

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

ESPEDITO DE PASCALE

PJOTR P. ZABREJKO

**New convergence criteria for the Newton-  
Kantorovich method and some applications  
to nonlinear integral equations**

*Rendiconti del Seminario Matematico della Università di Padova,*  
tome 100 (1998), p. 211-230

[http://www.numdam.org/item?id=RSMUP\\_1998\\_\\_100\\_\\_211\\_0](http://www.numdam.org/item?id=RSMUP_1998__100__211_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1998, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

**New Convergence Criteria  
for the Newton-Kantorovich Method  
and Some Applications  
to Nonlinear Integral Equations.**

ESPEDITO DE PASCALE(\*) - PJOTR P. ZABREJKO(\*\*)

**ABSTRACT** - This paper presents some new conditions for the convergence of Newton-Kantorovich approximations to solutions of nonlinear operator equations in Banach spaces. The derivatives of the nonlinear operators involved are required to satisfy a rather mild continuity condition in a ball centered at the initial approximation. The abstract results are illustrated by applications to nonlinear integral equations of Uryson type in the Chebyshev space  $C$ , the Lebesgue space  $L_p$  ( $1 \leq p \leq \infty$ ), and the Orlicz space  $L_M$ .

## 1. Introduction.

The purpose of this article is two-fold. First, we are going to generalize some results about the convergence of certain Newton-Kantorovich approximations discussed in [1-2]. Second, we give new applications of both the old and new results to Uryson integral equations in Lebesgue and Orlicz spaces.

As it was shown in [2], when applying the classical Newton-Kantorovich method to Uryson integral equations, one meets a strange situ-

(\*) Indirizzo dell'A.: Università della Calabria, Dipartimento di Matematica, I-87036 Arcavacata di Rende (CS), Italy.

(\*\*) Indirizzo dell'A.: Belgosuniversitet, Matematicheskij Fakultet, BR-220050 Minsk, Belarus.

AMS (MOS) Classification: 65J15, 47H17, 47H19, 47H30, 45G10, 65R20.

ation. Namely, the standard theorem on the convergence of the Newton-Kantorovich approximations does not apply in some spaces, for instance, in  $L_p$ -spaces for  $1 \leq p \leq 2$ . This unpleasant phenomenon can be overcome by modifying the smoothness hypotheses in a suitable way. Some modifications of this type have been suggested in [1]; here we propose some new variants which works «well» if the character of continuity of the derivative at the initial approximation is essentially better (in a sense that will become evident in the sequel) than the general continuity of the derivative.

The new conditions proposed below may be checked rather effectively for Uryson integral equations in the spaces  $C$  and  $L_\infty$ . On the other hand, the situation is worse in the spaces  $L_p$  for finite  $p$  and in Orlicz spaces. Here we describe some class of kernel functions for which the application of our convergence results to the corresponding Uryson equations is natural and sufficiently effective.

Generalizations of the convergence conditions introduced in this paper are in a different direction from that exposed in [1]. It is possible to give a unified treatment of the results in [1] and in this paper. Actually, for sake of simplicity, we prefer to confine ourselves to the variant presented here.

## 2. New convergence criteria for Newton-Kantorovich approximations

Let  $X$  and  $Y$  be two Banach spaces,  $B(x_0, R) := \{x: x \in X, \|x - x_0\| \leq R\}$  the closed ball centered at  $x_0 \in X$  with radius  $R > 0$ , and  $F: B(x_0, R) \rightarrow Y$  some (nonlinear) operator. The Newton-Kantorovich method is one of the basic tools for finding approximate solutions of the operator equation

$$(1) \quad F(x) = 0.$$

In the corresponding iterative scheme

$$(2) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n = 0, 1, 2, \dots)$$

one has to require, in particular, that the Fréchet derivative of  $F$  at all points  $x_n$  exists and is invertible in the Banach space  $\mathcal{L}(X, Y)$  of all bounded linear operators from  $X$  into  $Y$ . The nonnegative numbers

$$(3) \quad a := \|F'(x_0)^{-1}F(x_0)\|$$

and

$$(4) \quad b := \|F'(x_0)^{-1}\|$$

will be of particular interest to us in what follows.

We suppose that the Fréchet derivative  $F'(x)$  of  $F$  satisfies at each point of  $B(x_0, R)$  a condition of the form

$$(5) \quad \|F'(x_1) - F'(x_2)\| \leq \omega(\|x_1 - x_2\|) \quad (x_1, x_2 \in B(x_0, R)),$$

where  $\omega: [0, \infty) \rightarrow [0, \infty)$  is monotonically increasing with

$$(6) \quad \lim_{r \rightarrow 0} \omega(r) = 0.$$

Moreover, we assume that there is another monotonically increasing function  $\theta: [0, \infty) \rightarrow [0, \infty)$  such that  $0 \leq \theta(r) \leq \omega(r)$ , ( $0 \leq r \leq R$ ), and

$$(7) \quad \|F'(x)^{-1}\| \leq \frac{b}{1 - b\theta(r)} \quad (x \in B(x_0, r)).$$

We define three scalar functions on  $[0, R]$  by

$$(8) \quad \tilde{\omega}(r) := \sup \{ \omega(u) + \theta(v) : u + v = r \},$$

$$(9) \quad \phi(r) := \frac{a}{b} + \int_0^r \omega(t) dt - \frac{r}{b} \quad (0 \leq r \leq R),$$

and

$$(10) \quad \tilde{\phi}(r) := \frac{a}{b} + \int_0^r \tilde{\omega}(t) dt - \frac{r}{b} \quad (0 \leq r \leq R).$$

As a special case of the main theorem of [6], about the convergence of successive approximations, we get then the following:

**THEOREM 1.** *Suppose that the function (9) has a unique zero  $r_* \in [0, R]$  and that  $\phi(R) \leq 0$ . Then equation (1) has a solution  $x_* \in B(x_0, r_*)$ ; this solution is unique in the ball  $B(x_0, R)$ .*

As a matter of fact (see e.g. [1, 4]), under the hypotheses of Theorem 1 the Newton-Kantorovich approximations need not converge. However, the following is true.

LEMMA 1. Suppose that the function (10) has a unique zero  $\varrho_* \in [0, R]$  and that  $\tilde{\varphi}(R) \leq 0$ . Then the scalar sequence  $(r_n)_{n \in \mathbb{N}}$  defined by

$$(11) \quad r_0 = 0, \quad r_{n+1} = r_n + \frac{b\tilde{\varphi}(r_n)}{1 - b\theta(r_n)} \quad (n = 0, 1, 2, \dots).$$

converges monotonically to  $\varrho_*$ .

PROOF. For simplicity we consider only the case when all the scalar functions under consideration are differentiable; the general case can be reached with usual considerations involving monotonicity. Since  $\varrho_*$  is the only zero of the function (10), this function is strictly positive on  $[0, \varrho_*)$ . The same is true for the function  $r \mapsto -b\tilde{\varphi}'(r) = 1 - b\tilde{\omega}(r)$ , as may be seen from the following reasoning. First of all we have  $-b\tilde{\varphi}'(0) = 1$ . Suppose that  $\tilde{\varphi}'(\bar{\varrho}) = 0$  for some  $\bar{\varrho} \in (0, \varrho_*]$ . Since  $\tilde{\varphi}'(r) = \tilde{\omega}(r) - 1/b$  is increasing in  $[0, R]$ , we have  $\tilde{\varphi}'(0) = 0 \leq \tilde{\varphi}'(r)$  for  $r \in [\bar{\varrho}, R]$ . So  $\tilde{\varphi}$  is increasing and convex on  $[\bar{\varrho}, R]$ : the hypotheses  $\tilde{\varphi}(R) \leq 0$  together with  $\tilde{\varphi}(\varrho_*) = 0$  implies  $\bar{\varrho} = \varrho_* = R$ , i.e.  $-\tilde{\varphi}'(r) > 0$  on  $(0, \varrho_*)$ . Now, since  $0 < -b\tilde{\varphi}'(r) = 1 - b\tilde{\omega}(r) \leq 1 - b\theta(r)$ , the function  $\psi$  defined by  $\psi(r) := b\tilde{\varphi}(r)/(1 - b\theta(r))$  is also strictly positive on  $(0, \varrho_*)$ .

We claim that the map  $r \mapsto r + \psi(r)$  is increasing on  $[0, \varrho_*)$ . In fact, its derivative satisfies for  $r \in [0, \varrho_*)$

$$\begin{aligned} 1 + \psi'(r) &= 1 + b \frac{\tilde{\varphi}'(r)(1 - b\theta(r)) + \tilde{\varphi}(r)\theta'(r)}{(1 - b\theta(r))^2} = \\ &= 1 + \frac{b\tilde{\varphi}'(r)}{1 - b\theta(r)} + b \frac{\tilde{\varphi}(r)\theta'(r)}{(1 - b\theta(r))^2} \geq 0. \end{aligned}$$

Indeed condition  $-b\tilde{\varphi}'(r) \leq 1 - b\theta(r)$  implies that  $1 + b\tilde{\varphi}'(r)/(1 - b\theta(r)) \geq 0$ , and the monotonicity of  $\theta$  together with the positivity of  $\tilde{\varphi}$  imply the validity of inequality  $\tilde{\varphi}(r)\theta'(r) \geq 0$ .

Now the assertion of the lemma follows easily. In fact, the sequence (11) is monotonically increasing and bounded above by  $\varrho_*$ :

$$r_{n+1} = r_n + \psi(r_n) = r_n + \frac{b\tilde{\varphi}(r_n)}{1 - b\theta(r_n)} \leq \varrho_* + \frac{b\tilde{\varphi}(\varrho_*)}{1 - b\theta(\varrho_*)} = \varrho_*$$

for  $r_n \leq \varrho_*$ . Consequently, the sequence (11) converges to some  $r_* \in [0, \varrho_*]$ . But  $r_* = r_* + \psi(r_*)$  implies  $\psi(r_*) = 0$ , that is  $r_* = \varrho_*$ .

**THEOREM 2.** *Under the hypotheses of Lemma 1 the approximations (2) are defined for all  $n$ , belong to the ball  $B(x_0, \varrho_*)$ , are converging to a solution  $x_*$  of (1) and satisfy the estimates*

$$(12) \quad \|x_{n+1} - x_n\| \leq r_{n+1} - r_n \quad (n = 0, 1, 2, \dots)$$

and

$$(13) \quad \|x_* - x_n\| \leq \varrho_* - r_n \quad (n = 0, 1, 2, \dots).$$

**PROOF.** If for every  $n > 0$   $\|x_{n+1} - x_n\| \leq r_{n+1} - r_n$ , then for  $m > n$  we have  $\|x_m - x_n\| \leq r_m - r_n$ , and consequently  $(x_n)_{n \in N}$  is Cauchy sequence convergent to a limit  $x_*$  in  $X$ , which solves equation (1) by (2) and by our assumption on  $F$ . Clearly  $\|x_* - x_n\| \leq \varrho_* - r_n$  and it is sufficient to prove (12).

For  $n = 0$  the equality (12) simply reads

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| = a = r_1 - r_0 = r_1$$

which is obvious. Suppose that (12) holds for all  $n < k$ . This implies, in particular, that

$$\|x_k - x_0\| \leq \sum_{j=1}^k \|x_j - x_{j-1}\| \leq \sum_{j=1}^k (r_j - r_{j-1}) = r_k.$$

Now, by definition (2), we have:

$$\begin{aligned} \|x_{k+1} - x_k\| &= \|F'(x_k)^{-1}F(x_k)\| = \\ &= \|F'(x_k)^{-1}F(x_k) - F'(x_k)^{-1}F(x_{k-1}) + \\ &+ [F'(x_k)^{-1}F'(x_{k-1})][F'(x_{k-1})^{-1}F(x_{k-1})]\| = \\ &= \|F'(x_k)^{-1}[F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1})]\| \leq \\ &\leq \|F'(x_k)^{-1}\| \int_0^1 \|F'[x_{k-1} + t(x_k - x_{k-1})] - F'(x_{k-1})\| \|x_k - x_{k-1}\| dt. \end{aligned}$$

We conclude that

$$\|x_{k+1} - x_k\| \leq \frac{b}{1 - b\theta(r_k)} \int_0^1 \omega[r_{k-1} + t(r_k - r_{k-1}) - r_{k-1}](r_k - r_{k-1}) dt.$$

But the definition (8) of the function  $\tilde{\omega}$  implies that  $\omega(r - s) \leq \tilde{\omega}(r) - \theta(s)$ , hence

$$\begin{aligned} \|x_{k+1} - x_k\| &\leq \\ &\leq \frac{b}{1 - b\theta(r_k)} \int_0^1 \{\tilde{\omega}[r_{k-1} + t(r_k - r_{k-1})] - \theta(r_{k-1})\}(r_k - r_{k-1}) dt \leq \\ &\leq \frac{b\tilde{\varphi}(r_k) - b\tilde{\varphi}(r_{k-1}) - (b\theta(r_{k-1}) - 1)(r_k - r_{k-1})}{1 - b\theta(r_k)} = r_{k+1} - r_k. \end{aligned}$$

This shows that (12) holds for  $n = k$  as well, and so the proof is complete. ■

Theorem 2 can be extended in a standard way (see [1]) by replacing the assumption on the function (10) by the assumption that the sequence (11) converges. This extension, however, is not very effective; we therefore just formulate the corresponding result without proof:

**THEOREM 3.** *Suppose that the sequence  $(r_n)_n$  given by (11) converges to some limit  $r_\infty(a)$ . Then the approximations (2) are defined for all  $n$ , belong to the ball  $B(x_0, r_\infty(a))$ , and satisfy the estimates (12) and (13).*

Our Theorems 1-3 are generalizations of the Theorems 1-3 contained in [1] which may be obtained by the special choice  $\theta(r) \equiv \omega(r)$ .

When  $\theta(r) \equiv \omega(r)$  the approximations (2) coincide with the classical ones studied by Kantorovich, Vertgeim and others. When  $\theta(r)$  is a constant in  $(0, 1)$ , our conditions coincide with the so-called Mysovskikh conditions ([11]). We remark that the usefulness of Theorem 2 consists in reducing the (hard) problem of finding zeros of a nonlinear operator in a Banach space to the (possibly simpler) problem of finding zeros of a scalar function.

### 3. Integral equations of Uryson type.

The purpose of this section is to discuss and illustrate various aspects of our preceding Theorems 1-3 by means of the nonlinear integral equation

$$(14) \quad x(t) = \int_{\Omega} k(t, s, x(s)) ds .$$

We suppose that the Uryson integral operator

$$(15) \quad K(x)(t) = \int_{\Omega} k(t, s, x(s)) ds$$

defined by the right-hand side of (14) acts in some Banach space  $X$ ; thus we may put  $Y = X$  and write equation (14) in the form (1) with

$$(16) \quad F(x)(t) = x(t) - \int_{\Omega} k(t, s, x(s)) ds .$$

We shall assume throughout that  $\Omega$  is a compact set in the Euclidean space, and  $k: \Omega \times \Omega \times \mathbf{R} \rightarrow \mathbf{R}$  is a Carathéodory function (i.e.  $k(\cdot, \cdot, u)$  is measurable on  $\Omega \times \Omega$ , and  $k(t, s, \cdot)$  is continuous on  $\mathbf{R}$ ). Moreover, we suppose that the derivative

$$l(t, s, u) = \frac{\partial k(t, s, u)}{\partial u}$$

exists and is also a Carathéodory function. We point out, however, that the results of this section carry over, without essential changes, also to arbitrary sets  $\Omega$ , equipped with some  $\sigma$ -algebra  $\mathcal{A}$  of measurable subsets and some countably additive measure; in particular, our results are true for infinite systems of nonlinear equations.

For simplicity, we choose  $x_0 = 0$  in this and in the following sections; the case of arbitrary  $x_0 \in X$  may be formulated simply by «shifting» arguments.

In what follows, as Banach space  $X$  we take either the space  $C = C(\Omega)$ , the Lebesgue space  $L_p(\Omega)$  ( $1 \leq p \leq \infty$ ), or the Orlicz space  $L_M(\Omega)$ . We remark, however, that our results carry over the ideal spaces of measurable functions (see [6, 13]):

A Banach space of measurable functions is called ideal if  $x \in X$



implies  $|x| \in X$  and  $\| |x| \| = \|x\|$ , and the relations  $x \in X$  and  $|y(s)| \leq |x(s)|$  a.e. in  $\Omega$  imply  $y \in X$  and  $\|y\| \leq \|x\|$ .

For fixed  $x \in X$ , we define a linear integral operator  $L(x)$  by

$$(17) \quad L(x)h(t) = \int_{\Omega} l(t, s, x(s)) h(s) ds \quad (h \in X),$$

and put

$$(18) \quad G(x) = I - L(x).$$

It is natural to expect that the derivative of the operator (15) is related to the operator (17), and hence the derivative of (16) is related to (18). In fact, the following is true:

LEMMA 2. *Let  $X$  be an ideal space or the space  $C$ . Suppose that*

$$(19) \quad K(0)(t) = \int_{\Omega} k(t, s, 0) ds \in X,$$

and the operator  $L$  given by (17) is defined on the ball  $B(0, R)$ , takes its values in the space  $\mathcal{L}(X, X)$ , and satisfies a condition

$$(20) \quad \|L(x_1) - L(x_2)\| \leq \omega(\|x_1 - x_2\|) \quad (x_1, x_2 \in B(0, R)),$$

where

$$(21) \quad \lim_{r \rightarrow 0} \omega(r) = 0.$$

Then the operator (16) is differentiable, as an operator on  $X$ , at every point  $x \in B(0, R)$ , and

$$(22) \quad F'(x) = G(x) \quad (x \in B(0, R)).$$

We omit the (rather elementary) proof of the lemma (see e.g. [2]). The following lemma was been proved in [2] as well:

LEMMA 3. *Suppose that the Uryson integral operator (15) acts in an ideal space  $X$  and admits a derivative*

$$(23) \quad L(0)h(t) = \int_{\Omega} l(t, s, 0) h(s) ds$$

at zero. Assume that  $1 \notin \text{spec}(L(0))$ , and denote by  $r(t, s)$  the resolvent

kernel for  $l(t, s, 0)$ . Then

$$(24) \quad a = \left\| \int_{\Omega} \left[ k(t, s, 0) + \int_{\Omega} r(t, \xi) k(\xi, s, 0) d\xi \right] ds \right\|_X$$

and

$$(25) \quad b \leq 1 + \|r(t, s)\|_{\mathcal{K}(X, X)};$$

the inequality (25) turns into an equality if  $X = C$  or  $X = L_{\infty}$ .

Recall that the resolvent kernel  $r(t, s)$  for a given kernel  $l(t, s)$  is defined as a solution to the equation

$$r(t, s) = l(t, s) + \int_{\Omega} l(t, \xi) r(\xi, s) d\xi;$$

the condition  $1 \notin \text{spec}(L(0))$  is therefore necessary (and, usually, also sufficient) for the existence of  $r(t, s)$ . Furthermore, the norm  $\|r(t, s)\|_{\mathcal{K}(X, X)}$ , by definition, coincides with the norm of the corresponding integral operator. At least in the case  $X = C$  or  $X = L_{\infty}$  the equality

$$\|r(t, s)\|_{\mathcal{K}(X, X)} = \left\| \int_{\Omega} |r(t, s)| ds \right\|_{L_{\infty}}$$

holds true.

In the case of spaces  $X$  different from  $C$  and  $L_{\infty}$ , the calculation of the norm of  $r(t, s)$  in  $\mathcal{K}(X, X)$  leads to serious problems. We will return to this problem in Sections 5 and 6 for the cases  $X = L_p$  and  $X = L_M$ .

In order to apply the convergence results and error estimates obtained in Theorems 1-3 above, we have to calculate or estimate the constants  $a$  and  $b$  defined by (3) and (4), respectively, and to «catch» the functions  $\omega(r)$  and  $\theta(r)$  for which the inequalities (5) and (7) hold. It is easy to see that the «optimal» choice for these functions is

$$(26) \quad \omega(r) = \sup \{ \|L(x_1) - L(x_2)\| : \|x_1 - x_2\| \leq r, x_1, x_2 \in B(x_0, r) \}$$

and

$$(27) \quad \theta(r) = \sup \{ \|L(x) - L(x_0)\| : \|x - x_0\| \leq r \},$$

respectively. In fact

$$L(x) = L(x_0) - L(x) + L(x_0) = L(x_0)[I + L(x_0)^{-1}(L(x) - L(x_0))]$$

implies

$$L(x)^{-1} = [I + L(x_0)^{-1}(L(x) - L(x_0))]^{-1} L(x_0)^{-1}$$

and consequently

$$\|L(x)^{-1}\| \leq \frac{b}{1 - b\|L(x) - L(x_0)\|} .$$

The last inequality gives the optimality of  $\theta$  defined in (27). The problem of calculating  $a$  and  $b$  was studied in detail in [2]. As a matter of fact, the article [2] contains also some information about how to calculate the functions  $\omega(r)$  and  $\theta(r)$ . We return to this problem in the following sections; here we restrict ourselves only to mentioning the following important lemma whose proof can be obtained by reasoning as in the proof of Theorem 5 in [2].

Before to state the next lemma, we want to recall that an ideal space  $X$  is called rearrangement-invariant if any two equimeasurable functions  $f$  and  $g$  in  $X$  (i.e. for every  $h \geq 0$   $\mu(\{s \in \Omega, |f(s)| > h\}) = \mu(\{s \in \Omega, |g(s)| > h\})$ ) have the same norm.

LEMMA 4. *Let  $X$  be a rearrangement-invariant ideal space with fundamental function  $\chi(t)$ . ( $\chi(t)$  is defined by the formula  $\chi(t) = \|\chi_D\|_X$ , where  $\chi_D$  is the characteristic function of a measurable subset  $D$  of  $\Omega$ , with  $\text{mes } D = t$ ). Suppose that for every fixed  $\tau$*

$$(28) \quad \liminf_{t \rightarrow 0} \frac{\omega(\tau\chi(t))\chi(t)}{t} = 0$$

or more generally

$$(29) \quad \liminf_{t \rightarrow 0} \frac{\omega(\tau\chi(t))\chi(t)}{t} = \alpha(\tau) < \infty .$$

Then the condition (6) can be satisfied in case (28) only if the function  $l = \partial k / \partial u$  does not depend on  $u$ , and in case (29) only if  $l$  satisfies the condition

$$(30) \quad \|l(\cdot, s, u_1) - l(\cdot, s, u_2)\|_X \leq \alpha(|u_1 - u_2|) \quad (s \in \Omega; u_1, u_2 \in \mathbf{R}).$$

Roughly speaking, the statement of Lemma 4 means that the Newton-Kantorovich method, applied to the Uryson operator (15), works only in sufficiently «small» spaces. As was noticed in [2], any convergence result for the Newton-Kantorovich method in the classical Kantorovich setting (i.e.  $\omega(r) = kr$ ) is useless in the spaces  $L_p$  for  $1 \leq p \leq 2$ ; a similar situation occurs in Vertgeim's setting (i.e.  $\omega(r) = kr^\alpha$ ,  $0 < \alpha < 1$ ) in the spaces  $L_p$  for  $1 \leq p \leq 1 + \alpha$  (see [1])

#### 4. The case $L_\infty$ .

Now we return to the problem of calculating (or estimating) the functions  $\omega(r)$  and  $\theta(r)$ . The simplest case is that of the space  $X = L_\infty$ . The reason lies in the very pleasant fact that one may calculate both functions in explicit form.

**THEOREM 4.** *Suppose that the kernel  $k(t, s, u)$  satisfies the following three conditions:*

$$(31) \quad \int_{\Omega} k(t, s, 0) ds \in L_\infty ,$$

$$(32) \quad \int_{\Omega} \max_{|u| \leq R} |l(t, s, u)| ds \in L_\infty ,$$

$$(33) \quad \lim_{\delta \rightarrow 0} \left\| \int_{\Omega} \max_{\substack{|u_1|, |u_2| \leq R \\ |u_1 - u_2| \leq \delta}} |l(t, s, u_1) - l(t, s, u_2)| ds \right\| = 0 .$$

Then

$$(34) \quad \omega(r) = \left\| \int_{\Omega} \max_{\substack{|u_1|, |u_2| \leq R \\ |u_1 - u_2| \leq r}} |l(t, s, u_1) - l(t, s, u_2)| ds \right\|$$

and

$$(35) \quad \theta(r) = \left\| \int_{\Omega} \max_{|u| \leq r} |l(t, s, u) - l(t, s, 0)| ds \right\| .$$

PROOF. The inequality

$$(36) \quad \omega(r) \leq \left\| \int_{\Omega} \max_{\substack{|u_1|, |u_2| \leq R \\ |u_1 - u_2| \leq r}} |l(t, s, u_1) - l(t, s, u_2)| ds \right\|$$

is evident. The reverse inequality is a straightforward consequence of the important equality

$$(37) \quad \sup_{\substack{|x_1(s)|, |x_2(s)| \leq r \\ |x_1(s) - x_2(s)| \leq r}} \left| \int_{\Omega} [l(t, s, x_1(s)) - l(t, s, x_2(s))] ds \right| = \\ = \int_{\Omega} \max_{\substack{|u_1|, |u_2| \leq R \\ |u_1 - u_2| \leq r}} |l(t, s, u_1) - l(t, s, u_2)| ds$$

which is a modification of the general equality

$$(38) \quad \sup_{|\xi(s)| \leq u(s)} \left| \int_{\Omega} m(t, s, \xi(s)) ds \right| = \int_{\Omega} \max_{|u| \leq u(s)} |m(t, s, u)| ds .$$

Here one should notice that the sup in the left-hand side of (37) and (38) is meant as supremum in the space  $S$  of measurable functions. The proof of the formula (38) can be found in [14] (see also [2]). The formula (37) may be proved similarly. ■

As a matter of fact, the space  $L_{\infty}$  has rather «bad» properties. As a consequence, whenever possible, one tries to avoid the space  $L_{\infty}$  and to consider, instead, the space  $C$  of continuous functions.

Surprisingly, stating an analogue to Theorem 4 for the space  $C$ , is essentially more complicated than for the space  $L_{\infty}$ . The basic reason for this is the complexity of conditions under which the Uryson integral operator (15) (and even linear integral operators) acts in the space  $C$ .

**THEOREM 5.** *Suppose that the kernel  $k(t, s, u)$  satisfies the following three conditions:*

$$(39) \quad \int_{\Omega} k(t, s, 0) ds \in C ,$$

$$(40) \quad \int_{\Omega} \max_{|u| \leq R} |l(t, s, u)| ds \in L_{\infty} ,$$

$$(41) \quad \lim_{\delta \rightarrow 0} \left\| \int_{\Omega} \max_{\substack{|u_1|, |u_2| \leq R \\ |u_1 - u_2| \leq \delta}} |l(t, s, u_1) - l(t, s, u_2)| ds \right\| = 0.$$

Moreover, assume that, for each continuous function  $x(s)$  satisfying  $|x(s)| \leq R$ , the kernel  $l(t, s, x(s))$  defines a continuous linear integral operator  $L(x)$  in the space  $C$ . Then

$$(42) \quad \omega(r) = \left\| \int_{\Omega} \max_{\substack{|u_1|, |u_2| \leq R \\ |u_1 - u_2| \leq r}} |l(t, s, u_1) - l(t, s, u_2)| ds \right\|$$

and

$$(43) \quad \theta(r) = \left\| \int_{\Omega} \max_{|u| \leq r} |l(t, s, u) - l(t, s, 0)| ds \right\|.$$

Of course, Theorem 5 is only a repetition of Theorem 4 containing, in addition, some cumbersome assumption on the kernels  $l(t, s, x(s))$  for  $\|x\|_C \leq R$ .

We point out that the assumption on the kernels  $l(t, s, x(s))$  is complicated only because of its high generality: this assumption covers in fact all possible situations in which one can consider equation (14) in the space  $C$ . There are simple sufficient conditions under which the kernels  $l(t, s, x(s))$  generate, for  $\|x\|_C \leq R$ , continuous linear integral operators in  $C$ . One such condition is

$$(44) \quad \int_D l(t, s, u) ds \in C \quad (|u| \leq R, D \in \mathfrak{A}(\Omega)),$$

where  $\mathfrak{A}(\Omega)$  denotes the  $\sigma$ -algebra of Lebesgue-measurable subsets of  $\Omega$ . In fact, the condition (45) guarantees that all linear integral operators

$$L(x)h(t) = \int_{\Omega} l(t, s, x(s))h(s) ds \quad (x \in L_{\infty}, \|x\|_{\infty} \leq R)$$

and the Uryson integral operator (15) acts from  $L_{\infty}$  to  $C$ , and the identity

$$K'(x) = L(x) \quad (x \in L_{\infty}, \|x\| \leq R)$$

holds in the space  $\mathcal{L}(L_{\infty}, C)$  of bounded linear operators from  $L_{\infty}$  to  $C$ .

Another (rather strong) sufficient condition on the kernels  $l(t, s, x(s))$  to generate continuous (in fact, compact) linear integral operators in  $C$  is

$$\lim_{t' - t'' \rightarrow 0} \int_{\Omega} \max_{|u| \leq R} |l(t', s, u) - l(t'', s, u)| ds = 0.$$

We omit the proofs of these well-known facts which may be found, for example, in [9].

## 5. The case $L_p$ .

The simplicity of Theorem 4 and Theorem 5 is a strong motivation to study Uryson integral equations as operator equations in the space  $L_{\infty}$  or in the space  $C$ . Unfortunately, it may happen that the corresponding Uryson operator does not act in the space  $L_{\infty}$  or  $C$ , or that one is just interested in unbounded solutions. In these cases it is natural to study the given integral equation in some other Banach space  $X$ ; a classical choice is here  $X = L_p$  for  $1 \leq p < \infty$ . However, the use of these spaces leads to serious difficulties. As was shown in [2], the classical theorems on the convergence of the Newton-Kantorovich approximations [6, 8, and also 15-17] can be applied to the Uryson integral equation (4) in  $L_p$  only in case  $p \geq 2$ . Similarly, the generalizations of the classical theorems, where the Lipschitz condition for the derivative of the nonlinear operator involved is replaced by a Hölder condition with exponent  $\alpha$  ( $0 < \alpha < 1$ ), apply only in  $L_p$  for  $p \geq 1 + \alpha$  [1]. In Section 3 we have seen that it is impossible to use the space  $L_1$  even in the case when the derivative of the operator involved is uniformly continuous on the ball  $B(0, R)$  (see Lemma 4 and following paragraph). The last assumption seems to be «maximal» in this kind of theorems.

Unfortunately, the utilization of the space  $L_p$  ( $1 < p < \infty$ ) in the analysis of the Uryson equation (14) is connected with yet another difficulty. As a matter of fact, in the spaces  $L_p$  for  $1 < p < \infty$  one cannot give explicit formulas for the functions  $\omega(r)$  and  $\theta(r)$ , for the simple reason that one does not know, except for trivial special cases, explicit formulas for the norm of a linear integral operator in these spaces.

All this emphasizes the need of changing the statement of the problem in general. Below we will restrict ourselves to the description of a class of nonlinearities  $k(t, s, u)$  for which the corresponding

Uryson integral equation (14) can be studied successfully in the space  $L_p$  for  $1 < p < \infty$ .

To this end, we need some standard definitions and notation. Let  $1 \leq p, q \leq \infty$ . As above, by  $\mathcal{L}(L_p, L_q)$  we denote the space of all bounded linear operators with the usual norm, and by  $\mathcal{K}(L_p, L_q)$  the kernel space with norm induced from  $\mathcal{L}(L_p, L_q)$ . Moreover, by  $\mathcal{Z}(L_p, L_q)$  we denote the (Zaanen) space of kernels which generate regular (see [9] for the definition) integral operators from  $L_p$  into  $L_q$ , endowed with the norm

$$(45) \quad \|k(t, s)\|_{\mathcal{Z}(L_p, L_q)} = \| \|k(t, s)\| \|_{\mathcal{K}(L_p, L_q)}.$$

Calculating these norms for given kernels is, in general, a difficult problem. In most application, however, one may use some simple classical inequalities (see [6, 9, 12, 13]), namely the direct Hille-Tamarkin inequality

$$\|k(t, s)\|_{\mathcal{Z}(L_p, L_q)} \leq \left[ \int_{\Omega} \left( \int_{\Omega} |k(t, s)|^{p/(p-1)} ds \right)^{(q(p-1))/p} dt \right]^{1/q},$$

the dual Hille-Tamarkin inequality

$$\|k(t, s)\|_{\mathcal{Z}(L_p, L_q)} \leq \left[ \int_{\Omega} \left( \int_{\Omega} |k(t, s)|^q dt \right)^{p/q(p-1)} ds \right]^{(p-1)/p},$$

or the Schur-Kantorovich inequality

$$\|k(t, s)\|_{\mathcal{Z}(L_p, L_q)} \leq \left[ \int_{\Omega} \left( \int_{\Omega} |k(t, s)|^{r_0} ds \right)^{q_0/r_0} dt \right]^{(1-\lambda)/q_0} \left[ \int_{\Omega} \left( \int_{\Omega} |k(t, s)|^{r_1} dt \right)^{q_1/r_1} ds \right]^{\lambda/q_1};$$

here the numbers  $r_0, r_1, q_0, q_1 \in (0, \infty)$  and  $\lambda \in (0, 1)$  are connected by the conditions

$$\frac{1-\lambda}{r_0} + \frac{\lambda}{r_1} \leq 1, \quad \frac{1}{p} = 1 - \frac{1-\lambda}{r_0} - \frac{\lambda}{q_1}, \quad \frac{1}{q} = 1 - \frac{1-\lambda}{q_0} - \frac{\lambda}{r_1}.$$

There are more complicated inequalities based on sophisticated interpolation theorems like the classical Marcinkiewicz-Stein-Weiss theorem; we refer the reader to [9].

Suppose that  $k(t, s, u)$  and  $l(t, s, u)$  satisfy a Carathéodory condi-



tion, and let  $0 < \alpha < 1$ . We define a function  $h_\alpha$  of four variables by

$$(46) \quad h_\alpha(t, s, u_1, u_2) = \frac{l(t, s, u_1) - l(t, s, u_2)}{|u_1 - u_2|^\alpha} \quad (u_1 \neq u_2);$$

this may be considered as a Hölder analogue to the Hadamard function. Assume that the corresponding (generalized) superposition operator

$$(47) \quad \mathbf{h}_\alpha(x_1, x_2)(t, s) = h_\alpha(t, s, x_1(s), x_2(s))$$

is bounded from the space  $L_p \times L_p$  into the kernel space  $\mathfrak{K}(L_{p/(1+\alpha)}, L_p)$ . We have then

$$[L(x_1) - L(x_2)](t) = \int_{\Omega} h_\alpha(t, s, x_1(s), x_2(s)) |x_1(s) - x_2(s)|^\alpha h(s) ds,$$

and therefore

$$\|L(x_1) - L(x_2)\|_{\mathfrak{K}(L_p, L_p)} \leq \|\mathbf{h}_\alpha\|_{\mathfrak{K}(L_{p/(1+\alpha)}, L_p)} \|x_1 - x_2\|^\alpha.$$

The boundedness of the operator (47) implies that

$$c(R) = \sup_{\|x_1\|_{L_p}, \|x_2\|_{L_p} \leq R} \|\mathbf{h}_\alpha(x_1, x_2)\|_{\mathfrak{K}(L_{p/(1+\alpha)}, L_p)} < \infty$$

and

$$\|L(x_1) - L(x_2)\|_{\mathfrak{K}(L_p, L_p)} \leq c(R) \|x_1 - x_2\|^\alpha.$$

These simple calculations are useful and effective in our problem discussed above; however, they require the computation of the norms of integral operators between  $L_{p/(1+\alpha)}$  and  $L_p$  in terms of their kernels which, as we already remarked, is not possible.

Here is a pleasant exception, where this can be done; the following theorem is in fact a direct consequence of the definition of the kernel class  $\mathfrak{Z}(L_p, L_q)$ :

**THEOREM 6.** *Suppose that the kernel  $k(t, s, u)$  satisfies the following three conditions:*

$$(48) \quad \int_{\Omega} k(t, s, 0) ds \in L_p,$$

$$(49) \quad \partial_u k(t, s, 0) \in \mathfrak{Z}(L_p, L_p),$$

$$(50) \quad |\partial_u k(t, s, u_1) - \partial_u k(t, s, u_2)| \leq \sum_{j=0}^m c_j(t, s) |u_1|^{\mu_j} |u_2|^{\nu_j} |u_1 - u_2|^\alpha,$$

where

$$\mu_j, \nu_j \geq 0, \quad \mu_j + \nu_j \leq p - 1 - \alpha, \quad c_j(t, s) \in \mathcal{Z}(L_{p(1+p+\mu_j+\nu_j)}, L_p)$$

for  $j = 0, 1, \dots, m$ . Then

$$\omega(r) \leq \left[ \sum_{j=0}^m \|c_j(t, s)\|_{\mathcal{Z}(L_{p(1+\alpha+\mu_j+\nu_j)}, L_p)} R^{\mu_j + \nu_j} \right] r^\alpha$$

and

$$\theta(r) \leq \sum_{\{j: \nu_j=0\}} \|c_j(t, s)\|_{\mathcal{Z}(L_{p(1+\alpha+\mu_j)}, L_p)} r^{\mu_j}.$$

### 6. The case $L_M$ .

Whenever one has to deal with operator equations involving strong nonlinearities (for example, of exponential growth), it is a useful device to consider these equations not in Lebesgue spaces, but in Orlicz spaces. The constructions and results of the preceding section carry over to Orlicz spaces almost without changes. However, since Orlicz spaces have a more complicated structure than Lebesgue spaces, it is not surprising that the corresponding calculations become more tedious. Therefore, we omit the technical details in the following discussion, and present only some important facts. Detailed information on the definition and the basic properties of Orlicz spaces may be found in [7, 12].

Given two Orlicz spaces  $L_M$  and  $L_N$ , the spaces  $\mathcal{L}(L_M, L_N)$ ,  $\mathcal{X}(L_M, L_N)$  and  $\mathcal{Z}(L_M, L_N)$  are defined analogously as before for Lebesgue spaces. Let us call a pair of Young functions  $M_1(u)$  and  $M_2(u)$  admissible if the product  $x_1 x_2$  of any two functions  $x_1 \in L_{M_1}$  and  $x_2 \in L_{M_2}$  is integrable. In this case the Young function  $M = M_1 \odot M_2$  defined by the formula

$$M(u) = \inf \{M_1(u_1) + M_2(u_2): 0 < u_1, u_2 < \infty, u_1 u_2 = |u|\}$$

generates an Orlicz space  $L_M$  which is called the  $\odot$ -product of the Orlicz spaces  $L_{M_1}$  and  $L_{M_2}$ . In the same way one can define admissible triples, quadruples, and  $n$ -tuples of Young functions and corresponding  $\odot$ -products of several factors.

Let  $\omega$  be a concave function staisfying the condition

$$\lim_{u \rightarrow 0} \omega(u) = 0 .$$

We associate with  $\omega$  the function of four variables

$$(51) \quad h_{(\omega)}(t, s, u_1, u_2) = \frac{\partial_u k(t, s, u_1) - \partial_u k(t, s, u_2)}{\omega(|u_1 - u_2|)} \quad (u_1 \neq u_2);$$

in case  $\omega(u) = |u|^\alpha$  we get of course the function (46). Assume that the corresponding (generalized) superposition operator

$$(52) \quad \mathbf{h}_{(\omega)}(x_1, x_2)(t, s) = h_{(\omega)}(t, s, x_1(s), x_2(s))$$

is bounded from the space  $L_M \times L_M$  into the kernel space  $\mathfrak{X}(L_{M_\omega}, L_M)$ , where  $M_\omega(u) = M(\omega^{(-1)}(u))$ . Then we have

$$\|L(x_1) - L(x_2)\|_{\mathfrak{X}(L_M, L_M)} \leq c(R) \omega(\|x_1 - x_2\|),$$

where

$$c(R) = \sup_{\|x_1\|_{L_M}, \|x_2\|_{L_M} \leq R} \|\mathbf{h}_{(\omega)}(x_1, x_2)\|_{\mathfrak{X}(L_{M_\omega}, L_M)} < \infty .$$

Of course, here we encounter again the problem of computing the norm of an integral operator in terms of its kernel, but this time in an Orlicz space which is still more difficult than in a Lebesgue space (see e.g. [7, 12, 13]). However, also in this case we may formulate sufficient conditions which give a parallel result to Theorem 6 for Orlicz spaces:

**THEOREM 7.** *Suppose that the kernel  $k(t, s, u)$  satisfies the following three conditions:*

$$(53) \quad \int_{\Omega} k(t, s, 0) ds \in L_M ,$$

$$(54) \quad l(t, s, 0) \in \mathfrak{Z}(L_M, L_M) ,$$

$$(55) \quad |l(t, s, u_1) - l(t, s, u_2)| \leq \sum_{j=0}^m c_j(t, s) \phi_j(u_1) \psi_j(u_2) \omega(|u_1 - u_2|) ,$$

where  $\phi_j(u)$  and  $\psi_j(u)$  are continuous functions such that the corresponding superposition operators satisfy the acting conditions

$$(56) \quad \phi_j: L_M \rightarrow L_{P_j}, \quad \psi_j: L_M \rightarrow L_{Q_j} ,$$

the Young functions  $M(u)$ ,  $M_\omega(u)$ ,  $P_j(u)$  and  $Q_j(u)$  are admissible, and

$$c_j(t, s) \in \mathcal{Z}(L_{P_j} \circ Q_j \circ M_\omega \circ M, L_M)$$

for  $j = 0, 1, \dots, m$ . Then

$$\omega(r) \leq \left[ \sum_{j=0}^m \|c_j(t, s)\|_{\mathcal{Z}(L_{P_j} \circ Q_j \circ M_\omega \circ M, L_M)} \phi_j(R) \psi_j(R) \right] \omega(r)$$

and

$$\theta(r) \leq \left[ \sum_{\{j: \psi_j(0) > 0\}} \|c_j(t, s)\|_{\mathcal{Z}(L_{P_j} \circ Q_j \circ M_\omega \circ M, L_M)} \phi_j(r) \psi_j(0) \right] \omega(r).$$

Here

$$(57) \quad \phi(R) = \sup_{\|x\|_{L_M} \leq R} \|\phi_j(x)\|_{L_{P_j}}, \quad \psi_j(R) = \sup_{\|x\|_{L_M} \leq R} \|\psi_j(x)\|_{L_{Q_j}}.$$

The acting conditions (56) are equivalent to the relations

$$\limsup_{x \rightarrow \infty} \frac{P_j(\alpha_j \phi_j(x))}{M(x)} < \infty, \quad \limsup_{x \rightarrow \infty} \frac{Q_j(\beta_j \psi_j(x))}{M(x)} < \infty$$

for suitable  $\alpha_j$  and  $\beta_j$  ( $j = 0, 1, \dots, m$ ). Some formulas useful to calculate the functions (57) can be found in [3].

*Acknowledgments.* This work was performed under the auspices of CNR-Italia and MURST-Italia. The second author was also supported by the Belorussian Foundation of Fundamental Scientific Research and the International Soros Science Education Program. We wish to thank our friend Jurgen Appell who read and improved all the paper, and Julia V. Lysenko, who discussed with us the results obtained in the first part.

We are obliged to the referee for many suggestions and remarks, which have improved the paper.

#### REFERENCES

- [1] J. APPELL - E. DE PASCALE - JU. V. LYSENKO - P. P. ZABREJKO, *New results on Newton-Kantorovich approximations with applications to nonlinear integral equations*, Numer. Funct. Anal. and Optimiz., 18 (1997), pp. 1-17.

- [2] J. APPELL - E. DE PASCALE - P.P. ZABREJKO, *On the application of the Newton-Kantorovich method to nonlinear integral equations of Uryson type*, Numer. Funct. Anal. Optim., **12** (1991), pp. 271-283.
- [3] J. APPELL - P. P. ZABREJKO, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge (1990).
- [4] N. T. DEMIDOVICH - P. P. ZABREJKO - JU. V. LYSENKO, *Some remarks on the Newton-Kantorovich method for nonlinear equations with Hölder continuous linearizations* (in Russian), Izv. Akad. Nauk Beloruss., **3** (1993), pp. 22-26.
- [5] E. HILLE - R. S. PHILLIPS, *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Publ., New York (1957).
- [6] L. V. KANTOROVICH - G. P. AKILOV, *Functional Analysis*, Pergamon Press, Oxford (1982).
- [7] M. A. KRASNOSEL'SKIJ - YA. B. RUTICKII, *Convex Functions and Orlicz Spaces*, Noordhoff (1981).
- [8] M. A. KRASNOSEL'SKIJ - G. M. VAJNIKKO - P. P. ZABREJKO - JA. B. RUTITSKIJ - V. JA. STETSSENKO, *Approximate Solutions of Operator Equations*, Noordhoff, Groningen (1972).
- [9] M. A. KRASNOSEL'SKIJ - P. P. ZABREJKO - JE. I. PUSTYL'NIK - P. JE. SOBOLEVSKIJ, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leyden (1976).
- [10] JU. V. LYSENKO, *Conditions for the convergence of the Newton-Kantorovich method for nonlinear equations with Hölder linearizations nonlinearities* (Russian), Dokl. Akad. Nauk BSSR, **38**, No. 30 (1994), pp. 20-24.
- [11] I. P. MYSOVSIKIH, *On the convergence of L.V. Kantorovich's method of solution of functional equations and its applications* (in Russian), Dokl. Akad. Nauk. SSSR, **70**, No. 4 (1950), pp. 565-568.
- [12] A. C. ZAAENEN, *Linear Analysis*, North-Holland Publ., Amsterdam (1953).
- [13] P. P. ZABREJKO, *Nonlinear Integral Operators* (in Russian), Izd. Voron. Gos. Univ., Voronezh (1966).
- [14] P. P. ZABREJKO - N. L. MAJOROVA, *On the solvability of nonlinear Uryson integral equation* (in Russian), Kach. Pribl. Methody Issled. Oper. Uravn., **3** (1978), pp. 61-73.
- [15] P. P. ZABREJKO - D. F. NGUYEN, *The majorant method in the theory of Newton-Kantorovich approximations and the Pták estimates*, Numer. Funct. Anal. Optim., **9** (1987), pp. 671-684.
- [16] P. P. ZABREJKO - D. NGUYEN, *The Pták estimates in the Newton-Kantorovich method for operator equations* (in Russian), Vestnik Akad. Nauk BSSR **3** (1989), pp. 8-13.
- [17] P. P. ZABREJKO - D. F. NGUYEN, *Estimating the rate of convergence of the Newton-Kantorovich method for equations with Hölder linearizations* (in Russian), Izv. Akad. Nauk BSSR, **2** (1991), pp. 8-14.

Manoscritto pervenuto in redazione il 15 maggio 1996  
e, in forma revisionata, il 18 febbraio 1997.