REND. SEM. MAT. UNIV. PADOVA, Vol. 111 (2004)

AE-rings.

MANFRED DUGAS (*) - SHALOM FEIGELSTOCK (**)

ABSTRACT - E-rings are a well known notion in the theory of abelian groups. They are those rings R such that $\mathrm{End}(R^+)$, the ring of endomorphisms of the additive group of R, is as small as possible, i.e. $\mathrm{End}(R^+) = R_\ell$, where $R_\ell = \{x \mapsto ax : a \in R\}$. We generalize the notion of E-rings by calling a ring R an almost-E-ring, or AE-ring for short, if $\mathrm{End}(R^+)$ is a radical extension of R_ℓ, i.e. for each $\varphi \in \mathrm{End}(R^+)$ there is some natural number n such that $\varphi^n \in R_\ell$. We will show that this notion does not lead to a new class of rings. It turns out that all AE-rings are actually E-rings. Our proof utilizes Herstein's Hypercenter Theorem.

0. Introduction.

The notion of an E-ring was introduced by P. Schultz [13] some 30 years ago and has attracted a lot of attention ever since. We refer to the survey article [14] for a guide to the literature and a discussion of generalizations of E-rings as proposed by several authors. A question raised in [1] was only recently answered in [6]: There exist generalized E-rings R, i.e. R is not an E-ring, but $R \approx \mathrm{End}(R^+)$ as rings. The existence of arbitrarily large E-rings, c.f. [4], is also of interest in category theory as discussed in [1], see also [11]. In [14], several new generalizations of the notion of E-rings were proposed. For example, a ring R is called a two-sid-

(*) Indirizzo dell'A.: Department of Mathematics, Baylor University, Waco, Texas 76798. E-mail: manfred_dugas@baylor.edu

(**) Indirizzo dell'A.: Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel. E-mail: feigel@macs.biu.ac.il

Research partially supported by Baylor University's Summer Sabbatical Program.

ed E-ring, if $\mathrm{End}(R^+)$ is generated as a ring by $R_\ell \cup R_r$, where $R_\ell = \{x \mapsto ax : a \in R\}$ is the set of all left multiplications by elements of R, and $R_r = \{x \mapsto xa : a \in R\}$ is the set of all right multiplications. In [5] torsion-free two-sided E-rings were studied and large two-sided E-rings that are not E-rings were constructed. Another generalization introduced in [3] is obtained by restricting attention to the automorphisms of R^+. A ring R is called an A-ring, if $\mathrm{Aut}(R^+)$, the group of automorphisms of R^+, is contained in R_ℓ, i.e. $\mathrm{Aut}(R^+) = U(R_\ell)$ the group of units of R_ℓ. Large A-rings R were constructed in [3] that are not E-rings, indeed $\mathrm{End}(R^+)$ is a commutative polynomial ring in a single variable over R_ℓ. Moreover, A-rings whose additive groups are torsion-free of finite rank (tffr) were investigated, but the obvious question about the existence of tffr A-rings that are not E-rings was left open. This question was answered in the negative in [2]: All tffr A-rings are indeed E-rings. The main tool in the proof was yet another generalization of E-rings, one that is preserved under quasi-isomorphism. A unital ring R is called an AA-ring if R is «almost an A-ring» in the sense that for each $\alpha \in \mathrm{Aut}(R^+)$ there is some natural number n such that $\alpha^n \in U(R_\ell)$.

In the present paper we define a ring R to be an AE-ring, if $\mathrm{End}(R^+)$ is radical over R_ℓ, i.e. for each $\varphi \in \mathrm{End}(R^+)$ there is some natural number n such that the n-th power of φ belongs to R_ℓ, i.e. $\varphi^n \in R_\ell$. While all generalizations of E-rings described so far have led to new classes of rings, it is a little surprising that this one does not. We will proof the following:

MAIN THEOREM. *All AE-rings are E-rings.*

We will use some classical commutativity results from ring theory. Jacobson [9, p. 218] calls a ring R a *K-ring* if R is radical over $Z(R)$, the center of R. Kaplansky [10] proved that each semi-simple K-ring is, indeed, commutative. In [9, p. 219] it is shown that each K-ring R has a nil ideal N such that R/N is commutative. Herstein [7] proved that the commutator ideal of any K-ring is a nil ideal. This result was extended by Lihtman [12] who showed that if the ring R is radical over a commutative subring A, then $\mathrm{Nil}(R) = \{a \in R : a \text{ nilpotent}\}$ is an ideal of R and $R/\mathrm{Nil}(R)$ is commutative. Given a ring R, Herstein [8] defines the *hypercenter* $T(R)$ of R to be the set $T(R) = \{a \in R : (\forall x \in R)(\exists n = n(a, x) \in \mathbb{N})(ax^n = x^n a)\}$, i.e. the hypercenter consists of all those elements of the ring that commute with some power of each element of R. Theorem 2 in [8] states that if R is a ring with no nil ideals, then $Z(R) = T(R)$, i.e. center and hypercenter coincide.

1. Proof of the Main Theorem.

1.1 PROPOSITION. *Let R be an AE-ring. Then $1 \in R$.*

PROOF. There is an element $e \in R$ such that $id_R = e_\ell$, i.e. $x = ex$ for all $x \in R$ and thus e is an idempotent element of R. Now there is some element $a \in R$ and an integer n such that $a_\ell = (e_r)^n = (e^n)_r = e_r$ and we have $xe = ax$ for all $x \in R$. This implies $e = ee = ae = a^2$ and we obtain $xe = (xe)\, e = (ax)\, e = a(xe) = a(ax) = a^2 x = ex = x$ for all $x \in R$. This implies $1 = e \in R$. ■

After having established that AE-rings have an identity we will consider AE-rings with decomposable additive groups.

1.2 LEMMA. *Let R be an AE-ring such that $R^+ = H \oplus K$. Then the following hold:*

(1) *$R = H \oplus K$ is a ring direct sum.*
(2) *H and K are both AE-rings.*
(3) $\mathrm{Hom}(H, K) = 0 = \mathrm{Hom}(K, H)$.

PROOF. Let π_H be the natural projection of R^+ onto H. Since π_H is an idempotent endomorphism, there is an element $e_H \in R$ such that $\pi_H = (e_H)_\ell$ and it follows that $\pi_H(1) = e_H \in H$ and thus $e_H = \pi_H(e_H) = (e_H)^2$ is idempotent. This shows that $(e_H)_r$ is idempotent and $(e_H)_r = a_\ell$ for some $a \in R$. We conclude $xe_H = ax$ for all $x \in R$ and $e_H = a$ is in the center of R. This shows that $H = e_H R$ is a subring of R. Define the element $e_K \in K$ in the same fashion for K. Then $K = e_K R$ and $0 = \pi_H(e_K) = e_H e_K$ and it follows that $HK = \{0\} = KH$. This shows (1).

Let $\varphi : H \to H$ be an endomorphism of H. Then $\varphi \circ \pi_H$ is an endomorphism of R with $(\varphi \circ \pi_H)^n = \varphi^n \circ \pi_H$ for all $n \in \mathbb{N}$. Now there is some $a \in R$ and $n \in \mathbb{N}$ such that $\varphi^n \circ \pi_H = a_\ell$ Since $a = a_\ell(1)$ we have $a \in H$ and $\varphi^n = (a_\ell)\restriction_H$. This shows that H is an AE-ring.

Let $\varphi \in \mathrm{Hom}(H, K)$ and define $\psi \in \mathrm{End}(R^+)$ by $\psi = \varphi \circ \pi_H + \pi_K$. Then $\psi^2 = (\varphi \circ \pi_H) \circ (\varphi \circ \pi_H) + (\varphi \circ \pi_H) \circ \pi_K + \pi_K \circ (\varphi \circ \pi_H) + (\pi_K)^2 = 0 + 0 + \varphi \circ \pi_H + \pi_K = \psi$ is idempotent and thus $\psi = a_\ell$ for some $a \in R$

and $a = \psi(1) \in K$. Since, for all $h \in H$ we have that $ah = \psi(h) = \varphi(h) \in \in KH = \{0\}$. Thus $\varphi = 0$. ■

1.3 COROLLARY. *Let R be an AE-ring and R_p the p-primary torsion part of R^+. Moreover, let $P = \{p : R_p \neq 0\}$ and $t(R) = \oplus_{p \in P} R_p$. Then $R_p = \mathbb{Z}(p^{e_p})$ for all $p \in P$, $e_p \geqslant 1$ and $R/t(R)$ is P-divisible, i.e. p-divisible for all $p \in P$.*

PROOF. R_p has to be reduced because of Lemma 1.2 (2). Thus R_p splits off cyclic summands and Lemma 1.2 (3) implies that R_p is cyclic and $R = R_p \oplus H^{(p)}$ with $H^{(p)}$ a p-divisible subgroup because of 1.2 (3). This implies $R/t(R)$ is P-divisible. ■

If R is torsion, then $R \approx \oplus_{p \in P} R_p$ with $1 \in R$. This shows that P is finite and we obtain:

1.4 COROLLARY. If R is a torsion AE-ring, then $R \approx \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$, i.e. R is an E-ring.

We will now prove that non-torsion AE-rings are E-rings.

Let R be a ring. Recall that $\mathrm{Nil}(R) = \{a \in R : a \text{ nilpotent}\}$ is the set of nilpotent elements of the ring R. We let $Z(R)$ denote the center of the ring R and if R is a subring of a ring E, we define $Z(R)_{*E} = \{r \in E : (\exists n \in \mathbb{N})(nr \in Z(R)\}$. Then $Z(R)_{*E}$ is a pure subring of E. Recall that $T(R) = \{x \in R : (\forall a \in R)(\exists n \in \mathbb{N})(xa^n = a^n x)\}$ is the hypercenter of R.

From now on, let R be an AE-ring and $P = \{p \in \mathbb{P} : R_p \neq \{0\}\}$, where $\mathbb{P}$ is the set of natural primes and R_p is the p-torsion part of R^+. Let $t(R) = \oplus_{p \in P} R_p$ be the torsion part of R. For shorter notation, let $E = \mathrm{End}(R^+)$. Note that R has a fully invariant ideal H, namely $H = = \cap_{p \in P}(\cap_{n \in \mathbb{N}} p^n R)$ and R/H is isomorphic to a subring of $\Pi_{p \in P} R_p$, the $\mathbb{Z}$-adic completion of $t(R)$. Note that R/H is an E-ring, c.f. [13]. Recall that there is an example in [1, Example 3.24] of a mixed E-ring R such that H is not a summand of R. We will proceed by steps.

(1.5) The ring E is radical over $Z(R)_\ell = Z(R_\ell)$. Moreover, R is radical over $Z(R)$.

Let $\varphi \in E$. Then $\varphi^n = a_\ell \in R_\ell$ for some natural number n since R is an AE-ring and $a_r \in E$. This implies that there is a natural number m such that $(a_r)^m = b_\ell \in R_\ell$. This implies $xa^m = bx$ for all $x \in R$, which implies $a^m = b \in Z(R)$. We infer $\varphi^{nm} \in Z(R_\ell)$.

(1.6) $\mathrm{Nil}(E) \subseteq Z(R_\ell)_{*E}$:

Let $\varphi \in \mathrm{Nil}(E)$. Then there is a least $k \geq 1$ such that $\varphi^j \in Z(R_\ell)_{*E}$ for all $j \geq k$, because φ is nilpotent. By (1.5), there is a natural number n such that $(id_R + \varphi^{k-1})^n = \sum_{j=0}^{n} \binom{n}{j} \varphi^{(k-1)j} \in Z(R_\ell) \subseteq Z(R_\ell)_{*E}$. Assuming $k \geq 2$, we have $(k-1)j \geq k$ for $j \geq 2$ and we infer $n\varphi^{k-1} \in Z(R_\ell)_{*E}$. Thus there is some natural number m such that $mn\varphi^{k-1} \in Z(R)_\ell$ and $\varphi^{k-1} \in Z(R_\ell)_{*E}$, a contradiction to the choice of k. Thus $k = 1$.

(1.7) Let $\varphi \in E$ such that $\varphi(R) \subseteq t(R)$. Then $\varphi \in (t(R))_\ell$.

We have $\varphi : R \to t(R)$ and H is P-divisible and $t(R)$ is P-reduced. Thus $\varphi(H) = \{0\}$ and φ induces an endomorphism $\varphi\restriction_{R/H} = (t+H)_\ell$ of the E-ring R/H where $t \in t(R)$. Note that $\varphi(x) + H = \varphi\restriction_{R/H}(x+H) = tx + H$ and therefore $\varphi(x) - tx \in H \cap t(R) = \{0\}$. We conclude $\varphi(x) = tx$ for all $x \in R$ and $\varphi = t_\ell \in (t(R))_\ell$.

(1.8) $Z(R_\ell)_{*E} \subseteq R_\ell$.

Let $\psi \in Z(R_\ell)_{*E}$. Then there is some natural number n such that $n\psi = r_\ell \in Z(R_\ell)$. For $s = \psi(1) \in Z(R)$ we have $r = ns$ and it follows $n(\psi - s_\ell) = 0$, i.e. $\psi - s_\ell : R \to t(R)$. By (1.7), we have $\psi - s_\ell \in (t(R))_\ell$ and it follows that $\psi \in R_\ell$.

(1.9) $t(R) \subseteq Z(R)$.

This follows from the fact that R/H is commutative and $H \cdot t(R) = \{0\} = t(R) \cdot H$.

(1.10) $\mathrm{Nil}(E)$ is an ideal of E. Moreover, $\mathrm{Nil}(E) = (\mathrm{Nil}(R))_\ell$.

Let $\varphi \in E$ and $\eta \in \mathrm{Nil}(E)$. By (1.6) we have $\eta \in Z(R_\ell)_{*E}$ and $\eta \in R_\ell$ by (1.8). There is some natural number m such that $(\varphi\eta)^m \in Z(R_\ell)$ since E is radical over $Z(R_\ell)$ by (1.5). We claim that for all $k \geq 1$ we have $(\varphi\eta)^{mk} = [(\varphi\eta)^{m-1}\varphi]^k \eta^k$:

If $k=1$, there is nothing to show. Consider $(\varphi\eta)^{m(k+1)} = (\varphi\eta)^{mk}(\varphi\eta)^m = ([(\varphi\eta)^{m-1}\varphi]^k\eta^k)(\varphi\eta)^m = [(\varphi\eta)^{m-1}\varphi]^k(\varphi\eta)^m\eta^k = [(\varphi\eta)^{m-1}\varphi]^k \cdot [(\varphi\eta)^{m-1}\varphi]\,\eta\eta^k = [(\varphi\eta)^{m-1}\varphi]^{k+1}\eta^{k+1}$. Here we used that $\eta \in R_\ell$ and $(\varphi\eta)^m \in Z(R_\ell)$. We infer $\varphi\eta \in \mathrm{Nil}(E)$ and by symmetry, also $\eta\varphi \in \mathrm{Nil}(E)$. This shows that $\mathrm{Nil}(E)$ is an ideal of E.

(1.11) $E/(\mathrm{Nil}(R))_\ell$ is commutative.

Since E is radical over $Z(R_\ell)$, we have that $R_\ell \subseteq T(E)$. Let $N = (\mathrm{Nil}(R))_\ell$. Then $R_\ell/N \subseteq (T(E)+N)/N \subseteq T(E/N)$ and 0 is the only nilpotent element of E/N. By [8, Theorem 2] we have $T(E/N) = Z(E/N) \supseteq R_\ell/N$. Since E is radical over R_ℓ, we have that E/N is radical over $Z(E/N)$. By [9, Theorem 2 on page 219] we have that E/N is commutative.

We can now finish our proof for mixed AE-rings: By Corollary 1.4 we may assume that R is not torsion. Let P be defined as above and assume $P \neq \infty$. Let $N = (\mathrm{Nil}(R))_\ell$. Then N is an ideal of $E = \mathrm{End}(R^+)$ by (1.10) and E/N is commuatative by (1.11.) Let $a \in R$, $\varphi \in E$. Then $\varphi a_r - a_r \varphi = b_\ell \in N$ and $b = \varphi(a) - \varphi(1)a \in \mathrm{Nil}(R)$. This shows that $\varphi - (\varphi(1))_\ell : R \to \mathrm{Nil}(R)$. Define $K = \{\varphi \in E : \varphi(R) \subseteq N\}$. We have just seen that $E = R_\ell + K$. Let $\eta \in K - N$. Then η is not nilpotent. On the other hand, there is a natural number m and $b \in R$ such that $\eta^m = b_\ell : R \to \mathrm{Nil}(R)$. This shows that $b \in \mathrm{Nil}(R)$ and b is nilpotent. Thus η^m is nilpotent and so is η, a contradiction. This implies $E = R_\ell$ and R is an E-ring.

If R is torsion-free, things simplify a little. Now we can work with $Z(R)$ in place of $Z(R)_{*E} (= Z(R))$ and $t(R) = \{0\} = H$. Again, we can run through the steps (1.5) - (1.11) and it follows that R is an E-ring.

2. Constructing large AA-rings that are not A-rings.

We want to present a brief hint about a construction of large torsion-free AA-rings that are not A-rings. First, we need a tool to construct rings with only a few, but non-trivial automorphisms.

2.1 Proposition. *Let S be an integral domain such that $U(S) = \{1, -1\}$. Let $\gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define $S[\gamma] = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in S \right\}$. Then $U(S[\gamma]) = \{1, -1, \gamma, -\gamma\}$ is a group of order* 4.

Proof. Let QS be the field of quotients of the integral domain S and $\mu = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in S[\gamma]$. Then μ has an inverse μ^{-1} over QS if and only if $a^2 - b^2 \neq 0$. Note that $\mu^{-1} = \frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}$ and $\mu^{-1} \in S[\gamma]$ if and only if $\frac{a}{a^2-b^2}, \frac{b}{a^2-b^2} \in S$. Assume $\mu^{-1} \in S[\gamma]$. Then $\frac{a}{a^2-b^2} \pm \frac{b}{a^2-b^2} \in S$. Using the factorization $a^2 - b^2 = (a-b)(a+b)$ we derive $\frac{1}{a-b}, \frac{1}{a+b} \in S$, which means $a-b$, $a+b \in U(S) = \{1, -1\}$. Considering the four possible cases it turns out that one of the parameters a, b has to be 0 while the other is equal to 1 or -1. This shows that $\mu \in \{1, -1, \gamma, -\gamma\}$. ∎

Let μ be a cardinal such that $\mu^{\aleph_0} = \mu$ and $\lambda = \mu^+$ is the successor cardinal of μ. Let $P = \mathbb{Z}[x_{\alpha, \varepsilon}: \alpha < \lambda, \varepsilon \in \{0, 1\}]$ be a commutative polynomial ring with variables $x_{\alpha, \varepsilon}$. Define $\gamma \in \mathrm{Aut}(P)$ be $\gamma(x_{\alpha, \varepsilon}) = x_{\alpha, \delta}$ such that $\{\varepsilon, \delta\} = \{0, 1\}$, $\alpha < \lambda$. Now one can run through the Black Box construction in section 3 of [3], or the proof in [4], without any significant changes, and obtain an integral domain S sandwiched between P and its p-adic completion $\widehat{P}$, where p is some fixed prime number, such that $\mathrm{End}(S^+) = S_\ell[\gamma] \approx S[\gamma]$ and $U(S) = \{1, -1\}$. Therefore $\mathrm{Aut}(S^+) \approx$ $\approx U(S[\gamma])$ has order 4 by Proposition 2.1. Thus S is an AA-ring, but not an A-ring since $\gamma \notin S_\ell$. Therefore we have the following:

2.2 THEOREM. *There exist arbitrarily large torsion-free integral domains of infinite rank that are AA-rings but not A-rings.*

In conclusion, we summarise:

Our Main Theorem shows that {AE-rings} = {E-rings}. Moreover, {E-rings} $\subsetneqq$ {A-rings} $\subsetneqq$ {AA-rings}. It was shown in [3] that the first inclusion is proper and our Theorem 2.2 shows that the second inclusion is proper. Restricting attention to torsion-free rings of finite rank (tffr) it was shown in [2] that {tffr E-rings} = {tffr A-rings} $\subsetneqq$ {tffr AA-rings}. To demonstrate that the last inclusion is proper, just pick any tffr group N such that $\mathrm{End}(N) = \mathbb{Z}$ and $\mathrm{Hom}(N, \mathbb{Z}) = \{0\}$. Define $NN = \{0\}$ and use this to introduce the natural ring structure on $R = \mathbb{Z} \oplus N$. It is easy to verify that R is an AA-ring but not an A-ring.

REFERENCES

[1] R. A. BOWSHELL - P. SCHULTZ, *Unital rings whose additive endomorphisms commute*, Math. Ann., **228** (1977), pp. 197-214.

[2] M. DUGAS, *AA-Rings*, to appear in Comm. Alg.

[3] M. DUGAS - S. FEIGELSTOCK, *A-rings*, to appear in Coll. Math.

[4] M. DUGAS - A. MADER - C. VINSONHALER, *Large E-rings exist*, J. Algebra, **108** (1987), pp. 88-101.

[5] M. DUGAS - C. VINSONHALER, *Two-sided E-rings*, to appear in J. Pure Appl. Algebra.

[6] R. GÖBEL - S. SHELAH - L. STRÜNGMANN, *Generalized E-rings*, to appear in Conf. Proc. of the Algebra Conf. Venezia (2003).
[7] I. N. HERSTEIN, *A theorem on rings*, Canadian J. Math., **5** (1953), pp. 238-241.
[8] I. N. HERSTEIN, *On the hypercenter of a ring*, J. Algebra, **36**, no. 1 (1975), pp. 151-157.
[9] N. JACOBSON, *Structure of rings*, Amer. Math. Soc., Providence, RI, 1956.
[10] I. KAPLANSKY, *A theorem on division rings*, Canadian J. Math., **3** (1951), pp. 290-292.
[11] A. LIBMAN, *Cardinality and nilpotencey of localizations of groups and G-modules*, Israel J. Math., **117** (2000), pp. 221-236.
[12] A. I. LIHTMAN, *Rings that are radical over a commutative subring*, Math. Sbornik (N.S.), **83** (1970), pp. 513-523.
[13] P. SCHULTZ, *The endomorphism ring of the additive group of a ring*, J. Austral. Math. Soc., **15** (1973), pp. 60-69.
[14] C. VINSONHALER, *E-rings and related structures, in: Non-Noethereian Commutative Ring Theory*, Math. Appl., **520**, Kluwer, Dordrecht (2002), pp. 387-402.

Manoscritto pervenuto in redazione il 22 luglio 2003.