$\text{Adj} \, (A + B) = \text{Adj} \, A + D_A \, \text{Adj}.B + \text{Adj} \, B \,.$$

Then :

$$\text{Adj} \, F_\varepsilon = \text{Adj} \, F_0 + D_{F_0} \, \text{Adj} \, (\varepsilon F_1 + o(\varepsilon)) + \text{Adj} \, (\varepsilon F_1 + o(\varepsilon))$$

$$= \text{Adj} \, F_0 + \varepsilon D_{F_0} \, \text{Adj}(F_1 + O(\varepsilon)) + \varepsilon^2 \, \text{Adj}(F_1 + O(\varepsilon)) \,.$$

Since $p > 3$, $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $\text{Adj}(F_1 + O(\varepsilon))$ is bounded in $W^{1,p}(\Omega)^9$ ($W^{1,p}(\Omega)$ is a Banach algebra [1]), therefore the lemma clearly holds. $\square$

LEMMA II.4 : *The first Piola-Kirchhoff stress tensor* $T_R^\varepsilon(F)$ *admits the asymptotic expansion* :

$$T_R^\varepsilon(F_\varepsilon) = \frac{1}{\varepsilon} h'(\det F_0) \text{ Adj } F_0^T +$$

$$+ (T^D(F_0) + (h''(\det F_0) \text{ Adj } F_0^T : F_1 + t_1(F_0)) \text{ Id}) \text{ Adj } F_0^T$$
$$+ h'(\det F_0) \text{ Adj } F_0^T(D_{F_0} \text{ Adj.} F_1)^T + o(1) \quad in \quad W^{1,p}(\Omega)^9 .$$

Proof : Since $W^{1,p}(\Omega)$ is a Banach algebra, we obtain the asymptotic expansions of products by multiplying the asymptotic expansions of the factors. We have :

$$T^D(F_\varepsilon) = T^D(F_0) + o(1), \quad \text{by continuity} .$$

$$h'(\det F) = h'(\det F_0) + \varepsilon h''(\det F_0) \text{ Adj } F_0^T : F_1 + o(\varepsilon),$$

by Taylor's formula.

Therefore, using lemma II.3, we get :

$$T_R^\varepsilon(F_\varepsilon) = \left(\frac{1}{\varepsilon} h'(\det F_0) \text{ Id } + T^D(F_0) + \right.$$

$$+ (h''(\det F_0) \text{ Adj } F_0^T : F_1 + t_1(F_0)) \text{ Id } + o(1))$$
$$\times (\text{Adj } F_0^T + \varepsilon(D_{F_0} \text{ Adj.} F_1)^T + o(\varepsilon)),$$

$$= \frac{1}{\varepsilon} h'(\det F_0) \text{ Adj } F_0^T + (T^D(F_0) +$$

$$+ (h''(\det F_0) \text{ Adj } F_0^T : F_1 + t_1(F_0)) \text{ Id}) \text{ Adj } F_0^T$$
$$+ h'(\det F_0) (D_{F_0} \text{ Adj.} F_1)^T + o(1) . \quad \square$$

Proof of theorem II.2 i) : The mapping div is linear continuous from $W^{1,p}(\Omega)^9$ to $L^p(\Omega)^3$, therefore the equilibrium equations (4_ε) can be expanded in the following way :

$$\begin{cases} \dfrac{1}{\varepsilon} \text{ div } (h'(\det F_0) \text{ Adj } F_0^T) + \text{ div } ((T^D(F_0) + \\[2mm] + (h''(\det F_0) \text{ Adj } F_0^T : F_1 + t_1(F_0)) \text{ Id}) \text{ Adj } F_0^T) \\[2mm] + \text{ div } (h'(\det F_0) \text{ Adj } F_0^T(D_{F_0} \text{ Adj.} F_1)^T) + b + o(1) = 0 \quad \text{in} \quad \Omega, \\[2mm] \text{corresponding expansion on } \partial\Omega_2 . \end{cases} \quad (8_\varepsilon)$$

We multiply equations (8_ε) by ε, and then let $\varepsilon = 0$. This yields :

$$\begin{cases} \text{div } (h'(\det F_0) \text{ Adj } F_0^T) = 0 & \text{in} \quad \Omega \\[2mm] h'(\det F_0) \text{ Adj } F_0^T.n = 0 & \text{on} \quad \partial\Omega_2 \\[2mm] \phi_0 = \overline{\phi} & \text{on} \quad \partial\Omega_1 . \end{cases} \quad (9)$$

Recall that by the Sobolev imbedding theorem, ϕ_0 is of class C^1 and F_0, Adj F_0, det F_0 are continuous on $\overline{\Omega}$.

The first equation in (9) gives :

$$\text{Adj } F_0^T \nabla(h'(\det F_0)) = 0 \text{ in } \Omega, \quad \text{since} \quad \text{div Adj } F_0^T = 0 \text{ (Piola identity)}.$$
(10)

We now separate the pure displacement case and the mixed case.

* mixed case :

By hypothesis I.2, ϕ_0 is a global diffeomorphism of $\overline{\Omega}$ onto $\phi_0(\overline{\Omega})$, therefore Adj F_0^T is everywhere nonsingular. Consequently :

$$\nabla(h'(\det F_0)) = 0 \quad \text{in } \Omega, \text{ and since } \Omega \text{ is connex :}$$
$$h'(\det F_0) = C, \quad \text{for some constant } C, \text{ in } \Omega.$$

The second equation in (9) now gives :

$$h'(\det F_0) = 0 \quad \text{on } \partial\Omega_2, \quad \text{therefore :}$$
$$h'(\det F_0) = 0 \quad \text{in } \overline{\Omega}, \quad \text{or equivalently by H5}', \quad \det F_0 = 1 \text{ in } \overline{\Omega}.$$

* pure displacement case :

By hypothesis I.3, $\overline{\phi}$ is a global diffeomorphism of $\overline{\Omega}$ onto $\overline{\phi}(\overline{\Omega})$. Let us extend $\overline{\phi}^{-1}$ to $\mathbb{R}^3$. Then $\overline{\phi}^{-1} \circ \phi_{0|\partial\Omega} = \text{Id}$. We thus reduce ourselves to the case $\overline{\phi} = \text{Id}$. We then reduce ourselves to the unit ball by hypothesis I.3 again. Let us set :

$$\mathcal{O} = \{ x \in \Omega, \quad \text{Adj } F_0^T(x) \text{ is nonsingular} \}.$$

Then $\mathcal{O}$ is an open subset of Ω, and as in the mixed case, we see that $h'(\det F_0)$ is constant on each connex component of $\mathcal{O}$. Let us index the connex components of $\mathcal{O}$ by the values of $h'(\det F_0)$:

$\mathcal{O}_c$ is a connex component of Ω where $h'(\det F_0) = c$. Fix c, and let $U_c = \cup \mathcal{O}_c$ the union of all such components. Then, U_c is open as a union of open sets. Moreover U_c is closed. Indeed, $U_c = h'(\det F_0)^{-1} \{ c \} \cap \mathcal{O}$. If $U_c \neq \varnothing$, then $c \neq h'(0)$ by the definition of $\mathcal{O}$. Let x be a point of $h'(\det F_0)^{-1} \{ c \}$. Then, $h'(\det F_0(x)) \neq h'(0)$ and therefore $\det F_0(x) \neq 0$ or $x \in \mathcal{O}$. Consequently, $h'(\det F_0)^{-1} \{ c \} \subset \mathcal{O}$ and $U_c = h'(\det F_0)^{-1} \{ c \}$. Since $h'(\det F_0)$ is continuous, U_c is closed. Now, since Ω is connex, we see that the following alternative holds :

α) there exists a unique c in $\mathbb{R}$, $c \neq h'(0)$, with $\Omega = U_c$, or,

β) $\mathcal{O} = \varnothing$, or Adj F_0^T is singular over Ω.

Now, if β) holds, by Sard's theorem we get :

$$\text{meas } \phi_0(\Omega) = 0 .$$

Moreover $\phi_0(\overline{\Omega})$ is compact and has also measure zero. Therefore, $\phi_0(\overline{\Omega}) \cap \overline{\Omega}$ is not dense in $\overline{\Omega}$. So there exists an open subset U of $\overline{\Omega}$ such that $U \cap \phi_0(\overline{\Omega}) = \varnothing$. Since $\phi_{0|\partial\Omega} = \text{Id}$, we can take $\overline{U} \subset \Omega$. Let us consider a ball $B \subset U$. Then there exists a retraction of $\overline{\Omega} \setminus B$ onto $\partial\Omega$, i.e. a continuous mapping p from $\overline{\Omega} \setminus B$ to $\partial\Omega$, such that $p_{|\partial\Omega} = \text{Id}$. By construction, $p \circ \phi_0$ is continuous and is actually a retraction of $\overline{\Omega}$ onto $\partial\Omega$. This is impossible, therefore α) holds. We now return to the initial Ω and $\overline{\phi}$. Let us denote by $d(\phi, \Omega, b)$ the topological degree of the mapping ϕ at the point b with respect to Ω. Then :

for all $b \notin \phi_0(\partial\Omega)$, $d(\phi_0, \Omega, b) = d(\overline{\phi}, \Omega, b)$,

since $\phi_{0|\partial\Omega} = \overline{\phi}_{|\partial\Omega}$ [11].

By hypothesis I.3, $\overline{\phi}$ is a global diffeomorphism, therefore :

for $b \in \overline{\phi}(\overline{\Omega}) \setminus \overline{\phi}(\partial\Omega)$, $d(\phi_0, \Omega, b) = 1$, and

for $b \notin \overline{\phi}(\overline{\Omega})$, $d(\phi_0, \Omega, b) = 0$.

Since $\det F_0 = \delta \neq 0$, by the definition of the degree :

$$d(\phi_0, \Omega, b) = \sum_{x \in \phi_0^{-1}(b)} \text{sign}(\delta) = \text{sign}(\delta) \text{ Card } \{ \phi_0^{-1}(b) \} .$$

Then, $\delta > 0$ and :

for $b \in \overline{\phi}(\overline{\Omega}) \setminus \overline{\phi}(\partial\Omega)$, Card $\{ \phi_0^{-1}(b) \} = 1$ and,

for $b \notin \overline{\phi}(\overline{\Omega})$, Card $\{ \phi_0^{-1}(b) \} = 0$.

Now $\phi_{0|\partial\Omega} = \overline{\phi}_{|\partial\Omega}$, and we see that ϕ_0 is a global diffeomorphism from $\overline{\Omega}$ onto $\overline{\phi}(\overline{\Omega})$. Finally, thanks to the change of variable formula for integrals, we get :

$$\text{meas } \Omega = \int_\Omega dx = \int_{\phi_0(\Omega)} \det F_0 \, dy = \delta \int_{\overline{\phi}(\Omega)} dy = \delta \text{ meas } \overline{\phi}(\Omega),$$

and hence, $\det F_0 = \delta = 1$. $\quad\square$

Remarks : In the mixed case, we have to assume a priori the global invertibility of ϕ_0. In the pure displacement case, this is a consequence of the same assumption made on the boundary values $\overline{\phi}$ only, which is very natural.

Proof of theorem II.2 ii) *and* iii) : From the preceding step, we have seen that in any case, $h'(\det F_0) = 0$. We let ε tend to zero in the remaining terms of equations (8_ε). This yields :

$$
\begin{cases}
\text{div}\,((T^D(F_0) + (h''(\det F_0)\,\text{Adj}\,F_0^T : F_1 + t_1(F_0))\,\text{Id})\,\text{Adj}\,F_0^T) + b = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{in}\quad \Omega\,, \\
(T^D(F_0) + (h''(\det F_0)\,\text{Adj}\,F_0^T : F_1 + t_1(F_0))\,\text{Id})\,\text{Adj}\,F_0^T\,n = t \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{on}\quad \partial\Omega_2\,,\quad (11) \\
\phi_0 = \bar{\phi}\quad\text{on}\quad \partial\Omega_1\,.
\end{cases}
$$

Now, $T^D(F_0)$ belongs to $W^{1,p}(\Omega)^9 \hookrightarrow H^1(\Omega)^9$, as well as the other terms. We can clearly multiply equations (11) by tests functions ψ of $H^1(\Omega)^3$ such that :

$$
\psi_{|\partial\Omega_1} = 0\,,\quad \text{Adj}\,F_0^T : \nabla\psi = 0\quad\text{almost everywhere}\,.
$$

Integrating the resulting equalities by parts, we get :

$$
-\int_\Omega T^D(F_0)\,\text{Adj}\,F_0^T : \nabla\psi + \int_\Omega b.\psi + \int_{\partial\Omega_1} t.\psi = 0\,.
$$

Therefore ϕ_0 satisfies the restricted principle of virtual works (6), and is a solution of the equilibrium equations for the limit incompressible body. By proposition II.1, we see that the associated hydrostatic pressure p_0 satisfies :

$$
p_0 = \frac{1}{3}\lim_{\varepsilon\to 0}\text{tr}\,T^\varepsilon(F) = h''(\det F_0)\,\text{Adj}\,F_0^T : F_1 + t_1(F_0)\,.\quad\square
$$

Remarks : Let us point out again that the results of theorem II.2 are quite formal. The general convergence result we have proved in part A gives us only a convergence in $W^{1,p}(\Omega)^3$. Here, we have assumed more regularity of the solutions $(W^{2,p}(\Omega)^3)$, we have assumed that they satisfy strongly the equilibrium equations and finally we have assumed a strong C^1 dependence of the solution ϕ_ε on the parameter ε. In the mixed case, all thoses hypotheses surely are too strong. However, we shall see in part C that they can be satisfied for the pure displacement problem, provided that the body force b is small enough.

C. EXISTENCE OF THE ASYMPTOTIC EXPANSION OF THE SOLUTION IN THE PURE DISPLACEMENT CASE

I. Hypotheses

In this section, we shall assume that the function h is of class C^3, and we normalize it by taking $h''(1) = 1$ without loss of generality. Therefore, the equations to solve are :

$$\begin{cases} \text{div} \left((T^D(F_\varepsilon) + t_1(F_\varepsilon) \text{ Id}) \text{ Adj } F_\varepsilon^T \right) + \dfrac{1}{\varepsilon} \text{ div } (h'(\det F_\varepsilon) \text{ Adj } F_\varepsilon^T) + b = 0 \text{ in } \Omega \\ \phi_\varepsilon = \text{Id} \quad \text{on} \quad \partial\Omega . \end{cases} \tag{12}$$

where $\quad \phi_\varepsilon = \phi_0 + \varepsilon\phi_1(\varepsilon) + \varepsilon^2 \phi_2(\varepsilon) \quad$ with $\quad \phi_0, \phi_1, \phi_2 \quad$ in $\quad W^{2,p}(\Omega)$.

$$\tag{13}$$

The idea is to start the expansion by a known solution ϕ_0 of the incompressible problem. To do this, we make on the tensor T^D suitable ellipticity assumptions at $\phi = \text{Id}$, see [8], so that the following proposition holds :

let us denote by :

$$\Sigma_{2,p}(\Omega) = \{ \phi \in W^{2,p}(\Omega)^3 , \quad \phi_{|\partial\Omega} = \text{Id}, \quad \det \nabla\phi = 1 \quad \text{in} \quad \Omega \},$$

Since $p > 3$, this set has a manifold structure, [8].

$$W^{1,p,0}(\Omega) = \left\{ q \in W^{1,p}(\Omega); \int_\Omega q = 0 \right\}$$

We define the incompressible elasticity operator by :

$$E : \Sigma_{2,p}(\Omega) \times W^{1,p,0}(\Omega) \to L^p(\Omega)^3$$

$$(\phi, q) \to \text{div} \left(T^D(F) \text{ Adj } F^T + q \text{ Adj } F^T \right).$$

PROPOSITION I.1 [8] : *The operator E is a local diffeomorphism from a neighborhood of* (Id, 0) *in* $\Sigma_{2,p}(\Omega) \times W^{1,p,0}(\Omega)$ *to a neighborhood of* 0 *in* $L^p(\Omega)^3$.

This result is proved *via* the inverse function theorem. We have implicitly assumed that $\phi = \text{Id}$ is a natural (stress-free) configuration i.e. : $T^D(\text{Id}) = 0$, which implies $E(\text{Id}, 0) = 0$.

COROLLARY I.2 : *If the body force b is small enough in $L^p(\Omega)^3$, there exists a unique solution* (ϕ_0, p_0) *of the incompressible problem close to* (Id, 0) *in* $\Sigma_{2,p}(\Omega) \times W^{1,p,0}(\Omega)$.

Moreover, if b is chosen in $W^{1,p}(\Omega)^3$, then ϕ_0 belongs to $W^{3,p}(\Omega)$ and therefore ϕ_0 belongs to $C^2(\overline{\Omega})^3$ [8]. Again in [8], we show that ϕ_0 is a global diffeomorphism, therefore the linear mapping :

$$\theta : W^{2,p}(\Omega) \to W^{2,p}(\Omega),$$

$$\phi \mapsto \phi \circ \phi_0,$$

is an isomorphism, [1]. In this way, we can reduce ourselves to $\phi_0 = \mathrm{Id}$, to simplify the calculations. Moreover, although the first Piola-Kirchhoff stress tensor has to be taken relatively to this new reference configuration, by proposition I.1 and the remark above, we see that the differential at $(\phi, p) = (\mathrm{Id}, p_0)$ of the corresponding elasticity operator (still denoted by E) is an isomorphism.

LEMMA I.3 : *The differential of the elasticity operator E at (Id, p_0) is expressed as follows* :

$$D_{(\mathrm{Id},p_0)} E : T_{\mathrm{Id}} \Sigma_{2,p}(\Omega) \times W^{1,p,0}(\Omega) \to L^p(\Omega)^3,$$

$$(\psi, q) \mapsto \mathrm{div}\,(D_{\mathrm{Id}}\,T^D.\psi - T^D(\mathrm{Id})\,\nabla\psi^T) + \nabla q - \nabla\psi^T\,\nabla p_0.$$

where the tangent space to $\Sigma_{2,p}(\Omega)$ at $\phi = \mathrm{Id}$ is the space :

$$T_{\mathrm{Id}}\Sigma_{2,p}(\Omega) = \{\,\psi \in (W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega))^3,\quad \mathrm{div}\,\psi = 0\ in\ \Omega\,\}.$$

Proof : Making use of the Piola identity $(\mathrm{div}\,\mathrm{Adj}\,F^T = 0)$, we get

$$E(\phi, p) = \mathrm{div}\,(T^D(F)\,\mathrm{Adj}\,F^T) + \mathrm{Adj}\,F^T\,\nabla p.$$

By direct computation, we have :

$$D_{\mathrm{Id}}\,\mathrm{Adj}.H = \mathrm{tr}\,H\,\mathrm{Id} - H,\quad \text{for all } H \text{ in } M_3.$$

Therefore, since $\mathrm{tr}\,\nabla\psi = \mathrm{div}\,\psi = 0$ for any ψ in $T_{\mathrm{Id}}\Sigma_{2,p}(\Omega)$, the result clearly holds. □

Let us pose :

$$L(\nabla\psi) = D_{\mathrm{Id}}\,T^D.\psi + T^D(\mathrm{Id})\,(\mathrm{div}\,\psi\,\mathrm{Id} - \nabla\psi^T).$$

II. CONSTRUCTION OF THE ASYMPTOTIC EXPANSION (13)

We shall seek ϕ_1 and ϕ_2 in supplementary

closed subspaces of $(W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega))^3$,

since we hope to find them by the implicit function theorem and with the help of proposition I.1. Actually, ϕ_1 will somehow play the role of the displacement in the differential of the elasticity operator E and ϕ_2 (or more precisely div ϕ_2), the role of the hydrostatic pressure. For reasons to be cleared up later on, we choose :

$$\begin{cases} \phi_1 \in E_1 = \left\{ \phi \in (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^3 , \right. \\ \\ \qquad\qquad\qquad\qquad \left. \operatorname{div} \phi_1 = p_0 - t_1(\operatorname{Id}) - \frac{1}{\operatorname{meas} \Omega} \int_\Omega t_1(\operatorname{Id}) \right\}, \quad (14) \\ \\ \phi_2 \in E_2 = \operatorname{div}^{-1} \left\{ W^{1,p,0}(\Omega) \right\}, \end{cases}$$

where div^{-1} is a continuous right inverse of the mapping :

$$\operatorname{div} : (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^3 \rightarrow W^{1,p,0}(\Omega) .$$

That such a right inverse exists follows for example from the results in [8]. The space E_1 is an affine subspace, a translation of the space $T_{\operatorname{Id}} \Sigma_{2,p}(\Omega)$:

$$E_1 = T_{\operatorname{Id}} \Sigma_{2,p}(\Omega) + \operatorname{div}^{-1} \left(p_0 - t_1(\operatorname{Id}) - \frac{1}{\operatorname{meas} \Omega} \int_\Omega t_1(\operatorname{Id}) \right).$$

Therefore one clearly has :

$$(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))^3 = E_1 \oplus E_2 .$$

Let us now turn to the effective construction of the asymptotic expansion (13).

THEOREM II.2 : *There exists a neighborhood V of 0 in $\mathbb{R}$ and two C^1-functions :*

$$\phi_1 : V \rightarrow E_1 \quad and \quad \phi_2 : V \rightarrow E_2 , \quad such\ that :$$

$$\phi(\varepsilon) = \operatorname{Id} + \varepsilon\phi_1(\varepsilon) + \varepsilon^2 \phi_2(\varepsilon) ,$$

solves equations (12) for ε in V.

Proof : Let us pose :

$$G(\varepsilon, \phi_1, \phi_2) = \nabla\phi_1 + \varepsilon \nabla\phi_2 .$$

Then G is in the space $C^1(\mathbb{R} \times E_1 \times E_2, W^{1,p}(\Omega)^9)$. For brevity, we let :

$$T(F) = T^D(F) + t_1(F) \operatorname{Id} ,$$

although this tensor is not the Cauchy stress tensor. Let us define :

$$H(\varepsilon, \phi) = \varepsilon \operatorname{div}\left(T(F) \operatorname{Adj} F^T\right) + \operatorname{Adj} F^T \nabla(h'(\det F)) + \varepsilon b. \qquad (15)$$

Now, we take ϕ to be :

$$\phi(\varepsilon, \phi_1, \phi_2) = \operatorname{Id} + \varepsilon \phi_1 + \varepsilon^2 \phi_2.$$

Then :

$$F = \operatorname{Id} + \varepsilon G(\varepsilon, \phi_1, \phi_2).$$

We now expand equation (15). Since $h''(1) = 1$, by Taylor's formula, we get :

$$\operatorname{Adj} F^T \nabla(h'(\det F)) = (\operatorname{Id} + \varepsilon(\operatorname{tr} G \operatorname{Id} - G^T) + \varepsilon^2 \operatorname{Adj} G^T) \times$$
$$\times \nabla(\varepsilon \operatorname{tr} G + \varepsilon^2 r_1(\varepsilon, G)), \quad (16)$$

where $r_1(., .)$ is of class C^1. Therefore :

$$\operatorname{Adj} F^T \nabla(h'(\det F)) = \varepsilon \nabla \operatorname{tr} G + \varepsilon^2 r_2(\varepsilon, G),$$

where r_2 shares the same properties as r_1. Putting this last equation into (15), we get :

$$H(\varepsilon, \phi) = \varepsilon \operatorname{div}\left(T(F) \operatorname{Adj} F^T\right) + \varepsilon \nabla \operatorname{tr} G + \varepsilon^2 r_2(\varepsilon, G) + \varepsilon b.$$

We are thus led to define :

$$\overline{H}(\varepsilon, \phi_1, \phi_2) = \operatorname{div}\left(T(F) \operatorname{Adj} F^T\right) + \nabla \operatorname{tr} G + \varepsilon r_2(\varepsilon, G) + b. \qquad (17)$$

Using Taylor's formula again, we obtain :

$$\operatorname{div}\left(T(F) \operatorname{Adj} F^T\right) = \operatorname{div}\left(T^D(\operatorname{Id}) + t_1(\operatorname{Id}) \operatorname{Id} + \varepsilon L(G)\right) +$$
$$+ \varepsilon(\operatorname{tr} G \operatorname{Id} - G^T) \nabla t_1(\operatorname{Id}) + \varepsilon \nabla(D_{\operatorname{Id}} t_1 . G) + \varepsilon^2 r_3(\varepsilon, G),$$
$$= \operatorname{div}\left(T^D(\operatorname{Id}) + t_1(\operatorname{Id}) \operatorname{Id}\right) + \varepsilon(\operatorname{div} L(G) +$$
$$+ (\operatorname{tr} G \operatorname{Id} - G^T) \nabla t_1(\operatorname{Id}) + \nabla(D_{\operatorname{Id}} t_1 . G)) + \varepsilon^2 r_3(\varepsilon, G).$$

Now, by corollary I.2 :

$$\operatorname{div} T^D(\operatorname{Id}) = - \nabla p_0 - b.$$

Therefore, equation (17) becomes :

$$\overline{H}(\varepsilon, \phi_1, \phi_2) = \nabla(t_1(\operatorname{Id}) - p_0) + \varepsilon \operatorname{div} L(G) +$$
$$+ \varepsilon((\operatorname{tr} G \operatorname{Id} - G^T) \nabla t_1(\operatorname{Id}) + \nabla(D_{\operatorname{Id}} t_1 . G))$$
$$+ \varepsilon^2 r_3(\varepsilon, G) + \nabla(\operatorname{div} \phi_1) + \varepsilon \nabla(\operatorname{div} \phi_2) + \varepsilon r_2(\varepsilon, G).$$

Recall now that ϕ_1 belongs to the space E_1. Therefore, the zero order terms vanish, and we define :

$$\overline{\overline{H}}(\varepsilon, \phi_1, \phi_2) = \frac{1}{\varepsilon} \overline{H}(\varepsilon, \phi_1, \phi_2) = \operatorname{div} L(G) + (\operatorname{tr} G \operatorname{Id} - G^T) \nabla t_1(\operatorname{Id}) +$$
$$+ \nabla(D_{\operatorname{Id}} t_1 . G) + \nabla(\operatorname{div} \phi_2) + r_2(\varepsilon, G) + \varepsilon r_3(\varepsilon, G) . \quad (18)$$

For $\varepsilon = 0$, we have $G = \nabla\phi_1$, and equation (18) becomes :

$$\overline{\overline{H}}(0, \phi_1, \phi_2) = \operatorname{div} L(\nabla\phi_1) + (\operatorname{div} \phi_1 \operatorname{Id} - \nabla\phi_1^T) \nabla t_1(\operatorname{Id}) +$$
$$+ \nabla(D_{\operatorname{Id}} t_1 . \nabla\phi_1) + \nabla(\operatorname{div} \phi_2) + r_2(0, \nabla\phi_1) . \quad (19)$$

We henceforth decompose the proof into several lemmas.

To proceed further, we have to compute explicitly the remainder $r_2(0, \nabla\phi_1)$. We do this in the following lemma :

LEMMA II.3 : *The following relation holds :*

$$r_2(0, \nabla\phi_1) = \frac{1}{2} \nabla((1 + h'''(1)) (\operatorname{div} \phi_1)^2 + \operatorname{tr} (\nabla\phi_1)^2) +$$
$$+ (\operatorname{div} \phi_1 \operatorname{Id} - \nabla\phi_1^T) \nabla(\operatorname{div} \phi_1) .$$

Proof of lemma II.3 : The term $r_2(., .)$ comes from the second order terms in expansion (16). We have :

$$\det F = 1 + \varepsilon \operatorname{tr} G + \frac{\varepsilon^2}{2} D_{\operatorname{Id}}^2 \det.(G, G) + o(\varepsilon^2) .$$

Therefore

$$h'(\det F) = \varepsilon \operatorname{tr} G + \frac{\varepsilon^2}{2} (D_{\operatorname{Id}}^2 \det.(G, G) + h'''(1) (\operatorname{tr} G)^2) + o(\varepsilon^2) .$$

Now, it is easily checked that :

$$D_{\operatorname{Id}}^2 \det (G, H) = \operatorname{tr} G . \operatorname{tr} H - \operatorname{tr} (G.H) ,$$

and thus :

$$h'(\det F) = \varepsilon \operatorname{tr} G + \frac{\varepsilon^2}{2} ((1 + h'''(1)) (\operatorname{tr} G)^2 - \operatorname{tr} (G^2)) + o(\varepsilon^2) .$$

We finally obtain :

$$\text{Adj } F^T \, \nabla h'(\det F) = \varepsilon \, \nabla \text{ tr } G + \varepsilon^2 \left[\frac{1}{2} \nabla((1 + h'''(1))(\text{tr } G)^2 - \text{tr }(G^2)) + \right.$$

$$\left. + (\text{tr } G \text{ Id} - G^T) \, \nabla \text{ tr } G \right] + o(\varepsilon^2) .$$

Letting $\varepsilon = 0$ yields $G = \nabla\phi_1$, and tr $G = \text{div } \phi_1$, and the lemma holds. $\quad\square$

Despite its unpleasant look, the term $r_2(0, \nabla\phi_1)$ will fit with the preceding terms in equation (19), so as to allow us to use the implicit function theorem with proposition I.1 and lemma I.3. Let us state the following lemma :

LEMMA II.4 : *There exists a unique couple* $(\overline{\phi}_1, \overline{\phi}_2)$ *in* $E_1 \times E_2$ *such that* $\overline{\overline{H}}(0, \overline{\phi}_1, \overline{\phi}_2) = 0$.

Proof of lemma II.4 : Let us pose :

$$K(\phi_1) = D_{\text{Id}} \, t_1.\nabla\phi_1 + \frac{1}{2}((1 + h'''(1)) \text{ div } \phi_1^2 - \text{tr }(\nabla\phi_1)^2) .$$

Then K is a given nonlinear operator. Using lemma II.3, we get :

$$\overline{\overline{H}}(0, \phi_1, \phi_2) = \text{div } L(\nabla\phi_1) + (\text{div } \phi_1 \text{ Id} - \nabla\phi_1^T) \, \nabla(t_1(\text{Id}) + \text{div } \phi_1) +$$

$$+ \nabla(K(\phi_1)) + \nabla(\text{div } \phi_2) .$$

Since ϕ_1 belongs to E_1, we have :

$$\overline{\overline{H}}(0, \phi_1, \phi_2) = \text{div } L(\nabla\phi_1) + (\text{div } \phi_1 \text{ Id} - \nabla\phi_1^T) \, \nabla p_0 + \nabla(K(\phi_1)) +$$

$$+ \nabla(\text{div } \phi_2) . \quad (20)$$

Now, the affine space E_1 is parallel to the space $T_{\text{Id}} \, \Sigma_{2,p}(\Omega)$. Therefore, the operator $D_{\text{Id},p_0} E$ is still an isomorphism when acting between $E_1 \times W^{1,p,0}(\Omega)$ and $L^p(\Omega)^3$. Thus, there exists a unique $(\overline{\phi}_1, \overline{q})$ in $E_1 \times W^{1,p,0}(\Omega)$ such that :

$$\text{div } L(\nabla\overline{\phi}_1) + \nabla\overline{q} - \nabla\overline{\phi}_1^T \, \nabla p_0 = - \text{div } \overline{\phi}_1.\nabla p_0 .$$

Indeed, div $\overline{\phi}_1$ is independent of $\overline{\phi}_1$ for $\overline{\phi}_1$ in E_1 by the definition of E_1. Moreover div $\phi_1.\nabla p_0$ belongs to $L^p(\Omega)^3$. Then :

$$\overline{\overline{H}}(0, \overline{\phi}_1, \phi_2) = - \nabla\overline{q} + \nabla(K(\overline{\phi}_1)) + \nabla(\text{div } \phi_2) .$$

Since ϕ_2 belongs to div^{-1} $(W^{1,p,0}(\Omega))$, there exists a unique $\overline{\phi}_2$ such that :

$$\text{div } \overline{\phi}_2 = - \overline{q} + K(\overline{\phi}_1) - \frac{1}{\text{meas } \Omega} \int_\Omega K(\overline{\phi}_1) .$$

Hence :

$$\overline{\overline{H}}(0, \overline{\phi}_1, \overline{\phi}_2) = 0 .$$

Tracing back through the proof, we easily see that the solution is unique.   □
    To use the implicit function theorem, we only have to show that the partial
differential $D_{(\phi_1,\phi_2)} \overline{\overline{H}}(0, \overline{\phi}_1, \overline{\phi}_2)$ is an isomorphism between $T_{\mathrm{Id}} \Sigma_{2,p}(\Omega) \times E_2$
and $L^p(\Omega)^3$. This is the aim of the next lemma.

LEMMA II.5 : *The partial differential* :

$$D_{(\phi_1,\phi_2)} \overline{\overline{H}}(0, \phi_1, \phi_2) : T_{\mathrm{Id}} \Sigma_{2,p}(\Omega) \times E_2 \to L^p(\Omega)^3 ,$$

is an isomorphism.

Proof of lemma II.5 : Let us express this differential from equation (20) :

$$B(\psi_1, \psi_2) = D_{(\phi_1,\phi_2)} \overline{\overline{H}}(0, \overline{\phi}_1, \overline{\phi}_2)(\psi_1, \psi_2) =$$
$$= \mathrm{div}\, L(\nabla\psi_1) - \nabla\psi_1^T \nabla p_0 + \nabla(D_{\phi_1} K.\psi_1) + \nabla(\mathrm{div}\, \psi_2) .$$

Let b be an element of $L^p(\Omega)^3$. Then by lemma I.3, there exists a unique
$(\overline{\psi}_1, \overline{q})$ such that :

$$\mathrm{div}\, L(\nabla\overline{\psi}_1) - \nabla\overline{\psi}_1^T \nabla p_0 + \nabla\overline{q} = b .$$

Then :

$$B(\overline{\psi}_1, \psi_2) = b - \nabla\overline{q} + \nabla(D_{\phi_1} K.\overline{\psi}_1) + \nabla(\mathrm{div}\, \psi_2) .$$

And again there exists a unique $\overline{\psi}_2$ such that

$$\nabla(\mathrm{div}\, \overline{\psi}_2) - \nabla\overline{q} + \nabla(D_{\phi_1} K.\overline{\psi}_1) = 0 .$$

Therefore $B(\overline{\psi}_1, \overline{\psi}_2) = b$ and B is onto. Taking $b = 0$ and tracing back through
the proof shows that B is one-to-one as in lemma II.4. Finally since B is conti-
nuous, by the Banach theorem, we see that B is an isomorphism.   □

    End of the proof of theorem II.2 : By construction, $\overline{\overline{H}}$ belongs to $C^1(\mathbb{R} \times E_1 \times$
$E_2, L^p(\Omega)^3)$. For $\varepsilon = 0$, we have seen that there exists a unique $(\overline{\phi}_1, \overline{\phi}_2)$ in
$E_1 \times E_2$ such that :

$$\overline{\overline{H}}(0, \overline{\phi}_1, \overline{\phi}_2) = 0 .$$

Moreover : $D_{(\phi_1,\phi_2)} \overline{\overline{H}}(0, \overline{\phi}_1, \overline{\phi}_2)$ is an isomorphism between $T_{\mathrm{Id}} \Sigma_{2,p}(\Omega) \times E_2$
and $L^p(\Omega)^3$.
    The result thus follows from a straightforward application of the implicit
function theorem.   □

As $\phi_1(\varepsilon)$ and $\phi_2(\varepsilon)$ are C^1-functions of ε, expansion (13) actually takes the (less precise) form :

$$\phi(\varepsilon) = \mathrm{Id} + \varepsilon\overline{\phi}_1 + o(\varepsilon^2) \quad \text{in} \quad W^{2,p}(\Omega)^3,$$

which agrees with the expansions in part B, therefore justifying the results there in the special case of the pure displacement problem, with suitably elliptic elasticity tensor and sufficiently small body forces.

Let us end up the paper with a picture of what happens in the space $W^{2,p}(\Omega)^3$.
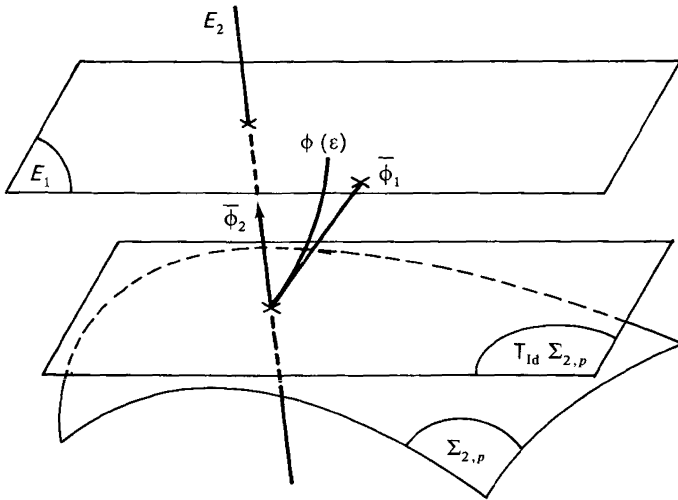
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