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MAXIMUM NORM ERROR ESTIMATES IN THE FINITE ELEMENT METHOD WITH ISOPARAMETRIC QUADRATIC ELEMENTS AND NUMERICAL INTEGRATION (*) (1)

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Communiqué par V. THOMÉE

Abstract. — *We consider a second order real elliptic Dirichlet problem in a bounded plane smooth domain,* $Lu = f$ *in* $\Omega \subset\subset R^2$, $u = 0$ *on* $\partial\Omega$. *For its numerical solution we employ the finite element method with quadratic triangular isoparametric elements combined with a numerical integration procedure involving function values at three nodal points in each element.*

We prove that if f *has third order derivatives in* L_1 *and if the sequence of finite element partitions is quasi-uniform with the maximum diameter of any element for a certain partition being essentially* h, *then with* u_h *denoting the approximate solution we have*

$$\max_{x \in \Omega} | u(x) - u_h(x) | \leqq C_\varepsilon h^{3-\varepsilon} \| f \|_{W_1^3}.$$

Here ε *is an arbitrarily small positive number, and* C_ε *does not depend on* h *or* f.

1. THE NUMERICAL PROCEDURE AND THE MAIN RESULT

In this section we shall present the problem which we want to solve, define precisely our method for finding an approximate solution, and state our error estimates (in Theorem 1.1). After giving some references to related work we proceed to prove Theorem 1.1 using certain auxiliary results which will be verified in the remaining sections of the paper.

The following conventions will be used throughout this paper. The letter C will denote a generic constant, and ε will be used for an arbitrarily small positive number, often subject to a non-essential change. Generic constants C may depend on ε without explicit mention.

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Let $\Omega \subset\subset R^2$ be a bounded domain with boundary $\partial\Omega$ of class $\mathscr{C}^3$, and consider the Dirichlet problem

$$\left.\begin{aligned} Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i}+c(x)u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

Here it is assumed that the real coefficients a_{ij}, b_i and c belong to $\mathscr{C}^3(\bar{\Omega})$, and that the symmetric matrix function $\{ a_{ij}(x) \}$ is uniformly positive definite. We remark that the smoothness of the coefficients will be used mainly in estimates for the numerical quadrature; for various regularity results needed for solutions of (1.1) or its adjoint, less smoothness suffices. We postulate that the problem (1.1) has a unique weak solution in $\mathring{W}_2^1(\Omega)$ for any f in $L_2(\Omega)$. Here $\mathring{W}_p^k(\Omega)$ for $k \geqq 0$ an integer and $1 \leqq p \leqq \infty$ stands for the closure of $\mathscr{C}_0^\infty(\Omega)$ in $W_p^k(\Omega)$, the Sobolev space with norm

$$\| v \|_{W_p^k(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leqq k} \| D^\alpha v \|_{L_p(\Omega)}^p\right)^{1/p} & \text{for } 1 \leqq p < \infty, \\ \max_{|\alpha| \leqq k} \| D^\alpha v \|_{L_\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

For $p = \infty$ we shall also use non-integral k; $W_\infty^k(\Omega) = \mathscr{C}^k(\Omega)$ is then the appropriate Hölder space.

We shall now describe the numerical solution of the problem (1.1) using the finite element method with isoparametric quadratic elements and a second order accurate numerical integration scheme.

We follow the work of Ciarlet [5], and Ciarlet-Raviart [6, 7, 8].

We consider a sequence of finite element partitions associated with Ω. Let $N = N_0, N_0+1, \ldots$, (or a subsequence thereof) and let

$$h = N^{-1/2}.$$

In a preliminary way, let points on $\partial\Omega$ and in Ω be given, inducing a certain straight-edged triangulation of a domain close to Ω (*fig.* 1). Each triangle

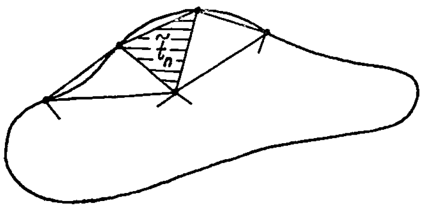

Figure 1.

$\tilde{t}_n$, $n = 1, \ldots, N$, can be thought of as the image under an affine map of the two-simplex $\hat{t}$ with vertices $\hat{q}_1 = (0, 0)$, $\hat{q}_2 = (1, 0)$, and $\hat{q}_3 = (0, 1)$. We shall refer to $\hat{t}$ as the reference element. Assume that the sequence of triangulations $\{ \{ \tilde{t}_n \}_1^N \}_{N_0}^\infty$ is quasi-uniform, i. e., if $\tilde{\rho}_n = \operatorname{diam}(\tilde{t}_n)$ and $\tilde{\sigma}_n = \max \{$ diameter of inscribed circles of $\tilde{t}_n \}$, then there exist constants $\tilde{c}_i > 0$, $i = 1, 2, 3$ independent of n and N such that for any N,

$$\tilde{c}_1 h \leqq \tilde{\rho}_n \leqq \tilde{c}_2 \tilde{\sigma}_n \leqq \tilde{c}_3 h \qquad \text{for} \quad n = 1, \ldots, N. \tag{1.2}$$

For large N, no triangle has more than two vertices on $\partial\Omega$. If $\tilde{t}_n$ does have two vertices on the boundary, we modify it in the following way in order to obtain a better approximation to $\partial\Omega$. Let q_1, q_2, and q_3 be the vertices of $\tilde{t}_n$, with $q_1, q_2 \in \partial\Omega$. Denote by q_{13} and q_{23} the midpoints of the segments $[q_1, q_3]$ and $[q_2, q_3]$, respectively, and by $\tilde{q}_{12}$ the midpoint of $[q_1, q_2]$. Let q_{12} be the (closest) point of intersection between $\partial\Omega$ and the normal to $[q_1, q_2]$ through $\tilde{q}_{12}$, (cf. *fig.* 2).

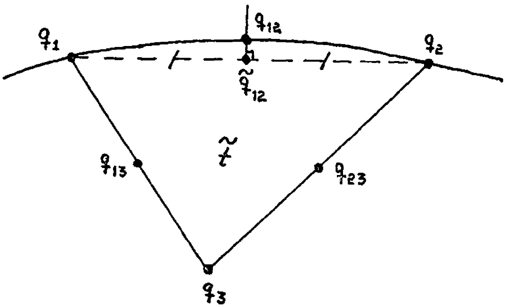

Figure 2.

Introduce $\hat{q}_{ij}$, $1 \leqq i < j \leqq 3$ as the midpoint of the segments $[\hat{q}_i, \hat{q}_j]$ of the reference element $\hat{t}$, and let

$$F_{(n)} = (F_1, F_2) : \quad \hat{t} \to R^2,$$

be the map with F_l, $l = 1, 2$, quadratic polynomial functions of the coordinates $\hat{x}_1$, $\hat{x}_2$ of $\hat{t}$, and such that

$$F_{(n)}(\hat{q}_i) = q_i \qquad 1 \leqq i \leqq 3,$$

$$F_{(n)}(\hat{q}_{ij}) = q_{ij} \qquad 1 \leqq i < j \leqq 3.$$

Set

$$t_n = F_{(n)}(\hat{t}).$$

The notation above is extended in the obvious way to the unaltered triangles; then $t_n \equiv \tilde{t}_n$, and $F_{(n)}$ is affine. When no ambiguity is possible, we shall often write t for t_n and F for $F_{(n)}$.

Since the sequence of triangulations $\{ \{ \tilde{t}_n \}_1^N \}_{N_0}^\infty$ is quasi-uniform, the maps $F_{(n)}$ are uniquely determined and invertible if h is small enough (see [5, 7, 8]), and if c_i, ρ_n and σ_n have the same meaning as $\tilde{c}_i$, $\tilde{\rho}_n$ and $\tilde{\sigma}_n$ in (1.2) but relative to t_n, then

$$c_1 h \leqq \rho_n \leqq c_2 \sigma_n \leqq c_3 h. \tag{1.2'}$$

We set

$$\Omega_h = \bigcup_1^N \text{closure } (t_n),$$

and note that since $\partial\Omega$ is of class $\mathscr{C}^3$, we may assume that

$$\max_{x \in \partial\Omega_h} \operatorname{dist}(x, \partial\Omega) = O(h^3) \quad \text{as } h \to 0. \tag{1.3}$$

In general, neither is Ω_h included in Ω, nor Ω included in Ω_h.

We next define the sequence of spaces of approximating functions, $\{ \mathring{S}^h \}$. Let first S^h consist of all functions χ on Ω_h such that

$$\chi \in \mathscr{C}(\Omega_h), \quad \chi\,|_{t_n} = \hat{P} \circ F_{(n)}^{-1},$$

where $\hat{P}$ is a quadratic polynomial on the reference element $\hat{t}$. We note, [5, 6, 7, 8] that χ is determined in each element t_n by its values at the six points q_i, q_{ij}. We set now

$$\mathring{S}^h = \{ \chi \in S^h : \chi\,|_{\partial\Omega_h} = 0 \}.$$

Whenever necessary, functions in $\mathring{S}^h$ will be extended by zero to larger domains.

In order to solve the problem (1.1) numerically, consider first its weak formulation: Find $u \in \mathring{W}_2^1(\Omega)$ such that

$$B_\Omega(u, v) \equiv \int_\Omega \left\{ \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \left(\sum_{i=1}^2 b_i \frac{\partial u}{\partial x_i} + cu \right) v \right\} dx$$

$$= (f, v)_\Omega \equiv \int_\Omega fv\,dx \quad \text{for all } v \in \mathring{W}_2^1(\Omega). \tag{1.4}$$

Roughly speaking, to obtain an approximate computable problem, we replace here Ω by Ω_h and $\mathring{W}_2^1(\Omega)$ by $\mathring{S}^h$. In general we must use numerical integration to evaluate the integrals involved. We proceed now to describe the method of approximate quadrature.

Consider first an integral $\int_{\hat{t}} \hat{g}(\hat{x})\, d\hat{x}$ over the reference element. For this, let

$$I(\hat{t}, \hat{g}) \equiv \frac{1}{6} \sum_{1 \leq i < j \leq 3} \hat{g}(\hat{q}_{ij})$$

be an approximation using the values of the function only at the three midpoints of the sides. We then have

$$I(\hat{t}, \hat{P}) = \int_{\hat{t}} \hat{P}(\hat{x})\, d\hat{x} \tag{1.5}$$

for P a quadratic polynomial.

Now, for t an arbitrary element,

$$\int_t g(x)\, dx = \int_{\hat{t}} g(F(\hat{x}))\, J_F(\hat{x})\, d\hat{x},$$

where

$$J_F(\hat{x}) = \det\left\{\frac{\partial}{\partial x_i} F_j(\hat{x})\right\}.$$

We put

$$I(t, g) = I(\hat{t}, g(F(.))J_F(.)) = \frac{1}{6} \sum_{1 \leq i < j \leq 3} g(q_{ij}) J_F(\hat{q}_{ij}).$$

Let

$$B_{\Omega_h}^{(h)}(v, w) = \sum_{n=1}^{N} I\left(t_n, \sum_{i,j=1}^{2} a_{ij} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i} + \left(\sum_{i=1}^{2} b_i \frac{\partial v}{\partial x_i} + cv\right) w\right) \tag{1.6}$$

and

$$(f, w)_{\Omega_h}^{(h)} = \sum_{i=1}^{N} I(t_n, fw). \tag{1.7}$$

Note that even if $\Omega_h \not\subset \Omega$, in order to evaluate the terms in (1.6) and (1.7) it suffices to have the coefficients of L and the functions f, v and w defined on $\bar{\Omega} \cap \bar{\Omega}_h$. Thus, no extension of the coefficients or of the functions will be necessary in the numerical work.

The sequence of approximations $u_h \in \mathring{S}^h$ to the solution u of the problem (1.1), or (1.4), is given by the requirement that

$$B_{\Omega_h}^{(h)}(u_h, \chi) = (f, \chi)_{\Omega_h}^{(h)} \quad \text{for all } \chi \in \mathring{S}^h. \tag{1.8}$$

Our main result can nowbe stated as follows.

THEOREM 1.1: *Assume that* $\partial\Omega$ *is of class* $\mathscr{C}^3$, *that the coefficients of the operator L belong to* $\mathscr{C}^3(\bar{\Omega})$, *and that* $\{a_{ij}(x)\}$ *is uniformly positive definite. Assume furthermore that the problem* (1.4) *has a unique solution for every f in* $L_2(\Omega)$. *Let the sequence of partitions* $\{\{t_n\}_1^N\}_{N_0}^\infty$ *be quasi-uniform and satisfy* (1.3).

Then, given any $\varepsilon > 0$ *there exists a constant* $C = C_\varepsilon$ *such that if h is sufficiently small, then for any* $f \in W_1^3(\Omega)$, u_h *is uniquely determined by* (1.8) *and*

$$\|u - u_h\|_{L_\infty(\Omega)} \leqq C h^{3-\varepsilon} \|f\|_{W_1^3(\Omega)}. \tag{1.9}$$

Here u_h *is taken as zero outside of* Ω_h.

We note that for $f \in W_1^3(\Omega)$, it follows from Sobolev's inequalities and regularity results for (1.1) (*cf.* Section 2) that $u \in \mathscr{C}^{3-\varepsilon}(\bar{\Omega})$. Furthermore, essentially no more smoothness in the maximum norm can be guaranteed for u, so that the estimate (1.9) is in a certain sense sharp.

Error estimates in the L_2 and W_2^1 norms for the problem (1.1) (with $b_i = c \equiv 0$), and also other classes of isoparametric elements and integration methods, were given in Ciarlet and Raviart [8], and, in the case of W_2^1 estimates, in sharper form in Ciarlet [5]. For maximum norm estimates in the case when the upcoming integrals are assumed to be evaluated exactly, *cf.* Nitsche [12], Schatz and Wahlbin [15, 16] and Scott [17], and references given there.

In the case of piecewise linear functions combined with the midpoint rule for numerical integration, an error estimate of the form

$$\|u - u_h\|_{L_\infty(\Omega)} \leqq C h^{2-\varepsilon} \|f\|_{W_2^2(\Omega)} \tag{1.10}$$

can be inferred from the work of Nitsche [12] and Fix [9]. In [12] it is shown that an estimate $C h^2 \ln 1/h \, \|u\|_{W_\infty^2}$ holds for the error in the approximation $\tilde{u}_h$ "calculated" without use of numerical integration. The result of [9] is that $\|\tilde{u}_h - u_h\|_{W_2^1} \leqq C h^2 \|f\|_{W_2^2}$, and (1.10) easily obtains. The techniques of the present paper would give the somewhat sharper error estimate $C h^{2-\varepsilon} \|f\|_{W_1^2}$ for the piecewise linear case with the midpoint rule.

In our quadratic situation, the estimates in W_2^1 of [5] would only give an $O(h^{2-\varepsilon})$ estimate in the maximum norm.

Theorem 1.1 can readily be extended to the case of cubic isoparametric simplicial elements, using then an integration method which is exact for quartics on the reference element (*cf.* [5, 8]). The result is that under appropriate smoothness assumptions,

$$\|u - u_h\|_{L_\infty(\Omega)} \leqq C h^{4-\varepsilon} \|f\|_{W_1^4(\Omega)}.$$

We shall now give the proof of Theorem 1.1. For this purpose we need to modify a number of results from [5, 6, 7, 8 and 15, 16] to fit the present situation. In connection with maximum norm estimates, new difficulties arise from the numerical integration, and from the "boundary layer" $(\Omega\setminus\Omega_h) \cup (\Omega_h\setminus\Omega)$. In the proof immediately below, only the relevant final modifications will be quoted – their proofs will occupy the remaining sections of the paper.

Proof of Theorem 1.1: Note that if we can establish (1.9) for any u_h satisfying (1.8), then unique solvability of (1.8) follows from the alternative theorem of linear algebra, and the uniqueness in the continuous problem.

We shall first shift our point of view slightly and consider u_h as an approximation to a function $u(h)$ defined on a domain $\Omega(h)$ containing Ω_h. Although $u(h)$, as indicated, will depend on h, its dependence will be "uniform" for all our purposes, and having $\Omega_h \subset \Omega(h)$ will be convenient.

Let c_4 be such that

$$\max_{x \in \partial\Omega_h} \operatorname{dist}(x, \partial\Omega) \leqq c_4 h^3,$$

cf. (1.3), and let

$$\Omega(h) = \{x : \operatorname{dist}(x, \Omega) \leqq c_4 h^3\}. \tag{1.11}$$

It will be shown in Lemma 2.1 that if the coefficients of L are extended to the outside of Ω (continuously in W_p^k norms) then for $h_0 > 0$ small enough, for $h \leqq h_0$ the problem

$$\left.\begin{aligned} Lv &= g \quad \text{in } \Omega(h), \\ v &= 0 \quad \text{on } \partial\Omega(h), \end{aligned}\right\} \tag{1.12}$$

has a unique solution $v = v(h)$ for any $g \in L_2(\Omega(h))$. Let now f be as in (1.1), and extend it continuously in W_p^k norms to the outside of Ω. Let $u(h)$ be the solution of (1.12) with $g = f$. Then Lemma 2.1 also shows that

$$\| u - u(h) \|_{L_\infty(\Omega)} \leqq C h^3 \| f \|_{W_1^2(\Omega)}. \tag{1.13}$$

We note that $\partial\Omega(h)$ is of class $\mathscr{C}^3$ with the third derivatives of functions occuring in local chart representations of the boundary being uniformly bounded. The modulus of ellipticity for the extended operator L is uniformly positive for h small, and the $\mathscr{C}^1$ norms of the coefficients are uniformly bounded. It follows that the constants occuring in *a priori* estimates involving up to third order derivatives, and in Sobolev inequalities, are uniformly bounded.

The domains Ω_h which were based on the original domain Ω are unchanged, and what were previously the boundary nodes are now not necessarily on $\partial\Omega(h)$. Figure 3 depicts the general perturbed situation.

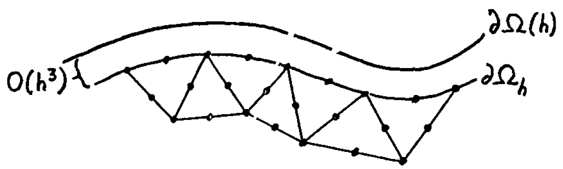

Figure 3.

It remains, by (1.13), to estimate $u(h)-u_h$. Let first $\tilde{u}_h \in \overset{\circ}{S}{}^h$ be such that

$$B_{\Omega_h}(\tilde{u}_h, \chi) = (f, \chi)_{\Omega_h} = B_{\Omega_h}(u(h), \chi) \qquad \text{for} \quad \chi \in \overset{\circ}{S}{}^h; \tag{1.14}$$

here the forms do not involve numerical integration. That $\tilde{u}_h$ is uniquely determined will follow from Lemma 5.3 (applied with L^* replaced by L).

In Lemma 6.1 we shall show that

$$\| u(h)-\tilde{u}_h \|_{L_\infty(\Omega(h))} \leqq C h^{3-\varepsilon} \| f \|_{W_1^2(\Omega)}. \tag{1.15}$$

The proof of (1.15) consists to a large extent in modifying arguments from [15]. The major novelty is to take into consideration the "boundary layer" $\Omega(h) \setminus \Omega_h$. In particular, Lemma 5.9 is crucial in this context. Following [16], certain simplifications of the proof in [15] are used, see in particular the proof of Lemma 5.7. These simplifications depend on the fact that our problem is two-dimensional and that we are content with a loss of ε in the rate of convergence in (1.15). We make essential use of the basic properties of S^h given in [5, 6, 7, 8].

For our present purposes, an important by-product of the proof of (1.15) is the following result, Lemma 5.8: Let v be supported in an element t, which we recall has diameter less than or equal $c_3 h$. Let $\psi_h \in \overset{\circ}{S}{}^h$ satisfy

$$B_{\Omega_h}(\chi, \psi_h) = (\chi, v) \qquad \text{for} \quad \chi \in \overset{\circ}{S}{}^h. \tag{1.16}$$

This problem has a unique solution for h sufficiently small, *cf.* Lemma 5.3, and we have

$$\| \psi_h \|_{L_\infty(\Omega_h)} + \| \psi_h \|_{W_2^1(\Omega_h)} + \| \psi_h \|^{(h)}_{W_1^2(\Omega_h)} \leqq C h^{1-\varepsilon} \| v \|_{L_2(t)}. \tag{1.17}$$

Here

$$\| w \|^{(h)}_{W_p^k(\Omega_h)} = \left(\sum_{n=1}^{N} \| w \|^p_{W_p^k(t_n)} \right)^{1/p} \qquad \text{for} \quad 1 \leqq p < \infty, \tag{1.18}$$

with the obvious modification for $p = \infty$. Let us remark that it is the estimation of the last term on the left of (1.17) that will be most troublesome.

By (1.15) we are now left with bounding $\tilde{u}_h - u_h$. Let $x_0 \in t_{n_0}$ be such that

$$|(\tilde{u}_h - u_h)(x_0)| = \| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)}.$$

By an inverse property, (3.11), we have

$$\| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)} \leqq C h^{-1} \| \tilde{u}_h - u_h \|_{L_2(t_{n_0})}. \tag{1.19}$$

To estimate the right hand side here we shall employ a duality argument,

$$\| \tilde{u}_h - u_h \|_{L_2(t_{n_0})} = \sup_{\substack{\| v \|_{L_2(t_{n_0})} = 1 \\ v \in \mathscr{C}_0^\infty(t_{n_0})}} (\tilde{u}_h - u_h, v). \tag{1.20}$$

For each fixed such v, let $\psi_h \in \overset{\circ}{S}{}^h$ satisfy (1.16). Note that we base the duality argument on use of the form B not involving numerical integration.

We now come to a crucial point in our argument, (1.21) below. Using (1.14) and (1.8) we obtain

$$\begin{aligned}(\tilde{u}_h - u_h, v) &= B_{\Omega_h}(\tilde{u}_h - u_h, \psi_h) \\ &= (f, \psi_h)_{\Omega_h} - B_{\Omega_h}(u_h, \psi_h) \\ &= (f, \psi_h)_{\Omega_h} + (B_{\Omega_h}^{(h)} - B_{\Omega_h})(u_h, \psi_h) - (f, \psi_h)_{\Omega_h}^{(h)}.\end{aligned}$$

Writing $u_h = (u_h - \tilde{u}_h) + \tilde{u}_h$ in the right hand side we thus have

$$\begin{aligned}(\tilde{u}_h - u_h, v) = &\{ (B_{\Omega_h}^{(h)} - B_{\Omega_h})(u_h - \tilde{u}_h, \psi_h) \} \\ &+ \{ (B_{\Omega_h}^{(h)} - B_{\Omega_h})(\tilde{u}_h, \psi_h) \} \\ &+ \{ (f, \psi_h)_{\Omega_h} - (f, \psi_h)_{\Omega_h}^{(h)} \} \equiv I_1 + I_2 + I_3.\end{aligned} \tag{1.21}$$

The reason for this last step is, loosely speaking, that we know much more about the properties of $\tilde{u}_h$ than about u_h. The term I_1 will be handled *via* a "kick-back" argument.

Using (1.5) and the Bramble-Hilbert lemma, and mapping back and forth to the reference element, relying on the results of [5, 6, 7, 8] we establish in Lemma 4.1 that for $v_h, w_h \in \overset{\circ}{S}{}^h$,

$$| B_{\Omega_h}^{(h)}(v_h, w_h) - B_{\Omega_h}(v_h, w_h) | \leqq C h^3 \| v_h \|_{W_\infty^2(\Omega_h)}^{(h)} \| w_h \|_{W_1^2(\Omega_h)}^{(h)}, \tag{1.22}$$

$$\begin{aligned}|(f, w_h)_{\Omega_h}^{(h)} - (f, w_h)_{\Omega_h}| \leqq &C h^{3-\varepsilon} \| f \|_{W_1^3(\Omega)} \\ &\times \{ \| w_h \|_{L_\infty(\Omega_h)} + \| w_h \|_{W_2^1(\Omega_h)} + \| w_h \|_{W_1^2(\Omega_h)}^{(h)} \}.\end{aligned} \tag{1.23}$$

As we shall now show, these results enable us to estimate I_1, I_2 and I_3 in (1.21).

For I_1 we obtain, employing (1.22), the inverse property (3.10), and (1.17),

$$\begin{aligned} |I_1| &\leqq C h^3 \| \tilde{u}_h - u_h \|^{(h)}_{W^2_\infty(\Omega_h)} \| \psi_h \|^{(h)}_{W^2_1(\Omega_h)} \\ &\leqq C h^3 h^{-2} \| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)} C h^{1-\varepsilon} \\ &\leqq C h^{2-\varepsilon} \| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)}. \end{aligned} \tag{1.24}$$

To estimate I_2, we first note that

$$\| \tilde{u}_h \|^{(h)}_{W^2_\infty(\Omega_h)} \leqq C h^{-\varepsilon} \| f \|_{W^3_1(\Omega)}. \tag{1.25}$$

This follows from (1.15), see Lemma 6.2 for the additional details. From (1.22), (1.17) and (1.25) we get

$$\begin{aligned} |I_2| &\leqq C h^3 \| \tilde{u}_h \|^{(h)}_{W^2_\infty(\Omega_h)} \| \psi_h \|^{(h)}_{W^2_1(\Omega_h)} \\ &\leqq C h^3 h^{-\varepsilon} \| f \|_{W^3_1(\Omega)} C h^{1-\varepsilon} \\ &\leqq C h^{4-2\varepsilon} \| f \|_{W^3_1(\Omega)}. \end{aligned} \tag{1.26}$$

Finally, (1.23) and (1.17) give

$$\begin{aligned} |I_3| &\leqq C h^{3-\varepsilon} \| f \|_{W^3_1(\Omega)} \{ \| \psi_h \|_{L_\infty(\Omega_h)} + \| \psi_h \|_{W^1_2(\Omega_h)} + \| \psi_h \|^{(h)}_{W^2_1(\Omega_h)} \} \\ &\leqq C h^{4-2\varepsilon} \| f \|_{W^3_1(\Omega)}. \end{aligned} \tag{1.27}$$

Inserting the estimates (1.24), (1.26) and (1.27) into (1.21), and the result of that operation into (1.20) and (1.19) we have (changing ε for convenience in notation),

$$\| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)} \leqq C h^{1-\varepsilon} \| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)} + C h^{3-\varepsilon} \| f \|_{W^3_1(\Omega)}.$$

Taking h sufficiently small, this proves that

$$\| \tilde{u}_h - u_h \|_{L_\infty(\Omega(h))} = \| \tilde{u}_h - u_h \|_{L_\infty(\Omega_h)} \leqq C h^{3-\varepsilon} \| f \|_{W^3_1(\Omega_h)}. \tag{1.28}$$

The desired result (1.10) now follows from (1.13), (1.15), (1.28) and the triangle inequality. This proves Theorem 1.1.

The remainder of this paper consists of proving results used in the proof of Theorem 1.1 above. In Section 2 we give the perturbation argument that enables us to assume that Ω_h is contained in the domain of the problem we are approximating. In Section 3 we collect basic results concerning the spaces S^h, relying on [5, 6, 7, 8]. Error estimates for the numerical integration scheme are considered in Section 4, again following [5] and [8] with some modifications. In Section 5 we derive, *cf.* [13, 14, 15, 16], certain estimates in L_2 and L_1 based norms for the projection with respect to the form B not involving numerical integration. Finally, the results of Section 5 are used in Section 6 to derive maximum norm estimates in the case without numerical integration – the considerations here follow those of [15] and [16], with some additional arguments necessary in order to handle the difficulties introduced in the boundary layer.

2. A PERTURBATION RESULT

In this section we shall prove the perturbation result used in the proof of Theorem 1.1.

Let Ω be as in Section 1, and for $\delta > 0$,

$$\Omega^\delta = \{ x : \text{there exists } y \in \Omega \text{ such that } |x-y| \leqq \delta \}.$$

(In the proof of Theorem 1.1, $\Omega(h) = \Omega^{c_4 h^3}$.) Note that for sufficiently small $\delta_1 > 0$, there exists a constant $C = C(\delta_1, \Omega, p, q, l, k)$ such that for $0 < \delta \leqq \delta_1$,

$$\| w \|_{W_p^l(\Omega^\delta)} \leqq C \| w \|_{W_q^k(\Omega^\delta)} \left\{ \begin{array}{ll} \text{for } 0 \leqq l < k, \text{ with} & \\ q \geqq \dfrac{2p}{2+p(k-l)} & \text{if } \; p < \infty, \\ q > \dfrac{2}{k-l} & \text{if } \; p = \infty. \end{array} \right\} \tag{2.1}$$

That the constant will depend only on the indicated quantities may be seen from any standard proof of the Sobolev inequalities, see e. g. [11], since the boundary of Ω^δ has, for small δ, essentially the same smoothness properties as $\partial\Omega$.

Assume that the coefficients of the operator L have been extended to Ω^{δ_1} for some $\delta_1 > 0$, and that this extension is continuous in W_p^k norms, see [18] (or [3, 11] for $k \leqq 3$, which are the only cases we shall use). We may also assume that the extended matrix $\{ a_{ij}(x) \}$ is uniformly positive definite on Ω^{δ_1}.

Consider now for $\delta \leqq \delta_1$ the family of problems

$$\left. \begin{array}{rl} Lv = g & \text{in } \Omega^\delta, \\ v = 0 & \text{on } \partial\Omega^\delta. \end{array} \right\} \tag{2.2}$$

LEMMA 2.1: (i) *there exists* $0 < \delta_0 \leqq \delta_1$ *such that for* $\delta \leqq \delta_0$, *the problem* (2.2) *has a unique weak solution for every* g *in* $L_2(\Omega^\delta)$;

(ii) *given* $1 < p < \infty$ and $k = 2,3$ *there exists a constant* $C = C(\delta_0, \Omega, p, k)$ *such that for* $\delta \leqq \delta_0$,

$$\| v \|_{W_p^k(\Omega^\delta)} \leqq C \| g \|_{W^{k-2}(\Omega^\delta)}; \tag{2.3}$$

(iii) *there exists a constant* $C = C(\delta_0, \Omega)$ *such that the following holds: If* $f \in W_1^2(\Omega)$ *and* g *is an extension of* f *to* Ω^{δ_0}, *continuous in* W_p^k *norms, then* *with u the solution of* (1.1) *and* u^δ *the solution of* (2.2),

$$\| u^\delta - u \|_{L_\infty(\Omega)} \leqq C \delta \| f \|_{W_1^2(\Omega)}. \tag{2.4}$$

Proof: We first show the statement concerning the existence of a unique solution. Assume this were false. Then there would exist a sequence $\delta^i \to 0$, $\delta^i \leqq \delta_1$ and functions v_i on $\Omega_i \equiv \Omega^{\delta_i}$ with $\| v_i \|_{L_2(\Omega_i)} = 1$ such that

$$\begin{aligned} Lv_i &= 0 \quad \text{on } \Omega_i, \\ v_i &= 0 \quad \text{on } \partial\Omega_i. \end{aligned}$$

By well known *a priori* estimates, *cf.* [1] and references there,

$$\| v_i \|_{W_2^2(\Omega_i)} \leqq C \| v_i \|_{L_2(\Omega_i)} \leqq C. \tag{2.5}$$

The constant C here is independent of i since the proof of (2.5) can be accomplished by locally flattening the boundary *via* a mapping that has its third derivatives bounded uniformly in δ.

By Sobolev's inequality (2.1),

$$\| v_i \|_{L_\infty(\Omega_i)} \leqq C \| v_i \|_{W_2^2(\Omega_i)} \leqq C$$

so that for i large,

$$\| v_i \|_{L_2(\Omega)} = (1 - \| v_i \|^2_{L_2(\Omega_i \setminus \Omega)})^{1/2} \geqq \frac{1}{2}. \tag{2.6}$$

Since the inclusions $W_2^2(\Omega) \subset \mathscr{C}(\Omega)$ and $W_2^2(\Omega) \subset W_2^1(\Omega)$ are compact, *cf.* [11], we may assume that $v_i \to \tilde{v}$ in $\mathscr{C}(\Omega) \cap W_2^1(\Omega)$. For $w \in \overset{\circ}{W}{}_2^1(\Omega)$ we have $B_\Omega(v_i, w) = 0$ and thus $\tilde{v}$ satisfies

$$B_\Omega(\tilde{v}, w) = 0 \quad \text{for all } w \in \overset{\circ}{W}{}_2^1(\Omega).$$

Since for $x \in \partial\Omega$,

$$| v_i(x) | \leqq C \delta_i^{1/2} \| v_i \|_{\mathscr{C}^{1/2}(\Omega_i)} \leqq C \delta_i^{1/2} \| v_i \|_{W_2^2(\Omega_i)} \leqq C \delta_i^{1/2},$$

we see that $\tilde{v} \in \overset{\circ}{W}{}_2^1(\Omega)$.

But by (2.6), $\| \tilde{v} \|_{L_2(\Omega)} \geqq 1/2$ and this contradicts the assumed uniqueness for the problem (1.1).

Thus, for δ small enough the problem (2.2) has a unique solution.

To see that (2.3) holds one may again consult [1], reasoning as for (2.5) to see that the constant is uniform in δ.

For the estimate (2.4), note first that

$$\left. \begin{aligned} L(u^\delta - u) &= 0 \quad \text{in } \Omega, \\ u^\delta - u &= u^\delta \quad \text{on } \partial\Omega. \end{aligned} \right\} \tag{2.7}$$

By the mean-value theorem and (2.3), (2.1),

$$\begin{aligned}|u^\delta\|_{\partial\Omega} &\leqq \delta\|u^\delta\|_{W^1_\infty(\Omega^\delta)} \leqq C\delta\|u^\delta\|_{W^2_p(\Omega^\delta)}\\ &\leqq C\delta\|f\|_{L_p(\Omega)} \leqq C\delta\|f\|_{W^2_1(\Omega)}\end{aligned} \tag{2.8}$$

where $2 < p < \infty$.

Let now k be a non-negative number so large that the operator $\tilde{L}$,

$$\tilde{L}v \equiv Lv + kv$$

satisfies the maximum principle on Ω, *cf.* e. g. [3]. Let $\tilde{u}$ be determined by

$$\begin{aligned}\tilde{L}\tilde{u} &= 0 \quad \text{in } \Omega,\\ \tilde{u} &= u^\delta \quad \text{on } \partial\Omega.\end{aligned}$$

By the maximum principle and (2.8) we obtain

$$\|\tilde{u}\|_{L_\infty(\Omega)} \leqq C\delta\|f\|_{W^2_1(\Omega)}. \tag{2.9}$$

Next, using (2.7) we have

$$\begin{aligned}L(u+\tilde{u}-u^\delta) &= -k\tilde{u} \quad \text{in } \Omega,\\ u+\tilde{u}-u^\delta &= 0 \qquad \text{on } \partial\Omega.\end{aligned}$$

Thus from (2.1), (2.3) and (2.9),

$$\|u+\tilde{u}-u^\delta\|_{L_\infty(\Omega)} \leqq C\|u+\tilde{u}-u^\delta\|_{W^2_2(\Omega)} \leqq C\|\tilde{u}\|_{L_2(\Omega)} \leqq C\delta\|f\|_{W^2_1(\Omega)}.$$

From this and (2.9) it follows that

$$\|u-u^\delta\|_{L_\infty(\Omega)} \leqq \|u+\tilde{u}-u^\delta\|_{L_\infty(\Omega)} + \|\tilde{u}\|_{L_\infty(\Omega)} \leqq C\delta\|f\|_{W^2_1(\Omega)},$$

which is the desired estimate (2.4).

This completes the proof of Lemma 2.1.

3. THE FINITE ELEMENT SPACES

In this section we collect results from Ciarlet [5], and Ciarlet-Raviart [6, 7, 8] and simple consequences thereof that will be needed in the sequel. Most of the notation in this section was introduced in Section 1.

We shall use the seminorms

$$|v|_{W^l_p(\Omega)} = \begin{cases} \left(\sum_{|\alpha|=l} \|D^\alpha v\|^p_{L_p(\Omega)}\right)^{1/p} & \text{for } 1 \leqq p < \infty,\\ \max_{|\alpha|=l} \|D^\alpha v\|_{L_\infty(\Omega)} & \text{for } p = \infty.\end{cases}$$

Recall that the sequence of meshes is quasi-uniform in the sense of (1.2′). Then for h small enough, $F : \hat{t} \to t$ is one-to-one (we suppress the index n in the notation). Letting $\hat{v}(\hat{x}) = v(F\hat{x})$ we have the following fundamental relations for the behavior of seminorms under this mapping. Corresponding results hold with v replaced by $\hat{v}$ and F by F^{-1}:

$$|v|_{L_p(t)} \leqq C|J_F|_{L_\infty(\hat{t})}^{1/p}|\hat{v}|_{L_p(\hat{t})}, \tag{3.1}$$

$$|v|_{W_p^1(t)} \leqq C|J_F|_{L_\infty(\hat{t})}^{1/p}|F^{-1}|_{W_\infty^1(t)}|\hat{v}|_{W_p^1(\hat{t})}, \tag{3.2}$$

$$|v|_{W_p^2(t)} \leqq C|J_F|_{L_\infty(\hat{t})}^{1/p}\{|F^{-1}|_{W_\infty^1(t)}^2|\hat{v}|_{W_p^2(\hat{t})} + |F^{-1}|_{W_\infty^2(t)}|\hat{v}|_{W_p^1(\hat{t})}\}, \tag{3.3}$$

$$\begin{aligned} |v|_{W_p^3(t)} \leqq C|J_F|_{L_\infty(\hat{t})}^{1/p} & \\ \times\{|F^{-1}|_{W_\infty^1(t)}^3|\hat{v}|_{W_p^3(\hat{t})} &+ |F^{-1}|_{W_\infty^1(t)}|F^{-1}|_{W_\infty^2(t)}|\hat{v}|_{W_p^2(\hat{t})} \\ &+ |F^{-1}|_{W_\infty^3(t)}|\hat{v}|_{W_p^1(t)}\}. \end{aligned} \tag{3.4}$$

Here and below the generic constant C is independent of t, v, and h, and depends on the constants in (1.2′), and on p. The notation J stands for the Jacobian determinant.

When applying the estimates (3.1)-(3.4) the following results are needed:

$$|F|_{W_\infty^l(\hat{t})} \leqq \left\{\begin{array}{c} Ch^l,\ l = 1, 2, \\ 0,\ l \geqq 3, \end{array}\right\} \tag{3.5}$$

$$|F^{-1}|_{W_\infty^l(t)} \leqq Ch^{-1},\ l = 1, 2, 3, 4, \tag{3.6}$$

$$|J_F|_{W_\infty^l(\hat{t})} \leqq \left\{\begin{array}{c} Ch^{2+l},\ l = 0, 1, 2, \\ 0,\ l \geqq 3, \end{array}\right\} \tag{3.7}$$

$$|J_{F^{-1}}|_{L_\infty(t)} = \left|\frac{1}{J_F}\right|_{L_\infty(\hat{t})} \leqq Ch^{-2}. \tag{3.8}$$

If F is affine, certain of these estimates can trivially be improved.

We next consider certain approximation properties of S^h that will be important. Let Π denote the interpolation operator which takes a continuous function u on t into the function Πu of $S^h(t)$ which has the same values at the points q_i, $1 \leqq i \leqq 3$, q_{ij}, $1 \leqq i < j \leqq 3$. If $\hat{\Pi}$ denotes the corresponding interpolation operator on $\hat{t}$, we note that $\hat{\Pi}$ is exact on quadratic polynomials. Since $\widehat{(\Pi v)} = \hat{\Pi}\hat{v}$ we obtain the following results by mapping back and forth from t to $\hat{t}$ and applying the Bramble-Hilbert lemma [4] on $\hat{t}$. The results (3.1)-(3.8) above are used to evaluate the influence of the mappings on various norms and seminorms.

Let $1 \leqq p,\ q \leqq \infty,\ 0 \leqq l \leqq k \leqq 3\ (l = 0, 1, 2)$, and

(i) $q > 2/k$,

(ii) if $p < \infty : q \geqq 2p/(2+p(k-l))$,
if $p = \infty : q > 2/(k-l)$.

Then

$$\| v - \Pi v \|_{W_p^l(t)} \leqq C h^{2((1/p)-(1/q))+k-l} \| v \|_{W_q^k(t)}. \tag{3.9}$$

We shall have later use for various estimates for functions χ in S^h. We easily derive the following.

$$\| \chi \|_{W_p^k(t)} \leqq C h^{-(k-l)} \| \chi \|_{W_p^l(t)} \quad \text{for} \quad 0 \leqq l < k \leqq 2, \quad 1 \leqq p \leqq \infty. \tag{3.10}$$

$$\| \chi \|_{W_p^l(t)} \leqq C h^{-2((1/q)-(1/p))} \| \chi \|_{W_q^l(t)} \quad \text{for} \quad 1 \leqq q, p \leqq \infty. \tag{3.11}$$

$$\| \chi \|_{L_\infty(t)} \leqq C \max_{\substack{x = q_i,\ 1 \leqq i \leqq 3 \\ q_{ij},\ 1 \leqq i \leqq j \leqq 3}} | \chi(x) |. \tag{3.12}$$

As a preparation for our next and final result in this section, note that for quadratic polynomials, the third order derivatives vanish. Correspondingly for $\chi \in S^h$, we use (3.4), the fact that $\hat{\chi}$ is a quadratic polynomial, and (3.3) (with F replaced by F^{-1}) to see that

$$\| \chi \|_{W_p^3(t)} \leqq C \| \chi \|_{W_p^2(t)} \quad \text{for} \quad 1 \leqq p \leqq \infty. \tag{3.13}$$

This estimate enables us to prove the following result, which is well known for various other finite element spaces.

Let $\chi \in S^h$ and $\omega \in \mathscr{C}^\infty(\bar{\Omega})$. Then

$$\| \omega\chi - \Pi(\omega\chi) \|_{W_2^1(\mathscr{D})} \leqq C h \| \chi \|_{W_2^1(\mathscr{D})}, \tag{3.14}$$

where $\mathscr{D} = \{ \bigcup \bar{t}_n : t_n \cap \operatorname{supp} \omega \neq \emptyset \}$. To see this, use (3.9), (3.13) and (3.10) to obtain

$$\begin{aligned} \| \omega\chi - \Pi(\omega\chi) \|_{W_2^1(t)} &\leqq C h^2 \| \omega\chi \|_{W_2^3(t)} \\ &\leqq C h^2 \| \chi \|_{W^2(t)} \leqq C h \| \chi \|_{W_2^1(t)}. \end{aligned}$$

The desired result (3.14) follows by squaring and summing over the elements in $\mathscr{D}$.

4. ERROR ESTIMATES IN THE NUMERICAL INTEGRATION

The notation in this section will be as in Section 1.

Using ideas from Ciarlet [5] and Ciarlet-Raviart [8] we shall prove the following result.

LEMMA 4.1: *For any* $\varepsilon > 0$ *there exists a constant* C *such that for* $v, w \in S^h$ *and* $f \in W_1^3(\Omega)$,

$$|B_{\Omega_h}(v, w) - B_{\Omega_h}^{(h)}(v, w)| \leqq C h^3 \|v\|_{W_\infty^2(\Omega_h)}^{(h)} \|w\|_{W_1^2(\Omega_h)}^{(h)} \tag{4.1}$$

and

$$|(f, w)_{\Omega_h} - (f, w)_{\Omega_h}^{(h)}|$$
$$\leqq C h^{3-\varepsilon} \|f\|_{W_1^3(\Omega)} \{ \|w\|_{L_\infty(\Omega_h)} + |w|_{W^1(\Omega_h)} + \|w\|_{W_1^2(\Omega_h)}^{(h)} \}. \tag{4.2}$$

Here f *is assumed to be extended to the outside of* Ω, *continuously in* W_p^k *norms.*

Proof: We shall first consider the highest order terms occuring in the form B. Let t be a typical element and set

$$E_2(t) = \int_t a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx - I\left(t, a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}\right).$$

Letting $\hat{g}(\hat{x}) = g(F\hat{x})$ and $(F^{-1})_k$ as in denote the kth component of the map F^{-1}, we have

$$\int_t a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} = \int_{\hat{t}} \hat{a}_{ij}(\hat{x}) \left(\sum_{k=1}^{2} \frac{\partial \hat{v}}{\partial \hat{x}_k} \frac{\partial (F^{-1})_k}{\partial x_i} \right) \left(\sum_{l=1}^{2} \frac{\partial \hat{w}}{\partial \hat{x}_l} \frac{\partial (F^{-1})_l}{\partial x_j} \right) J_F \, d\hat{x}.$$

Observe that $D(F^{-1}) = (DF)^{-1} = (1/J_F)(\text{co } DF)$, where co DF is the matrix of cofactors of DF. Thus

$$\frac{\partial}{\partial x_i}(F^{-1})_k = \frac{1}{J_F}(-1)^{\sigma+1} \frac{\partial}{\partial \hat{x}_{i+\sigma}} F_{k+\sigma}$$

where $\sigma = 1$ for $i = k = 1$, $\sigma = -1$ for $i = k = 2$, and $\sigma = 0$ otherwise. By the Bramble-Hilbert lemma and (1.5) we have that

$$\left| \int_{\hat{t}} \hat{g}(\hat{x}) \, d\hat{x} - I(\hat{t}, \hat{g}) \right| \leqq C |\hat{g}(\hat{x})|_{W_1^3(\hat{t})}.$$

It follows that

$$|E_2(t)| \leqq C \sum_{k, l=1}^{2} \left| \hat{a}_{ij}(\hat{x}) \frac{\partial \hat{v}}{\partial \hat{x}_k} \frac{\partial \hat{w}}{\partial \hat{x}_l} K_{i,j,k,l}(\hat{x}) \right|_{W_1^3(\hat{t})}$$

where

$$K_{i,j,k,l}(\hat{x}) = (-1)^{\sigma+\sigma'} \frac{\partial F_{k+\sigma}}{\partial \hat{x}_{i+\sigma}} \frac{\partial F_{l+\sigma'}}{\partial \hat{x}_{j+\sigma'}} \frac{1}{J_F}.$$

Thus,

$$|E_2(t)| \leqq C \sum_{k,l=1}^{2} \sum_{\substack{\alpha+\beta+\gamma+\delta=3 \\ \beta+1\leqq 2 \\ \gamma+1\leqq 2}} |\hat{a}_{ij}(\hat{x})|_{W_\infty(\hat{t})} \times |\hat{v}|_{W_\infty^{\beta+1}(\hat{t})} |\hat{w}|_{W_1^{\gamma+1}(\hat{t})} |K_{i,j,k,l}(\hat{x})|_{W_\infty^{\delta}(\hat{t})}. \tag{4.3}$$

Now, using (3.5) and (3.8) combined with (3.7),

$$|K_{i,j,k,l}(\hat{x})|_{W_\infty^{\delta}(\hat{t})} \leqq C \sum_{\delta_1+\delta_2+\delta_3=\delta} |F|_{W_\infty^{\delta_1+1}(\hat{t})} |F|_{W_\infty^{\delta_2+1}(\hat{t})} \left|\frac{1}{J_F}\right|_{W_\infty^{\delta_3}(\hat{t})}$$
$$\leqq C \sum_{\delta_1+\delta_2+\delta_3=\delta} h^{\delta_1+1} h^{\delta_2+1} h^{-2+\delta_3} \leqq C h^{\delta}.$$

Considering (3.1)-(3.4) with F replaced by F^{-1}, and using (3.5) and (3.8), we obtain from (4.3) that

$$|E_2(t)| \leqq C \sum_{\substack{\alpha+\beta+\gamma+\delta=3 \\ \beta+1\leqq 2 \\ \gamma+1\leqq 2}} h^{\alpha} |a_{ij}|_{W_\infty^{\alpha}(t)} h^{\beta+1} |v|_{W_\infty^{\beta+1}(t)} h^{\gamma+1-2} |w|_{W_1^{\gamma+1}(t)} h^{\delta}$$
$$\leqq C h^3 \|v\|_{W_\infty^2(t)} \|w\|_{W_1^2(t)}. \tag{4.4}$$

Next let

$$E_1(t) = \int_t b_i \frac{\partial v}{\partial x_i} w dx - I\left(t, b_i \frac{\partial v}{\partial x_j} w\right).$$

Reasoning as above we see that

$$|E_1(t)| \leqq C \sum_{k=1}^{2} \left| \hat{b}_i \frac{\partial \hat{v}}{\partial \hat{x}_k} \hat{w} \frac{\partial (F)_{k+\sigma}}{\partial \hat{x}_{i+\sigma}} \right|_{W_1^3(\hat{t})},$$
$$\leqq C \sum_{\substack{\alpha+\beta+\gamma+\delta=3 \\ \beta+1\leqq 2 \\ \gamma\leqq 2}} |\hat{b}_i|_{W_\infty^{\alpha}(\hat{t})} |\hat{v}|_{W_\infty^{\beta+1}(\hat{t})} |\hat{w}|_{W_1^{\gamma}(\hat{t})} |F|_{W_\infty^{\delta+1}(\hat{t})}$$
$$\leqq C h^3 \|v\|_{W_\infty^2(t)} \|w\|_{W_1^2(t)}. \tag{4.5}$$

Finally, if

$$E_0(t) = \int_t cvw\, dx - I(t, cvw)$$

then

$$|E_0(t)| \leqq C |\hat{c}\hat{v}\hat{w} J_F|_{W_1^3(\hat{t})} \leqq C h^3 \|v\|_{W_\infty^2(t)} \|w\|_{W_1^2(t)}. \tag{4.6}$$

Summing the above estimates (4.4)-(4.6) over all elements t in Ω_h we obtain the desired estimate (4.1).

To prove (4.2), we have as before with $E(t) = \int_t f w \, dx - I(t, f w)$.

$$|E(t)| \leqq C |\hat{f} \hat{w} J_F|_{W_1^3(\hat{t})}.$$

Using (3.7) to estimate derivatives of J_F, and using also Hölder's inequality with $1/p+1/p' = 1$ ($p = 2-\delta$, δ small positive) and $1/q+1/q' = 1$ (q large but $< \infty$), we have

$$\begin{aligned} |E(t)| \leqq C h^2 \{ |\hat{f}|_{W_1^3(\hat{t})} |\hat{w}|_{L_\infty(\hat{t})} + |\hat{f}|_{W_p^2(\hat{t})} |\hat{w}|_{W_{p'}^1(\hat{t})} + |\hat{f}|_{W_q^1(\hat{t})} |\hat{w}|_{W_{q'}^2(\hat{t})} \} \\ + C h^3 \{ |\hat{f}|_{W_p^2(\hat{t})} |\hat{w}|_{L_{p'}(\hat{t})} + |\hat{f}|_{W_q^1(\hat{t})} |\hat{w}|_{W_{q'}^1(\hat{t})} + |\hat{f}|_{L_\infty(\hat{t})} |\hat{w}|_{W_1^2(\hat{t})} \} \\ + C h^4 \{ |\hat{f}|_{W_q^1(\hat{t})} |\hat{w}|_{L_{q'}(\hat{t})} + |\hat{f}|_{L_\infty(\hat{t})} |\hat{w}|_{W_1^1(\hat{t})} \}. \end{aligned}$$

Using (3.1)-(3.4) with F replaced by F^{-1}, and (3.6), (3.8) we obtain

$$\begin{aligned} |E(t)| \leqq C h^3 (\| f \|_{W_1^3(t)} \| w \|_{L_\infty(t)} \\ + \| f \|_{W_p^2(t)} \| w \|_{W_{p'}^1(t)} + \| f \|_{W_q^1(t)} \| w \|_{W_{q'}^2(t)}). \end{aligned}$$

Sum this inequality over all elements t in Ω_h. Using that

$$\| f \|_{W_p^1(\Omega_h)} \leqq C \| f \|_{W_p^1(\Omega)}$$

and Sobolev's inequality, *cf.* (2.1), we get

$$\begin{aligned} &|(f, w)_{\Omega_h} - (f, w)_{\Omega_h}^{(h)}| \\ &\leqq C h^3 (\| f \|_{W_1^3(\Omega_h)} \| w \|_{L_\infty(\Omega_h)} + \| f \|_{W_p^2(\Omega_h)} \| w \|_{W_{p'}^1(\Omega_h)} \\ &\qquad + \| f \|_{W_q^1(\Omega_h)} \| w \|_{W_{q'}^2(\Omega_h)}^{(h)}) \\ &\leqq C h^3 \| f \|_{W_1^3(\Omega)} (\| w \|_{L_\infty(\Omega_h)} + \| w \|_{W_{p'}^1(\Omega_h)} + \| w \|_{W_{q'}^2(\Omega_h)}^{(h)}). \end{aligned}$$

Employing the inverse property (3.11) we may replace p' by 2 and q' by 1 at the expense of a factor $h^{-\varepsilon}$, if p' and q' were suitably chosen close to 2 and 1, respectively. The inequality (4.2) obtains.

This completes the proof of Lemma 4.1.

5. SOME AUXILIARY ESTIMATES

Let L^* be the adjoint operator of L, taken on the domain $\Omega(h)$, *cf.* (1.11). Consider the problem of finding ψ such that for v given,

$$\left. \begin{aligned} L^* \psi &= v \quad \text{in } \Omega(h), \\ \psi &= 0 \quad \text{on } \partial\Omega(h). \end{aligned} \right\} \tag{5.1}$$

Let $\psi_h \in \overset{\circ}{S}{}^h$ be such that

$$B_{\Omega_h}(\chi, \psi_h) = (\chi, v)_{\Omega_h} \qquad \text{for} \quad \chi \in \overset{\circ}{S}{}^h. \tag{5.2}$$

Then, with functions in $\overset{\circ}{S}{}^h$ extended by zero,

$$B_{\Omega(h)}(\chi, \psi - \psi_h) = 0 \qquad \text{for} \quad \chi \in \overset{\circ}{S}{}^h \tag{5.3}$$

so that ψ_h is the projection of ψ with respect to the bilinear form B without numerical integration. In this section we shall give various estimates for $\psi - \psi_h$, ψ_h and ψ. In Lemmas 5.7-5.9, v shall furthermore be supported only in an element t.

We shall first need to investigate approximation of functions which vanish on $\partial\Omega(h)$ by functions in $\overset{\circ}{S}{}^h$. Compared to Section 3, we need to take some care of what happens in the "boundary layer" $\Omega(h) \setminus \Omega_h$.

LEMMA 5.1: *Let $R > 0$ be given. There exists a constant C depending on R and the W_1^∞-norms of functions occurring in local chart representations of $\partial\Omega$, such that for h sufficiently small the following holds.*

Let D be a disc of radius R with center in Ω_h, and let $D_1 = D \cap \Omega(h)$, $D_2 = 2D \cap \Omega(h)$ where $2D$ denotes the disc of radius $2R$ concentric with D. For any $w \in \overset{\circ}{W}{}_2^1(\Omega(h)) \cap W_2^2(\Omega(h))$ there exists $\chi \in \overset{\circ}{S}{}^h$ such that

$$\| w - \chi \|_{W_2^1(D_1)} \leqq C h \| w \|_{W_2^2(D_2)}. \tag{5.4}$$

Proof: Let $\chi \in \overset{\circ}{S}{}^h$ be such that

$$\chi = \left\{ \begin{matrix} w \text{ at nodes interior to } \Omega, \\ 0 \text{ at boundary nodes.} \end{matrix} \right\} \tag{5.5}$$

Consider first an element t without nodes on the boundary of Ω. By (3.9) then,

$$\| w - \chi \|_{W_2^1(t)} \equiv \| w - \Pi w \|_{W_2^1(t)} \leqq C h \| w \|_{W_2^2(t)}. \tag{5.6}$$

For t an element with nodes on the boundary we obtain using (3 9) and (3.10), (3.11),

$$\begin{aligned} \| w - \chi \|_{W_2^1(t)} &\leqq \| w - \Pi w \|_{W_2^1(t)} + \| \Pi w - \chi \|_{W_2^1(t)} \\ &\leqq C h \| w \|_{W_2^2(t)} + C \| \Pi w - \chi \|_{L_\infty(t)}. \end{aligned} \tag{5.7}$$

By (3.12) we have

$$\| \Pi w - \chi \|_{L_\infty(t)} \leqq C \max_{x \in \partial\Omega \cap t} | w(x) |. \tag{5 8}$$

Assume now that w has been suitably extended from $\Omega(h)$ so as to be defined on $(3/2)\,D$; this can be accomplished using only the values of w on D_2. Since dist $(\partial\Omega, \partial\Omega(h)) \leqq C h^3$ we have for $\alpha < 1$, using Sobolev's inequality on $(3/2)\,D$ (so that the constant in (2.1) depends only on R),

$$\max_{x \in \partial\Omega \cap t} |w(x)| \leqq C h^{3\alpha} \| w \|_{\mathscr{C}^\alpha((3/2)\,D)} \leqq C h^{3\alpha} \| w \|_{W_2^2((3/2)\,D)} \leqq C h^{3\alpha} \| w \|_{W_2^2(D_2)}.$$

Taking e. g. $\alpha = 2/3$ we obtain from this and (5.7), (5.8) that for a boundary element,

$$\| w - \chi \|_{W_2^1(t)} \leqq C h \| w \|_{W_2^2(t)} + C h^2 \| w \|_{W_2^2(D_2)}. \tag{5.9}$$

Finally, we consider $D_1 \setminus \Omega_h$. Here $\chi \equiv 0$ and with $1/p + 1/p' = 1$, $p < \infty$,

$$\| w \|_{W_2^1(D_1 \setminus \Omega_h)} \leqq C h^{3/2p'} \| w \|_{W_{2p}^1((3/2)\,D)} \leqq C h^{3/2p'} \| w \|_{W_2^2(D_2)}. \tag{5.10}$$

Choose here $p' = 3/2$.

By (5.6), (5.9) and (5.10) we obtain, since the number of boundary elements is proportional to h^{-1},

$$\| w - \chi \|_{W_2^1(D_1)} \leqq C h \| w \|_{W_2^2(D_2)}.$$

This completes the proof of the lemma.

We note the following consequence:

COROLLARY 5.1: *There exists a constant C such that the following holds. For any* $w \in \overset{\circ}{W}{}_2^1(\Omega(h)) \cap W_2^2(\Omega(h))$, *there exists* $\chi \in \overset{\circ}{S}{}^h$ *such that*

$$\| w - \chi \|_{W_2^1(\Omega(h))} \leqq C h \| w \|_{W_2^2(\Omega(h))}.$$

The next lemma, concerning approximation in the maximum norm, is proven in a similar way.

LEMMA 5.2: *Given* $\varepsilon > 0$ *there exist* C *and* $p < \infty$ *such that for* $w \in W_p^3(\Omega(h)) \cap \overset{\circ}{W}{}_2^1(\Omega(h))$ *there exists* $\chi \in \overset{\circ}{S}{}^h$ *such that*

$$\| w - \chi \|_{L_\infty(\Omega(h))} + h \| w - \chi \|_{W_\infty^1(\Omega_h)} \leqq C h^{3-\varepsilon} \| w \|_{W_p^3(\Omega(h))}.$$

Proof: Let χ be as in (5.5), and for an element t, let

$$I(t) = \| w - \chi \|_{L_\infty(t)} + h \| w - \chi \|_{W_\infty^1(t)}.$$

For t an element without boundary nodes, by (3.9),

$$I(t) \leqq C h^{3-(2/p)} \| w \|_{W_p^3(t)}. \tag{5.11}$$

For t a boundary element we have

$$\begin{aligned} I(t) &\leqq \{ \| w-\Pi w \|_{L_\infty(t)} + h \| w-\Pi w \|_{W^1_\infty(t)} \} \\ &\quad + \{ \| \Pi w-\chi \|_{L_\infty(t)} + h \| \Pi w-\chi \|_{W^1_\infty(t)} \} \\ &= I_1(t)+I_2(t). \end{aligned}$$

By (3 9),

$$I_1(t) \leqq C h^{3-(2/p)} \| w \|_{W^3_p(t)}.$$

Using (3 10) and (3.12),

$$\begin{aligned} I_2(t) &\leqq C \| \Pi w-\chi \|_{L_\infty(t)} \leqq C \max_{x \in \partial\Omega} | w(x) | \leqq C h^{3\alpha} \| w \|_{\mathscr{C}^\alpha(\Omega(h))} \\ &\leqq C h^{3\alpha} \| w \|_{W^3_1(\Omega(h))}, \end{aligned}$$

for $\alpha < 1$. Thus, for t a boundary element, for p large,

$$I(t) \leqq C h^{3-(2/p)} \| w \|_{W^3_p(\Omega(h))}. \tag{5.12}$$

By (5.11) and (5.12) it remains only to consider

$$\| w \|_{L_\infty(\Omega(h) \setminus \Omega_h)}.$$

As above, this is bounded by $C h^{3\alpha} \| v \|_{W^3_1(\Omega(h))}$ for any $\alpha < 1$.

This concludes the proof of the lemma.

Note that for any $\chi \in \mathring{S}^h$,

$$\| w-\chi \|_{W^1_\infty(\Omega(h) \setminus \Omega_h)} \equiv \| w \|_{W^1_\infty(\Omega(h) \setminus \Omega_h)}$$

and thus, in the term $h \| w-\chi \|_{W^1_\infty(\Omega_h)}$ in Lemma 5.2, Ω_h cannot be replaced by $\Omega(h)$. (This fact accounts for the need of Lemma 5.9 in the proof of Lemma 6.1.)

We now return to the problems (5.1), (5.2). Note that by Lemma 2.1 and Fredholm's alternative, the problem (5.1) has a unique solution. Since $\mathring{S}^h \subset \mathring{W}^1_2(\Omega(h))$, using Corollary 5.1 one obtains, following Schatz [14], that (5.2) has a unique solution for h sufficiently small. Furthermore [14] gives

LEMMA 5.3: *There exists a constant C such that*

$$\begin{aligned} \| \psi-\psi_h \|_{L_2(\Omega(h))} &+ h \| \psi-\psi_h \|_{W^1_2(\Omega(h))} \\ &\leqq C h^2 \| \psi \|_{W^2_2(\Omega(h))} \leqq C h^2 \| v \|_{L_2(\Omega(h))}. \end{aligned}$$

We shall also need local estimates for $\psi-\psi_h$. Using Lemma 5.1, the inverse estimate (3.10), and (3.14), one obtatns the next result, following Nitsche-

Schatz [13] with some minor modifications (which we shall not give) when the domain D_2 below intersects the boundary of Ω.

LEMMA 5.4: *Let $R > 0$ be given. There exists a constant C depending on R, the W_3^∞ norms of functions occurring in local chart representations of $\partial\Omega$, the modulus of ellipticity for the operator L^*, and the W_1^∞ norms of the coefficients of L^*, such that for h sufficiently small the following holds.*

Let D be a disc of radius R with center in Ω_h, and $2D$ the concentric disc of radius $2R$. Let $D_1 = D \cap \Omega(h)$, $D_2 = 2D \cap \Omega(h)$. Then for ψ, ψ_h as in (5.1)-(5.3),

$$\|\psi-\psi_h\|_{W_2^1(D_1)} \leqq Ch\|\psi\|_{W_2^2(D_2)}+C\|\psi-\psi_h\|_{L_2(D_2)}.$$

As in [15], to determine the dependence of C on R in more detail for R small, we map D linearly to a disc of unit size. There we apply Lemma 5.3 (with h replaced by h/R – for h/R sufficiently small), noting that the modulus of ellipticity is unchanged, while the W_3^∞ and W_1^∞ norms mentioned in Lemma 5.4 are decreased. Thereafter we map back to D, utilizing the obvious counterparts of (3.1)-(3.8) to account for the influence of the linear maps. We obtain then:

COROLLARY 5.4: *There exist constants $c_5 > 0$ and C such that for $c_5 h \leqq R$,*

$$\|\psi-\psi_h\|_{W_2^1(D_1)} \leqq Ch\{|\psi|_{W_2^2(D_2)}+R^{-1}|\psi|_{W_2^1(D_2)}+R^{-2}\|\psi\|_{L_2(D_2)}\} + CR^{-1}\|\psi-\psi_h\|_{L_2(D_2)}.$$

We shall next consider the case when the function v in (5 1) has small support. We start with two simple preliminary results.

LEMMA 5.5: *For any $\varepsilon > 0$ there exists a constant C such that if* $\operatorname{diam}(\operatorname{supp} v) \leqq R$, *then*

$$\|\psi\|_{W_2^1(\Omega(h))} \leqq CR^{1-\varepsilon}\|v\|_{L_2(\Omega(h))}.$$

Proof: By Sobolev's inequality and the regularity results (2.3) we have for p close to 1,

$$\|\psi\|_{W_2^1(\Omega(h))} \leqq C\|\psi\|_{W_p^2(\Omega(h))} \leqq C\|v\|_{L_p(\Omega(h))} \leqq CR^{(2-p)/p}\|v\|_{L_2(\Omega(h))}.$$

This proves the lemma.

LEMMA 5.6: *For any $\varepsilon > 0$ there exists a constant C such that if $\mathcal{D} \subseteq \Omega(h)$ with* $\operatorname{diam}(\mathcal{D}) \leqq R$, *then for $\psi \in W_2^1(\Omega(h))$,*

$$\|\psi\|_{L_2(\mathcal{D})} \leqq CR^{1-\varepsilon}\|\psi\|_{W_2^1(\Omega(h))}.$$

Proof: Using Sobolev's inequality for p large, we have

$$\| \psi \|_{L_2(\mathscr{D})} \leqq C R^{1-(2/p)} \| \psi \|_{L_p(\mathscr{D})} \leqq C R^{1-(2/p)} \| \psi \|_{W_2^1(\Omega(h))}$$

which proves the lemma.

For the rest of the section, let the function v in (5.1) have its support in an element t. Following [15, 16] we have next

LEMMA 5.7: *For any* $\varepsilon > 0$ *there exists a constant* C *such that*

$$\| \psi - \psi_h \|_{W_1^1(\Omega(h))} \leqq C h^{2-\varepsilon} \| v \|_{L_2(t)}.$$

Proof: Let $x_0 \in t$, and

$$\tilde{\Omega}_j = \{ x : 2^{-j-1} \leqq | x - x_0 | \leqq 2^{-j} \}, \qquad j = J_0, \ldots, J_1,$$

where $\Omega(h) \subset \{ x : | x - x_0 | \leqq 2^{-J_0} \}$, and $2^{-J_1-3} \leqq c_5 h \leqq 2^{-J_1-2}$, with c_5 as in Corollary 5.4. Put

$$\Omega_j = \tilde{\Omega}_j \cap \Omega(h),$$

$$\Omega_j^l = (\Omega_{j-l} \cup \Omega_{j-l+1} \cup \ldots \cup \Omega_{j+l}), \qquad l = 1, 2,$$

and

$$\Omega_I = \Omega(h) \setminus \bigcup_{j=J_0}^{J_1} \Omega_j.$$

Letting $e = \psi - \psi_h$ we have

$$\| e \|_{W_1^1(\Omega(h))} \leqq \sum_{J_0}^{J_1} \| e \|_{W_1^1(\Omega_j)} + \| e \|_{W_1^1(\Omega_I)}. \tag{5.13}$$

With $d_j = 2^{-j}$,

$$\| e \|_{W_1^1(\Omega_j)} \leqq \left(\frac{3}{4} \pi \right)^{1/2} d_j \| e \|_{W_2^1(\Omega_j)} \tag{5.14}$$

and using Corollary 5.4,

$$\| e \|_{W_2^1(\Omega_j)} \leqq C h \{ | \psi |_{W_2^2(\Omega_j^1)} + d_j^{-1} | \psi |_{W_2^1(\Omega_j^1)} + d_j^{-2} \| \psi \|_{L_2(\Omega_j^1)} \} + C d_j^{-1} \| e \|_{L_2(\Omega_j^1)}. \tag{5.15}$$

One next finds that

$$| \psi |_{W_2^2(\Omega_j^1)} \leqq C \{ d_j^{-1} \| \psi \|_{W_2^1(\Omega_j^2)} + d_j^{-2} \| \psi \|_{L_2(\Omega_j^2)} \}. \tag{5.16}$$

This can be seen by using a suitable function $\omega \in \mathscr{C}^\infty(\Omega_j^2)$ with $\omega \equiv 1$ on Ω_j^1, and which vanishes on the part of $\partial\Omega_j^2$ which does not coincide with

$\partial\Omega(h) \cap \tilde{\Omega}_j^{1.5}$, where $\tilde{\Omega}_j^{1.5} = \{x : 2^{-j-2.5} \leqq |x-x_0| \leqq 2^{-j+1.5}\}$. By the counterpart of (2.3),

$$|\omega\psi|_{W_2^2(\Omega_j^1)} \leqq C \| L^*(\omega\psi)\|_{L_2(\Omega(h))},$$

and using that $L^*\psi = 0$ on Ω_j^2 (assuming, as we may, that $c_5 \geqq \max_n \rho_n$, *cf.* (1.2')), one easily obtains (5.16).

Utilizing now (5.16) and Lemma 5.6 in (5.15),

$$\| e \|_{W_2^1(\Omega_j)} \leqq C\{hd_j^{-1-\varepsilon}\|\psi\|_{W_2^1(\Omega(h))} + d_j^{-1}\| e \|_{L_2(\Omega(h))}\},$$

and by Lemmas 5.5 and 5.3,

$$\| e \|_{W_2^1(\Omega_j)} \leqq C h^{2-\varepsilon} d_j^{-1-\varepsilon} \| v \|_{L_2(t)}.$$

Thus, from (5.14),

$$\| e \|_{W_1^1(\Omega_j)} \leqq C h^{2-\varepsilon} d_j^{-\varepsilon} \| v \|_{L_2(t)}.$$

One also has, using Lemma 5.3,

$$\| e \|_{W_1^1(\Omega_I)} \leqq C h \| e \|_{W_2^1(\Omega_I)} \leqq C h \| e \|_{W_2^1(\Omega(h))} \leqq C h^2 \| v \|_{L_2(t)}.$$

The desired result now follows from (5.13), and this completes the proof of the lemma.

The following result was used in the proof of Theorem 1.1. It is essentially a consequence of Lemma 5.7.

LEMMA 5.8: *For any $\varepsilon > 0$ there exists a constant C such that*

$$\| \psi_h \|_{L_\infty(\Omega_h)} + \| \psi_h \|_{W_2^1(\Omega_h)} + \| \psi_h \|_{W_1^2(\Omega_h)}^{(h)} \leqq C h^{1-\varepsilon} \| v \|_{L_2(t)}.$$

Proof: For p slightly larger than 2 we have using Sobolev's inequality and the inverse property (3.11),

$$\| \psi_h \|_{L_\infty(\Omega_h)} \leqq C \| \psi_h \|_{W_p^1(\Omega_h)} \leqq C h^{-2((1/2)-(1/p))} \| \psi_h \|_{W_2^1(\Omega_h)}$$

and by Lemmas 5.3 and 5.5,

$$\| \psi_h \|_{W_2^1(\Omega_h)} \leqq \| \psi \|_{W_2^1(\Omega(h))} + C h \| v \|_{L_2(t)} \leqq C h^{1-\varepsilon} \| v \|_{L_2(t)}.$$

Thus it suffices to estimate $\| \psi_h \|_{W_1^2(\Omega_h)}^{(h)}$.

For T an element,

$$\| \psi_h \|_{W_1^2(T)} \leqq \| \psi_h - \Pi \psi \|_{W_1^2(T)} + \| \Pi \psi - \psi \|_{W_1^2(T)} + \| \psi \|_{W_1^2(T)}.$$

By (3.11),

$$\| \psi_h - \Pi \psi \|_{W_1^2(T)} \leqq C h^{-1} \| \psi_n - \Pi \psi \|_{W_1^1(T)} \leqq C h^{-1} \| \psi_h - \psi \|_{W_1^1(T)} + C h^{-1} \| \psi - \Pi \psi \|_{W_1^1(T)}.$$

Thus, summing and using Lemma 5.7,

$$\| \psi_h \|^{(h)}_{W_1^2(\Omega_h)} \leqq C h^{1-\varepsilon} \| v \|_{L_2(t)} + C h^{-1} \| \Pi \psi - \psi \|_{W_1^1(\Omega_h)} + \| \Pi \psi - \psi \|^{(h)}_{W_1^2(\Omega_h)} + \| \psi \|_{W_1^2(\Omega_h)}.$$

By (3.9) we have

$$h^{-1} \| \Pi \psi - \psi \|_{W_1^1(\Omega_h)} + \| \Pi \psi - \psi \|^{(h)}_{W_1^2(\Omega_h)} \leqq C \| \psi \|_{W_q^2(\Omega(h))}, \qquad q > 1,$$

and since from (2.3) it follows for q close to 1 that

$$\| \psi \|_{W_q^2(\Omega(h))} \leqq C \| v \|_{L_q(t)} \leqq C h^{(2-q)/q} \| v \|_{L_2(t)}.$$

we obtain the desired result.

This proves the lemma.

Finally, we shall prove the following result which will be needed in Section 6 to handle the discrepancy between Ω and Ω_h. Recall that ψ is the solution of (5.1) with v supported in an element t.

LEMMA 5.9: *For any* $\varepsilon > 0$ *there exists a constant* C *such that*

$$\| \psi \|_{W_1^1(\Omega(h) \setminus \Omega_h)} \leqq C h^{4-\varepsilon} \| v \|_{L_2(t)}.$$

Proof: Let $S = \Omega(h) \setminus \Omega_h$. Partition S into pieces S_j,

$$S_j = \{ x \in S ; 2^{-j-1} < \operatorname{dist}(x, t) \leqq 2^{-j} \},$$

for $j = J_0, \ldots, J_1$. Here if $2^{-J_1} \leqq h$, then redefine J_1 to be such that $2^{-J_1-1} \leqq h < 2^{-J_1}$ and let

$$S_I = S \setminus \bigcup_{J_0}^{J_1} S_j.$$

We have

$$\| \psi \|_{W_1^1(S)} \leqq \| \psi \|_{W_1^1(S_I)} + \sum_{J_0}^{J_1} \| \psi \|_{W_1^1(S_j)}, \tag{5.17}$$

where, if dist $(t, \partial\Omega(h)) \geqq h$, the first term on the right is missing.

If the first term on the right of (5.17) is there, then for p large and using Sobolev's inequality,

$$\begin{aligned} \|\psi\|_{W_1^1(S_I)} &\leqq C(h^4)^{1-(1/p)}\|\psi\|_{W_p^1(S_I)} \\ &\leqq C(h^4)^{1-(1/p)}\|\psi\|_{W_p^1(\Omega(h))} \\ &\leqq C(h^4)^{1-(1/p)}\|\psi\|_{W_2^2(\Omega(h))} \\ &\leqq C(h^4)^{1-(1/p)}\|v\|_{L_2(t)}. \end{aligned} \tag{5.18}$$

To estimate the contributions from S_j, put $d_j = 2^{-j}$. With p large we have

$$\|\psi\|_{W_1^1(S_j)} \leqq C(h^3 d_j)^{1-(1/p)}\|\psi\|_{W_p^1(S_j)}. \tag{5.19}$$

Letting $S_j \subset \Omega_j$ where Ω_j is a suitable domain of diameter less than $C\,d_j$, and dist $(\Omega_j, t) \geqq d_j/4$, we have, employing an affine transformation $x \to x/d_j$ in Sobolev's inequality and using (3.1)-(3.8) that

$$\|\psi\|_{W_p^1(\Omega_j)} \leqq C\,d_j^{2/p}\{|\psi|_{W_2^2(\Omega_j)} + d_j^{-1}|\psi|_{W_2^1(\Omega_j)} + d_j^{-2}\|\psi\|_{L_2(\Omega_j)}\}.$$

The constant C here can be made independent of j by a suitable choice of Ω_j. By Lemma 5.6 and the obvious counterpart of (5.16) we obtain

$$\|\psi\|_{L_2(\Omega_j)} \leqq C d_j^{1-\varepsilon}\|\psi\|_{W_2^1(\Omega(h))},$$

$$|\psi|_{W_2^2(\Omega_j)} \leqq C\,d_j^{-1-\varepsilon}\|\psi\|_{W_2^1(\Omega(h))}.$$

Thus, using Lemma 5.5,

$$\|\psi\|_{W_p^1(\Omega_j)} \leqq C\,d_j^{-1-\varepsilon}h^{1-\varepsilon}\;\|v\|_{L_2(t)}.$$

Combining this with (5.19) and inserting the result (and (5.18) if appropriate) into (5.17), we get for p large,

$$\|\psi\|_{W_1^1(S)} \leqq C h^{4-\varepsilon}\left(1 + \sum_{J_0}^{J_1} d_j^{-\varepsilon}\right)\|v\|_{L_2(t)}.$$

This proves the lemma.

REMARK: If we demand more smoothness of the coefficients of L^* and of the boundary $\partial\Omega$, then the above Lemma 5.9 can be proven in a somewhat more straightforward fashion by representing ψ in terms of v *via* the Green's function and using the estimates of [2] or [10] for derivatives of the Green's function (when the singularity is close to the boundary, in particular).

6. MAXIMUM NORM ESTIMATES IN THE CASE WITHOUT NUMERICAL INTEGRATION

Let $u(h)$ be the solution of the problem

$$\begin{cases} Lu(h) = f & \text{in } \Omega(h), \\ u(h) = 0 & \text{on } \partial\Omega(h), \end{cases}$$

(where f and the coefficients of L are extended from Ω), and let $\tilde{u}_h \in \mathring{S}^h$ be such that

$$B_{\Omega_h}(\tilde{u}_h - u(h), \chi) = 0 \qquad \text{for} \quad \chi \in \mathring{S}^h.$$

Relying on the results of Section 5, in particular Lemmas 5.7 and 5.9, we shall show the following.

LEMMA 6.1: *For any $\varepsilon > 0$ there exists a constant C such that for h sufficiently small,*

$$\| \tilde{u}_h - u(h) \|_{L_\infty(\Omega(h))} \leqq C h^{3-\varepsilon} \| f \|_{W_1^3(\Omega)}.$$

Proof: Let x_0 be such that

$$\| \tilde{u}_h - u(h) \|_{L_\infty(\Omega(h))} = | (\tilde{u}_h - u(h))(x_0) |.$$

If $x_0 \notin \Omega_h$, then (2.8) (which holds on $\Omega(h) \setminus \Omega_h$) shows the desired result. Assume hence that x_0 belongs to an element t. Using (3.11) we have for $\chi \in \mathring{S}^h$,

$$\begin{aligned} &\| \tilde{u}_h - u(h) \|_{L_\infty(t)} \\ &\leqq \| \tilde{u}_h - \chi \|_{L_\infty(t)} + \| \chi - u(h) \|_{L_\infty(t)} \\ &\leqq C h^{-1} \| \tilde{u}_h - \chi \|_{L_2(t)} + \| \chi - u(h) \|_{L_\infty(t)} \\ &\leqq C h^{-1} \| \tilde{u}_h - u(h) \|_{L_2(t)} + C h^{-1} \| u(h) - \chi \|_{L_2(t)} + \| \chi - u(h) \|_{L_\infty(t)} \\ &\leqq C h^{-1} \| \tilde{u}_h - u(h) \|_{L_2(t)} + C \| u(h) - \chi \|_{L_\infty(t)}. \end{aligned}$$

By Lemma 5.2, and (2.3) and Sobolev's inequality for large $p = p(\varepsilon) < \infty$, we may choose χ so that

$$\begin{aligned} \| u(h) - \chi \|_{L_\infty(t)} &\leqq C h^{3-\varepsilon} \| u(h) \|_{W_p^3(\Omega(h))} \\ &\leqq C h^{3-\varepsilon} \| f \|_{W_p^1(\Omega(h))} \leqq C h^{3-\varepsilon} \| f \|_{W_1^3(\Omega)}. \end{aligned}$$

It remains therefore to show that

$$C h^{-1} \| \tilde{u}_h - u(h) \|_{L_2(t)} \leqq C h^{3-\varepsilon} \| f \|_{W_1^3(\Omega)}. \qquad (6.1)$$

We have

$$\| \tilde{u}_h - u(h) \|_{L_2(t)} = \sup_{\substack{v \in \mathscr{C}_0^\infty(t) \\ \| v \|_{L_2(t)} = 1}} (\tilde{u}_h - u(h), v).$$

For fixed v as above let

$$L^* \psi = v \quad \text{in} \quad \Omega(h),$$
$$\psi = 0 \qquad \text{on } \partial\Omega(h).$$

Then with ψ_h as in (5.2) we get for arbitrary χ in $\overset{\circ}{S}{}^h$,

$$\begin{aligned}(\tilde{u}_h - u(h), v) &= B_{\Omega(h)}(\tilde{u}_h - u(h), \psi) \\ &= B_{\Omega(h)}(\tilde{u}_h - u(h), \psi - \psi_h) \\ &= B_{\Omega(h)}(\chi - u(h), \psi - \psi_h).\end{aligned}$$

Thus, with $S = \Omega(h) \setminus \Omega_h$,

$$\begin{aligned}&|(\tilde{u}_h - u(h), v)| \\ &\quad \leqq C \| u(h) \|_{W^1_\infty(S)} \| \psi \|_{W^1_1(S)} + C \| \chi - u(h) \|_{W^1_\infty(\Omega_h)} \| \psi - \psi_h \|_{W^1_1(\Omega_h)}. \quad (6.2)\end{aligned}$$

We proceed to estimate the four norms occuring on the right.

Clearly,

$$\| u(h) \|_{W^1_\infty(S)} \leqq C \| f \|_{W^3_1(\Omega)}. \tag{6.3}$$

Furthermore, by Lemma 5.9,

$$\| \psi \|_{W^1_1(S)} \leqq C h^{4-\varepsilon} \| v \|_{L_2(t)}. \tag{6.4}$$

By Lemma 5.2 we may choose χ such that with $p < \infty$, using also Sobolev's inequality and (2.3),

$$\| \chi - u(h) \|_{W^1_\infty(\Omega_h)} \leqq C h^{2-\varepsilon} \| u(h) \|_{W^3_p(\Omega(h))} \leqq C h^{2-\varepsilon} \| f \|_{W^3_1(\Omega)}. \tag{6.5}$$

Lastly, by Lemma 5.7,

$$\| \psi - \psi_h \|_{W^1_1(\Omega_h)} \leqq C h^{2-\varepsilon} \| v \|_{L_2(t)}. \tag{6.6}$$

Inserting (6.3)-(6.6) into (6.2) we obtain the desired result (6.1). This proves the lemma.

Finally we shall derive the following simple (and not very sharp) consequence of Lemma 6.1. The result was used in the proof of Theorem 1.1.

LEMMA 6.2: *For any* $\varepsilon > 0$ *there exists a constant* C *such that*

$$\| \tilde{u}_h \|^{(h)}_{W^2_\infty(\Omega_h)} \leqq C h^{-\varepsilon} \| f \|_{W^3_1(\Omega)}.$$

Proof: Consider a typical element T. For $\chi = \Pi u(h) \in S^h$ and for p large we have by (3.11),

$$\begin{aligned}&\| \tilde{u}_h \|_{W^2_\infty(T)} \\ &\quad \leqq C h^{-2/p} \| \tilde{u}_h \|_{W^2_p(T)} \\ &\quad \leqq C h^{-2/p} \| \tilde{u}_h - \chi \|_{W^2_p(T)} + C h^{-2/p} \| \chi - u(h) \|_{W^2_p(T)} + C h^{-2/p} \| u(h) \|_{W^2_p(T)}.\end{aligned}$$

The two last terms on the right are easily seen to be bounded by the correct quantities, by (3.9), (2.3) and Sobolev's inequality.

Next, by (3.10),

$$\|\tilde{u}_h-\chi\|_{W_p^2(T)} \leqq C h^{-2}\|\tilde{u}_h-\chi\|_{L_p(T)}$$
$$\leqq C h^{-2}\|\tilde{u}_h-u(h)\|_{L_p(T)}+C h^{-2}\|u(h)-\chi\|_{L_p(T)},$$

and it follows from Lemma 6.1, (3.9), (2.3) and Sobolev's inequalities that

$$\|\tilde{u}_h-\chi\|_{W_p^2(T)} \leqq C h^{1-\varepsilon}\|f\|_{W_1^2(\Omega)}.$$

This proves the lemma.

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