REND. SEM. MAT. UNIV. PADOVA, Vol. 107 (2002)

Periodicity in K-groups of Certain Fields.

PAUL ARNE ØSTVÆR (*)

ABSTRACT - Let k be a field of characteristic different from p. We study the p-torsion and the p-cotorsion in the higher algebraic K-groups of k. Under a certain hypothesis we find that these groups are periodic. Some (co)-descent properties are also pointed out.

1. Introduction.

Let k be a field of characteristic different from p. In the main part of this paper we will assume that the p-cohomological dimension $\mathrm{cd}_p(k)$ of k is less than three. Additionally, we will assume that the group $H^2_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(i))$ is trivial for $i \geqslant 2$. For such a k we first prove some periodicity results for its algebraic K-groups. Second we discuss some (co)-descent properties for the same groups. These results are easily deduced from the Bloch-Lichtenbaum spectral sequence, denoted by BLSS from now on, with finite coefficients. We claim no originality whatsoever for this part. The BLSS for a field such as above resembles the BLSS for a complex surface. That example was first considered by Suslin [Su2].

There are several versions of the BLSS, cf. [BL], [FS], [Le2], [RW] and [We]. Assume k has characteristic zero. The mod p^ν BLSS for k is a third quadrant cohomological spectral sequence with input the higher Chow groups of k with mod p^ν coefficients, and abutment the mod p^ν algebraic K-groups of k. Suslin [Su3] has proved that the higher Chow groups of k are isomorphic to the motivic cohomology groups of k. We let

(*) Indirizzo dell'A.: Department of Mathematical Sciences, The Norwegian University of Science and Technology, Trondheim, Norway. E-mail: ostvar@math.ntnu.no

subscript $\mathcal{M}$ indicate motivic cohomology. From the mentioned results, the mod p^ν BLSS for k takes the form:

$$E_2^{m,n} = H_{\mathcal{M}}^{m-n}(k;\, \mathbb{Z}/p^\nu(-n)) \;\Rightarrow\; K_{-m-n}(k;\, \mathbb{Z}/p^\nu)\,.$$

The outcome of Weibel's valuation trick from [We] is a mod p^ν BLSS for fields of positive characteristic. The idea is to replace k by a field $F(k)$ of characteristic zero, and whose motivic cohomology and algebraic K-theory groups are naturally isomorphic to the same groups for k. Assume k has positive characteristic l, where $l \neq p$. Define $R_0(k)$ to be the Cohen l-ring of k, and define inductively $R_n(k)$ to be $R_{n-1}(k)[t]/(t^l - \pi)$ where π is a uniformizing parameter for $R_{n-1}(k)$ and $n \geqslant 1$. The quotient field of the union

$$\operatorname{colim}\,(R_0(k) \subset R_1(k) \subset R_2(k) \subset \dots)$$

has the desired properties of $F(k)$.

Next we explain the relation between the motivic cohomology groups and the étale cohomology groups of k. The Bloch-Kato conjecture [BK] at the prime p predicts that the Galois symbol

$$K_n^M(F)/p^\nu \longrightarrow H_{\text{ét}}^n(F;\, \mathbb{Z}/p^\nu(n))$$

is an isomorphism for every field F of characteristic different from p. Voevodsky proved this conjecture in [Vo] for the prime $p = 2$. For $p = 2$ the Bloch-Kato conjecture was originally formulated by Milnor [Mi]. Suslin and Voevodsky proved in [SV] that if the Bloch-Kato conjecture is true at the prime p, then there exists natural isomorphisms

$$H_{\mathcal{M}}^n(k;\, \mathbb{Z}/p^\nu(i)) \cong \begin{cases} H_{\text{ét}}^n(k;\, \mathbb{Z}/p^\nu(i)) & \text{for } 0 \leqslant n \leqslant i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

By specialization we get the following result (for two groups A and B we let $A \rtimes\!\!\!\ltimes B$ denote an Abelian extension of B by A).

THEOREM 1.1. *Assume* $\operatorname{cd}_p(k) \leqslant 2$. *If p is an odd prime, we also assume that the Bloch-Kato conjecture holds at p.*

(*a*) *The mod p^ν algebraic K-groups of k are given up to extensions by*

$$K_n(k;\, \mathbb{Z}/p^\nu) \cong \begin{cases} H_{\text{ét}}^1(k;\, \mathbb{Z}/p^\nu(i)) & \textit{for } n = 2i-1\,, \\ H_{\text{ét}}^2(k;\, \mathbb{Z}/p^\nu(i+1)) \rtimes\!\!\!\ltimes H_{\text{ét}}^0(k;\, \mathbb{Z}/p^\nu(i)) & \textit{for } n = 2i > 0\,. \end{cases}$$

(*b*) *The extension above is split by the anti-Chern classes of Kahn if p is odd, or* $p=2$ *and k contains a primitive fourth root of unity.*

REMARK 1.2. *Part* (*b*) *of Theorem* 1.1 *is due to Kahn, see Theorem* 3.1 *in* [Ka2]. *The results from* [FS] *and* [Le2] *make it plain that Theorem* 1.1, *and hence some of the results in this paper may be generalized to certain schemes with mod p étale cohomological dimension less than three.*

In Section 2 we prove results which appear to be new. For this we will only consider fields with the properties stated in the beginning of the introduction. The assumptions on k can often be checked in practice. Our results reveal a periodicity phenomena for the p-torsion and the p-cotorsion in the algebraic K-groups of such a field. The proofs are very elementary and straightforward. However, the results might be useful in specific examples. The same remarks apply to the results in Section 3. Let k'/k be a Galois extension of fields as above. In Proposition 3.3 we point out the connection between the Galois (co)-invariants of the algebraic K-groups of k' and the algebraic K-groups of k.

2. Periodicity in K-groups.

Assume $\mathrm{cd}_p(k) \leqslant 2$. Then the long exact sequence in étale cohomology induced by the coefficient extension $0 \to \mathbb{Z}/p(n) \to \mathbb{Q}_p/\mathbb{Z}_p(n) \to \to \mathbb{Q}_p/\mathbb{Z}_p(n) \to 0$ shows that the group $H^2_{\text{ét}}(k;\, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is divisible. We impose the additional assumption that the latter group is trivial for $n \geqslant \geqslant 2$. For an Abelian group A we let $A\{p\} = \bigcup_{\nu} {}_{p^\nu}A$ be its maximal p-torsion subgroup. Let $\bar{k}$ be an algebraic closure of k.

First we translate the additional assumption into a statement about the K-groups of k. Consider the diagram

$$\begin{array}{ccc} K_{2n}(k;\, \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & K_{2n}(\bar{k};\, \mathbb{Q}_p/\mathbb{Z}_p) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ K_{2n-1}(k)\{p\} & \longrightarrow & K_{2n-1}(\bar{k})\{p\} \end{array}$$

where the vertical maps are the Bockstein maps. From Theorem 1.1; the upper horizontal map is injective, since it can be identified with the natural injective map $H^0_{\text{ét}}(k;\, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^0_{\text{ét}}(\bar{k};\, \mathbb{Q}_p/\mathbb{Z}_p(n))$. We know the

Bockstein map for $\bar{k}$ is an isomorphism from [Su]. Hence the Bockstein map for k is an isomorphism, and it follows that $K_{2n}(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ is the trivial group for all $n\geqslant 1$. Note also that $K_{2n-1}(k)\{p\}$ injects into $K_{2n-1}(\bar{k})\{p\}$.

The previous remarks combined with Theorem 1.1 give an isomorphism:

$$H^0_{\text{ét}}(k;\,\mathbb{Q}_p/\mathbb{Z}_p(n))\xrightarrow{\cong}K_{2n-1}(k)\{p\}\,. \tag{2.1}$$

Let e_ν denote the exponent of the multiplicative group $(\mathbb{Z}/p^\nu)^\times$, and let $\mu_n(k)$ denote the group of nth roots of unity in k.

Lemma 2.2. *Let $m,n\geqslant 1$. Then ${}_{p^\nu}K_{2n-1}(k)$ is isomorphic to ${}_{p^\nu}K_{2(n+me_\nu)-1}(k)$ and there is an exact sequence*

$$0\to H^0_{\text{ét}}(k;\,\mathbb{Z}/p^\nu(n))\to K_{2n-1}(k)\xrightarrow{p^\nu}K_{2n-1}(k)\,. \tag{2.3}$$

In particular, the group $K_{2me_\nu-1}(k)$ contains an element of order p^ν.

Proof. From (2.1) we find an isomorphism $H^0_{\text{ét}}(k;\mathbb{Z}/p^\nu(n))\xrightarrow{\cong}{}_{p^\nu}K_{2n-1}(k)$. Now employ the $\mathrm{Gal}(k^s/k)$-module isomorphism $\mathbb{Z}/p^\nu(n)\cong\mathbb{Z}/p^\nu(n+e_\nu)$ where k^s is a separable closure of k. The last claim follows from ${}_{p^\nu}K_{2me_\nu-1}(k)\cong{}_{p^\nu}K_{2e_\nu-1}(k)\cong H^0_{\text{ét}}(k;\,\mathbb{Z}/p^\nu(0))$ and the fact that the absolute Galois group of k acts trivially on $\mathbb{Z}/p^\nu(0)$ by definition of the Tate twist. ■

Remark 2.4. *If k contains a primitive p^νth root of unity, then:*

$$\mu_{p^\nu}(k)\cong{}_{p^\nu}K_3(k)\cong{}_{p^\nu}K_5(k)\cong\dots$$

This follows since $\mathbb{Z}/p^\nu(i)$ is independent of the twist i under the given assumption.

We claim the Bockstein exact sequence in K-theory and Theorem 1.1 combine to make a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2n}(k)/p^\nu & \longrightarrow & K_{2n}(k;\,\mathbb{Z}/p^\nu) & \longrightarrow & {}_{p^\nu}K_{2n-1}(k) & \longrightarrow & 0\\
 & & \downarrow & & \| & & \downarrow\cong & & \\
0 & \to & H^2_{\text{ét}}(k;\,\mathbb{Z}/p^\nu(n+1)) & \to & K_{2n}(k;\,\mathbb{Z}/p^\nu) & \to & H^0_{\text{ét}}(k;\,\mathbb{Z}/p^\nu(n)) & \to & 0
\end{array}$$

For $\bar{k}$ there is a unique choice of isomorphism on the right hand side that makes the diagram commutative. For k we choose the isomorphism that is compatible with the inclusion into $\bar{k}$. This gives a natural isomorphism:

$$K_{2n}(k)/p^{\nu} \xrightarrow{\cong} H^2_{\text{ét}}(k;\mathbb{Z}/p^{\nu}(n+1)) \tag{2.5}$$

LEMMA 2.6. *Let* $m,n \geqslant 1$. *Then* $K_{2n}(k)/p^{\nu}$ *is isomorphic to* $K_{2(n+me_{\nu})}(k)/p^{\nu}$ *and there is an exact sequence*

$$K_{2n-2}(k) \xrightarrow{p^{\nu}} K_{2n-2}(k) \rightarrow H^2_{\text{ét}}(k;\mathbb{Z}/p^{\nu}(n)) \rightarrow 0\,. \tag{2.7}$$

PROOF. Given (2.5), the proof is a verbatim copy of the argument for Lemma 2.2. The periodicity can be decreased according to Remark 2.4. ■

The mod p^{ν} Bockstein exact sequence in K-theory and Theorem 1.1 give the short exact sequence

$$0 \rightarrow K_{2n-1}(k)/p^{\nu} \rightarrow H^1_{\text{ét}}(k;\mathbb{Z}/p^{\nu}(n)) \rightarrow {}_{p^{\nu}}K_{2n-2}(k) \rightarrow 0\,. \tag{2.8}$$

The sequence (2.8) splits if n is a multiple of e_{ν} and k is a number field which satisfies the assumptions in Theorem 1.1. These assumptions are satisfied unless k is real and $p=2$, cf. Theorem 4.5 [RW]. Indeed, Lemma 2.2 shows that the mod p^{ν} reduction of $K_{2me_{\nu}-1}(k)$ is a full subgroup of $H^1_{\text{ét}}(k;\mathbb{Z}/p^{\nu}(me_{\nu}))$, hence a direct summand. These remarks motivate the following observation.

LEMMA 2.9. *If* (2.8) *splits for* n *and* $n+me_{\nu}$, *then:*

$$K_{2n-1}(k)/p^{\nu} \oplus {}_{p^{\nu}}K_{2n-2}(k) \cong K_{2(n+me_{\nu})-1}(k)/p^{\nu} \oplus {}_{p^{\nu}}K_{2(n+me_{\nu})-2}(k)\,.$$

In particular, if $K_{2n-1}(k)/p^{\nu}$ *is finite and isomorphic to* $K_{2(n+me_{\nu})-1}(k)/p^{\nu}$, *then* ${}_{p^{\nu}}K_{2n-2}(k) \cong {}_{p^{\nu}}K_{2(n+me_{\nu})-2}(k)$. *Likewise, if* ${}_{p^{\nu}}K_{2n-2}(k)$ *is finite and isomorphic to* ${}_{p^{\nu}}K_{2(n+me_{\nu})-2}(k)$, *then* $K_{2n-1}(k)/p^{\nu} \cong K_{2(n+me_{\nu})-1}(k)/p^{\nu}$.

PROOF. The first claim is clear from periodicity of $H^1_{\text{ét}}(k;\mathbb{Z}/p^{\nu}(n))$. The remaining claims follow from the cancellation property of finite groups, see [Hi]. ■

The exact sequences (2.3), (2.7) and (2.8) imply the next result.

THEOREM 2.10. *Let $n \geqslant 2$. Then we have the exact sequence*

$$0 \to H^0_{\text{ét}}(k; \mathbb{Z}/p^\nu(n)) \to K_{2n-1}(k) \xrightarrow{p^\nu} K_{2n-1}(k) \to H^1_{\text{ét}}(k; \mathbb{Z}/p^\nu(n)) \to \tag{2.11}$$

$$\to K_{2n-2}(k) \xrightarrow{p^\nu} K_{2n-2}(k) \to H^2_{\text{ét}}(k; \mathbb{Z}/p^\nu(n)) \to 0 .$$

REMARK 2.11. *Sequence* (2.11) *inserted $n = 2$ and with $K_2(k)$ replaced with its indecomposable part is known from* [Le1] *and* [MS].

3. (Co)-descent.

Let k'/k be a Galois extension of fields with group Γ. We keep the assumptions that $\mathrm{cd}_p(k) \leqslant 2$ and $H^2_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$ for all $n \geqslant 2$, and likewise for k'. Consider the Hochschild-Serre spectral sequence

$$E_2^{s,t} = H^s(\Gamma, H^t_{\text{ét}}(k'; \mathbb{Q}_p/\mathbb{Z}_p(n))) \Rightarrow H^{s+t}_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) \tag{3.1}$$

and the Tate spectral sequence:

$$E_2^{-s,t} = H_s(\Gamma, H^t_{\text{ét}}(k'; \mathbb{Q}_p/\mathbb{Z}_p(n)) \Rightarrow H^{-s+t}_{\text{ét}}(k; \mathbb{Q}_p/\mathbb{Z}_p(n)) . \tag{3.2}$$

Here (3.1) is a first quadrant cohomological spectral sequence. Moreover, (3.2) is discussed in Chapter I Appendix 1 [Se] and in Proposition 3.1.1 [Ka1]. This is a second quadrant cohomological spectral sequence. The following result is now trivial to prove.

PROPOSITION 3.3. *Let M^q denote $H^q_{\text{ét}}(k'; \mathbb{Q}_p/\mathbb{Z}_p(n))$, and let $n \geqslant 2$. We have the exact sequences*

$$0 \to H^1(\Gamma, M^0) \to K_{2n-1}(k; \mathbb{Q}_p/\mathbb{Z}_p) \to K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma \to H^2(\Gamma, M^0) \to 0$$

and:

$$0 \to H_2(\Gamma, M^1) \to K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p)_\Gamma \to K_{2n-2}(k; \mathbb{Q}_p/\mathbb{Z}_p) \to H_1(\Gamma, M^1) \to 0 .$$

In addition we have the naturally induced isomorphisms

$$K_{2n-2}(k; \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\cong} K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma$$

and:

$$K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p)_\Gamma \xrightarrow{\cong} K_{2n-1}(k; \mathbb{Q}_p/\mathbb{Z}_p) .$$

The d^2-differentials in (3.1) *and* (3.2) *give isomorphisms*

$$H^q(\Gamma, K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p)) \xrightarrow{\cong} H^{q+2}(\Gamma, K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p))$$

and

$$H_{q+2}(\Gamma, K_{2n-1}(k'; \mathbb{Q}_p/\mathbb{Z}_p)) \xrightarrow{\cong} H_q(\Gamma, K_{2n-2}(k'; \mathbb{Q}_p/\mathbb{Z}_p))$$

for all $q \geqslant 1$.

REMARK 3.4. *It follows that* $K_{2n-1}(k)\{p\} \xrightarrow{\cong} K_{2n-1}(k')\{p\}^\Gamma$, *and the transfer map induces a surjection* $K_{2n-2}(k')\{p\}_\Gamma \twoheadrightarrow K_{2n-2}(k)\{p\}$. *That surjection is an isomorphism if* $K_{2n-2}(k')\{p\}$ *is reduced. The first claim follows from the diagram displayed in the beginning of Section* 2, *and the second claim follows from an obvious Bockstein sequence argument.*

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