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On the Curves of Constant Relative Width.

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ABSTRACT - We study some geometric problems concerned with the ovals of constant width relative to certain oval. In particular, we give the six vertex theorem for such curves.

1. Introduction.

In this paper we consider the class of plane strictly convex closed curves of class at least C^2. Such curves will be called ovals throughout the paper. Let us fix an oval B. For any oval C, let $\omega_C(t)$ denote the width of C in the direction of the vector $e^{it} = (\cos t, \sin t)$ in the fixed coordinate system. Following the papers [Ch], [KH], [Oh], [Sh] we give the following definition.

DEFINITION 1.1. *An oval C is said to be of constant width d relative to the oval B if $\omega_C(t) = d \cdot \omega_B(t)$ for every $t \in [0, 2\pi)$.*

Note that the ovals of constant relative width are the natural generalization of the ovals of constant width since if the oval B is a circle then obviously C is an oval of constant width in an ordinary sense.

We start with a geometric characterization of such ovals. Next, we consider the ovals of the same constant width relative to B with the com-

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mon Steiner centroid. For these pairs, we prove among the others, the six vertex theorem for the relative curvature radius. Moreover, we prove that these ovals have at least six common tangents and six common normals with equal orientation and at least six normals with opposite orientation. We give also a simple construction of a counterpart of the Reuleaux triangle in the class of ovals of constant width relative to an ellipse.

2. Basic facts.

Let $p_B(t)$ (resp. $p_C(t)$) be the distance from the origin O to the support line $l_B(t)$ (resp. $l_C(t)$) to B (resp. C) perpendicular to the vector e^{it}. It is well known that the parametrization of B (resp. C) is given by $z_B(t) = p_B(t)\, e^{it} + \dot{p}_B(t)\, ie^{it}$ (resp. $z_C(t) = p_C(t)\, e^{it} + \dot{p}_C(t)\, ie^{it}$).

Let $q_B(t) = z_B(t) - z_B(t+\pi)$ and $q_C(t) = z_C(t) - z_C(t+\pi)$. We have

THEOREM 2.1. *The following conditions are equivalent:*

(1) *an oval C is of constant width d relative to B,*

(2) *for each $t \in [0, 2\pi)$ the angle between the vectors $\dot{z}_B(t)$ and $q_B(t)$ is equal to the angle between the vectors $\dot{z}_C(t)$ and $q_C(t)$.*

PROOF. The condition (1) means that

$$p_C(t) + p_C(t+\pi) = d \cdot (p_B(t) + p_B(t+\pi)) . \tag{2.1}$$

We have

$$\dot{z}_B(t) = (p_B(t) + \ddot{p}_B(t))\, ie^{it}$$

$$\dot{z}_C(t) = (p_C(t) + \ddot{p}_C(t))\, ie^{it}$$

and

$$\left\langle \frac{q_B(t)}{|q_B(t)|}\, ,\, ie^{it} \right\rangle = \left\langle \frac{q_C(t)}{|q_C(t)|}\, ,\, ie^{it} \right\rangle . \tag{2.2}$$

Let us suppose that the equality (2.2) holds. Then

$$(\dot{p}_B(t) + \dot{p}_B(t+\pi))(p_C(t) + p_C(t+\pi)) -$$

$$- (\dot{p}_C(t) + \dot{p}_C(t+\pi))(p_B(t) + p_B(t+\pi)) = 0 .$$

Hence, we have

$$\frac{p_C(t) + p_C(t+\pi)}{p_B(t) + p_B(t+\pi)} = \text{const.} \qquad \blacksquare$$

Let R_C and R_B denote the curvature radii of the ovals C and B, respectively. If the oval C is of constant width d relative to B then

$$p_C(t) + p_C(t+\pi) = d\cdot(p_B(t) + p_B(t+\pi))$$

$$\ddot{p}_C(t) + \ddot{p}_C(t+\pi) = d\cdot(\ddot{p}_B(t) + \ddot{p}_B(t+\pi))\,.$$

Hence

(2.3) $$R_C(t) + R_C(t+\pi) = d\cdot(R_B(t) + R_B(t+\pi))\,.$$

Then we have the following theorem:

THEOREM 2.2. *An oval C is of constant width d relative to B if and only if the condition* (2.3) *holds.*

PROOF. Suppose (2.3) holds. Let

$$q_B(t) = \lambda_B(t)\, ie^{it} + \mu_B(t)\, e^{it}.$$

Then

$$\lambda_B(t) = \dot{p}_B(t) + \dot{p}_B(t+\pi)$$

$$\mu_B(t) = p_B(t) + p_B(t+\pi)\,.$$

Similarly, if $q_C(t) = \lambda_C(t)\, ie^{it} + \mu_C(t)\, e^{it}$ then

$$\lambda_C(t) = \dot{p}_C(t) + \dot{p}_C(t+\pi)$$

$$\mu_C(t) = p_C(t) + p_C(t+\pi)\,.$$

From the formula (2.3) we have

$$(\mu_C(t) - d\cdot\mu_B(t)) + (\ddot{\mu}_C(t) - d\cdot\ddot{\mu}_C(t)) = 0\,.$$

All, the solutions of the above equation are of the form

$$\mu_C(t) - d\cdot\mu_B(t) = \alpha_1 \sin t + \alpha_2 \cos t\,.$$

But since

$$\mu_C(t) - d\cdot\mu_B(t) = \mu_C(t+\pi) - d\cdot\mu_B(t+\pi)$$

then $\alpha_1 = \alpha_2 = 0$, $\mu_C(t) = d\cdot\mu_B(t)$ and $\omega_C(T) = d\cdot\omega(t)$. ■

By integrating the equality $p_C(t) + p_C(t+\pi) = d\cdot(p_B(t) + p_B(t+\pi))$ we get

$$2dL_B = d\int_0^{2\pi} (p_B(t) + p_B(t))\,dt = \int_0^{2\pi} (p_C(t) + p_C(t+\pi))\,dt = 2L_C,$$

where L_B and L_C denote the lengths of B and C, respectively. Hence, we have obtained

THEOREM 2.3. (Barbier theorem)

$$L_C = d\cdot L_B.$$

REMARK. In the special case when B is a circle the above theorems give the well-known theorems for the constant width curves.

Note that among the curves of constant width the only centrally symmetric curves are circles. In our setting we have

THEOREM 2.4. *There exists exactly one centrally symmetric curve in the class of ovals of constant width d relative to B*

PROOF. From the formulas $p_C(t) = p_C(t+\pi)$ and $p_C(t) + p_C(t+\pi) = = d\cdot(p_B(t) + p_B(t+\pi))$, it follows that

$$p_C(t) = \frac{d\cdot(p_B(t) + p_B(t+\pi))}{2}\,. \qquad ■$$

EXAMPLE. Consider the ellipse $B : \frac{x^2}{25} + \frac{y^2}{9} = 1$. Its support function is given by the formula $p_B(t) = \sqrt{25\cos^2 t + 9\sin^2 t}$. Figure 2.1 presents

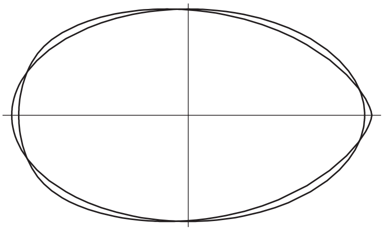

Fig. 2.1. – Oval of constant width 1 relative to the ellipse.

the curve of constant width 1 relative to B described by the support function

$$p_C(t) = \sqrt{25\cos^2 t + 9\sin^2 t} + 0.2\cos 3t\,.$$

3. Area of ovals of constant relative width.

We know from the isoperimetric inequality that among all sets of constant width d the circle has the largest area. In our context, we have the following

THEOREM 3.1. *In the class of all sets of constant width d relative to B the unique centrally symmetric curve has the largest area.*

PROOF. We can take for B the unique centrally symmetric curve in this class. Let C be any curve of constant width 1 relative to B. Consider the Fourier expansions

$$p_C(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$$

$$p_B(t) = A_0 + \sum_{n=1}^{\infty}(A_n\cos nt + B_n\sin nt)\,.$$

We can choose the origin O of the coordinate system at the point of symmetry of B. Then

$$p_B(t) = p_B(t+\pi)$$

and then we have $A_n = B_n = 0$ for n odd. From formula (2.1) we obtain

$$A_0 = a_0,\ A_n = a_n,\ B_n = b_n \qquad \text{for n even}.$$

From the well-known formula expressing the enclosed area in terms of Fourier coefficients of the support function (cf. [Gr]) we have

$$P(C) = \pi a_0^2 - \frac{\pi}{2} - \sum_{n\,-\,\text{even}} (n^2-1)(A_n^2+B_n^2) - \frac{\pi}{2} \sum_{n\,-\,\text{odd}} (n^2-1)(a_n^2+b_n^2)$$

$$P(B) = \pi a_0^2 - \frac{\pi}{2} \sum_{n\,-\,\text{even}} (n^2-1)(A_n^2+B_n^2).$$

Thus

$$P(C) \leqslant P(B)$$

and

$$P(C) = P(B) \ \Leftrightarrow\ B = C. \qquad \blacksquare$$

We note that in the papers [Ch], [KH], [Oh], [Sh] the counterpart of the Blaschke-Lebesgue theorem concerning the minimal area in this setting is given.

4. Extrema connected with ovals of constant relative width.

Consider two curves C_1, C_2 of the same constant width d relative to B. Assume that the origin of the coordinate system is the Steiner centroid of C_1 and C_2. If p_2 and p_2 are the support functions of C_1 and C_2 then

$$p_1(t) = a_0 + \sum_{n=2}^{\infty} (a_n \cos nt + b_n \sin nt),$$

$$p_2(t) = a_0 + \sum_{n=2}^{\infty} (c_n \cos nt + d_n \sin nt)$$

and $a_{2n} = c_{2n},\ b_{2n} = d_{2n}$. Thus, we have immediately

$$p_1(t) - p_2(t) = \sum_{\substack{n=3\\ n\,-\,\text{odd}}} ((a_n - c_n)\cos nt + (b_n - d_n)\sin nt)\,.$$

From the generalized Sturm theorem (cf. [Hu]) we have

Theorem 4.1. *Assume that C_1 and C_2 are ovals of the same constant width d relative to B with the common Steiner centroid O. Let p_1 and p_2 be their support functions with respect to O. Then there exist at least six points $t \in [0, 2\pi)$ such that $p_1(t) = p_2(t)$.*

Observe that the equality of the support functions is equivalent to the identity of the tangent lines. Thus, we have

Theorem 4.2. *Under the assumptions of theorem* 4.1 *the ovals C_1 and C_2 have at least six common tangents.*

Corollary 4.1. *Any two ovals of the same constant width d relative to B with the identical Steiner centriod have at least six common points.*

Similarly, we get

Theorem 4.3. *Two ovals satisfying the assumptions of Theorem* 4.1 *have at least six common normals with equal orientation and at least six normals with opposite orientation.*

Let us consider the curvature radii $R_1(t)$ and $R_1(t)$ of ovals C_1 and C_2 of the same constant width d relative to B. We can assume that C_1 and C_2 have common curvature centroid at the origin. Then

$$R_1(t) - R_2(t) = p_1(t) + \ddot{p}_1(t) - p_2(t) - \ddot{p}_2(t) =$$

$$= \sum_{\substack{n=3 \\ n - \text{odd}}} ((1-n^2)(a_n - c_n)\cos nt + (1-n^2)(b_n - d_n)\sin nt)\,.$$

Thus, we obtained

Theorem 4.4. *Under the above assumptions there exist at least six values $t \in [0, 2\pi)$ such that $R_1(t) = R_2(t)$.*

As a consequence of Theorem 4.4 we get

Theorem 4.5. *The function $g(t) = \frac{R_1(t)}{R_2(t)}$ has at least six extrema in the interval $[0, 2\pi)$.*

Proof. It is sufficient to compare the graphs of the function $g(t)$ and the constant function $y = 1$. ■

REMARK. When the model curve B is a circle then C_1 and C_2 are curves of equal constant width. If C_1 is a circle then theorem 4.5 gives the well-known six-vertex theorem for curves of constant width.

5. Curves of constant width relative to an ellipse.

In the general case, in the papers [Ch], [KH], [Oh], [Sh] there is given a construction of curve of a constant relative width of the minimal area. In the class of curves of constant width 1 relative to an ellipse $B: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have a very simple method to obtain such a curve.

Consider the affine transformation $f: x' = x,\ y' = \frac{a}{b}y$. It is easy to check that f maps the class of curves of constant width 1 relative to B onto the class of curves of constant width $2a$. Thus the inverse map f^{-1} transforms the Reuleaux triangle of width 2 onto the elliptic Reuleaux triangle of constant width 1 relative to B.

PROPOSITION 5.1. *Each elliptic Reuleaux triangle of width* 1 *relative to B has the area equal to* $2\pi ab - \frac{ab\sqrt{3}}{2}$.

Let us consider an arbitrary oval C. For a fixed direction k we consider the longest chord parallel to k. The tangent lines to C at the ends of this chord are parallel and their direction k' is called the conjugate direction to k.

REMARK. In general, this relation is not symmetric.

The next theorem is probably well-known, so we give it without proof.

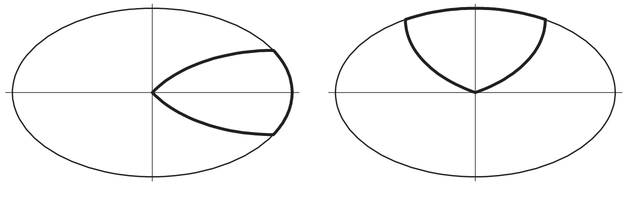

Fig. 5.1. – Elliptic Reuleaux triangles.

THEOREM 5.1. *(Apollonius theorem) All parallelograms circumscribed to an oval with sides parallel to conjugate directions have the same area.*

In the class of ovals of constant width 1 relative B we have

THEOREM 5.2.

(1) *The conjugacy relation for ovals of constant width d relative B is symmetric,*

(2) *The area of any parallelogram circumscribed to an oval of constant width* 1 *relative B with sides parallel to conjugate directions is equal to* $4ab$.

REMARK For any oval of the constant width 1 relative a certain oval B the area of the parallelogram with sides parallel to conjugate directions is constant and the same is true for any oval in this class.

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