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Differential and Geometric Structure for the Tangent Bundle of a Projective Limit Manifold.

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ABSTRACT - The tangent bundle of a wide class of Fréchet manifolds is studied here. A vector bundle structure is obtained with structural group a topological subgroup of the general linear group of the fiber type. Moreover, basic geometric results, known form the classical case of finite dimensional manifolds, are recovered here: Connections can be defined and are characterized by a generalized type of Christoffel symbols while, at the same time, parallel displacements of curves are possible despite the problems concerning differential equations in Fréchet spaces.

Introduction.

The study of infinite dimensional manifolds and bundles has been seriously developed the last decades with a number of applications which surpass the borders of Differential Geometry. However, several questions remain open, especially for the case of infinite dimensional manifolds whose model are not of Banach type.

To be more precise, two problems seem to be of fundamental importance: The lack of a general solvability theory of differential equations in non-Banach topological vector spaces and the pathological structure of general linear groups in this framework. Both, have serious impacts even at the first steps of the study of the corresponding manifolds.

Concerning, for example, the tangent bundle TM of a smooth manifold M modeled on a topological vector space $\mathbb{F}$ of the aforementioned

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type, we cannot even be sure for the existence of a vector bundle structure on it. Indeed, the classical folklore cannot be patterned here since the general linear group $GL(\mathbb{F})$, that serves as the structural group in the finite dimensional case, does not admit a reasonable Lie-group structure. On the other hand, even if this obstacle is overcame in some way, the study of many basic geometric entities of M (e.g. connections, parallel translations, holonomy groups, etc.) seems also to be under question due to problems with the solvability of the involved differential equations.

Despite the abovementioned difficulties, the study of certain types of infinite dimensional manifolds modeled on non-Banach topological vector spaces is in several cases inevitable because of the extended use of such type of manifolds in modern Differential Geometry and Mathematical Physics. This need led a number of authors (e.g. [8], [10], [11]) to study certain types of infinite dimensional manifolds using, in most of the cases, rather algebraic approaches.

In this paper we study a certain sub-category of infinite dimensional non-Banach manifolds: Those that are modeled on Fréchet (i.e. locally convex, metrizable, Hausdorff and complete) spaces. Taking advantage of the fact that any Fréchet space can be realized as a projective limit of an appropriate sequence of Banach spaces, we focus further our attention on those Fréchet-modeled manifolds M that are, correspondingly, projective limits of Banach manifolds. In a previous work M.C. Abbati and A. Manià ([6]) worked on projective limits of manifolds by using a pure algebraic approach. No real differential structure is determined on them and several fundamental notions are treated only through the properties of projective limits. So, the tangent bundle of such a manifold $M = \varprojlim M^i$ was determined to be the projective limit of its counterparts on the factors: $TM := \varprojlim TM^i$ without any other assumptions referring or providing any differential or vector bundle structure. As a result, the obtained space is only a topological one since even the local triviality of it cannot be ensured. Analogous difficulties one faces also in several other subjects (differentiability of mappings, additional structures) thus the corresponding geometric properties are set also under question.

Throughout our note, based on a new approach, we overcome the abovementioned problems proving not only that any projective limit manifold $M = \varprojlim M^i$ can be endowed with a differential structure in the classical way but also that a vector bundle structure can be defined on its tangent space TM having a new structural group that replaces the pathological $GL(\mathbb{F})$. On the other hand, the geometric study of this bundle goes surprisingly far: Connections can be defined and are characterized by a generalized type of Christoffel symbols, parallel translation along curves of M is succeeded and a study of the corresponding holonomy groups can be attempted.

1. Preliminaries.

In this first Section we introduce the basic notions and all the preliminary material needed for a complete and self-contained presentation of the paper.

As already discussed in the Introduction, we are going to approach and study infinite dimensional manifolds modeled on Fréchet spaces based on the algebraic, but also compatible in several cases with differential tools, notion of projective limits. Our motivation was, at least partly, the fact that every Fréchet space $\mathbb{F}$ can be always realized as a projective limit of a sequence of Banach spaces $\{\mathbb{E}^i; \varrho^{ji}\}_{i,j\in\mathbb{N}}$: $\mathbb{F} \cong \varprojlim \mathbb{E}^i$ (see [9]). Using this interpretation we have already studied some fundamental problems on $\mathbb{F}$, e.g. solution of linear differential equations ([1]), Floquet-type theorems in corresponding manifolds ([2]), etc.

On the other hand, this approach gives us the opportunity to partially overcome one of the main drawbacks in the study of Fréchet spaces with several reflections in Differential Geometry. Namely, it is well known that the general linear group $GL(\mathbb{F})$ of $\mathbb{F}$, which keeps a fundamental role in a number of issues of Analysis and Differential Geometry, is very difficult to be handled within the framework of Fréchet spaces and manifolds since it cannot be endowed not only with a smooth Lie group structure but even with a reasonable topological group structure. Here we are going to replace $GL(\mathbb{F})$ with a topological and, in a generalized sense smooth Lie, group defined as follows:

$$H_0(\mathbb{F}) := \left\{ (f^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{E}^i) : \varprojlim f^i \text{ exists} \right\}. \tag{1.1}$$

This is a topological group, since it coincides with the projective limit of the Banach Lie groups

$$H_0^i(\mathbb{F}) := \left\{ (f_1, \dots, f_i) \in \prod_{k=1}^{i} GL(\mathbb{E}^k) : \varrho^{jk} \circ f^j = f^k \circ \varrho^{jk} \ (i \geq j \geq k) \right\}$$

via the identification

$$(f^i)_{i \in \mathbb{N}} \cong \big((f^1), (f^1, f^2), (f^1, f^2, f^3), \dots\big).$$

$H_0(\mathbb{F})$ may also considered as a *«generalized» Lie group* if we thought of it as embedded in the Fréchet space

$$H(\mathbb{F}) := \left\{ (f^i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} \mathcal{L}(\mathbb{E}^i) : \varprojlim f^i \text{ exists} \right\}.$$

Similarly, the natural continuous mapping

$$\varepsilon : H(\mathbb{F}) \mapsto \mathcal{L}(\mathbb{F}) : (f^i)_{i \in \mathbb{N}} \mapsto \varprojlim f^i, \tag{1.2}$$

can be also thought of as a generalized smooth mapping if we restrict it onto $H_0(\mathbb{F})$.

2. A certain type of Fréchet manifolds and the corresponding tangent bundles.

Based on the thoughts presented in the previous Section, we study here a certain type of Fréchet manifolds: Those which can be obtained as projective limits of Banach corresponding manifolds.

Definition 2.1. *Let $\{M^i; \phi^{ji}\}_{i,j \in \mathbb{N}}$ be a projective system of smooth manifolds modeled on the Banach spaces $\{\mathbb{E}^i\}_{i \in \mathbb{N}}$ respectively. We assume that:*

(1) *The models $\{\mathbb{E}^i\}_{i \in \mathbb{N}}$ form a projective system with connecting morphisms $\{\varrho^{ji} : \mathbb{E}^j \to \mathbb{E}^i;\ j \geq i\}$ and limit the Fréchet space $\mathbb{F} = \varprojlim \mathbb{E}^i$.*

(2) *For any element $x = (x^i)_{i \in \mathbb{N}} \in M = \varprojlim M^i$ there exists a family of charts $\{(U^i, \psi^i)\}_{i \in \mathbb{N}}$ of M^i's such that the limits $\varprojlim U^i$, $\varprojlim \psi^i$ can be defined and the sets $\varprojlim U^i$, $\varprojlim \psi^i(\varprojlim U^i)$* are open in M, $\mathbb{F}$ respectively.

Then the limit $M = \varprojlim M^i$ is called a PLB-manifold.

Under the assumptions of the previous Definition, we easily check that a PLB-manifold M turns to be a Fréchet manifold modeled on $\mathbb{F}$. The corresponding local structure is fully determined by the charts $(\varprojlim U^i, \varprojlim \psi^i)$. The differentiability of mappings involved can be either this of J. A. Leslie ([5]) or that of A.Kriegl-P.Michor ([7]).

Characteristic examples of such type of manifolds are the PLBL-groups presented in [3] as well as the space of infinite jets of a Banach modeled manifold.

Using now the notion of projective limits and a new technique, we are going to study the tangent bundle of a smooth Fréchet manifold of the above type. This study in the general case seems to encounter serious obstacles from the very beginning. Indeed, although the definition of a smooth manifold structure on the tangent bundle can be finally achieved, any try to endow it with a vector bundle structure following the classical procedure is doomed to failure due to the pathological structure of general linear groups of Fréchet spaces. However, focusing on a PLB-manifold $M = \varprojlim M^i$ we obtain the next, necessary for the sequel, result.

PROPOSITION 2.2. *The tangent bundles* $\{TM^i\}_{i \in \mathbb{N}}$ *form a projective system with limit set-theoretically isomorphic to* TM: $TM \simeq \simeq \varprojlim TM^i$.

PROOF. For any indexes (i, j), with $j \geqslant i$, we define the mapping:

$$q^{ji}: TM^j \to TM^i : [\alpha, x]^j \mapsto [\phi^{ji} \circ \alpha, \phi^{ji}(x)]^i,$$

where the brackets $[\cdot, \cdot]^j$, $[\cdot, \cdot]^i$ stands for the equivalence classes of TM^j, TM^i with respect to the classical equivalence relations

$$\alpha \sim_x \beta \Leftrightarrow \alpha(0) = \beta(0) = x \text{ and } \alpha'(0) = \beta'(0). \tag{2}$$

Here by α' we denote the first derivative of α:

$$\alpha': \mathbb{R} \to TM^j: t \mapsto [d\alpha(t)](1).$$

We may check that each q^{ji} is well-defined since two equivalent curves α, β on M^j will give

$$(\phi^{ji} \circ \alpha)(0) = (\phi^{ji} \circ \beta)(0),$$

$$(\phi^{ji} \circ \alpha)'(0) = d\phi^{ji}(\alpha(0))(\alpha'(0)) = d\phi^{ji}(\beta(0))(\beta'(0)) = (\phi^{ji} \circ \beta)'(0),$$

where by $d\phi^{ji}: TM^j \to TM^i$ we denote the first differential of ϕ^{ji}.

Moreover, $q^{ik} \circ q^{ji} = q^{jk}$ $(j \geqslant i \geqslant k)$ holds, as a consequence of the corresponding relations for $\{\varphi^{ji}\}_{i, j \in \mathbb{N}}$. Therefore, $\{TM^i; q^{ji}\}_{i, j \in \mathbb{N}}$ is a projective system and $\varprojlim TM^i$ can be defined.

Let now $\phi^i: M \to M^i$ be the canonical projections of the limit $M =$

$= \varprojlim M^i$. Then, we set

(3.12) $\quad Q^i\colon TM \to TM^i\colon [\alpha, x] \to [\phi^i \circ \alpha, \phi^i(x)]^i \quad (i \in \mathbb{N})$

and we readily verify that $q^{ji} \circ Q^j = Q^i$ holds for any $j \geqslant i$. As a result

$$Q = \varprojlim Q^i\colon TM \to \varprojlim TM^i\colon [\alpha, x] \to ([\phi^i \circ \alpha, \phi^i(x)]^i)_{i\in\mathbb{N}}$$

can be defined. This is a one to one mapping because $Q([\alpha, x]) = Q([\beta, x])$ gives

$$d\phi^i(\alpha(0))(\alpha'(0)) = (\phi^i \circ \alpha)'(0) = (\phi^i \circ \beta)'(0) = d\phi^i(\beta(0))(\beta'(0)),$$

and, therefore, $\alpha'(0) = \beta'(0)$ since $\alpha = \varprojlim (\phi^i \circ \alpha)$ and $\beta = \varprojlim (\phi^i \circ \beta)$.

Moreover, Q is surjective since if $a = ([\alpha^i, x^i]^i)_{i\in\mathbb{N}}$ is an arbitrarily chosen element of $\varprojlim TM^i$ we obtain:

(3) $\quad [\phi^{ji} \circ \alpha^j, \phi^{ji}(x^j)]^i = [\alpha^i, x^i]^i, \quad$ for $j \geqslant i$.

As a result, $x = (x^i) \in M = \varprojlim M^i$ and if $\{(U^i, \psi^i)\}_{i\in\mathbb{N}}$ is a system of charts on M through x, as in Definition 2.1 and $((\pi^i)^{-1}(U^i), \varprojlim T\psi^i)$ the corresponding charts of $\{TM^i\}_{i\in\mathbb{N}}$ ($\pi^i\colon TM^i \to M^i$ denoting the classical projections of TM^i), we obtain:

$$((\psi^i \circ \phi^{ji} \circ \alpha^j)(0), T\psi^i((\phi^{ji} \circ \alpha^j)'(0))) = ((\psi^i \circ \alpha^i)(0), T\psi^i((\alpha^i)'(0))) \Rightarrow$$
$$(\varrho^{ji}((\psi^j \circ \alpha^j)(0)), T\psi^i(T\phi^{ji}((\alpha^j)'(0)))) = ((\psi^i \circ \alpha^i)(0), T\psi^i((\alpha^i)'(0))) \Rightarrow$$
$$(\varrho^{ji}((\psi^j \circ \alpha^j)(0)), \varrho^{ji}((\psi^j \circ \alpha^j)'(0)) = ((\psi^i \circ \alpha^i)(0), (\psi^i \circ \alpha^i)'(0)).$$

As a result, $u = ((\psi^i \circ \alpha^i)(0))_{i\in\mathbb{N}}$, $v = ((\psi^i \circ \alpha^i)'(0))_{i\in\mathbb{N}}$ are elements of $\mathbb{F} = \varprojlim \mathbb{E}^i$. If we consider the curve h of $\mathbb{F}$ with $h(t) = u + t\cdot v$, as well as the corresponding one a of M with respect to the chart ($U = \varprojlim U^i$, $\psi = \varprojlim \psi^i$), we may check that

$$(\phi^i \circ a)(0) = \phi^i(x) = x^i = a^i(0),$$
$$\begin{aligned}(\phi^i \circ a)'(0) &= ((\psi^i)^{-1} \circ \varrho^i \circ h)'(0) = (T\psi^i)^{-1}((\varrho^i \circ h)'(0)) = \\ &= (T\psi^i)^{-1}(\varrho^i(v)) = (T\psi^i)^{-1}((\psi^i \circ \alpha^i)'(0)) = \\ &= (\alpha^i)'(0),\end{aligned}$$

for any i, j with $j \geqslant i$. Therefore, the curves $\phi^i \circ \alpha$, α^i are equivalent on M^i $(i \in \mathbb{N})$ and $Q([\alpha, x]) = ([a^i, x^i]^i)_{i\in\mathbb{N}} = a$.

We have proved in this way that TM, $\varprojlim TM^i$ are isomorphic sets with respect to the mapping Q. ■

Based now on the previous result, we may proceed with the definition of a vector bundle structure on the tangent bundle TM of M.

THEOREM 2.3. *TM admits a Fréchet type vector bundle structure over M with structural group $H_0(\mathbb{F})$.*

PROOF Let $\{(U_n = \varprojlim U_n^i,\ \psi_n = \varprojlim \psi_n^i)\}_{n \in I}$ be an atlas of M consisting of charts that can be obtained as projective limits in the form explained in Definition 2.1. Then, the atlas $\{(U_n^i,\ \psi_n^i)\}_{n \in I}$ of M^i can be used to define a vector bundle structure of fiber type $\mathbb{E}^i$ on TM^i over M^i in the classical way with trivializations:

$$\tau_n^i : (\pi^i)^{-1}(U_n^i) \to U_n^i \times \mathbb{E}^i : [\alpha,\, x]^i \mapsto (x, (\psi_n^i \circ \alpha)'(0)); \quad n \in I.$$

Taking into account that the families $\{q^{ji}\}_{i,j \in \mathbb{N}}$, $\{\varphi^{ji}\}_{i,j \in \mathbb{N}}$, $\{\varrho^{ji}\}_{i,j \in \mathbb{N}}$ are connecting morphisms of the projective systems $TM = \varprojlim TM^i$, $M = \varprojlim M^i$, $\mathbb{F} = \varprojlim \mathbb{E}^i$ respectively, we check that the projections $\{\pi^i\}_{i \in \mathbb{N}}$ satisfy

$$\varphi^{ji} \circ \pi^j = \pi^i \circ q^{ji} \qquad (j \geqslant i)$$

and the trivializations $\{\tau_n^i\}_{i \in \mathbb{N}}$

$$(\varphi^{ji} \times \varrho^{ji}) \circ \tau_n^j = \tau_n^i \circ q^{ji} \qquad (j \geqslant i).$$

Therefore, $\pi = \varprojlim \pi^i$: $TM \to M$ exists and is a surjective mapping,

$$\tau_n = \varprojlim \tau_n^i : \pi^{-1}(U_n) \to U_n \times \mathbb{F} \qquad (n \in I)$$

are smooth, as projective limits of smooth mappings (see also [3] for a detailed proof), and $pr_1 \circ \tau_n = \pi$, if pr_1 denotes the projection to the first factor.

Moreover, the restrictions of τ_n to the fibers $\pi^{-1}(x)$ are linear isomorphisms since $\tau_{n,x} := pr_2 \circ \tau_n \,|_{\pi^{-1}(x)} = \varprojlim (pr_2 \circ \tau_n^i \,|_{(\pi^i)^{-1}(x)})$.

Concerning, finally, the transition functions $\{T_{nm} = \tau_{n,x} \circ \circ \tau_{m,x}^{-1}\}_{n,m \in I}$, we see that they can be considered as taking values in the group $H_0(\mathbb{F})$, since $T_{nm} = \varepsilon \circ T_{nm}^*$, where $\{T_{nm}^*\}_{n,m \in I}$ are the smooth mappings

$$T_{nm}^* : U_n \cap U_m \to H_0(\mathbb{F}) : x \mapsto (pr_2 \circ \tau_n^i \,|_{(\pi^i)^{-1}(x)} \circ (\tau_m^i)^{-1} \,|_{(\pi^i)^{-1}(x)})_{i \in \mathbb{N}}$$

and ε the natural inclusion $\varepsilon : H^0(\mathbb{F}) \to \mathcal{L}(\mathbb{F}) : (f^i)_{i \in \mathbb{N}} \mapsto \varprojlim f^i$.

As a result, TM admits, indeed, a vector bundle structure over M with fibers of type $\mathbb{F}$ and structural group $H_0(\mathbb{F})$. ■

3. Projective limits of connections.

Here we are going to study some basic geometric properties of the infinite dimensional manifolds presented in Section 2. To be more precise, we will focus on the characterization of connections via Christoffel symbols as well as on the possibility of having parallel translation of curves with respect to a connection. Both issues are under question in the general case of Fréchet manifolds due to intrinsic difficulties of their models: The lack of a general solvability theory for ordinary differential equations dooms to failure any try to apply the classical theory in order to obtain parallel translations. On the other hand, the fact that the space of linear mappings between Fréchet spaces does not remain in the same category, sets serious obstacles in the relationship between connections and Christoffel symbols.

However, if we restrict our study to those Fréchet manifolds that can be obtained as projective limits of Banach manifolds, in the sense described in the previous Sections, we may recover a great number of fundamental results of the theory of finite dimensional manifolds and connections.

To this end we consider $M = \varprojlim M^i$ a PLB-manifold and a corresponding atlas $\{(U_n = \varprojlim U_n^i,\ \psi_n = \varprojlim \psi_n^i)\}_{n \in I}$ as in Definition 2.1. We denote by $\{(\pi^{-1}(U_n),\ \Phi_n)\}_{n \in I}$ the corresponding local structure of the tangent bundle TM with:

$$\Phi_n \colon \pi^{-1}(U_n) \to \psi_n(U_n) \times \mathbb{F} : [\alpha, x] \mapsto (\psi_n(x), (\psi_n \circ \alpha)'(0)); \quad n \in I,$$

and by $\{(\pi_{TM}^{-1}(\pi^{-1}(U_n)),\ \Psi_n)\}_{n \in I}$ the obtained atlas on the tangent bundle $T(TM)$ of TM. Clearly, all these charts can be realized as projective limits of their counterparts on the factors TM^i, $T(TM^i)$: $\Phi_n = \varprojlim \Phi_n^i$, $\Psi_n = \varprojlim \Psi_n^i$.

If now

$$D^i \colon T(TM^i) \to TM^i$$

is a connection on the manifold M^i, in the sense of J. Vilms ([12]), for any $i \in \mathbb{N}$, and we assume that they form a projective system:

$$T\varphi^{ji} \circ D^j = D^i \circ T(T\varphi^{ji}), \quad j \geqslant i, \tag{3.3}$$

then the corresponding limit $D = \varprojlim D^i \colon T(TM) \to TM$, which can be naturally defined as a smooth mapping according to the results of the previous Section, seems to be far from being called a connection of M due to the difficulties emerged in the study of the corresponding local forms:

$$\omega_n\colon \psi_n(U_n)\times\mathbb{F}\to\mathcal{L}(\mathbb{F},\mathbb{F})$$

defined by relations $D_n(y,u,v,w)=(y,w+\omega_n(y,u)(v))$, where D_n stands for the restriction of D onto the local charts of $T(TM)$ and TM: $D_n=\Phi_n\circ D\circ\Psi_n^{-1}$ $(n\in I)$. Indeed, since the space of linear continuous mappings of any Fréchet space does not remain into the same category of topological vector spaces, it is not possible to realize the forms $\{\omega_n\}$ as projective limits of their counterparts obtained by the connections D^i and, therefore, as smooth maps. However, making again use of the space $H(\mathbb{F}):=\{(f^i)_{i\in\mathbb{N}}\in\prod_{i=1}^{\infty}\mathcal{L}(\mathbb{E}^i)\colon \varprojlim f^i \text{ exists}\}$, defined in Section 1, we may overcome these difficulties obtained the following main result.

THEOREM 3.1. *The projective limit $D=\varprojlim D^i$ of a system of connections on the PLB-manifold M is a connection characterized by a generalized type of Christoffel symbols with values into the Fréchet space $H(\mathbb{F})$.*

PROOF We observe first that relations (3.3) and the fact that the charts $\{(\pi^{-1}(U_n),\Phi_n)\}_{n\in I}$, $\{(\pi_{TM}^{-1}(\pi^{-1}(U_n)),\Psi_n)\}_{n\in I}$ have been chosen to be projective limits of corresponding charts on TM^i and $T(TM^i)$ respectively, ensure that the mappings $\{D_n\}_{n\in I}$ can be also realized as projective limits. Indeed, if $\{D_n^i\}_{n\in I}$ are the corresponding restrictions of the connections D^i on the charts of M^i $(i\in\mathbb{N})$ and $\varrho^{ji}\colon\mathbb{E}^j\to\mathbb{E}^i$ $(j\geqslant i)$ the connecting morphisms of the model $\mathbb{F}=\varprojlim\mathbb{E}^i$, we check that:

$$(\varrho^{ji}\times\varrho^{ji})\circ D_n^j=(\varrho^{ji}\times\varrho^{ji})\circ(\Phi_n^j\circ D^j\circ(\Psi_n^j)^{-1})=$$

$$=\Phi_n^i\circ T\phi^{ji}\circ D^j\circ(\Psi_n^j)^{-1}=\Phi_n^i\circ D^i\circ T(T\phi^{ji})\circ(\Psi_n^j)^{-1}=$$

$$=\Phi_n^i\circ D^i\circ(\Psi_n^i)^{-1}\circ(\varrho^{ji}\times\varrho^{ji}\times\varrho^{ji}\times\varrho^{ji})=D_n^i\circ(\varrho^{ji}\times\varrho^{ji}\times\varrho^{ji}\times\varrho^{ji}).$$

Similarly, we prove that $(\varrho^i\times\varrho^i)\circ D_n=D_n^i\circ(\varrho^i\times\varrho^i\times\varrho^i\times\varrho^i)$, for any $i\in\mathbb{N}$, where $\varrho^i\colon\mathbb{F}\to\mathbb{E}^i$ are the canonical projections of $\mathbb{F}$. As a result, $D_n=\varprojlim D_n^i$ for each $n\in I$.

Concerning now the local forms $\omega_n\colon\psi_n(U_n)\times\mathbb{F}\to\mathcal{L}(\mathbb{F},\mathbb{F})$ of D as well as the corresponding ones $\omega_n^i\colon\psi_n^i(U_n^i)\times\mathbb{E}^i\to\mathcal{L}(\mathbb{E}^i,\mathbb{E}^i)$ of D^i's $(i\in\in\mathbb{N})$, we cannot demand of them to be related by a similar manner, since the projective limit of the spaces $\{\mathcal{L}(\mathbb{E}^i, \mathbb{E}^i)\}_{i\in\mathbb{N}}$ can not be defined. However, we may check that for any $y=(y^i)\in\psi_n(U_n)$ and $u=(u^i)\in\mathbb{F}$, the family $(\omega_n^i(x^i, y^i))_{i\in\mathbb{N}}$ belongs to the Fréchet space $H(\mathbb{F})$. Indeed, relations $(\varrho^{ji}\times\varrho^{ji})\circ D_n^j = D_n^i\circ(\varrho^{ji}\times\varrho^{ji}\times\varrho^{ji}\times\varrho^{ji})$, $j\geqslant i$, obtained above, give rise to

$$(\varrho^{ji}\times\varrho^{ji})(D_n^j(y^j, u^j, v^j, w^j)) = D_n^i(\varrho^{ji}(y^j), \varrho^{ji}(u^j), \varrho^{ji}(v^j), \varrho^{ji}(w^j)) \Leftrightarrow$$

$$(\varrho^{ji}\times\varrho^{ji})(y^j, w^j+\omega_n^j(y^j, u^j)(v^j)) = (y^i, w^i+\omega_n^i(y^i, u^i)(v^i)) \Leftrightarrow$$

$$\varrho^{ji}\circ\omega_n^j(y^j, u^j) = \omega_n^i(y^i, u^i)\circ\varrho^{ji}.$$

As a result, we can define the mapping

$$\omega_n^*\colon \psi_n(U_n)\times\mathbb{F}\to H(\mathbb{F}):((y^i),(u^i))\mapsto(\omega_n^i(y^i, u^i))_{i\in\mathbb{N}}$$

which is smooth, as the projective limit of the smooth (Banach) local forms

$$\widetilde{\omega}_n^i\colon \psi_n^i(U_n)\times\mathbb{E}^i\to H^i(\mathbb{F}):(y, u)\mapsto(\omega_n^k(\varrho^{ik}(y), \varrho^{ik}(u)))_{k=1,2,\dots,i}.$$

These generalized local forms of D are connected to the classical ones via the relation

$$\omega_n = \varepsilon\circ\omega_n^*,$$

where ε stands for the natural inclusion $\varepsilon: H(\mathbb{F})\mapsto\mathcal{L}(\mathbb{F}):(f^i)_{i\in N}\mapsto \varprojlim f^i$, defined in Section 1. This exactly is the fact that allows us to conclude that the local forms $\{\omega_n\}_{n\in I}$ are smooth and that D is a connection of M under J.Vilms' point of view ([12]).

Concerning finally the Christoffel symbols $\{\Gamma_n\colon \psi_n(U_n)\to \to\mathcal{L}(\mathbb{F}, \mathcal{L}(\mathbb{F}, \mathbb{F}))\}_{n\in I}$ that characterize the above connection we check that

$$\Gamma_n(y)[u] = \omega_n(y, u) = (\varepsilon\circ\omega_n^*)(y, u) = \varprojlim(\omega_n^i(y^i, u^i)) =$$

$$= \varprojlim(\Gamma_n^i(y^i)[u^i]),$$

for any $y=(y^i)\in\psi_n(U_n)$ and $u=(u^i)\in\mathbb{F}$. Therefore, these Christoffel symbols may be considered as taking values into the Fréchet space $H(\mathbb{F})$ with respect to the inclusion ε. ■

Remark 3.2. *It is worth noticing here that in the case where the initial connections $\{D^i\}$ are linear the same holds for the correspond-*

ing limit connection $D = \varprojlim D^i$. Indeed, the linearity of D^i's is equivalent to the linearity of the second variable of the local forms $\{\omega^i_n : \psi^i_n(U^i_n) \times \mathbb{E}^i \to \mathcal{L}(\mathbb{E}^i, \mathbb{E}^i)\}$ which, in turns, gives rise to the corresponding property for the generalized local forms $\{\tilde{\omega}^i_n : \psi^i_n(U_n) \times \mathbb{E}^i \to \to H^i(\mathbb{F})\}$. Taking into account that the notion of projective limits is compatible with linear structures, we conclude that the same holds for $\{\omega^*_n : \psi_n(U_n) \times \mathbb{F} \to H(\mathbb{F})\}$ and, thus, connection D is also linear.

Taking now one step further, we are going to study the possibility of having parallel translations of curves of the PLB manifold $M = \varprojlim M^i$ with respect to a linear connection $D = \varprojlim D^i$ of the above type. In the general case of arbitrarily chosen Fréchet manifolds and connections, this basic geometric property fails to be recovered. Indeed, such a procedure demands the existence of horizontal lifts with respect to the chosen connection, thus the solution of linear differential equations on the fiber type. However, this is not possible on Fréchet spaces where even trivial linear differential equations may not be solved or may have more than one solution through the same initial condition (see for more details and corresponding counterexamples [4]).

By restricting our attention to the case of Fréchet manifolds and connections obtained as projective limits, we may overcome the previous difficulties avoiding to interfere at all with differential equations. Namely:

Proposition 3.3. *Let $M = \varprojlim M^i$ be a PLB manifold, as defined in Section 2, and D a linear connection on M that can be realized as a projective limit of connections: $D = \varprojlim D^i$. Then, for any smooth curve $\alpha : [0, 1] \to M$ and for any element $u \in TM$, there exists a unique horizontal lift $\xi_{\alpha, u}$ of α on TM satisfying the initial condition $\xi_{\alpha, u}(0) = u$.*

Proof. We observe first that α can be thought of as the projective limit of the curves $\alpha^i := \phi^i \circ \alpha : [0, 1] \to M^i$ ($i \in \mathbb{N}$), where $\phi^i : M \to M^i$ are the canonical projections of the limit $M = \varprojlim M^i$. On the other hand, the element u of $TM = \varprojlim TM^i$ will have the form: $u = (u^i)$, with $u^i \in \in M^i$ and $T\phi^{ji}(u^j) = u^i, j \geq i$. Taking into account that each manifold M^i is of Banach type, where the aforementioned problems concerning the solvability of linear differential equations are not longer present, we obtain, using the classical folklore, a horizontal lift $\xi^i_{\alpha^i, u^i} : [0, 1] \to TM^i$ through

$(0, u^i)$, for any $i \in \mathbb{N}$. Moreover, and for any indexes $j \geqslant i$, we check that:

$$\pi^i \circ (T\phi^{ji} \circ \xi^j_{\alpha^j, u^j}) = \phi^{ji} \circ \pi^j \circ \xi^j_{\alpha^j, u^j} = \phi^{ji} \circ \alpha^j = \alpha^i,$$

$$D^i \circ T(T\phi^{ji} \circ \xi^j_{\alpha^j, u^j}) \circ \partial = T\phi^{ji} \circ D^j \circ T\xi^j_{\alpha^j, u^j} \circ \partial = T\phi^{ji} \circ 0 = 0,$$

where ∂ stands for the basic vector field of $\mathbb{R}$. As a result, the vector field $T\phi^{ji} \circ \xi^j_{\alpha^j, u^j}$ of M^i is also a horizontal lift of the smooth curve α^i satisfying the initial condition $(T\phi^{ji} \circ \xi^j_{\alpha^j, u^j})(0) = T\phi^{ji}(u^j) = u^i$. Thus, having in mind the uniqueness of horizontal lifts on Banach manifolds, we conclude that

$$T\phi^{ji} \circ \xi^j_{\alpha^j, u^j} = \xi^i_{\alpha^i, u^i}, \qquad j \geqslant i.$$

As a result, $\xi_{\alpha, u} := \varprojlim \xi^i_{\alpha^i, u^i} : [0, 1] \to TM$ can be defined. This the desired horizontal lift of the smooth curve α on TM since

$$\pi \circ \xi_{\alpha, u} = \varprojlim (\pi^i \circ \xi^i_{\alpha^i, u^i}) = \varprojlim \alpha^i = \alpha,$$

$$D \circ T\xi_{\alpha, u} \circ \partial = \varprojlim (D^i \circ T\xi^i_{\alpha^i, u^i} \circ \partial) = \varprojlim 0 = 0,$$

$$\xi_{\alpha, u}(0) = (\xi^i_{\alpha^i, u^i}(0)) = (u^i) = u.$$

The uniqueness of $\xi_{\alpha, u}$ can be proved following analogous thoughts and observing that if ξ^* is a second horizontal lift of α on TM satisfying the same initial condition $(0, (u^i))$, then the factors $T\phi^i \circ \xi_{\alpha, u}$, $T\phi^i \circ \xi^*$ would coincide as horizontal lifts of $\phi^i \circ \alpha$ on TM^i through the same initial condition $(0, u^i)$. ■

A direct consequence of the previous Proposition is the following main result.

THEOREM 3.4. *If $M = \varprojlim M^i$ is a PLB manifold and $D = \varprojlim D^i$ a linear connection on it, then, for any smooth curve $\alpha : [0, 1] \to M$ a parallel translation*

$$\tau_\alpha : \pi^{-1}(\alpha(0)) \to \pi^{-1}(\alpha(1)) : u \mapsto \xi_{\alpha, u}(1)$$

can be defined. ■

REMARK 3.5. *It is also remarkable here that the corresponding holonomy group $\Phi_x = \{\tau_\alpha,\ \alpha \in C^\infty([0, 1], M)$ and $\alpha(0) = \alpha(1) = x\}$ is* *now a subgroup of the topological group* $H_0(\mathbb{F})$ *and not of the (pathological in our framework) general linear group* $GL(\mathbb{F})$ *as in the classical case. Indeed, as we readily verify using the thoughts presented in the proof of Proposition* 3.3, $\tau_\alpha = \varprojlim \tau^i_{\alpha^i}$ holds for any smooth curve α of M if by $\tau^i_{\alpha^i}$ we denote the parallel translation of $\alpha^i = \phi^i \circ \alpha$ on the factor manifolds M^i.

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