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A MIXED FINITE ELEMENT METHOD FOR A WEIGHTED ELLIPTIC PROBLEM (*)

par Marie-Noëlle Le Roux ([1])

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Abstract — *In this paper, we study a Dirichlet's weighted problem, we give a mixed formulation which has a unique solution and we obtain error bounds in weighted Sobolev spaces for a mixed finite element approximation*

Résumé — *Cet article est consacre a l'etude d'un problème de Dirichlet avec poids, nous donnons une formulation mixte de ce problème admettant une solution unique, nous obtenons ensuite des estimations d'erreur dans des espaces de Sobolev avec poids pour une approximation par des elements finis mixtes*

I. INTRODUCTION

Let Ω be a bounded domain of $\mathbb{R}^2$ with a regular boundary Γ. We consider the Dirichlet's problem : Find a function u defined over Ω such that :

$$\left.\begin{aligned} &-\operatorname{div}(D^{-1}\,\overrightarrow{\operatorname{grad}}\,u) = f \text{ in } \Omega\,, \\ &u = 0 \text{ on } \Gamma\,, \end{aligned}\right\} \tag{1.1}$$

where D is a given function $\in C^\infty(\overline{\Omega})$, positive over Ω and null over Γ such that there are two constants $a_1, a_2 > 0$ such that

$$0 < a_1 \leqslant \frac{D(x)}{d(x,\Gamma)} \leqslant a_2 \quad \text{for} \quad x \in \overline{\Omega} \text{ such that } d(x,\Gamma) \leqslant a \quad (a > 0)\,. \tag{1.2}$$

This problem is occurring in oceanography ; D is the deepthness of the water, which is null on the shore ; f is the vorticity and the components of the hori-

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zontal velocity $\vec{V}$ on the surface can be obtained from u. As a matter of fact, we have

$$\vec{V} = \begin{pmatrix} D^{-1} \dfrac{\partial u}{\partial y} \\ - D^{-1} \dfrac{\partial u}{\partial x} \end{pmatrix}.$$

This kind of problem is found at each time step for vorticity method. So, in order to calculate $\vec{V}$, we shall use a mixed finite element method.

An outline of the paper is as follows : in section 2, we introduce a mixed formulation of problem (1.1); in section 3, we give a formulation of the finite element approximation. Section 4 will be devoted to the derivation of error bounds and section 5 is devoted to the proof of a result used in section 4.

II. A MIXED FORMULATION OF PROBLEM 1.1

Introduce the space :

$$H^1_{-1/2}(\Omega) = \{ v \in \mathscr{D}'(\Omega)/D^{-3/2}\, v \in L^2(\Omega) \; ; \; D^{-1/2}\, \overrightarrow{\text{grad}}\, v \in (L^2(\Omega))^2 \} \qquad (2.1)$$

provided with the norm :

$$\| v \|_{1,-1/2,\Omega} = (| D^{-3/2}\, v |^2_{L^2(\Omega)} + | D^{-1/2}\, \overrightarrow{\text{grad}}\, v |^2_{(L^2(\Omega))^2})^{1/2} . \qquad (2.2)$$

We have the following result (*cf.* Bolley-Camus [1]).

THEOREM 2.1 : *$\mathscr{D}(\Omega)$ is dense in $H^1_{-1/2}(\Omega)$ and the semi-norm*

$$| v |_{1,-1/2,\Omega} = | D^{-1/2}\, \overrightarrow{\text{grad}}\, v |_{(L^2(\Omega))^2} \qquad (2.3)$$

is a norm over $H^1_{-1/2}(\Omega)$ which is equivalent to the norm $\| \; \|_{1,-1/2,\Omega}$.

Then, a variational formulation of problem (1.1) is as follows : *given a function* $f \in L^2_{3/2}(\Omega) = \{ f \in \mathscr{D}'(\Omega)/D^{3/2}\, f \in L^2(\Omega) \}$, *find a function* $u \in H^1_{-1/2}(\Omega)$ *such that* :

$$\int_\Omega D^{-1}\, \overrightarrow{\text{grad}}\, u . \overrightarrow{\text{grad}}\, v \, dx = \int_\Omega fv \, dx \text{ for all } v \in H^1_{-1/2}(\Omega) . \qquad (2.4)$$

Clearly, from theorem 2.1, this problem is coercive in $H^1_{-1/2}(\Omega)$ and then it has a unique solution $u \in H^1_{-1/2}(\Omega)$.

Introduce now the space :

$$H^2_{1/2}(\Omega) = \{ v \in \mathcal{D}'(\Omega)/D^{-3/2} v \in L^2(\Omega), D^{-1/2} \partial^\alpha v \in L^2(\Omega), \quad |\alpha| = 1, \\ D^{1/2} \partial^\alpha v \in L^2(\Omega), \quad |\alpha| = 2 \} \quad (2.5)$$

provided with the norm :

$$\| u \|_{2,1/2,\Omega} = \left(\sum_{|\alpha| \leqslant 2} | D^{-3/2+|\alpha|} \partial^\alpha v |^2_{L^2(\Omega)} \right)^{1/2} \quad (2.6)$$

THEOREM 2.2 : *Problem* (2.4) *has a unique solution u which is in* $H^2_{1/2}(\Omega)$ *if f is in* $L^2_{3/2}(\Omega)$; *moreover, this solution satisfies* :

$$\| u \|_{2,1/2,\Omega} \leqslant C \, | D^{3/2} f |_{L^2(\Omega)} \quad (2.7)$$

where C is a positive constant depending only on Ω.

Proof : The dual space of $H^1_{-1/2}(\Omega)$ is given by :

$$(H^1_{-1/2}(\Omega))' = \left\{ f \in \mathcal{D}'(\Omega)/f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}; \quad D^{3/2} f_0 \in L^2(\Omega), \\ D^{1/2} f_1 \in L^2(\Omega), \quad D^{1/2} f_2 \in L^2(\Omega) \right\}$$

and the problem : given a function $f \in (H^1_{-1/2}(\Omega))'$, find $u \in H^1_{-1/2}(\Omega)$ such that :

$$\int_\Omega D^{-1} \, \overrightarrow{\text{grad}}\, u \, \overrightarrow{\text{grad}}\, v \, dx = \langle f, v \rangle_{(H^1_{-1/2}(\Omega))', H^1_{-1/2}(\Omega)} \text{ for all } v \in H^1_{-1/2}(\Omega)$$

has a unique solution.

Assume now that f is in $L^2_{3/2}(\Omega)$; we note

$$\Psi = D \frac{\partial u}{\partial x_1}.$$

Then, we obtain

$$\text{div}\,(D^{-1} \, \overrightarrow{\text{grad}}\, \Psi) = \text{div}\left(D^{-1} \frac{\partial u}{\partial x_1} \overrightarrow{\text{grad}}\, D \right) + \frac{\partial}{\partial x_1} (\Delta u).$$

Besides,

$$\Delta u = D^{-1} \, \overrightarrow{\text{grad}}\, D . \overrightarrow{\text{grad}}\, u - Df,$$

hence, we get

$$D^{+1/2}\,\Delta u \in L^2(\Omega) \quad \text{and} \quad \frac{\partial}{\partial x_1}(\Delta u) \in (H^1_{-1/2}(\Omega))' .$$

Similarly, we have

$$D^{1/2}\left(D^{-1}\,\frac{\partial u}{\partial x_1}\,\overrightarrow{\operatorname{grad}}\,D\right) \in L^2(\Omega)$$

and then

$$\operatorname{div}\left(D^{-1}\,\frac{\partial u}{\partial x_1}\,\overrightarrow{\operatorname{grad}}\,D\right) \in (H^1_{-1/2}(\Omega))' ;$$

so, we get

$$\operatorname{div}(D^{-1}\,\overrightarrow{\operatorname{grad}}\,\Psi) \in (H^1_{-1/2}(\Omega))' .$$

Besides we have

$$D^{1/2}\,u \in H^2(\Omega)\,(\Delta(D^{1/2}\,u) \in L^2(\Omega)) \quad \text{and} \quad D^{-1/2}\,u \in H^1(\Omega)$$

so, we get

$$D^{1/2}\,\frac{\partial u}{\partial x_1}\bigg|_\Gamma \in L^2(\Gamma)\,, \quad \text{hence } \Psi = 0 \text{ on } \Gamma .$$

Then, we obtain

$$\Psi \in H^1_{-1/2}(\Omega) .$$

Hence, we get

$$D^{-1/2}\,\frac{\partial \Psi}{\partial x_1} = D^{1/2}\,\frac{\partial^2 u}{\partial x_1^2} + D^{-1/2}\,\frac{\partial D}{\partial x_1}\,\frac{\partial u}{\partial x_1} \in L^2(\Omega)\,,$$

$$D^{-1/2}\,\frac{\partial \Psi}{\partial x_2} = D^{1/2}\,\frac{\partial^2 u}{\partial x_1\,\partial x_2} + D^{-1/2}\,\frac{\partial D}{\partial x_2}\,\frac{\partial u}{\partial x_1} \in L^2(\Omega)$$

and then

$$D^{1/2}\,\frac{\partial^2 u}{\partial x_1^2} \in L^2(\Omega)\,, \quad D^{1/2}\,\frac{\partial^2 u}{\partial x_1\,\partial x_2} \in L^2(\Omega)\,,$$

$$D^{1/2}\,\frac{\partial^2 u}{\partial x_2^2} = D^{1/2}\,\Delta u - D^{1/2}\,\frac{\partial^2 u}{\partial x_1^2} \in L^2(\Omega)$$

that is $u \in H^2_{1/2}(\Omega)$ and we get easily the estimate (2.7).

To obtain a mixed formulation of problem (1.5), we introduce the space X defined by :

$$X = \{ \vec{q} \in (\mathscr{D}'(\Omega))^2 / D^{1/2}\, \vec{q} \in (L^2(\Omega))^2, D^{3/2} \operatorname{div} \vec{q} \in L^2(\Omega) \} \qquad (2.8)$$

and provided with the norm :

$$\| \vec{q} \|_X = (\| D^{1/2}\, \vec{q} \|^2_{(L^2(\Omega))^2} + | D^{3/2} \operatorname{div} \vec{q} |^2_{L^2(\Omega)})^{1/2} . \qquad (2.9)$$

THEOREM 2.3 : $(\mathscr{D}(\Omega))^2$ *is dense in* X.

Proof : Let $\vec{q} \in X$. First, let us show that there exists a sequence of functions of X, with compact support in Ω that tends to $\vec{q}$.

Let θ be a function in $C^\infty([0, +\infty[)$ such that :

$$\begin{cases} \theta(x) = 0 , & 0 \leqslant x \leqslant 1 ; \\ 0 \leqslant \theta(x) \leqslant 1 , & 1 \leqslant x \leqslant 2 ; \\ \theta(x) = 1 , & x \geqslant 2 . \end{cases}$$

We set

$$\theta_\lambda(x) = \theta(\lambda D(x)) , \quad \lambda \in \mathbb{R}^+ , \quad x \in \overline{\Omega} ;$$

$$\vec{q}_\lambda(x) = \theta_\lambda(x)\, \vec{q}(x) .$$

Therefore, we have

$$| D^{1/2}(\vec{q}_\lambda - \vec{q}) |^2_{(L^2(\Omega))^2} = \int_{\Omega \cap \{x / D(x) \leqslant 2/\lambda\}} D(x)\, (\theta_\lambda(x) - 1)^2\, | \vec{q}(x) |^2\, dx$$

and from Lebesgue's theorem, this term tends to 0 when $\lambda \to +\infty$.

Similarly, we get

$$| D^{3/2} \operatorname{div} (\vec{q}_\lambda - \vec{q}) |^2_{L^2(\Omega)} = \int_{\Omega \cap \{x / D(x) \leqslant 2/\lambda\}} D^3(x)\, (\theta_\lambda(x) - 1)^2 \times$$

$$\times \, | \operatorname{div} \vec{q}(x) |^2\, dx + \int_\Omega D^3(x)\, (\overrightarrow{\operatorname{grad}}\, \theta_\lambda(x) . \vec{q}(x))^2\, dx .$$

But,

$$\begin{cases} \overrightarrow{\operatorname{grad}}\, \theta_\lambda(x) = 0 \quad \text{for} \quad D(x) \leqslant \dfrac{1}{\lambda} \quad \text{or} \quad D(x) \geqslant \dfrac{2}{\lambda} , \\ \overrightarrow{\operatorname{grad}}\, \theta_\lambda(x) = \lambda \theta'(\lambda D(x))\, \overrightarrow{\operatorname{grad}}\, D(x) , \quad \text{for} \quad 1 \leqslant \lambda D(x) \leqslant 2 . \end{cases}$$

Thus, we get

$$\int_\Omega D^3(x)\,(\overrightarrow{\operatorname{grad}}\,\theta_\lambda(x).\vec{q}(x))^2\,dx \leqslant C \int_{\Omega \cap \{x/\frac{1}{\lambda} \leqslant D(x) \leqslant \frac{2}{\lambda}\}} D(x)\,|\,q(x)\,|^2\,dx$$

and from Lebesgue's theorem ·

$$\lim_{\lambda \to +\infty} |\,D^{3/2}\operatorname{div}(\vec{q}_\lambda - \vec{q})\,|_{L^2(\Omega)} = 0\,.$$

Hence, $\vec{q}_\lambda$ is a sequence with compact support in Ω that tends to $\vec{q}$ in X.

From the above, we can assume that $\vec{q}$ has a compact support in Ω; then, $\vec{q} \in (L^2(\Omega))^2$ and $\operatorname{div}\vec{q} \in L^2(\Omega)$ Therefore there exists a sequence of functions $\vec{q}_n \in (\mathscr{D}(\Omega))^2$ such that

$$\lim_{n \to +\infty} (|\,\vec{q} - \vec{q}_n\,|^2_{(L^2(\Omega))^2} + |\operatorname{div}(\vec{q} - \vec{q}_n)\,|_{L^2(\Omega)}) = 0 \quad [7]$$

and then,

$$\lim_{n \to +\infty} \|\,\vec{q} - \vec{q}_n\,\|_X \leqslant C \lim_{n \to +\infty} (|\,\vec{q} - \vec{q}_n\,|^2_{(L^2(\Omega))^2} + |\operatorname{div}(\vec{q} - \vec{q}_n)\,|^2_{L^2(\Omega)})^{1/2} = 0\,.$$

LEMMA 2 1 · *For any $u \in H^1_{-1/2}(\Omega)$ and $\vec{q} \in X$, we have :*

$$\int_\Omega (\overrightarrow{\operatorname{grad}}\,u.\vec{q} + u.\operatorname{div}\vec{q})\,dx = 0\,. \tag{2.10}$$

This result follows immediately from theorems 2 1 and 2 3.

Now, we define a bilinear form a over $(L^2_{1/2}(\Omega))^2$

$$(L^2_{1/2}(\Omega)) = \{\,v \in \mathscr{D}'(\Omega)/D^{1/2}\,v \in L^2(\Omega)\,\}$$

$$a(\vec{p}, \vec{q}) = \int_\Omega D(x)\,\vec{p}(x).\vec{q}(x)\,dx\,, \quad \forall \vec{p}, \vec{q} \in (L^2_{1/2}(\Omega))^2 \tag{2.11}$$

Let us introduce the space M

$$M = \{\,v \in \mathscr{D}'(\Omega)/D^{-3/2}\,v \in L^2(\Omega)\,\} \tag{2.12}$$

provided with the norm

$$\|\,v\,\|_M = |\,D^{-3/2}\,v\,|_{L^2(\Omega)}\,. \tag{2.13}$$

We define a bilinear form b over $X \times M$ by

$$b(\vec{q}, v) = \int_\Omega v.\mathrm{div}\,\vec{q}\,dx \tag{2.14}$$

and we consider the following problem :

Given a function $f \in L^2_{3/2}(\Omega) = M'$, find functions $\vec{p} \in X$, $u \in M$ such that :

$$\left.\begin{aligned} &\forall \vec{q} \in X,\ a(\vec{p}, \vec{q}) + b(\vec{q}, u) = 0 \\ &\forall v \in M,\ b(\vec{p}, v) + \int_\Omega fv\,dx = 0 . \end{aligned}\right\} \tag{2.15}$$

THEOREM 2.4 : *Problem* (2.15) *has a unique solution* $(\vec{p}, u) \in X \times M$; *besides, u is the solution of problem* (2.4) *and* $\vec{p}$ *is given by* :

$$\vec{p} = D^{-1}\,\overrightarrow{\mathrm{grad}}\,u . \tag{2.16}$$

Proof : From a result of Brezzi [2], the problem (2.15) has a unique solution if the following conditions are satisfied :

i) There is some positive constant $\alpha > 0$ such that :

$$a(\vec{q}, \vec{q}) \geqslant \alpha \,\|\vec{q}\|^2_X ; \quad \forall \vec{q} \in V = \{ \vec{q} \in X / b(\vec{q}, v) = 0 \quad \forall v \in M \} . \tag{2.17}$$

ii) There is some positive constant $\beta > 0$ such that :

$$\sup_{\vec{q} \in X} \frac{b(\vec{q}, v)}{\|\vec{q}\|_X} \geqslant \beta \,\|v\|_M , \quad \forall v \in M . \tag{2.18}$$

Clearly, V is also defined by

$$V = \{ \vec{q} \in X / \mathrm{div}\,\vec{q} = 0 \} .$$

Therefore, if $\vec{q} \in V$, we have

$$\|\vec{q}\|_X = (a(\vec{q}, \vec{q}))^{1/2}$$

and hypothesis i) is satisfied with $\alpha = 1$.

Now, if $v \in M$, let us consider φ solution of the problem :

$$\begin{cases} \mathrm{div}\,(D^{-1}\,\overrightarrow{\mathrm{grad}}\,\varphi) = D^{-3}\,v, \text{ over } \Omega , \\ \varphi = 0 \text{ on } \Gamma . \end{cases}$$

From theorem 2.2, $\varphi \in H^2_{1/2}(\Omega)$ and $\| \varphi \|_{2,1/2,\Omega} \leqslant C \mid D^{-3/2} v \mid_{L^2(\Omega)}$.
We set,

$$\vec{q} = D^{-1} \overrightarrow{\text{grad}} \varphi .$$

Then,

$$\| \vec{q} \|_X \leqslant \| \varphi \|_{2,1/2,\Omega} \leqslant C \| v \|_M$$

and

$$b(\vec{q}, v) = \| v \|^2_M .$$

Thus, hypothesis ii) is satisfied with $\beta = 1/C$.
Besides, if u is the solution of problem (2.4) and $\vec{p}$ defined by (2.16) we get :

$$- \operatorname{div} \vec{p} = f$$

and

$$b(\vec{p}, v) = - \int_\Omega fv \, dx , \quad \forall v \in M ;$$

$$\forall \vec{q} \in X a(\vec{p}, \vec{q}) + b(\vec{q}, u) = \int_\Omega (\overrightarrow{\text{grad}} \, u . \vec{q} + \operatorname{div} \vec{q} . u) \, dx = 0$$

from lemma 2.1.
Therefore, $(\vec{p}, u)$ is the unique solution of problem (2.15).

III. A MIXED FINITE ELEMENT APPROXIMATION OF PROBLEM (2.15)

In order to approximate problem (2.15), we first construct a set $\overline{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K$ as a finite union of triangles with vertices in $\overline{\Omega}$. For any $K \in \mathcal{T}_h$, we set

$$h(K) = \text{diameter of } K, \ h(K) \leqslant h ,$$
$$\rho(K) = \text{diameter of the inscribed sphere of } K .$$

We assume that this triangulation is regular, that is :

$$\sup_{K \in \mathcal{T}_h} \frac{h(K)}{\rho(K)} \leqslant \sigma \quad (\sigma > 0) . \tag{3.1}$$

We also assume that a triangle K has not three vectices on the boundary Γ.

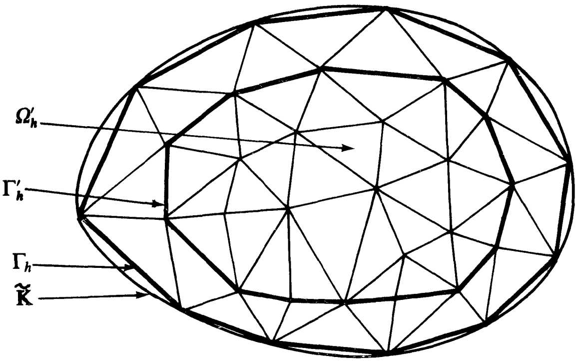
Notations

We denote :

- Γ_h the boundary of Ω_h ; it is clear that Γ_h does not coincide with Γ,

- $\mathcal{T}'_h$ the union of the triangles of $\mathcal{T}_h$ which have no vertices on Γ
- $\Omega'_h = \bigcup_{K \in \mathcal{T}'_h} K$
- Γ'_h is the boundary of Ω'_h
- $\partial\mathcal{T}_h = \mathcal{T}_h - \mathcal{T}'_h$.

If $K \in \partial\mathcal{T}_h$ has two vertices on the boundary Γ, let us denote by $\tilde{K}$ the surface limited by the two sides of K which are in Ω and the part of Γ between the two vertices of K.



Now, we define an approximation X_h of X.

If $\hat{K}$ is the reference finite element with vertices ($\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (0, 0)$), we define the space $\hat{X}$ by :

$$\hat{X} = \{ \hat{\vec{q}} = (\hat{q}_1, \hat{q}_2)/\hat{q}_1 = \mu_1 + \alpha x_1, \hat{q}_2 = \mu_2 + \alpha x_2 \,;\, \mu_1, \mu_2, \alpha \in P_0 \} \qquad (3.2)$$

(P_0 is the space of the constants).

Let us denote by F

$$F : \hat{x} \mapsto F(\hat{x}) = B\hat{x} + b, \; B \in \mathcal{L}(\mathbb{R}^2), \quad b \in \mathbb{R}^2$$

an affine invertible mapping such that $K = F(\hat{K})$. To any scalar function $\hat{\varphi}$ and any vectorial function $\hat{\vec{q}}$ defined over $\hat{K}$, we associate the functions φ and $\vec{q}$ defined over K by :

$$\left.\begin{aligned} \varphi &= \hat{\varphi} \circ F^{-1} \\ \vec{q} \circ F &= J^{-1} B\hat{\vec{q}}\,, \quad J = |\det B| = 2 \operatorname{mes} K\,. \end{aligned}\right\} \qquad (3.3)$$

For any $K \in \mathcal{T}_h$, we set :

$$X_K = \{ \vec{q}_h / \hat{\vec{q}}_h \in \hat{X} \}\,. \qquad (3.4)$$

We have the equalities :

$$\operatorname{div} \vec{q}_h = J^{-1} \operatorname{div} \hat{\vec{q}}_h , \tag{3.5}$$

$$\vec{q}_h . \vec{n}_K = J_n^{-1} \hat{\vec{q}}_h . \hat{\vec{n}}_K , \quad J_n = J \mid B^{-1} \hat{\vec{n}} \mid \tag{3.6}$$

where $\vec{n}_K$ denotes the outward unit vector normal to the boundary ∂K of K.

Besides, for $K \in \partial \mathcal{T}_h$, $\vec{q}_h$ can be defined over $\tilde{K}$ by the same expression as over K.

We note :

$$H(\operatorname{div}, \Omega) = \{ \vec{q} \in (L^2(\Omega))^2 / \operatorname{div} \vec{q} \in L^2(\Omega) \} \subset X$$

and we set

$$X_h = \{ \vec{q}_h \in H(\operatorname{div}; \Omega_h) / \forall K \in \mathcal{T}_h \ \vec{q}_{h|K} \in X_K \} . \tag{3.7}$$

Then, $\vec{q}_h \in X_h$ if $\vec{q}_{h|K} \in X_K$, $\forall K \in \mathcal{T}_h$ and satisfies :

$$\forall K_1, K_2 \in \mathcal{T}_h, \ \vec{q}_{h|K_1} . \vec{n}_{K_1} + \vec{q}_{h|K_2} . \vec{n}_{K_2} = 0 \quad \text{over} \quad \partial K_1 \cap \partial K_1 \quad [8] . \tag{3.8}$$

Let M_h be an approximation of the space M defined by .

$$M_h = \{ v_h \in L^2(\Omega_h) / \forall K \in \mathcal{T}_h, v_{h|K} \in P_0 \} . \tag{3.9}$$

Then $M_h \not\subset M$; but to any $v_h \in M_h$, we associate $\tilde{v}_h$ defined, over Ω by :

$$\forall K \in \mathcal{T}_h, \tilde{v}_{h|K} = D^{3/2}(x) D^{-3/2}(a_K) v_{h|K} \quad (K' = K \text{ if } K \in \mathcal{T}'_h$$
$$K' = \tilde{K} \text{ if } K \in \partial \mathcal{T}_h) \tag{3.10}$$

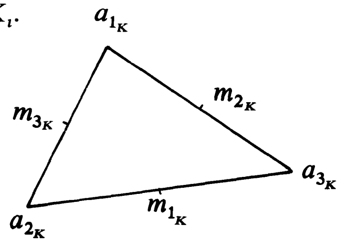
where a_K is the barycenter of K if a_K lies in Ω, otherwise a_K is the vertex of K inside Ω.

Notations

We denote by : a_{i_K}, $i = 1, 2, 3$ the vertices of K ; a_K, the barycenter of K if a_K lies in Ω ; otherwise a_K is the vertex of K inside Ω ∂K the boundary of K

$$\partial K_i = [a_j, a_k] \quad j \neq k \neq i \qquad i = 1, 2, 3$$

m_{i_K} the midpoint of ∂K_i.



To define the discrete problem, we shall use numerical integration. Denote by $a_h(\vec{p}_h, \vec{q}_h)$ the bilinear form defined over $X_h \times X_h$ by

$$a_h(\vec{p}_h, \vec{q}_h) = \sum_{K \in \mathcal{T}_h} \frac{(\text{mes } K)}{3} \sum_{i=1}^{3} D(m_{iK})\, \vec{p}_h(m_{iK}).\vec{q}_h(m_{iK}) \qquad (3\ 11)$$

(if $m_{iK} \notin \Omega$, we set $D(m_{iK}) = 0$), b_h defined over $X_h \times M_h$ by ·

$$b_h(\vec{q}_h, v_h) = \sum_{K \in \mathcal{T}_h} (\text{mes } K)\, v_{h|K}.\text{div } \vec{q}_{h|K} \qquad (3\ 12)$$

and $f_h(v_h)$ is defined over M_h by

$$f_h(v_h) = \sum_{K \in \mathcal{T}_h} (\text{mes } K)\, v_{h|K}\, f(a_K) \qquad (3\ 13)$$

Then, the approximate problem is the following *Find a pair* $(\vec{p}_h, u_h) \in X_h \times M_h$ *such that*

$$\left.\begin{aligned} \forall \vec{q}_h \in X_h,\ a_h(\vec{p}_h, \vec{q}_h) + b_h(\vec{q}_h, u_h) = 0 \\ \forall v_h \in M_h,\ b_h(\vec{p}_h, v_h) + f_h(v_h) = 0 \end{aligned}\right\} \qquad (3\ 14)$$

THEOREM 3 1 *The approximate problem* (3 14) *has a unique solution* $(\vec{p}_h, u_h) \in X_h \times M_h$

Proof It is sufficient to prove the unicity of the homogeneous problem Find a pair $(\vec{p}_h, u_h) \in X_h \times M_h$ such that

$$\left.\begin{aligned} &\forall \vec{q}_h \in X_h,\ a_h(\vec{p}_h, \vec{q}_h) + b_h(\vec{q}_h, u_h) = 0 \\ &\forall v_h \in M_h,\ b_h(\vec{p}_h, v_h) = 0 \end{aligned}\right\}$$

Then, we get

$$a_h(\vec{p}_h, \vec{p}_h) = 0\,,$$

and $$\forall K \in \mathcal{T}_h,\quad D(m_{iK})\, \vec{p}_h^{\,2}(m_{iK}) = 0\,,\quad i = 1, 2, 3\,.$$

But $D(m_{iK}) \neq 0$ for at least two points m_{iK}, so, we get :

$$\vec{p}_h = 0 \quad \text{and,} \quad \forall \vec{q}_h \in X_h\,,\quad b_h(\vec{q}_h, u_h) = 0\,.$$

Therefore, $u_h = 0$ (*cf* Thomas [8])

IV. ERROR ESTIMATES

a) General error bounds

We provide the spaces M_h and X_h with the norms :

$$\| v_h \|_{M_h} = \left(\sum_{K \in \mathcal{T}_h} (\text{mes } K') \, D^{-3}(a_K) \, (v_{h|K})^2 \right)^{1/2}, \tag{4.1}$$

then $$\| v_h \|_{M_h} = \| \tilde{v}_h \|_M ;$$

$$| \vec{q}_h |_h = (a_h(\vec{q}_h, \vec{q}_h))^{1/2} = \left(\sum_{K \in \mathcal{T}_h} \frac{(\text{mes } K)}{3} \sum_{i=1}^{3} D(m_{iK}) \, (\vec{q}_h(m_{iK}))^2 \right)^{1/2} \tag{4.2}$$

$$| \operatorname{div} \vec{q}_h |_h = \left(\sum_{K \in \mathcal{T}_h} (\text{mes } K) \, D^3(a_K) \, (\operatorname{div} q_{h|K})^2 \right)^{1/2} \tag{4.3}$$

$$\| \vec{q}_h \|_h = (| \vec{q}_h |_h^2 + | \operatorname{div} \vec{q}_h |_h^2)^{1/2} \tag{4.4}$$

We note : $| \vec{q} |_X = | D^{1/2} \vec{q} |_{(L^2(\Omega))^2}$.

LEMMA 4.1 : *The mapping* : $\vec{q}_h \in X_h \to | \vec{q}_h |_h$ *is a norm over* X_h *and there are two positive constants* C_1, C_2 *such that*

$$C_1 | \vec{q} |_X \leqslant | \vec{q}_h |_h \leqslant C_2 | \vec{q} |_X . \tag{4.5}$$

Proof : We have :

$$| \vec{q}_h |_h^2 = \sum_{K \in \mathcal{T}_h} \frac{\text{mes } K}{3} \sum_{i=1}^{3} D(m_{iK}) \, (\vec{q}_h(m_{iK}))^2 ,$$

$$\vec{q}_{h|K} \in X_K \quad \text{and} \quad \hat{\vec{q}}_{h|K} \in \hat{X} .$$

So we get,

$$\forall K \in \mathcal{T}_h, \quad \sum_{i=1}^{3} D(m_{iK}) \, (\vec{q}_h(m_{iK}))^2 \leqslant J^{-2} \| B \|^2 \left(\sup_{x \in K} D(x) \right) \sum_{i=1}^{3} (\hat{\vec{q}}_h(\hat{m}_i))^2 .$$

But, $\hat{\vec{q}}_h \in X$ which is a space of finite dimension, so there is some constant C_2' such that :

$$\sum_{i=1}^{3} (\hat{\vec{q}}_h(\hat{m}_i))^2 \leqslant C_2' \| \hat{\vec{q}}_h \|_{(L^2(K))^2}^2$$

and then, by using (3.1)

$$\forall K \in \mathcal{T}_h, \frac{(\text{mes } K)}{3} \sum_{\iota=1}^{3} D(m_{\iota K})(\vec{q}_h(m_{\iota K}))^2 \leqslant C_2' \left(\sup_{x \in K} D(x) \right) \| \vec{q}_h \|^2_{(L^2(K))^2} \leqslant$$

$$\leqslant C_2 \| D^{1/2} \vec{q}_h \|^2_{(L^2(K))^2} .$$

Further, we have

$$\forall K \in \mathcal{T}_h, \quad D(m_{\iota K}) \neq 0 \quad \text{for at least two points } m_\iota ,$$

then $\forall K \in \mathcal{T}_h$, $\sum_{\iota=1}^{3} D(m_{\iota K}) \vec{q}_h^{\,2}(m_{\iota K})$ is a norm over X_K and we have :

$$\frac{\text{mes } K}{3} \sum_{\iota=1}^{3} D(m_{\iota K})(\vec{q}_h(m_{\iota K}))^2 \geqslant \frac{1}{6} \| B \|^2 J^{-1} \left(\inf_{D(m_{\iota K}) \neq 0} D(m_{\iota K}) \right) \times$$

$$\times \left(\sum_{\substack{\iota=1 \\ D(m_{\iota K}) \neq 0}}^{3} \hat{\vec{q}}_h^{\,2}(\hat{m}_{\iota K}) \right)^2 .$$

Hence, we get

$$\frac{\text{mes } K}{3} \sum_{\iota=1}^{3} D(m_{\iota K})(\vec{q}_h(m_{\iota K}))^2 \geqslant C_1 \| D^{1/2} \vec{q}_h \|^2_{(L^2(K))^2}, \quad \forall K \in \mathcal{T}_h$$

and we deduce (4.5).

Let us denote by :

$$V_h = \{ \vec{q}_h \in X_h / \text{div } \vec{q}_h = 0 \} ,$$

$$\overline{V}_h = \{ \vec{q}_h \in X_h / \forall K \in \mathcal{T}_h \text{ div } \vec{q}_{h|K} + f(a_K) = 0 \} .$$

We may write for all $\vec{q}_h \in \overline{V}_h$

$$a_h(\vec{p}_h - \vec{q}_h, \vec{p}_h - \vec{q}_h) = a(\vec{p} - \vec{q}_h, \vec{p}_h - \vec{q}_h) + a(\vec{q}_h, \vec{p}_h - \vec{q}_h) - a_h(\vec{q}_h, \vec{p}_h - \vec{q}_h).$$

Thus, we get by using lemme 4.1

$$| \vec{p} - \vec{p}_h |_X \leqslant C \inf_{\vec{q}_h \in \overline{V}_h} \left[| \vec{p} - \vec{q}_h |_X + \sup_{\vec{z}_h \in V_h} \frac{| a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) |}{| \vec{z}_h |_X} \right] . \Bigg\} \quad (4.6)$$

In order to evaluate these terms, we need the following theorem which shall be proved in section 5.

THEOREM 4 1 *If $\mathcal{T}_h$ is a regular triangulation of Ω, there is some positive constant β, independent of h such that*

$$\forall v_h \in M_h \sup_{\vec{q}_h \in X_h} \frac{b_h(\vec{q}_h, v_h)}{\| \vec{q}_h \|_h} \geqslant \beta \| v_h \|_{M_h} \tag{4 7}$$

Then, we have the following theorem (Thomas [8])

THEOREM 4 2 *There is some positive constant C depending only on β and C_1 such that*

$$\begin{aligned} | \vec{p} - \vec{p}_h |_X \leqslant \inf_{\vec{q}_h \in X_h} \Bigg[\| \vec{p} - \vec{q}_h \|_X + \sup_{\vec{z}_h \in V_h} \frac{| a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) |}{| \vec{z}_h |_X} + \\ + C \left(\sup_{v_h \in M_h} \frac{| b(\vec{p} - \vec{q}_h, \tilde{v}_h) |}{\| v_h \|_{M_h}} + \sup_{v_h \in M_h} \frac{| b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) |}{\| v_h \|_{M_h}} \right. \\ \left. + \sup_{v_h \in M_h} \frac{| f_h(v_h) - (f, \tilde{v}_h) |}{\| v_h \|_{M_h}} \right) \Bigg] \end{aligned} \tag{4 8}$$

and,

$$\begin{aligned} \| u - \tilde{u}_h \|_M \leqslant C \Bigg[| \vec{p} - \vec{p}_h |_X \\ + \inf_{\vec{q}_h \in X_h} \left(| \vec{p} - \vec{q}_h |_X + \sup_{\vec{z}_h \in X_h} \frac{| a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) |}{\| \vec{z}_h \|_h} \right) \\ + \inf_{v_h \in M_h} \left(\| u - \tilde{v}_h \|_M + \sup_{\vec{z}_h \in X_h} \frac{| b(\vec{z}_h, \tilde{v}_h) - b_h(\vec{z}_h, v_h) |}{\| \vec{z}_h \|_h} \right) \Bigg] \end{aligned} \tag{4 9}$$

b) Consistency error bounds

We estimate the consistency errors

$$\sup_{\vec{z}_h \in X_h} \frac{a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h)}{\| \vec{z}_h \|_h}, \quad \sup_{z_h \in X_h} \frac{b(\vec{z}_h, \vec{v}_h) - b_n(\vec{z}_h, v_h)}{\| \vec{z}_h \|_h},$$

$$\sup_{v_h \in M_h} \frac{b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h)}{\| v_h \|_{M_h}}, \quad \sup_{v_h \in M_h} \frac{| f_h(v_h) - (f, \tilde{v}_h) |}{\| v_h \|_{M_h}}$$

These are the object of the next three lemmas

LEMMA 4.2 : *There is a positive constant C independent of h such that*

$$\left.\begin{array}{l} \forall \vec{q}_h, \vec{z}_h \in X_h \\ | a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) | \leqslant Ch \, | \vec{z}_h |_X (| \vec{q}_h |_X + | D^{1/2} \operatorname{div} \vec{q}_h |_{L^2(\Omega)}) . \end{array}\right\} \quad (4.10)$$

Proof : We set :

$$E_K(\varphi) = \int_K \varphi(x)\, dx - \sum_{K \in \mathcal{T}_h} (\text{mes } K) \sum_{i=1}^{3} \varphi(m_{i_K}) .$$

Let $\tilde{\Omega}$ be an open set such that $\Omega \subset \tilde{\Omega}$ and $\Omega_h \subset \tilde{\Omega}$ for any h. D is a regular function defined over Ω : then there is an extension $\tilde{D}$ of D on $\tilde{\Omega}$ such that :

$$\tilde{D} \in W^{2,\infty}(\tilde{\Omega}) , \quad \| \tilde{D} \|_{2,\infty,\tilde{\Omega}} \leqslant C \, \| D \|_{2,\infty,\Omega} .$$

We have :

$$a(\vec{q}_h, \vec{z}_h) - a_h(\vec{q}_h, \vec{z}_h) = \sum_{K \in \mathcal{T}_h} E_K(\tilde{D}\vec{q}_h \, \vec{z}_h) +$$

$$+ \sum_{K \in \partial \mathcal{T}_h} \left[\int_K D(x)\, \vec{q}_h(x)\, \vec{z}_h(x)\, dx - \int_K \tilde{D}(x)\, \vec{q}_h(x)\, \vec{z}_h(x)\, dx \right] .$$

But, we have the estimate (Ciarlet [4]) :

$$| E_K(\tilde{D}\vec{q}_h \, \vec{z}_h) | \leqslant Ch^2 \, \| D \|_{2,\infty,K} \, \| \vec{q}_h \|_{1,K} \, \| z_h \|_{L^2(K)} .$$

Besides, $\vec{q}_h \in X_h$, so, we get,

$$| \vec{q}_h |_{1,K} \leqslant C \, | \operatorname{div} \vec{q}_h |_{L^2(K)} .$$

Thus, we obtain :

$$| E_K(\tilde{D}\vec{q}_h \, \vec{z}_h) | \leqslant Ch(| D^{1/2} \vec{q}_h |^2_{(L^2(K))^2} + | D^{1/2} \operatorname{div} \vec{q}_h |^2_{L^2(K)})^{1/2} \, | D^{1/2} \vec{z}_h |_{L^2(K)}$$

and

$$\left| \int_{\tilde{K}} D(x)\, \vec{q}_h(x)\, \vec{z}_h(x)\, dx - \int_K \tilde{D}(x)\, \vec{q}_h(x)\, \vec{z}_h(x)\, dx \right| \leqslant$$

$$\leqslant Ch \, | D^{1/2} \vec{q}_h |_{L^2(\tilde{K})} \, | D^{1/2} \vec{z}_h |_{L^2(\tilde{K})} .$$

Hence, we can deduce (4.10).

LEMMA 4 3 · *There is some positive constant C independent of h such that*

$\forall \vec{q}_h \in X_h$, $\forall v_h \in M_h$,

$$| b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) | \leqslant Ch \| v_h \|_{M_h} | D^{1/2} \operatorname{div} \vec{q}_h |_{L^2(\Omega)} \quad (4\ 11)$$

and

$$| b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) | \leqslant Ch | D^{3/2} \operatorname{div} \vec{q}_h |_{L^2(\Omega)} \left(\sum_{K \in \mathcal{T}_h} (\operatorname{mes} K) D^{-5}(a_K) v_{h|K} \right)^{1/2} \quad (4\ 12)$$

Proof We set

$$E_K(\varphi) = \int_K \varphi(x)\, dx - (\operatorname{mes} K)\, \varphi(a)$$

Then, we have

$$b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) = \sum_{K \in \mathcal{T}_h} D^{-3/2}(a_K)\, E_K(\tilde{D}^{3/2}) \operatorname{div} \vec{q}_{h|K}\, v_{h|K} +$$
$$+ \sum_{K \in \partial \mathcal{T}_h} \left(\int_{\tilde{K}} D^{3/2}(x)\, dx - \int_K \tilde{D}^{3/2}(x)\, dx \right) \operatorname{div} \vec{q}_{h|K}\, v_{h|K}\, D^{-3/2}(a_K)$$

Besides,

$$| E_K(D^{3/2}) | \leqslant Ch (\operatorname{mes} K) \| D^{1/2} \|_{0\ \infty\ K}$$

$$\left| \int_K D^{3/2}(x)\, dx - \int_K \tilde{D}^{3/2}(x)\, dx \right| \leqslant Ch^3 \| D^{3/2} \|_{0\ \infty\ K}$$

Hence, we get

$$| b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) | \leqslant Ch \| v_h \|_{M_h} \left(\sum_{K \in \mathcal{T}_h} (\operatorname{mes} K)\, D(a_K) \operatorname{div}^2 \vec{q}_{h|K} \right)^{1/2}$$

or

$$| b(\vec{q}_h, \tilde{v}_h) - b_h(\vec{q}_h, v_h) | \leqslant Ch | D^{3/2} \operatorname{div} \vec{q}_h |_{L^2(\Omega)} \left(\sum_{K \in \mathcal{T}_h} (\operatorname{mes} K)\, D^{-5}(a_K)\, v_{h|K}^2 \right)^{1/2}$$

LEMMA 4 4 *There is some positive constant C independent of h such that*

$$\forall v_h \in M_h, \quad | (f, \tilde{v}_h) - f_h(v_h) | \leqslant Ch \| v_h \|_{M_h} [| D^{1/2} \vec{f} |_{L^2(\Omega)} + | D^{3/2} \overrightarrow{\operatorname{grad}} \vec{f} |_{(L^2(\Omega))^2}] \quad (4\ 13)$$

Proof : We set, as in lemma 4.3,

$$E_K(\varphi) = \int_K \varphi(x)\,dx - (\text{mes}\,K)\,\varphi(a_K)\,.$$

Then, we get,

$$(f, \tilde{v}_h) - f_h(v_h) = \sum_{K \in \mathcal{C}_h} D^{-3/2}(a_K)\,E_K(D^{3/2}\,f)\,v_{h|K} +$$

$$+ \sum_{K \in \partial\mathcal{C}_h} \left(\int_{\tilde{K}} D^{3/2}(x)\,f(x)\,dx - \int_K \tilde{D}^{3/2}(x)\,f(x)\,dx \right) D^{-3/2}(a_K)\,v_{h|K}\,.$$

Besides,

$$|\,E_K(D^{3/2}\,f)\,| \leqslant Ch^2\,|\,\overrightarrow{\text{grad}}\,(D^{3/2}\,f)\,|_{(L^2(K))^2}$$

and

$$\left| \int_{\tilde{K}} D^{3/2}(x)\,f(x)\,dx - \int_K D^{3/2}(x)\,f(x)\,dx \right| \leqslant Ch^{7/2}\,|\,D^{1/2}\,f|_{L^2(K)}\,.$$

Hence, we deduce 4.13.

c) Interpolation error bounds

It remains to estimate :

$$\inf_{\vec{q}_h \in X_h} \|\,\vec{p} - \vec{q}_h\,\|_X \quad \text{and} \quad \inf_{v_h \in M_h} \|\,u - \tilde{v}_h\,\|_M\,.$$

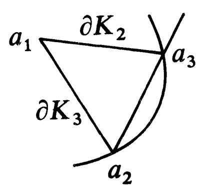
We define an interpolate $\overrightarrow{\rho_h q}$ of $\vec{q}$ with $\overrightarrow{\rho_h q} \in X_h$ as follows :

1. If $K \in \mathcal{C}'_h$, $\vec{q} \in H\,(\text{div}\,;K)$, and if we also assume that $\vec{q} \in (H^1(K))^2$, we can define the interpolate $\overrightarrow{\rho_K q}$ of $\vec{q}$ by [8] :

$$\left.\begin{aligned} &\overrightarrow{\rho_K q} \in X_K \\ &\forall i = 1, 2, 3 \quad \int_{\partial K_i} \overrightarrow{\rho_K q}.\vec{n}_i\,d\gamma = \int_{\partial K_i} \vec{q}.\vec{n}_i\,d\gamma. \end{aligned}\right\} \tag{4.14}$$

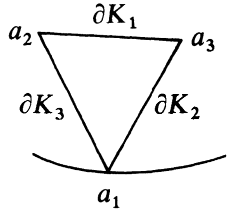
2. If $K \in \partial\mathcal{C}_h$ and has two vertices a_2^K and a_3^K on Γ, if $\vec{q} \in X_K$ such that $D^3\,\vec{q}.\vec{n} \in L^1(\partial K_i)$ ($i = 2, 3$), then we define the interpolate $\overrightarrow{\rho_K q}$ by :

$$\left.\begin{aligned} &\overrightarrow{\rho_K q} \in X_K \\ &\forall i = 2, 3 \quad \int_{\partial K_i} D^3\,\overrightarrow{\rho_K q}.\vec{n}_i\,d\gamma = \int_{\partial K_i} D^3\,\vec{q}.\vec{n}_i\,d\gamma \\ &\int_{\tilde{K}} D^3\,\text{div}\,\overrightarrow{\rho_K q}\,dx = \int_{\tilde{K}} D^3\,\text{div}\,\vec{q}\,dx\,. \end{aligned}\right\} \tag{4.15}$$



3 Therefore, because $\overrightarrow{\rho_h q}$ must be in $H(\operatorname{div}, \Omega)$, the interpolate $\overrightarrow{\rho_K q}$ of q on a triangle K which has one vertex a_1 on Γ is defined by

$$\left.\begin{array}{l} \overrightarrow{\rho_K q} \in X \\ \displaystyle\int_{\partial K_1} \overrightarrow{\rho_K q}\cdot\vec{n}_1 \, d\gamma = \int_{\partial K_1} \vec{q}\cdot\vec{n}_1 \, d\gamma \\ \displaystyle\forall i = 2, 3 \int_{\partial K_i} D^3 \, \overrightarrow{\rho_K q}\cdot\vec{n}_i \, d\gamma = \int_{\partial K_i} D^3 \, \vec{q}\cdot\vec{n}_i \, d\gamma \end{array}\right\} \tag{4 16}$$



Thus $\overrightarrow{\rho_h q}$ is in X_h and is defined for any $\vec{q} \in X$ such that $(D^{3/2} \vec{q}) \in (H^1(\Omega))^2$ We shall note

$$\overrightarrow{\operatorname{grad}} \, \vec{q} = \left(\frac{\partial q_1}{\partial x_1}, \frac{\partial q_1}{\partial x_2}, \frac{\partial q_2}{\partial x_1}, \frac{\partial q_2}{\partial x_2}\right)$$

THEOREM 4 3 *Let the triangulation $\mathcal{T}_h$ be regular Then, there is some positive constant C independent of h such that*

$$\forall \vec{q} \in X, \, D^{1/2} \, \overrightarrow{\operatorname{grad}} \, \vec{q} \in (L^2(\Omega))^4$$

$$| D^{1/2}(\vec{q} - \overrightarrow{\rho_h q}) |_{(L^2(\Omega))^2} \leqslant Ch(| D^{1/2} \vec{q} |_{(L^2(\Omega))^2} + | D^{1/2} \overrightarrow{\operatorname{grad}} \, \vec{q} |_{(L^2(\Omega))^4}) \tag{4 17}$$

$$\forall \vec{q} \in X, \, D^{1/2} \, \overrightarrow{\operatorname{grad}} \, \vec{q} \in (L^2(\Omega))^4 , \quad D^{3/2} \, \overrightarrow{\operatorname{grad}} \, (\operatorname{div} \vec{q}) \in (L^2(\Omega))^2$$

$$| D^{3/2} \operatorname{div} (\vec{q} - \overrightarrow{\rho_h q}) |_{L^2(\Omega)} \leqslant Ch(| D^{1/2} \vec{q} |_{L^2(\Omega)^2} + | D^{1/2} \overrightarrow{\operatorname{grad}} \, \vec{q} |_{(L^2(\Omega))^4} + + | D^{3/2} \overrightarrow{\operatorname{grad}} (\operatorname{div} \vec{q}) |_{(L^2(\Omega))^2}) \tag{4 18}$$

Proof If $K \in \mathcal{T}_h$, we have (*cf* Thomas [8])

$$| D^{1/2}(\vec{q} - \overrightarrow{\rho_K q}) |_{(L^2(K))^2} \leqslant Ch \, | D^{1/2} \, \overrightarrow{\operatorname{grad}} \, \vec{q} |_{(L^2(K))^4}$$

$$| D^{3/2} \operatorname{div} (\vec{q} - \overrightarrow{\rho_K q}) |_{L^2(K)} \leqslant Ch \, | D^{3/2} \, \overrightarrow{\operatorname{grad}} \, (\operatorname{div} \vec{q}) |_{(L^2(K))^2}$$

Now, if $K \in \partial\mathcal{T}_h$ and has two vertices on Γ, div $\overrightarrow{\rho_h q}$ is the orthogonal projection of div $\vec{q}$ on the space of constants with respect to the scalar product $(D^{3/2}, D^{3/2})$ and then we get

$$| D^{3/2} \operatorname{div} (\vec{q} - \overrightarrow{\rho_K q}) |_{L^2(K)} \leqslant Ch \, | D^{3/2} \, \overrightarrow{\operatorname{grad}} \, (\operatorname{div} \vec{q}) |_{(L^2(\tilde{K}))^2}$$

Besides, there is a positive constant C independent of K such that

$$| D^{1/2} \, \overrightarrow{\rho_K q} |_{(L^2(\tilde{K}))^2} \leqslant C(| D^{1/2} \vec{q} |_{L^2(\tilde{K})} + | D^{3/2} \, \overrightarrow{\operatorname{grad}} \, \vec{q} |_{(L^2(\tilde{K}))^4})$$

Then, we get

$$|\, D^{1/2}(\vec{q} - \overrightarrow{\rho_K q})\, |_{L^2(\tilde{K})} \leqslant C \inf_{\vec{a} \in (\rho_0)^2} |\, D^{1/2}(\vec{q} - \vec{a})\, |_{(L^2(\tilde{K}))^2} \leqslant$$
$$\leqslant Ch\, |\, D^{1/2}\, \overrightarrow{\text{grad}}\, \vec{q}\, |_{(L^2(\tilde{K}))^4}\,.$$

It remains to estimate $|\, D^{1/2}(\vec{q} - \overrightarrow{\rho_K q})\, |_{(L^2(K))^2}$ and $|\, D^{3/2}\, \text{div}\, (\vec{q} - \overrightarrow{\rho_K q})\, |_{L^2(K)}$ when K has one vertex a_1 on Γ.

Let us denote by $\overrightarrow{\tilde{\rho}_K q}$ the function of X_K defined as follows

$$\left.\begin{aligned} &\overrightarrow{\tilde{\rho}_K q}.\vec{n}_i = \overrightarrow{\rho_K q}.\vec{n}_i\,, \quad i = 2, 3\,; \\ &\int_K D^3\, \text{div}\, \overrightarrow{\tilde{\rho}_K q}\, dx = \int_K D^3\, \text{div}\, \vec{q}\, dx\,. \end{aligned}\right\} \tag{4.19}$$

Hence, we get

$$|\, D^{1/2}(\vec{q} - \overrightarrow{\tilde{\rho}_K q})\, |_{(L^2(K))^2} \leqslant Ch\, |\, D^{1/2}\, \overrightarrow{\text{grad}\, \vec{q}}\, |_{(L^2(K))^2}\,,$$
$$|\, D^{3/2}\, \text{div}\, (\vec{q} - \overrightarrow{\tilde{\rho}_K q})\, |_{L^2(K)} \leqslant Ch\, |\, D^{3/2}\, \overrightarrow{\text{grad}\, (\text{div}\, \vec{q})}\, |_{L^2(K)}\,.$$

Further, we have :

$$|\, D^{1/2}(\overrightarrow{\rho_K q} - \overrightarrow{\tilde{\rho}_K q})\, |_{(L^2(K))^2} \leqslant Ch^{3/2}\, |\, \overrightarrow{\rho_K q}.\vec{n}_1 - \overrightarrow{\tilde{\rho}_K q}.\vec{n}_1\, |\,;$$
$$|\, D^{3/2}\, \text{div}\, (\overrightarrow{\rho_K q} - \overrightarrow{\tilde{\rho}_K q})\, |_{L^2(K)} \leqslant Ch^{3/2}\, |\, \overrightarrow{\rho_K q}.\vec{n}_1 - \overrightarrow{\tilde{\rho}_K q}.\vec{n}_1\, |\,.$$

We have the equality :

$$\int_{\partial K_1} D^3(\overrightarrow{\tilde{\rho}_K q}.\vec{n}_1 - \overrightarrow{\rho_K q}.\vec{n}_1)\, d\gamma = \int_K (D^3\, \text{div}\, \overrightarrow{\tilde{\rho}_K q} + \overrightarrow{\text{grad}}\, D^3.\overrightarrow{\tilde{\rho}_K q})\, dx -$$
$$- \int_{\partial K_2 \cup \partial K_3} D^3\, \overrightarrow{\tilde{\rho}_K q}.\vec{n}\, d\gamma - \int_{\partial K_1} D^3\, \overrightarrow{\rho_K q}.\vec{n}_1\, d\gamma$$

and from (4.16) and (4.19), we get

$$\int_{\partial K_1} D^3(\overrightarrow{\tilde{\rho}_K q}.\vec{n}_1 - \overrightarrow{\rho_K q}.\vec{n}_1)\, d\gamma = \int_K \overrightarrow{\text{grad}}\, D^3(\overrightarrow{\tilde{\rho}_K q} - \vec{q})\, dx +$$
$$+ \int_{\partial K_1} D^3(\vec{q} - \overrightarrow{\rho_K q}).\vec{n}_1\, d\gamma\,.$$

Hence, we deduce

$$| \overrightarrow{\tilde{\rho}_K q}.\vec{n}_1 - \overrightarrow{\rho_K q}.\vec{n}_1 | \leqslant C(h^{-3/2} | \vec{q} - \overrightarrow{\tilde{\rho}_K q} |_X + h^{-1/2} | \vec{q}.\vec{n}_1 - \overrightarrow{\rho_K q}.\vec{n}_1 |_{L^2(\partial K_1)}).$$

Thus, we get

$$| \vec{q} - \overrightarrow{\rho_K q} |_X \leqslant C(h | D^{1/2} \overrightarrow{\text{grad}}\, \vec{q} |_{(L^2(K))} + h | \vec{q}.\vec{n}_1 - \overrightarrow{\rho_K q}.\vec{n}_1 |_{L^2(\partial K_1)})$$

$$| D^{3/2} \operatorname{div} (\vec{q} - \overrightarrow{\rho_K q}) |_{L^2(K)} \leqslant Ch[| D^{1/2} \overrightarrow{\text{grad}}\, \vec{q} |_{(L^2(K))^4} + | D^{3/2} \overrightarrow{\text{grad}} (\operatorname{div} \vec{q}) |_{(L^2(K))^2} + | \vec{q}.\vec{n}_1 - \overrightarrow{\rho_K q}.\vec{n}_1 |_{L^2(\partial K_1)}]$$

and, finally,

$$| D^{1/2}(\vec{q} - \overrightarrow{\rho_h q}) |_{(L^2(\Omega))^2} \leqslant Ch(| D^{1/2} \overrightarrow{\text{grad}}\, \vec{q} |_{(L^2(\Omega))^4} + | \vec{q}.\vec{n} - \overrightarrow{\rho_h q}.\vec{n} |_{L^2(\Gamma_h)}) .$$

$$| D^{3/2} \operatorname{div} (\vec{q} - \overrightarrow{\rho_h q}) |_{L^2(\Omega)} \leqslant Ch(| D^{1/2} \overrightarrow{\text{grad}}\, \vec{q} |_{(L^2(\Omega))^4} + | D^{3/2} \overrightarrow{\text{grad}} (\operatorname{div} \vec{q}) |_{(L^2(\Omega))^2} + | \vec{q}.\vec{n} - \overrightarrow{\rho_h q}.\vec{n} |_{L^2(\Gamma_h)}) .$$

Further, we have [5]

$$| \vec{q}.\vec{n} - \overrightarrow{\rho_h q}.\vec{n} |_{L^2(\Gamma_h)} \leqslant Ch^{1/2} \| \vec{q} \|_{H^{1/2}(\Gamma_h)} \leqslant Ch^{1/2} \| \vec{q} \|_{H^1(\Omega_h)} .$$

Then, we deduce (4.17) and (4.18).

Now we consider the M_h-interpolation operator π_h.

Given a function $v \in M$, the M_h-interpolant $\pi_h v$ is the unique function which satisfies :

$$\left. \begin{array}{l} \pi_h v \in M_h \\ \forall K \in \mathcal{C}'_h,\ \pi_h v |_K = \pi_K v = \dfrac{1}{\text{mes}\, K} \displaystyle\int_K v\, dx \\ \forall K \in \partial\mathcal{C}_h,\ \pi_K v = 0 . \end{array} \right\} \tag{4.20}$$

THEOREM 4.4 : *Let the triangulation $\mathcal{C}_h$ be regular. Then, there is some positive constant C independent of h such that :*

$$\forall v \in H^1_{-1/2}(\Omega),\ \| v - \tilde{\pi}_h v \|_M \leqslant C \{ h[| D^{-5/2} v |_{L^2(\Omega_h)} + | D^{-3/2} \overrightarrow{\text{grad}}\, v |_{L^2(\Omega_h)}] + | D^{-1/2} \overrightarrow{\text{grad}}\, v |_{L^2(\Omega - \Omega_h)} \} . \tag{4.21}$$

Proof : If $K \in \mathcal{C}'_h$, we get

$$| D^{-3/2}(v - \tilde{\pi}_K v) |_{L^2(K)} = | D^{-3/2} v - D^{-3/2}(a) \pi_K v |_{L^2(K)}$$

and then,

$$| D^{-3/2}(v - \tilde{\pi}_K v) |_{L^2(K)} \leqslant Ch(| D^{-3/2} \overrightarrow{\text{grad}}\, v |_{L^2(K)} + | D^{-5/2} v |_{L^2(K)}) .$$

Now, we assume that $K \in \partial \mathcal{T}_h$

Then, we have the estimate (*cf* Bolley-Camus [1])

$$| D^{-3/2} v |_{L^2(\tilde{K})} \leqslant C | D^{-1/2} \overrightarrow{\text{grad}} \, v |_{L^2(\tilde{K})}$$

(where C is a constant independent of $\tilde{K}$)

Then we deduce (4 21)

d) Error estimates

THEOREM 4 5 *Assume that the solution* $(\vec{p}, u)$ *of* (2 15) *satisfies the smoothness properties*

$$\left.\begin{array}{l} D^{1/2} \overrightarrow{\text{grad}} \, p \in (L^2(\Omega))^4 , \quad D^{-3/2}(\text{Log } R/D)^{-1} \overrightarrow{\text{grad}} \, u \in (L^2(\Omega))^2 \\ D^{-5/2}(\text{Log } R/D)^{-1} \, u \in L^2(\Omega) \end{array}\right\} \quad (4\ 22)$$

with $R > \max\left(\max_{x \in \bar{\Omega}} D(x), e^2\right)$

Then, we have the estimates

$$| \vec{p} - \vec{p}_h |_X \leqslant Ch(| \vec{p} |_X + | D^{1/2} \overrightarrow{\text{grad}} \, \vec{p} |_{(L^2(\Omega))^4} + | D^{1/2} f |_{L^2(\Omega)} + \\ + | D^{3/2} \overrightarrow{\text{grad}} \, f |_{(L^2(\Omega))^2}) \quad (4\ 23)$$

$$\| u - \tilde{u}_h \|_M \leqslant C \{ h(| \vec{p} |_X + | D^{1/2} \overrightarrow{\text{grad}} \, \vec{p} |_{(L^2(\Omega))^4} + \\ + h \text{ Log } h(| D^{-3/2}(\text{Log } R/D)^{-1} \overrightarrow{\text{grad}} \, u |_{(L^2(\Omega))^2} + | D^{-5/2}(\text{Log } R/D)^{-1} \, u |_{L^2(\Omega)}) \} \quad (4\ 24)$$

Proof In the estimate (4 8), we choose $\vec{q}_h = \rho_h \vec{p}$, then by using lemmas 4 2, 4 3, 4 4, we get

$$| \vec{p} - \vec{p}_h |_X \leqslant \| \vec{p} - \overrightarrow{\rho_h p} \|_X + C \{ h(| \overrightarrow{\rho_h p} |_X + | D^{1/2} \operatorname{div} \overrightarrow{\rho_h p} |_{L^2(\Omega)}) + \\ + | D^{3/2} \operatorname{div} (\vec{p} - \overrightarrow{\rho_h p}) |_{L^2(\Omega)} + h(| D^{1/2} f |_{L^2(\Omega)} + | D^{3/2} \overrightarrow{\text{grad}} \, f |_{(L^2(\Omega))^2}) \}$$

Then, from theorem (4 3), we deduce (4 23)

In the estimate (4 9), we choose $v_h = \pi_h u$, then, we get

$$\| u - \tilde{u}_h \|_M \leqslant C \left\{ | p - \vec{p}_h |_X + | \vec{p} - \overrightarrow{\rho_h p} |_X + h(| \overrightarrow{\rho_h p} |_X + | D^{1/2} \operatorname{div} \overrightarrow{\rho_h p} |_{L^2(\Omega)}) + \\ + \| u - \tilde{\pi}_h u \|_M + h \left(\sum_{K \in \mathcal{T}_h} (\text{mes } K) D^{-5}(a_K) (\pi_K u)^2 \right)^{1/2} \right\}$$

and by using theorem (4 4), we obtain (4 24)

Remark : If we assume that $D^{1/2}\,\vec{p} \in (L^2(\Omega))^2$, $D^{1/2}\overrightarrow{\text{grad}\,\vec{p}} \in (L^2(\Omega))^4$, then, we have the estimate (Bolley-Camus [1])

$$| D^{-1/2}(\text{Log}\, R/D)^{-1}\, \vec{p}\, |_{(L^2(\Omega))^2} \leqslant C\, |\, D^{1/2}\, \overrightarrow{\text{grad}\,\vec{p}}\, |_{(L^2(\Omega))^4}\,,$$

that is

$$| D^{-3/2}(\text{Log}\, R/D)^{-1}\, \overrightarrow{\text{grad}}\, u\, |_{(L^2(\Omega))^2} \leqslant C\, |\, D^{1/2}\, \overrightarrow{\text{grad}\,\vec{p}}\, |_{(L^2(\Omega))^4}\,.$$

Similarly, we have

$$| D^{-5/2}(\text{Log}\, R/D)^{-1}\, u\, |_{L^2(\Omega)} \leqslant C\, |\, D^{-3/2}(\text{Log}\, R/D)^{-1}\, \overrightarrow{\text{grad}}\, u\, |_{(L^2(\Omega))^2}\,.$$

V. PROOF OF THEOREM 4.1

Before proving theorem 4.1, we need several results.

Notations. We define the spaces S_K, $\mathcal{M}_h$, $\mathcal{H}_h$ by :

$$S_K = \{\, \mu \in L^2(\partial K)/\mu\, |_{\partial K_i} \in P_0\,, \quad i = 1, 2, 3\,\}$$

$$\mathcal{M}_h = \left\{ \mu_h \in \prod_{K \in \mathcal{T}_h} L^2(\partial K)/\mu_{h\,|\,\partial K} \in S_K\,;\ \ \forall K_1, K_2 \in \mathcal{T}_h,\ \mu_{h\,|\,K_1} + \mu_{h\,|\,K_2} = 0 \text{ over } K_1 \cap K_2 \right\}$$

$$\mathcal{H}_h = \{\, \varphi_h \in L^2(\Omega_h)/\forall K \in \mathcal{T}_h,\ \varphi_{h\,|\,K} \in P_1\,\}\,.$$

If Ψ is a function of $\prod_{K \in \mathcal{T}_h} L^2(\partial K)$, let us denote by $s_h\,\Psi$ the function such that $s_h\,\Psi \in \prod_{K \in \mathcal{T}_h} L^2(\partial K)$ and $s_h\,\Psi\, |_{\partial K_i}$ is the projection of Ψ onto P_0.

LEMMA 5.1 : *There exists a constant $C > 0$ independent of h such that*

$$\left.\begin{aligned} &\forall \varphi_h \in \mathcal{H}_h\,, \quad \forall \Psi \in H^1(\Omega_h')\,, \\ &\sum_{K \in \mathcal{T}_h'} D(a_K) \int_{\partial K} \varphi_h(\Psi - s_h\,\Psi)\, d\gamma\, | \leqslant Ch\Big(\sum_{K \in \mathcal{T}_h} D^{-1}(a_K)\, |\, \varphi_h\, |^2_{L^2(K)} \Big)^{1/2} \\ &\qquad\qquad (|\, D^{3/2}\, \Psi\, |^2_{L^2(\Omega_h)} + |\, D^{3/2}\, \overrightarrow{\text{grad}}\, \Psi\, |^2_{L^2(\Omega_h)})^{1/2}\,. \end{aligned}\right\} \quad (5.1)$$

Proof : Let $\varphi_h \in \mathscr{H}_h$ and $\Psi \in H^1(\Omega'_h)$; then we have the equality

$$\int_{\partial K} \varphi_h(\Psi - s_h \Psi)\, d\gamma = \int_{\partial K} (\varphi_h - s_h \varphi_h)(\Psi - s_h \Psi)\, d\gamma$$

and then

$$\left| \sum_{K \in \mathcal{C}_h} D(a_K) \int_{\partial K} \varphi_h(\Psi - s_h \Psi)\, d\gamma \right.$$

$$\leqslant C \sum_{K \in \mathcal{C}_h} D(a_K) \mid \varphi_h - s_h \varphi_h \mid_{L^2(\partial K)} \mid \Psi - s_h \Psi \mid_{L^2(\partial K)}$$

$$\leqslant C \left(\sum_{K \in \mathcal{C}_h} D^{-1}(a_K) \mid \varphi_h - s_h \varphi_h \mid^2_{L^2(\partial K)} \right)^{1/2} \left(\sum_{K \in \mathcal{C}_h} D^3(a_K) \mid \Psi - s_h \Psi \mid^2_{L^2(\partial K)} \right)^{1/2} .$$

Besides, we have the estimate [5]

$$\mid \Psi - s_h \Psi \mid_{L^2(\partial K)} \leqslant C h^{1/2} \parallel \Psi \parallel_{H^{1/2}(\partial K)} \leqslant C h^{1/2} \parallel \Psi \parallel_{H^1(K)}$$

hence, we get

$$\left(\sum_{K \in \mathcal{C}_h} D^3(a_K) \mid \Psi - s_h \Psi \mid^2_{L^2(\partial K)} \right)^{1/2} \leqslant$$

$$\leqslant C h^{1/2} \left(\mid D^{3/2} \Psi \mid^2_{L^2(\Omega_h)} + \mid D^{3/2}\, \overrightarrow{\operatorname{grad}}\, \Psi \mid^2_{L^2(\Omega_h)} \right)^{1/2} .$$

Further, we have

$$\mid \varphi_h - s_h \varphi_h \mid_{L^2(\partial K)} \leqslant C h \mid \operatorname{grad} \varphi_h \mid_{L^2(\partial K)} \leqslant C h^{1/2} \mid \overrightarrow{\operatorname{grad}}\, \varphi_h \mid_{L^2(K)}$$

and then

$$\left(\sum_{K \in \mathcal{C}_h} D^{-1}(a_K) \mid \varphi_h - s_h \varphi_h \mid^2_{L^2(\partial K)} \right)^{1/2} \leqslant$$

$$\leqslant C h^{+1/2} \left(\sum_{K \in \mathcal{C}_h} D^{-1}(a_K) \mid \overrightarrow{\operatorname{grad}}\, \varphi_h \mid^2_{L^2(K)} \right)^{1/2} .$$

So, we obtain 5.1.

THEOREM 5.1 : *Let φ_h be a function in $\mathscr{H}_h$ such that :*

$$\forall \mu_h \in \mathscr{M}_h , \quad \sum_{K \in \mathcal{C}_h} D(a_K) \int_{\partial K} \varphi_h\, \mu_h\, d\gamma = 0 . \tag{5.2}$$

Then, there is some positive constant C such that :

$$\sum_{K \in \mathcal{T}_h} D^{-1}(a_K) \mid \varphi_h \mid^2_{L^2(K)} \leqslant C \left(\sum_{K \in \mathcal{T}_h} D(a_K) \mid \overrightarrow{\text{grad}}\, \varphi_h \mid^2_{L^2(K)} \right). \quad (5.3)$$

Proof : Let $\varphi_h \in \mathcal{H}_h$ satisfying (5.2) and consider the problem. Find $u \in H^1_{-1/2}(\Omega)$ such that :

$$\left.\begin{aligned} &\text{div}\,(D^{-1}\,\overrightarrow{\text{grad}}\, u) = \bar{D}^{-2}\,\varphi_h\,, \\ &u \mid_\Gamma = 0 \end{aligned}\right\} \quad (5.4)$$

where $\bar{D}$ is the function defined over Ω by :

$$\bar{D} \mid_K = D(a_K)\,.$$

The solution of (6.4) is in $H^2_{1/2}(\Omega)$ and we have the estimate :

$$\| u \|_{2,1/2,\Omega} \leqslant C \mid D^{3/2}\, \bar{D}^{-2}\, \varphi_h \mid_{L^2(\Omega)} \leqslant C \left(\sum_{K \in \mathcal{T}_h} D^{-1}(a_K) \mid \varphi_h \mid^2_{L^2(K)} \right)^{1/2}. \quad (5.5)$$

Then, we may write :

$$\sum_{K \in \mathcal{T}_h} D^{-1}(a_K) \mid \varphi_h \mid^2_{L^2(K)} = \sum_{K \in \mathcal{T}_h} \int_K \text{div}\,(D^{-1}\,\overrightarrow{\text{grad}}\, u)\, D(a_K)\, \varphi_h\, dx + \\ + \sum_{K \in \partial \mathcal{T}_h} D^{-1}(a_K) \mid \varphi_h \mid^2_{L^2(K)}\,.$$

From (5.2), the function $\bar{D}\varphi_h$ is continue in the midpoints of each side ∂K_i and it is null in the midpoint of a side of the boundary Γ_h ; then, we get easily

$$\sum_{K \in \partial \mathcal{T}_h} D^{-1}(a_K) \mid \varphi_h \mid^2_{L^2(\partial K)} \leqslant C \sum_{K \in \partial \mathcal{T}_h} h^2\, D^{-1}(a_K) \mid \overrightarrow{\text{grad}}\, \varphi_h \mid^2_{L^2(K)} \leqslant \\ \leqslant C \sum_{K \in \partial \mathcal{T}_h} D(a_K) \mid \overrightarrow{\text{grad}}\, \varphi_h \mid^2_{L^2(K)}\,.$$

Besides, by using Green's formula over each $K \in \mathcal{T}'_h$, we obtain :

$$\sum_{K \in \mathcal{T}'_h} \int_K \text{div}\,(D^{-1}\,\overrightarrow{\text{grad}}\, u)\, D(a_K)\, \varphi_h\, dx = \\ = - \sum_{K \in \mathcal{T}'_h} \int_K D(a_K)\, D^{-1}\,\overrightarrow{\text{grad}}\, u . \overrightarrow{\text{grad}}\, \varphi_h\, dx + \sum_{K \in \mathcal{T}'_h} \int_{\partial K} D(a_K)\, D^{-1}\, \frac{\partial u}{\partial n}\, d\gamma\,.$$

Further, we have

$$\left| \sum_{K \in \mathcal{C}_h'} \int_K D(a_K)\, D^{-1}\, \overrightarrow{\operatorname{grad}}\, u\, \overrightarrow{\operatorname{grad}}\, \varphi_h\, dx \right| \leqslant$$

$$\leqslant C \left(\sum_{K \in \mathcal{C}_h'} D(a_K)\, | \overrightarrow{\operatorname{grad}}\, \varphi_h |^2_{L^2(K)} \right)^{1/2} | D^{-1/2}\, \overrightarrow{\operatorname{grad}}\, u |_{(L^2(\Omega))^2}$$

and from (5.5), we get

$$\left| \sum_{K \in \mathcal{C}_h'} \int_K D(a_K)\, D^{-1}\, \overrightarrow{\operatorname{grad}}\, u\, \overrightarrow{\operatorname{grad}}\, \varphi_h\, dx \right| \leqslant$$

$$\leqslant C \left(\sum_{K \in \mathcal{C}_h} D(a_K)\, | \overrightarrow{\operatorname{grad}}\, \varphi_h |^2_{L^2(K)} \right)^{1/2} \left(\sum_{K \in \mathcal{C}_h} D^{-1}(a_K)\, | \varphi_h |^2_{L^2(K)} \right)^{1/2} .$$

Consider now the term

$$\sum_{K \in \mathcal{C}_h} \int_{\partial K} D(a_K)\, D^{-1} \frac{\partial u}{\partial n}\, \varphi_h\, d\gamma .$$

We have

$$\frac{\partial u}{\partial n} = \sum_{\iota=1}^{2} \frac{\partial u}{\partial x_\iota}\, n_\iota \quad (n_\iota \text{ are the components of } \vec{n}) .$$

We note

$$\Psi_\iota(x) = D^{-1}(x) \frac{\partial u}{\partial x_\iota}(x) , \quad \iota = 1, 2 .$$

We may write

$$\sum_{K \in \mathcal{C}_h'} \int_{\partial K} D(a_K)\, D^{-1} \frac{\partial u}{\partial n}\, \varphi_h\, d\gamma = \sum_{K \in \mathcal{C}_h'} \sum_{\iota=1}^{2} D(a_K) \int_{\partial K} \Psi_\iota\, \varphi_h\, n_\iota\, d\gamma =$$

$$= \sum_{K \in \mathcal{C}_h'} \sum_{\iota=1}^{2} \int_{\partial K} D(a_K)\, \varphi_h (\Psi_\iota - s_h\, \Psi_\iota)\, n_\iota\, d\gamma +$$

$$+ \sum_{K \in \mathcal{C}_h'} \sum_{\iota=1}^{2} \int_{\partial K} D(a_K)\, s_h\, \Psi_\iota\, \varphi_h\, n_i\, d\gamma$$

and from lemma 5.1,

$$\left| \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \int_{\partial K} D(a_K)\, \varphi_h(\Psi_i - s_h \Psi_i)\, n_i \, d\gamma \right| \leqslant$$

$$\leqslant C\left(\sum_{K \in \mathcal{T}_h} D(a_K) \,|\, \overrightarrow{\text{grad}}\, \varphi_h \,|_{L^2(K)}^2 \right)^{1/2}$$

$$\times \sum_{i=1}^{2} (|\, D^{3/2}\, \Psi_i \,|_{L^2(\Omega_h)}^2 + |\, D^{3/2}\, \overrightarrow{\text{grad}}\, \Psi_i \,|_{L^2(\Omega_h)}^2)^{1/2} .$$

Then, by using (5 5) we get

$$\left| \sum_{K \in \mathcal{T}_h} \int_{\partial K} D(a_K)\, D^{-1} \frac{\partial u}{\partial n} \varphi_h \, d\gamma \right| \leqslant C\left(\sum_{K \in \mathcal{T}_h} D(a_K) \,|\, \overrightarrow{\text{grad}}\, \varphi_h \,|_{L^2(K)}^2 \right)^{1/2} \times$$

$$\times \left(\sum_{K \in \mathcal{T}_h} D^{-1}(a_K) \,|\, \varphi_h \,|_{L^2(K)}^2 \right)^{1/2}$$

Now, we estimate

$$\sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \int_{\partial K} D(a_K)\, s_h\, \Psi_i\, n_i\, \varphi_h \, d\gamma$$

Using hypothesis (5 2), we may write

$$\sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \int_{\partial K} D(a_K)\, s_h\, \Psi_i\, n_i\, \varphi_h \, d\gamma = - \sum_{K \in \partial \mathcal{T}_h} \sum_{i=1}^{2} \int_{\Gamma_h \cap \partial K} D(a_K)\, s_h\, \Psi_i\, n_i\, \varphi_h \, d\gamma$$

and therefore,

$$\left| \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \int_{\partial K} D(a_K)\, s_h\, \Psi_i\, n_i\, \varphi_h \, d\gamma \right| \leqslant \left(\sum_{K \in \partial \mathcal{T}_h} D^{-1}(a_K) \,|\, \varphi_h \,|_{L^2(\partial K)}^2 \right)^{1/2} \times$$

$$\times \left(\sum_{K \in \partial \mathcal{T}_h} \sum_{i=1}^{2} D^3(a_K) \,|\, s_h\, \Psi_i \,|_{L^2(\partial K \cap \Gamma_h)}^2 \right)^{1/2} .$$

But,

$$\left(\sum_{K \in \partial \mathcal{T}_h} D^3(a_K) \,|\, s_h\, \Psi_i \,|_{L^2(\partial K \cap \Gamma_h)}^2 \right)^{1/2} \leqslant$$

$$\leqslant C h^{1/2} \{ \,|\, D^{3/2}\, \Psi_i \,|_{L^2(\Omega_h)}^2 + |\, D^{3/2}\, \overrightarrow{\text{grad}}\, \Psi_i \,|_{L^2(\Omega_h)}^2 \}^{1/2}$$

$$\leqslant C h^{1/2} \left(\sum_{K \in \mathcal{T}_h} D^{-1}(a_K) \,|\, \varphi_h \,|_{L^2(K)}^2 \right)^{1/2}$$

and

$$\left(\sum_{K \in \partial \mathcal{T}_h}{}' D^{-1}(a_K) \mid \varphi_h \mid_{L^2(\partial K)}^2\right)^{1/2} \leqslant C h^{-1/2} \left(\sum_{K \in \partial \mathcal{T}_h} D(a_K) \mid \overrightarrow{\text{grad}}\, \varphi_h \mid_{(L^2(K))^2}^2\right)^{1/2}.$$

Then

$$\left| \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \int_{\partial K} D(a_K)\, s_h\, \Psi_i\, n_i\, \varphi_h\, d\gamma \right| \leqslant C \left(\sum_{K \in \mathcal{T}_h} D^{-1}(a_K) \mid \varphi_h \mid_{L^2(K)}^2\right)^{1/2} \times$$
$$\times \left(\sum_{K \in \partial \mathcal{T}_h} D(a_K) \mid \overrightarrow{\text{grad}}\, \varphi_h \mid_{L^2(K)}^2\right)^{1/2}.$$

Thus, we obtain (5.3).

Let us provide the space $\mathcal{M}_h$ with the norm :

$$||| \mu_h |||_h = \left(\sum_{K \in \mathcal{T}_h} h_K\, D(a_K) \parallel \mu_h \parallel_{L^2(\partial K)}^2\right)^{1/2}$$

and the space $\mathcal{H}_h$ with the norm :

$$||| \Psi_h |||_{\mathcal{H}_h} = \sum_{K \in \mathcal{T}_h} (D^{-3}(a_K) \mid \Psi_h \mid_{L^2(K)}^2 + D^{-1}(a_K) \mid \overrightarrow{\text{grad}}\, \Psi_h \mid_{(L^2(K))^2}^2)^{1/2}.$$

LEMMA 5.2 : *We have the estimate* :

$$\forall \mu_h \in \mathcal{M}_h, \quad ||| \mu_h |||_h \leqslant C \sup_{\Psi_h \in \mathcal{H}_h} \frac{\displaystyle\sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h\, \mu_h\, d\gamma}{||| \Psi_h |||_{\mathcal{H}_h}}. \tag{5.6}$$

Proof : Let us denote by :

$$\hat{\Psi}_h = \Psi_h \circ F, \quad \hat{\mu}_h = J_n\, \mu_h \circ F \quad (J_n = J \mid B^{-1} \vec{\hat{n}} \mid).$$

Then,

$$\int_{\partial K} \Psi_h\, \mu_h\, d\gamma = \int_{\partial \hat{K}} \hat{\Psi}_h\, \hat{\mu}_h\, d\hat{\gamma}$$

and the mapping

$$\hat{\mu}_h \mapsto \sup_{\hat{\Psi}_h \in P_1} \frac{\displaystyle\int_{\hat{K}} \hat{\Psi}_h\, \hat{\mu}_h\, d\hat{\gamma}}{\parallel \hat{\Psi} \parallel_{H^1(\hat{K})}}$$

is a norm over S_K; then, there is a positive constant C such that

$$\| \hat{\mu}_h \|_{L^2(\partial K)} \leqslant C \sup_{\hat{\Psi}_h \in P_1} \frac{\displaystyle\int_{\partial \hat{K}} \hat{\Psi}_h \hat{\mu}_h \, d\hat{\gamma}}{\| \hat{\Psi}_h \|_{H^1(\hat{K})}} .$$

Besides, we have [6]

$$\| \mu_h \|_{L^2(\partial K)} \leqslant C h^{-1/2} \| \hat{\mu}_h \|_{L^2(\partial \hat{K})}$$

and

$$\| \hat{\Psi}_h \|_{H^1(\hat{K})} \geqslant \frac{C}{h} (\| \Psi_h \|^2_{L^2(K)} + h^2 \, | \overrightarrow{\text{grad}} \, \Psi_h |^2_{(L^2(K))^2})^{1/2} .$$

Then, we get

$$\| \hat{\Psi}_h \|_{H^1(\hat{K})} \geqslant C D^{1/2}(a_K) \, (D^{-3}(a_K) \, | \Psi_h |^2_{L^2(K)} + D^{-1}(a_K) \, | \overrightarrow{\text{grad}} \, \Psi_h |^2_{(L^2(K))^2})^{1/2} .$$

Hence, we deduce

$$| \mu_h |_{L^2(\partial K)} \leqslant C h^{-1/2} \, D^{-1/2}(a_K) \, \times$$

$$\times \sup_{\Psi_h \in P_1} \frac{\displaystyle\int_{\partial K} \Psi_h \, \mu_h \, d\gamma}{(D^{-3}(a_K) \, | \Psi_h |^2_{L^2(K)} + D^{-1}(a_K) \, | \overrightarrow{\text{grad}} \, \Psi_h |^2_{(L^2(K))^2})^{1/2}} .$$

But, since the space $\mathscr{H}_h$ is isomorphic to the space $\prod_{K \in \mathcal{T}_h} P_1(K)$, the local estimate implies the global estimate.

LEMMA 5.3 : *There is a linear operator ϕ_h from $L^2(\Omega_h)$ into $\mathscr{M}_h$ such that if* $\mu_h = \phi_h \, v$

$$\forall K \in \mathcal{T}_h \int_{\partial K} \mu_h \, d\gamma = \int_K v \, dx \tag{5.7}$$

and

$$||| \mu_h |||_h \leqslant C \left(\sum_{K \in \mathcal{T}_h} D^3(a_K) \, | v |^2_{L^2(K)} \right)^{1/2} . \tag{5.8}$$

Proof : Let us consider the problem : given $v \in L^2(\Omega_h)$, find a pair

$$(\varphi_h, \mu_h) \in \mathscr{H}_h \times \mathscr{M}_h$$

such that

$$\left.\begin{aligned}&\forall \Psi_h \in \mathscr{H}_h, \sum_{K \in \mathcal{T}_h} \int_K \overrightarrow{\operatorname{grad}}\, \varphi_h\, \overrightarrow{\operatorname{grad}}\, \Psi_h\, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h\, \mu_h\, d\gamma = \\ &\qquad\qquad = - \sum_{K \in \mathcal{T}_h} \int_K v \Psi_h\, dx \\ &\forall \mu_h' \in \mathscr{M}_h, \sum_{K \in \mathcal{T}_h} D(a_K) \int_K \varphi_h\, \mu_h'\, d\gamma = 0\,.\end{aligned}\right\} \quad (5.9)$$

This problem has a unique solution (Thomas [8]).

Besides, from lemma 5.2, we have

$$|||\, \mu_h\, |||_h \leqslant C \sup_{\Psi_h \in \mathscr{H}_h} \frac{\displaystyle\sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h\, \mu_h\, d\gamma}{|||\, \Psi_h\, |||_{\mathscr{H}_h}}$$

and

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \Psi_h\, \mu_h\, d\gamma = \sum_{K \in \mathcal{T}_h} \int_K \overrightarrow{\operatorname{grad}}\, \varphi_h\, \overrightarrow{\operatorname{grad}}\, \Psi_h\, dx - \sum_{K \in \mathcal{T}_h} \int_K v \Psi_h\, dx\,.$$

Then, we obtain

$$|||\, \mu_h\, |||_h \leqslant \left(\sum_{K \in \mathcal{T}_h} D(a_K)\, |\, \overrightarrow{\operatorname{grad}}\, \varphi_h\, |_{L^2(K)}^2 \right)^{1/2} + \left(\sum_{K \in \mathcal{T}_h} D^3(a_K)\, |\, v\, |_{L^2(K)}^2 \right)^{1/2}.$$

Now, if in (5.9), we choose $\Psi_h = \overline{D} \varphi_h$, we obtain

$$\begin{aligned}\sum_{K \in \mathcal{T}_h} D(a_K)\, |\, \overrightarrow{\operatorname{grad}}\, \varphi_h\, |_{L^2(K)}^2 &= - \sum_{K \in \mathcal{T}_h} \int_K D(a_K)\, v \varphi_h\, dx \leqslant \\ &\leqslant \left(\sum_{K \in \mathcal{T}_h} D^3(a_K)\, |\, v\, |_{L^2(K)}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h} D^{-1}(a_K)\, |\, \varphi_h\, |_{L^2(K)}^2 \right)^{1/2}\end{aligned}$$

and, then from lemma 5.1, we get

$$\sum_{K \in \mathscr{T}_h} D(a_K)\, |\, \overrightarrow{\operatorname{grad}}\, \varphi_h\, |_{L^2(K)}^2 \leqslant C \sum_{K \in \mathcal{T}_h} D^3(a_K)\, |\, v\, |_{L^2(K)}^2\,.$$

Finally, we obtain (5.8).

LEMMA 5.4 : *There is a linear operator θ_h from M_h into X_h such that if* $\theta_h \vec{v}_h = \vec{q}_h$

$$\operatorname{div} \vec{q}_h = v_h \tag{5.10}$$

and

$$|||\, \vec{q}_h \,|||_h \leqslant C \sum_{K \in \mathcal{T}_h} (\text{mes } K)\, D^3(a_K)\, (v_{h|K})^2 . \tag{5.11}$$

Proof : Let v_h be in M_h ; we note

$$\mu_h = \phi_h\, v_h .$$

Then, there is $\vec{q}_h$ in X_h such that [8]

$$\begin{cases} \vec{q}_h . \vec{n}_K = \mu_h \\ \operatorname{div} \vec{q}_h = v_h . \end{cases}$$

Besides

$$| \vec{q}_h |_h^2 = \sum_{K \in \mathcal{T}_h} \frac{\text{mes } K}{3} \sum_{i=1}^{3} D(m_i)\, (\vec{q}_h(m_i))^2$$

$$\vec{q}_h(x) = J^{-1} B \hat{\vec{q}}_h(\hat{x}) .$$

Hence, we get

$$| \vec{q}_h |_h^2 \leqslant C \sum_{K \in \mathcal{T}_h} \frac{\text{mes } K}{3} \frac{D(a_K)}{h^2} \sum_{i=1}^{3} | \vec{q}_h . \vec{n}_i |^2$$

$$\leqslant C' \sum_{K \in \mathcal{T}_h} \frac{\text{mes } K}{3} D(a_K) \sum_{i=1}^{3} | \vec{q}_h . \vec{n}_i |^2 .$$

Thus, we obtain

$$| \vec{q}_h |_h \leqslant C \,|||\, \mu_h \,|||_h$$

and from lemma 5.3, we deduce

$$|||\, \vec{q}_h \,|||_h \leqslant C \sum_{K \in \mathcal{T}_h} (\text{mes } K)\, D^3(a_K)\, (v_{h|K})^2 .$$

Proof of theorem 4.1 : Let v_h be in M_h ; we note

$$w_h = \overline{D}^{-3}\, v_h \quad (\text{i.e. } w_{h|K} = D^{-3}(a_K)\, v_{h|K})$$

$$\vec{q}_h = \theta_h\, w_h .$$

Then,

$$b_h(\vec{q}_h, v_h) = \sum_{K \in \mathcal{T}_h} (\text{mes } K)\, v_{h|K} \operatorname{div} \vec{q}_h = \sum_{K \in \mathcal{T}_h} (\text{mes } K)\, D^{-3}(a_K)\,(v_{h|K})^2$$
$$= \| v_h \|_{M_h}^2 .$$

Further, we have

$$|||\, \vec{q}_h \,|||_h \leqslant C \sum_{K \in \mathcal{T}_h} D^3(a_K)\,(w_{h|K})^2 = \| v_h \|_{M_h}^2 .$$

Thus, we obtain

$$\frac{b_h(\vec{q}_h, v_h)}{\| \vec{q}_h \|_h} \geqslant C \| v_h \|_{M_h} .$$

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