

ROLF RANNACHER

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## SOME ASYMPTOTIC ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATION OF MINIMAL SURFACES (1)

par Rolf RANNACHER (2)

Communiqué par V. THOMÉE

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Abstract. — *The solution of a minimal surface problem over a plane domain is approximated by piecewise linear finite elements. Using related results of Johnson/Thomé [5] and a weighted Sobolev norm technique introduced by Nitsche [11], we show the  $L^\infty$ -convergence with rate  $0(h^2 |\ln h|)$  and the  $L^2$ -convergence with rate  $0(h^2)$ .*

### 1. INTRODUCTION

Let  $\Omega$  be a bounded, strictly convex domain in the plane  $R^2$  with smooth boundary  $\partial\Omega \in C^{2,\alpha}$ ,  $0 < \alpha < 1$ , and let  $g \in C^0(R^2)$  be a real function. We consider the following minimal surface problem for functions  $u \in C^{0,1}(\bar{\Omega})$ :

$$(V) \quad \int_{\Omega} (1 + |\nabla u|^2)^{1/2} dx = \text{Min.}, \quad u = g \text{ on } \partial\Omega.$$

If  $g$  satisfies the bounded slope condition on  $\partial\Omega$  and is in the Sobolev space  $W^{2,q}(\Omega)$  for some  $q \in ]2, \infty[$  or in the Hölder space  $C^{2,\alpha}(\bar{\Omega})$ , then it is known that there is a unique minimizing function

$$u \in W^{2,q}(\Omega) \quad \text{or} \quad u \in C^{2,\alpha}(\bar{\Omega}) \subset W^{2,\infty}(\Omega),$$

respectively, (see [7; 4.2.1]). This solution will be approximated by the simplest finite element method.

Let  $\hat{\Omega}_h \supset \bar{\Omega}$ ,  $0 < h \leq h_0 < 1$ , be polygonal domains, and let  $T_h = \{T_i\}$  be finite triangulations of  $\hat{\Omega}_h$ , such that the triangles have disjoint interiors and all their edges are the edge of another triangle or of the polygon  $\partial\hat{\Omega}_h$ . Further all vertices of the inscribed polygonal domain

$$\Omega_h := \cup \{T \in T_h \mid T \subset \bar{\Omega}\} \subset \bar{\Omega} \subset \hat{\Omega}_h$$

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(1) Manuscrit reçu le 22 mars 1976.

(2) Institut für Angewandte Mathematik der Universität Bonn. (Fed. Rep. Germany).

are on the boundary  $\partial\Omega$ . It is assumed that the triangulations  $T_h$  are quasi-regular :

(T) Each triangle  $T \in T_h$  contains a circle with radius  $c_1 h$  and is contained in a circle with radius  $c_2 h$ .

We define finite dimensional subspaces  $S_h \subset W^{1,2}(\hat{\Omega}_h)$  by

$$S_h := \{ v_h \in C(\hat{\Omega}_h) \mid v_h \text{ linear on each } T \in T_h \}$$

and  $S_h^0 \subset W_0^{1,2}(\Omega_h) \subset W_0^{1,2}(\Omega)$  by

$$S_h^0 := \{ v_h \in S_h \mid v_h = 0 \text{ on } \hat{\Omega}_h - \Omega_h \}.$$

The usual interpolant  $I_h g \in S_h$  of  $g$  is determined by

$$I_h g(x) = g(x) \quad \text{for each vertex } x \in T \in T_h. \quad (1)$$

Now the approximating functions  $u_h \in S_h$  are defined by the finite problems

$$(V_h) \quad \int_{\Omega_h} (1 + |\nabla u_h|^2)^{1/2} dx = \text{Min.}, \quad u_h - I_h g \in S_h^0.$$

The function  $F(\eta) := (1 + |\eta|^2)^{1/2}$  is strictly convex on  $R^2$ .

Thus the existence of unique solutions of  $(V_h)$  follows from the continuity and coerciveness of the functional.

Johnson/Thomee [5] have shown for  $u \in W^{2,2}(\Omega) \cap W^{1,\infty}(\Omega)$  the rate of convergence

$$\|\nabla(u - u_h)\|_2 = O(h) \quad (2)$$

and the estimate

$$\|\nabla u_h\|_\infty = O(1). \quad (3)$$

Further for  $u \in W^{2,q}(\Omega)$ ,  $q > 2$ , they derived, using a common duality argument,

$$\|u - u_h\|_p = O(h^2) \quad , \quad 1 \leq p < 2. \quad (4)$$

For uniform triangulations and  $u \in C^2(\bar{\Omega}) \cap W^{3,2}(\Omega)$  Mittelmann [6] has shown the  $L^\infty$ -estimate

$$\|u - u_h\|_\infty = O(h^{3/2} |\ln h|^{1/2}). \quad (5)$$

In the linear case  $F(u) = |\nabla u|^2 - 2fu$  with  $u \in W^{2,\infty}(\Omega)$  the analogue of the method  $(V_h)$  has the order of convergence (see [8], [13], [11], [12], [4])

$$\|u - u_h\|_\infty = O(h^2 |\ln h|). \quad (6)$$

Recently this was obtained by Frehse [3] for the general variational problem, too :

$$\int_{\Omega} F(\cdot, u, \nabla u) dx = \text{Min.}, \quad u \in W_0^{1,2}(\Omega). \quad (7)$$

He used a Morrey norm estimate

$$\|\nabla(u - u_h)\|_\alpha = O(h) \quad \text{with an } \alpha > 0, \quad (8)$$

which is proven in [2] for zero boundary conditions and uniformly convex functions  $F(x, \xi, \cdot)$ . These assumptions are not valid in the case of minimal surfaces. In this note we shall prove the  $L^\infty$ -estimate (6) for the nonlinear problem (V) without making use of (8). Furthermore the  $L^p$ -estimate (4) will be extended to the case  $p = 2$ .

**THEOREM :** Assume  $\partial\Omega \in C^{2,\alpha}$ ,  $0 < \alpha < 1$ , and  $g \in W^{2,q}(R^2) \cap W^{2,q}(\partial\Omega)$  and  $u \in W^{2,q}(\Omega)$  for some  $q$  with  $2 < q \leq \infty$ . Further let the triangulations  $T_h$  be quasi-regular in the sense of (T). Then

$$\|u - u_h\|_\infty = O(h^{2-2/q} |\ln h|), \quad (I)$$

$$\|u - u_h\|_2 = O(h^2). \quad (II)$$

**REMARKS :** The methods applied in the proof also work with slight modifications for the general variational problem (7) under the assumptions of [3]. This will be of its own interest since the estimate (8) is somewhat hard to prove. For this problem the above  $L^\infty$ -estimate can be extended to higher dimensions  $n \geq 3$ , too. This will be carried out in a forthcoming paper.

Moreover, there is no difficulty to obtain analogously to [11] the order of pointwise convergence  $O(h^m)$  for the finite element approximation of a solution  $u \in W^{m,\infty}(\Omega)$  of (7) with elements of order  $m \geq 3$ .

Here and below  $L^p(\Omega)$  and  $W^{m,p}(\Omega)$ ,  $W_0^{m,p}(\Omega)$  denote the usual real Lebesgue and Sobolev spaces with the corresponding norms

$$\|\cdot\|_p = \|\cdot\|_{p,\Omega} \quad , \quad \|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega} \quad , \quad 1 \leq p \leq \infty \quad , \quad m \in N_0.$$

Further we use the abbreviations

$$\partial_i v := \partial v / \partial x^i \quad , \quad i = 1, 2, \quad \nabla^1 v = \nabla v := \text{grad } v \quad , \quad \nabla^2 v := (\partial_i \partial_k v)_{i,k=1,2},$$

for the partial (generalized) derivatives, and  $c$  for a positive (generic) constant which is independent of the parameters  $h$  and  $\rho$ , defined below. Finally, we shall use the usual summation convention.

## 2. PROOF OF THE THEOREM

First we introduce some notations and technical facts.

The minimizing functions  $u \in W^{2,q}(\Omega) \subset C^1(\bar{\Omega})$ ,  $2 < q \leq \infty$ , of (V) and  $u_h \in S_h$  of  $(V_h)$  necessarily satisfy the Euler equations

$$\int_{\Omega} (1 + |\nabla u|^2)^{-1/2} \nabla u \cdot \nabla v \, dx = 0, \quad \forall v \in W_0^{1,2}(\Omega),$$

$$\int_{\Omega_h} (1 + |\nabla u_h|^2)^{-1/2} \nabla u_h \cdot \nabla v_h \, dx = 0, \quad \forall v_h \in S_h^0.$$

Denoting by  $F_{ik} := \partial_i \partial_k F$ ,  $i, k = 1, 2$ , the second derivatives of the function  $F(\eta) := (1 + |\eta|^2)^{1/2}$ ,  $\eta \in R^2$ , we obtain

$$F_{ik}(\eta)\xi_i\xi_k \geq (1 + |\eta|^2)^{-3/2} |\xi|^2 \quad , \quad \eta, \xi \in R^2.$$

Combination of the Euler equations yields

$$a^h(v_h, u - u_h) := \int_{\Omega_h} a_{ik}^h(\cdot) \partial_i v_h \partial_k (u - u_h) dx = 0, \quad \forall v_h \in S_h^0, \quad (9)$$

with the  $L^\infty(\Omega)$ -functions

$$a_{ik}^h(\cdot) := \int_0^1 F_{ik}(\nabla u_h(\cdot) + t \nabla(u - u_h)(\cdot)) dt, \quad i, k = 1, 2.$$

Further by the result (3),

$$a_{ik}^h \xi_i \xi_k \geq c |\xi|^2 \quad , \quad \xi \in R^2 \quad , \quad \text{on } \bar{\Omega}.$$

Since the  $a_{ik}^h$  are discontinuous we introduce the bilinear form

$$a(v, w) := \int_{\Omega} a_{ik}(\cdot) \partial_i v \partial_k w dx, \quad v, w \in W^{1,2}(\Omega),$$

with coefficients

$$a_{ik}(\cdot) := F_{ik}(\nabla u(\cdot)) \in W^{1,q}(\Omega) \quad , \quad i, k = 1, 2.$$

Then the differential operator

$$A := - \partial_k \{ a_{ik}(\cdot) \partial_i \} \quad (11)$$

is uniformly elliptic in  $\Omega$  and hence satisfies the well known a priori estimate (see [7; 5.2 ff]) :

$$\|v\|_{2,2} \leq c \|Av\|_2 \quad , \quad \forall v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega).$$

From the boundedness of the derivatives of  $F_{ik}$  it follows that

$$|a_{ik} - a_{ik}^h| \leq c |\nabla(u - u_h)| \quad \text{on } \Omega. \quad (12)$$

For functions  $v \in W^{2,p}(\Omega)$ ,  $2 \leq p \leq \infty$ , and  $w \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  and the corresponding interpolants  $I_h v \in S_h$  and  $I_h w \in S_h^0$  the following estimates are known (see [1] and [10]) :

$$\|v - I_h v\|_{j,p,\Omega_h} \leq ch^{2-j} \|\nabla^2 v\|_{p,\Omega_h} \quad , \quad 0 \leq j < 2 \quad , \quad 2 \leq p \leq \infty. \quad (13)$$

$$\|v - I_h v\|_{j,\infty,\Omega_h} \leq ch^{2-j-2/p} \|\nabla^2 v\|_{p,\Omega_h}$$

Using in addition Lemma A4 of the Appendix we find with (15)

$$\|w - I_h w\|_{j,2} \leq ch^{2-j} \|w\|_{2,2}, \quad 0 \leq j < 2. \quad (14)$$

*Proof of Proposition (I)*

In order to make the outline of the proof clear we write its main steps as lemmas.

Set  $e_h := u - u_h$  and let  $E_h := I_h e_h = I_h u - u_h$  be its interpolant on  $\Omega_h$ . Observe that  $E_h \in S_h^0$ .

First we estimate  $e_h$  on  $\Omega - \Omega_h$ :

The assumptions (T) and  $\partial\Omega \in C^2$  imply

$$d(\partial\Omega_h, \partial\Omega) := \sup_{x \in \partial\Omega_h} \text{dist}(x, \partial\Omega) \leq ch^2. \quad (15)$$

Thus, we conclude using  $u = g$  on  $\partial\Omega$ ,  $u_h = I_h g$  on  $\Omega - \Omega_h$  and the estimate (13)

$$\begin{aligned} \|e_h\|_{\infty; \Omega - \Omega_h} &\leq \|u - g\|_{\infty; \Omega - \Omega_h} + \|g - I_h g\|_{\infty; \Omega - \Omega_h} \\ &\leq ch^{2-2/q} (\|u\|_{2,q} + \|g\|_{2,q, \mathbb{R}^2}). \end{aligned}$$

Next, let  $z_h \in \Omega_h$  be points with the properties

$$|E_h(z_h)| = \|E_h\|_{\infty}.$$

Then, with any disk  $B := B_{\tau}(z_h)$ ,  $\tau > 0$ ,

$$\|E_h\|_{\infty} \leq c\tau^{-2} \int_B |E_h| dx + c\tau \|\nabla E_h\|_{\infty}.$$

Using the well known inverse relation  $\|\nabla E_h\|_{\infty} \leq ch^{-1} \|E_h\|_{\infty}$ , we find that for  $\tau := \delta h$ ,  $\delta > 0$  a constant sufficiently small,

$$\|E_h\|_{\infty} \leq ch^{-2} \int_B |E_h| dx.$$

Thus by the estimate (13)

$$\begin{aligned} \|e_h\|_{\infty; \Omega_h} &\leq \|u - I_h u\|_{\infty; \Omega_h} + \|E_h\|_{\infty} \\ &\leq ch^{2-2/q} \|u\|_{2,q} + ch^{-2} \int_B |e_h| dx. \end{aligned} \quad (17)$$

In the following our main tool will be a modification of the weighted norm technique by Nitsche [11]. With a real parameter  $0 < \rho \leq \rho_0$ , which will be appropriately coupled with  $h$  below, we define the weight function

$$\sigma_h(\cdot) := (|\cdot - z_h|^2 + \rho^2)^{1/2}$$

and the weighted norms ( $T$  denoting triangles of  $T_h$ )

$$\|\cdot\|_{(v)} := \left( \sum_{T \in \Omega_h} \int_T \sigma_h^v |\cdot|^2 dx \right)^{1/2}, v \in R. \tag{18}$$

Since the points  $z_h$  and the corresponding disks  $B := B_{\delta h}(z_h)$  will be fixed during the proof, we shall omit the index of  $\sigma$ .

Obviously with constants independent of  $\rho$

$$|\nabla\sigma| \leq c, \quad |\nabla^2\sigma| \leq c\sigma^{-1} \leq c\rho^{-1} \quad \text{on } R^2,$$

and for  $\rho \geq c_3 h$ ,  $c_3$  sufficiently large,

$$\max_{T \in T_h} \{ \max_{x \in T} \sigma^v(x) / \min_{x \in T} \sigma^v(x) \} \leq c, \quad -4 \leq v \leq 4.$$

From this and (13) we conclude the following interpolation estimate for functions  $v \in C(\Omega_h) \cap \bigoplus_{T \in \Omega_h} W^{2,2}(T)$  (see [11]) :

$$\|v - I_h v\|_{(v)} + h \|\nabla(v - I_h v)\|_{(v)} \leq ch^2 \|\nabla^2 v\|_{(v)}, \quad -4 \leq v \leq 4. \tag{19}$$

LEMMA 1 : *Let  $1 < \beta < 2$ . Then there are constants  $c_4, c_\beta$ , independent of  $h$  and  $\rho \geq c_3 h$ , so that*

$$\|e_h\|_{\infty; \Omega_h} \leq c_4 h^{2-2/q} |\ln h| + c_\beta \rho h^{-1} |\ln h| \|\nabla e_h\|_{(-\beta)}^2.$$

*Proof.* We shall estimate the integral in (17) making use of a common duality argument. Let  $G^h \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  be the solution of the boundary value problem (smoothed Green function)

$$AG^h - h^{-2} \text{sgn}(e_h) \chi_B \quad \text{in } \Omega, \quad G^h = 0 \quad \text{on } \partial\Omega, \tag{20}$$

with  $\chi_B$  the characteristic function of the disk  $B$ , and let  $G_h^h \in S_h^0$  be its Ritz projections defined by

$$a(v_h, G_h^h) = a(v_h, G^h), \quad \forall v_h \in S_h^0. \tag{21}$$

With this setting we obtain by integration by parts denoting

$$\partial_n := -n_k a_{ik} \partial_i, \quad (n_1, n_2) := \text{outward normal to } \partial\Omega,$$

$$h^{-2} \int_B |e_h| dx = \int_{\partial\Omega} e_h \partial_n G^h ds + a(e_h, G^h) \tag{22}$$

and by (9)

$$h^{-2} \int_B |e_h| dx = \int_{\partial\Omega} e_h \partial_n G^h ds + a(e_h, G^h - G_h^h) + (a - a^h)(e_h, G_h^h). \tag{23}$$

Now we shall estimate the three terms on the right.

Since  $e_h = g - I_h g$  on  $\partial\Omega$ , we find using Hölder's inequality and a well known trace theorem (see [9])

$$\left| \int_{\partial\Omega} e_h \partial_n G^h ds \right| \leq c \|g - I_h g\|_{q;\partial\Omega} \|G^h\|_{2,r}, \quad 1/r = 1 - 1/(2q) \quad (24)$$

For  $g \in W^{2,q}(R^2) \cap W^{2,q}(\partial\Omega)$  it is proven in [5] that

$$\|g - I_h g\|_{q;\partial\Omega} \leq ch^2.$$

Using Hölder's inequality (notice  $r = 2q/(2q - 1) < 2$ ), Poincaré's inequality for  $G^h \in W_0^{1,2}(\Omega)$  and Lemma A2(a) with  $\rho \geq c_3 h$ , we find (We note that the logarithm only appears for  $r = 1$ .)

$$\|G^h\|_{2,r} \leq ch^{-1/q} |\ln h|^{1/2} \{ \|\nabla G^h\|_2 + \|\nabla^2 G^h\|_{(2)} \} \leq ch^{-1/q} |\ln h|. \quad (25)$$

Thus

$$\left| \int_{\partial\Omega} e_h \partial_n G^h ds \right| \leq ch^{2-1/q} |\ln h|. \quad (26)$$

The second term in (23) is the same that occurs in the case of linear problems (see [4]). The modified interpolate of  $u$ , defined by

$$\hat{I}_h u := \begin{cases} I_h u & \text{on } \Omega_h, \\ I_h g & \text{on } \hat{\Omega}_h - \Omega_h, \end{cases}$$

obviously satisfies  $\hat{I}_h u - u_h \in S_h^0$ . Hence, we find using (21)

$$\begin{aligned} |a(e_h, G^h - G_h^h)| &= |a(u - \hat{I}_h u, G^h - G_h^h)| \\ &\leq c \|\nabla(u - I_h u)\|_{\infty;\Omega_h} \|\nabla(G^h - G_h^h)\|_1 \\ &\quad + c(\|u\|_{2,q} + \|g\|_{2,q;R^2}) \times \int_{\Omega - \Omega_h} |\nabla G^h| dx. \end{aligned}$$

By Lemma A4 (notice (15)) and Lemma A2(a) (with  $\varepsilon = 0$ ,  $\rho \geq c_3 h$ ) it follows in the same way as in (25) with  $r = 1$

$$\int_{\Omega - \Omega_h} |\nabla G^h| dx \leq ch^2 \|G^h\|_{2,1} \leq ch^2 |\ln h|.$$

Thus, we conclude using the estimate (13) and the result, stated in Lemma A2(b), concerning the convergence of Green functions :

$$|a(e_h, G^h - G_h^h)| \leq ch^{2-2/q} |\ln h|. \quad (27)$$



The third term in (23) comes from the nonlinearity of the problem (V), i.e. from the replacement of the discontinuous coefficients  $a_{ik}^h$  by  $a_{ik} \in W^{1,q}(\Omega)$ . By making use of (12) and Lemma A2(c) (with  $\rho \geq c_3 h$ ) we see (notice  $G_h^h = 0$  on  $\Omega - \Omega_h$ )

$$\begin{aligned} |(a - a^h)(e_h, G_h^h)| &\leq c \int_{\Omega_h} |\nabla e_h|^2 \sigma^{-\beta} |\sigma^\beta \nabla G_h^h| dx \\ &\leq c_\beta \rho h^{-1} |\ln h| \|\nabla e_h\|_{(-\beta)}^2. \end{aligned}$$

Together with (27) and (26) this establishes the desired estimate.

Since  $\beta < 2$  our proposition (I) suggests the estimate  $\|\nabla e_h\|_{(-\beta)} = O(h^{1-1/q})$  without any logarithmic term. Obviously this would complete the proof of (I). A first step in this direction will be the following :

LEMMA 2 : *Let  $1 < \beta < 2$ . Then there are constants  $c_6, c_\beta$ , independent of  $h$ , so that for  $\rho = c_6 h |\ln h|^{3/2}$*

$$\|\nabla e_h\|_{(-\beta)}^2 \leq c_\beta h^{3-\beta-2/q} |\ln h|.$$

*Proof.* The result (2) gives for  $\rho \geq h$

$$\|\nabla e_h\|_{(-\beta)}^2 \leq c \|\nabla e_h\|_{(-2)} \|\nabla e_h\|_2 h^{1-\beta} \leq c h^{2-\beta} \|\nabla e_h\|_{(-2)}. \quad (28)$$

The weighted norm on the right will be estimated in the same way as in [11] and [12]. By (10) we have

$$\begin{aligned} \|\nabla e_h\|_{(-2)}^2 &\leq c |a^h(e_h, \sigma^{-2} e_h)| + c \int_{\Omega_h} |\nabla e_h| |e_h| |\nabla \sigma^{-2}| dx \\ &\leq c |a^h(e_h, \sigma^{-2} e_h)| + c \|\nabla e_h\|_{(-2)} \|e_h\|_{(-4)}. \end{aligned}$$

Since  $v_h := I_h(\sigma^{-2} E_h) \in S_h^0$ ,  $E_h := I_h u - u_h$ , it follows by (9) that

$$\begin{aligned} |a^h(e_h, \sigma^{-2} e_h)| &= |a^h(e_h, \sigma^{-2}(u - I_h u) + \sigma^{-2} E_h - v_h)| \\ &\leq c \|\nabla e_h\|_{(-2)} \{ \|\nabla(\sigma^{-2}(u - I_h u))\|_{(2)} + \|\nabla(\sigma^{-2} E_h - v_h)\|_{(2)} \}. \end{aligned}$$

Using the estimate (19), we get for  $\rho \geq c_3 h$  (notice  $\nabla^2 E_h = 0$  on each  $T \in T_h$ )

$$\begin{aligned} \|\nabla(\sigma^{-2}(u - I_h u))\|_{(2)} &\leq c \{ \|u - I_h u\|_{(-4)} + \|\nabla(u - I_h u)\|_{(-2)} \} \\ &\leq c h \|\nabla^2 u\|_{(-2)} \end{aligned}$$

and

$$\begin{aligned} \|\nabla(\sigma^{-2} E_h - v_h)\|_{(2)} &\leq c h \|\nabla^2(\sigma^{-2} E_h)\|_{(2)} \\ &\leq c h \{ \|e_h\|_{(-6)} + \|\nabla e_h\|_{(-4)} + \|u - I_h u\|_{(-6)} + \|\nabla(u - I_h u)\|_{(-4)} \}. \\ &\leq c \{ \|e_h\|_{(-4)} + h \rho^{-1} \|\nabla e_h\|_{(-2)} + h \|\nabla^2 u\|_{(-2)} \}. \end{aligned}$$

Thus

$$\|\nabla e_h\|_{(-2)} \leq c \|e_h\|_{(-4)} + ch\rho^{-1} \|\nabla e_h\|_{(-2)} + ch \|\nabla^2 u\|_{(-2)}, \quad (29)$$

and with  $\rho \geq c_5 h$ ,  $c_5$  sufficiently large,

$$\|\nabla e_h\|_{(-2)} \leq c \|e_h\|_{(-4)} + ch^{1-2/q} |\ln h|^{1/2} \|u\|_{2,q}. \quad (30)$$

In order to estimate the first term on the right we use a duality argument. With the solution  $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  of the boundary value problem

$$Av = \sigma^{-4} e_h \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

we obtain by integration by parts

$$\|e_h\|_{(-4)}^2 = \int_{\partial\Omega} e_h \partial_n v \, ds + a(e_h, v)$$

and in addition with  $I_h v \in S_h^0$  by (9) (see (23))

$$\|e_h\|_{(-4)}^2 = \int_{\partial\Omega} e_h \partial_n v \, ds + a(e_h, v - I_h v) + (a - a^h)(e_h, I_h v). \quad (31)$$

The three terms on the right will be estimated analogously to those in (23). With  $1/r = 1 - 1/(2q)$  we get

$$\begin{aligned} \left| \int_{\partial\Omega} e_h \partial_n v \, ds \right| &\leq c \|g - I_h g\|_{q;\partial\Omega} \|v\|_{2,r} \\ &\leq ch^{2-1/q} |\ln h|^{1/2} \{ \|\nabla v\|_2 + \|\nabla^2 v\|_{(2)} \}. \end{aligned}$$

From the estimate (19) and Lemma A4 it follows that

$$\begin{aligned} |a(e_h, v - I_h v)| &\leq c \|\nabla e_h\|_{(-2)} \|\nabla(v - I_h v)\|_{(2)} + c \|\nabla(u - I_h g)\|_\infty \int_{\Omega - \Omega_h} |\nabla u| \, dx \\ &\leq ch \|\nabla e_h\|_{(-2)} \|\nabla^2 v\|_{(2)} + ch^2 \|v\|_{2,1} \\ &\leq c \{ h \|\nabla e_h\|_{(-2)} + h^2 |\ln h|^{1/2} \} \{ \|\nabla v\|_2 + \|\nabla^2 v\|_{(2)} \}. \end{aligned}$$

Using the estimate (12), Lemma A1 (with  $v = 2$ ) and the result (2), we find

$$\begin{aligned} |(a - a^h)(e_h, I_h v)| &\leq c \|\nabla e_h\|_{(-1)}^2 |\ln h| \{ \|\nabla v\|_2 + \|\nabla^2 v\|_{(2)} \} \\ &\leq ch |\ln h| \|\nabla e_h\|_{(-2)} \{ \|\nabla v\|_2 + \|\nabla^2 v\|_{(2)} \}. \end{aligned}$$

Thus, by Lemma A3, observing that  $\|Av\|_{(4)} = \|e_h\|_{(-4)}$ ,

$$\|e_h\|_{(-4)} \leq ch^{1-1/q} |\ln h| + ch\rho^{-1} |\ln h|^{3/2} \|\nabla e_h\|_{(-2)}.$$

We substitute this in (30), choose  $\rho = c_6 h |\ln h|^{3/2}$ ,  $c_6$  appropriately large, and obtain

$$\|\nabla e_h\|_{(-2)} \leq ch^{1-2/q} |\ln h|. \quad (32)$$

From this the desired estimate follows.

By combination of Lemma 1 and Lemma 2, we find as a first result for  $1 < \beta < 2$

$$\|e_h\|_{\infty; \Omega_h} \leq c_\beta h^{3-\beta-2/q} |\ln h|^4. \tag{33}$$

Now, this will be used to improve the estimate of Lemma 2.

LEMMA 3 : Let  $1 < \beta_q < 1 + (q - 2)/(3q)$ . Then there are constants  $c_7, c_8$  independent of  $h$ , so that for  $\rho = c_7 h$

$$\|\nabla e_h\|_{(-\beta_q)} \leq c_8 h^{1-1/q}.$$

*Proof.* Set  $\rho \geq c_3 h$  to guarantee the interpolation estimate (19). We start in the same way as in the proof of Lemma 2 concerning the term  $\|\nabla e_h\|_{(-2)}$ . With  $v_h := I_h(\sigma^{-\beta} E_h) \in S_h^0, E_h := I_h u - u_h$ , we find analogously to (29)

$$\|\nabla e_h\|_{(-\beta)} \leq c \|e_h\|_{(-\beta-2)} + ch\rho^{-1} \|\nabla e_h\|_{(-\beta)} + ch \|\nabla^2 u\|_{(-\beta)}$$

and for  $\rho = c_7 h, c_7$  appropriately large,

$$\|\nabla e_h\|_{(-\beta)} \leq c \|e_h\|_{(-\beta-2)} + ch \|\nabla^2 u\|_{(-\beta)}.$$

Thus, by the result (33)

$$\begin{aligned} \|\nabla e_h\|_{(-\beta)} &\leq ch^{3-\beta-2/q} |\ln h|^4 \|1\|_{(-\beta-2)} + ch^{2-\beta/2-2/q} \|u\|_{2,q} \\ &\leq ch^{1-1/q} \{ h^{2-3\beta/2-1/q} |\ln h|^4 + h^{1-\beta/2-1/q} \}. \end{aligned}$$

Since  $q > 2$ , we can find some  $\beta_q$  with  $1 < \beta_q < 1 + (q - 2)/(3q) < 2$ , so that

$$h^{(2-3\beta_q/2-1/q)} |\ln h|^4 + h^{(1-\beta_q/2-1/q)} \leq c.$$

This proves Lemma 3.

Finally, combination of Lemma 3 and Lemma 1 completes the proof of proposition (I).

*Proof of Proposition (II)*

The proof of Proposition (II) makes use of the estimate (I) for the given  $q > 2 (e_h := u - u_h)$  :

$$\|e_h\|_{\infty} \leq ch^{2-2/q} |\ln h| \leq c_q h. \tag{36}$$

Let  $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  be the solution of the problem

$$Av = e_h \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{37}$$

Using (9), we find analogously to (31) with  $I_h v \in S_h^0$

$$\|e_h\|_2^2 = \int_{\partial\Omega} e_h \partial_n v \, ds + a(e_h, v - I_h v) + (a - a^h)(e_h, I_h v). \tag{38}$$

The three terms on the right can be estimated in a similar way as those in (31). We get

$$\left| \int_{\partial\Omega} e_h \partial_n v \, ds \right| \leq c \|g - I_h g\|_{2; \partial\Omega} \|v\|_{2,2} \leq ch^2 \|v\|_{2,2}$$

and, using (14) and the result (2),

$$|a(e_h, v - I_h v)| \leq c \|\nabla e_h\|_2 \|\nabla(v - I_h v)\|_2 \leq ch^2 \|v\|_{2,2}.$$

Further

$$|(a - a^h)(e_h, I_h v)| \leq c \sum_{i,k} \int_{\Omega_h} |a_{ik} - a_{ik}^h| |\nabla e_h| |\nabla I_h v| \, dx.$$

It will be convenient to replace the discontinuous function  $|\nabla I_h v|$  by a continuous approximant  ${}_h := I_h |\nabla v|$  of  $|\nabla v| \in W^1(\Omega)$  on  $\Omega_h$ . By the boundedness of  $a_{ik}$ ,  $a_{ik}^h$  and the estimate (12), we have

$$|(a - a^h)(e_h, I_h v)| \leq c \int_{\Omega_h} \{ |\nabla e_h| |\nabla(I_h v - v)| + |\nabla e_h| | |\nabla v| - \psi_h | + |\nabla e_h|^2 \psi_h \} \, dx$$

and, using (13) and the result (2),

$$\begin{aligned} |(a - a^h)(e_h, I_h v)| &\leq c \|\nabla e_h\|_2 h \|v\|_{2,2} + c \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx \\ &\leq ch^2 \|v\|_{2,2} + c \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx. \end{aligned}$$

The integral on the right can be estimated in the same way as the term  $\|\nabla e_h\|_{(-2)}$  in the proof of Lemma 2. Defining  $v_h := I_h(\psi_h E_h) \in S_h^0$ ,  $E_h := I_h u - u_h$ , we have by (9)

$$\begin{aligned} \left| \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx \right| &\leq c |a^h(e_h, \psi_h(u - I_h u) + \psi_h E_h - v_h)| \\ &\quad + c \int_{\Omega_h} |\nabla e_h| |e_h| |\nabla \psi_h| \, dx \end{aligned}$$

and hence

$$\begin{aligned} \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx &\leq c \|\nabla e_h\|_2 \{ \|\nabla \psi_h(u - I_h u)\|_{2;\Omega_h} + \|\psi_h \nabla(u - I_h u)\|_{2;\Omega_h} \\ &\quad + \|\nabla(\psi_h E_h - v_h)\|_{2;\Omega_h} + \|\nabla \psi_h e_h\|_{2;\Omega_h} \} \\ &\leq c \|\nabla e_h\|_2 \{ \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} \}. \end{aligned}$$

Using Hölder’s inequality, the Sobolev embedding theorem and (13), it follows that

$$\begin{aligned} \text{(i)} \quad &\leq c \|u - I_h u\|_{\infty, \Omega_h} \|\nabla \psi_h\|_{2, \Omega_h} \leq ch^{2-2/q} \|u\|_{2, q} \|v\|_{2, 2}, \\ \text{(ii)} \quad &\leq c \|\nabla(u - I_h u)\|_{q; \Omega_h} \|\psi_h\|_{1; 2, \Omega_h} \leq ch \|u\|_{2, q} \|v\|_{2, 2}. \end{aligned}$$

Observing  $\psi_h \geq 0$ ,  $\nabla^2 \psi_h = \nabla^2 E_h = 0$  on each  $T \in T_h$  and the result (3), we find

$$\begin{aligned} \text{(iii)} \quad &\leq ch \left( \sum_{T \subset \Omega_h} \int_T |\nabla^2(\psi_h E_h)|^2 dx \right)^{1/2} \leq ch \|\nabla \psi_h\|_{2, \Omega_h} \|\nabla E_h\|_{\infty} \\ &\leq ch \|u\|_{2, q} \|v\|_{2, 2}. \end{aligned}$$

Finally, our result (36) applies to the crucial term (iv)

$$\text{(iv)} \quad \leq \|\nabla \psi_h\|_{2, \Omega_h} \|e_h\|_{\infty, \Omega_h} \leq ch \|v\|_{2, 2}.$$

This gives

$$|(a - a^h)(e_h, I_h v)| \leq ch \|v\|_{2, 2} \|\nabla e_h\|_2,$$

Thus, by  $\|v\|_{2, 2} \leq c \|e_h\|_2$  and the result (2),

$$\|e_h\|_2 = O(h^2).$$

This completes the proof of the theorem.

### 3. APPENDIX

Here we state some lemmas used in the proof of the theorem. Assume the condition (T) to be satisfied.

LEMMA A1 : Let  $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  and let  $I_h v \in S_h^0$  be its interpolant. Then

$$\|\nabla I_h v\|_{\infty} \leq c |\ln h| \|v\|_{2, 2}, \tag{a}$$

and with the weighted norms (18) for  $0 \leq \nu \leq 4$  and  $\rho \geq c_3 h$

$$\|\sigma^{\nu/2} \nabla I_h v\|_{\infty} \leq c |\ln h| (\|\nabla v\|_{(\nu-2)} + \|\nabla^2 v\|_{(\nu)}). \tag{b}$$

*Proof.* We shall prove (b). The proof of (a) is similar.

Let  $T \in T_h$  be any triangle with  $T \subset \Omega_h$  and let  $\xi \in T$  be the center of the inscribed circle with radius  $c_1 h$  (assumption (T)). The boundary  $\partial\Omega$  is of class  $C^2$  and hence satisfies a strong cone condition. The corresponding spherical cone  $K := K(\xi, \tau) \subset B(\xi, \tau)$  with vertex  $\xi$ , opening  $|\Sigma_K|$  and height

$\tau > 0$  (independent of  $\xi \in \Omega$  and  $h$ ) can be cut off to a cone  $K_T \subset T$  with volume  $|K_T| = ch^2$  and  $K_T \subset \subset B(\xi, \tau)$ . Then by (19)

$$\begin{aligned} |\sigma^{v/2} \nabla I_h v(\xi)| &\leq ch^{-2} \int_{K_T} \sigma^{v/2} |\nabla I_h v| dx \\ &\leq ch^{-1} \|\nabla(v - I_h v)\|_{(v)} + ch^{-2} \int_{K_T} \sigma^{v/2} |\nabla v| dx \\ &\leq c \|\nabla^2 v\|_{(v)} + ch^{-1} \left( \int_{K_T} \sigma^v |\nabla v|^2 dx \right)^{1/2}. \end{aligned}$$

Now choose a function  $\varphi \in C^\infty$  with support in the ball  $B(\xi, \tau)$  and the properties (independent of  $h$ )

$$0 \leq \varphi \leq 1, \quad |\nabla \varphi| \leq c, \quad \varphi = 1 \quad \text{on } K_T.$$

Using polar coordinates  $(r, \theta)$  centered in  $\xi$  we find with the function

$$p(|x - \xi|) := (|x - \xi|^2 + h^2)^{-1}$$

$$h^{-2} \int_{K_T} \sigma^v |\nabla v|^2 dx \leq c \int_{\Sigma_K} \int_0^\tau r p(r) |\varphi \sigma^{v/2} \nabla v|^2 dr d\theta.$$

According to the special choice of  $\varphi$  it follows by integration by parts

$$\int_0^\tau r p(r) |\varphi \sigma^{v/2} \nabla v|^2 dr = -2 \int_0^\tau \left\{ \int_0^r s p(s) ds \right\} \varphi \sigma^{v/2} |\nabla v| \partial_r (\varphi \sigma^{v/2} |\nabla v|) dr$$

and, using the inequality  $ab \leq \varepsilon a^2 + (4\varepsilon)^{-1} b^2$ ,  $\varepsilon > 0$ ,

$$\int_0^\tau r p(r) |\varphi \sigma^{v/2} \nabla v|^2 dr \leq 4 \int_0^\tau \left\{ \int_0^r s p(s) ds \right\}^2 r^{-2} p(r)^{-1} r |\nabla(\varphi \sigma^{v/2} |\nabla v|)|^2 dr.$$

The function  $\left\{ \int_0^r s p(s) ds \right\}^2 r^{-2} p(r)^{-1}$  is continuous and nondecreasing

for  $0 \leq r < h$  and hence uniformly bounded by  $c |\ln h|^2$  for  $r \leq \text{diam}(\Omega)$ . This gives

$$h^{-2} \int_{K_T} \sigma^v |\nabla v|^2 dx \leq c |\ln h|^2 (\|\nabla v\|_{(v-2)} + \|\nabla^2 v\|_{(v)})^2$$

and completes the proof.

LEMMA A2 : Let  $G^h \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$  be the smoothed Green functions defined by (20) and let  $G_h^h$  be its Ritz projections defined by (21). Then with the weighted norms (18) for  $\rho \geq c_3 h$

$$\|\nabla G^h\|_{(e)} + \|\nabla^2 G^h\|_{(2+\varepsilon)} \leq c_\varepsilon \rho h^{-1} \begin{cases} 1 & , \varepsilon > 0 \\ |\ln h|^{1/2} & , \varepsilon = 0, \end{cases} \tag{a}$$

$$\|\nabla(G^h - G_h^h)\|_1 \leq c |\ln h|^{1/2} \|\nabla(G^h - G_h^h)\|_{(2)} \leq c \rho |\ln h|, \tag{b}$$

$$\|\sigma^{1+\varepsilon} \nabla G_h^h\|_\infty \leq c_\varepsilon \rho h^{-1} |\ln h|, \quad \varepsilon > 0. \tag{c}$$

*Proof.* The estimate (a) and (b) are proven in [4].

Analogously to Lemma A1 we find

$$|\sigma^{1+\varepsilon} \nabla G_h^h(\xi)| \leq c h^{-1} \|\nabla(G_h^h - G^h)\|_{(2+2\varepsilon)} + c |\ln h| \left\{ \|\nabla G^h\|_{(2\varepsilon)} + \|\nabla^2 G^h\|_{(2+2\varepsilon)} \right\}$$

and, using (a) and (b),

$$|\sigma^{1+\varepsilon} \nabla G_h^h(\xi)| \leq c \rho h^{-1} |\ln h|^{1/2} + c_\varepsilon \rho h^{-1} |\ln h|.$$

This proves (c).

LEMMA A3 : Let  $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\tilde{\Omega})$  and let  $A$  be the uniformly elliptic differential operator defined by (11). Then with the weighted norms (18) for  $0 < \rho < \rho_0$

$$\|\nabla v\|_2 + \|\nabla^2 v\|_{(2)} \leq c \rho^{-1} (1 + |\ln \rho|^{1/2}) \|Av\|_{(4)}.$$

For the sake of completeness we shall sketch a proof of this important a priori estimate. It rests on ideas contained in [12] and [4]. A similar assertion is stated by Nitsche [11] for the Laplace operator.

*Proof.* Set  $\sigma(\cdot) := (|\cdot - z|^2 + \rho^2)^{1/2}$ ,  $0 < \rho \leq \rho_0$ ,  $z \in \Omega$ .

Denoting by  $y^j := x^j - z^j$ ,  $j = 1, 2$ , the components of the vector  $x - z$ ,  $x \in \Omega$ , we get

$$\|\nabla^2 v\|_{(2)}^2 = \sum_{j=1}^2 \|y^j \nabla^2 v\|_2^2 + \rho^2 \|\nabla^2 v\|_2^2,$$

and by the well known  $L^2$ -estimate,

$$\rho \|\nabla^2 v\|_2 \leq c \rho \|Av\|_2 \leq c \rho^{-1} \|Av\|_{(4)},$$

$$\|y^j \nabla^2 v\|_2 \leq c \{ \|\nabla^2(y^j v)\|_2 + \|\nabla v\|_2 \} \leq c \{ \|A(y^j v)\|_2 + \|\nabla v\|_2 \}.$$

Further, using  $M := \max_{i,k} \|a_{ik}\|_{1,q} < \infty$  and  $\max_{i,k} \|a_{ik}\|_\infty \leq cM$ , we find with  $p := 2q/(q - 2)$

$$\|A(y^j v)\|_2 \leq c \{ \|y^j Av\|_2 + M \|v\|_p + M \|\nabla v\|_2 \} \leq c \{ \|y^j Av\|_2 + \|v\|_{1,2} \},$$

and thus by Poincaré's inequality and  $\sigma^{-1} \leq c \rho^{-1}$

$$\|\nabla^2 v\|_{(2)} \leq c \{ \|Av\|_{(2)} + \|v\|_{1,2} \} \leq c \{ \rho^{-1} \|Av\|_{(4)} + \|\nabla v\|_2 \}. \tag{39}$$

By the inequality  $ab \leq (a^2 + b^2)/2$ , it follows that

$$\|\nabla v\|_2^2 \leq a(v, v) \leq c\rho^{-2} |\ln \rho| \|Av\|_{(4)}^2 + c\rho^2 |\ln \rho|^{-1} \|v\|_{(-4)}^2.$$

Denoting by  $g(\cdot, \cdot)$  the Green function of  $A$  over  $\Omega$  we obtain

$$\|v\|_{(-4)}^2 = \int \sigma^{-4}(x) \left| \int Av(\xi)g(x, \xi) d\xi \right|^2 dx,$$

and by Hölder's inequality and an interchange of the order of integration

$$\|v\|_{(-4)}^2 \leq \int \sigma^4(\xi) |Av(\xi)|^2 \left( \int \sigma^{-4}(x)g(x, \xi) \left\{ \int \sigma^{-4}(\eta)g(x, \eta) d\eta \right\} dx \right) d\xi.$$

It is well known that the Green function  $g$  can be estimated on  $\Omega$  by (see [4])

$$0 \leq g(x, y) \leq c(1 + |\ln |x - y||).$$

From this we conclude

$$\int \sigma^{-4}(\eta)g(x, \eta) d\eta \leq c\rho^{-2}(1 + |\ln \rho|).$$

It follows that

$$\|v\|_{(-4)}^2 \leq c\rho^{-4}(1 + |\ln \rho|)^2 \|Av\|_{(4)}^2,$$

and thus

$$\|\nabla v\|_2 \leq c\rho^{-1}(1 + |\ln \rho|) \|Av\|_{(4)}.$$

Together with (39) this completes the proof.

Finally, we state a simple boundary estimate, which can be proven by locally reduction to one dimensional integrations.

LEMMA A4. *Let  $v \in W^{2,p}(\Omega)$ ,  $1 \leq p < \infty$ . Then (see (15))*

$$\int_{\Omega - \Omega_h} |v|^p dx \leq c_p d(\partial\Omega, \partial\Omega_h) \int_{\Omega} \{ |v|^p + |\nabla v|^p \} dx.$$

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