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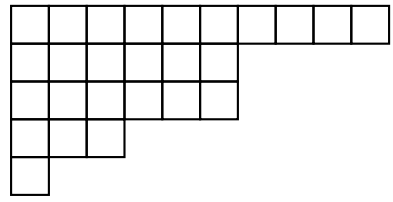
par Jeremy LOVEJOY et Robert OSBURN

Résumé. Nous prouvons des formules pour les fonctions génératrices des différences de rang M_2 pour les partitions où les parts impaires sont distinctes. Ces formules sont en termes de formes modulaires et de séries de Lambert généralisées.

Abstract. We prove formulas for the generating functions for M_2-rank differences for partitions without repeated odd parts. These formulas are in terms of modular forms and generalized Lambert series.

1. Introduction

A partition of a non-negative integer n is a non-increasing sequence whose sum is n. One of the most useful ways to represent a partition is with the Ferrers diagram. For example, the partition $(10, 6, 6, 3, 1)$ is represented by the diagram



MacMahon [18] generalized the Ferrers diagram to an M-modular diagram of a partition. A special case of his construction, the 2-modular diagram is a Ferrers diagram where all of the boxes are filled with 2's except possibly the last box of a row, which may be filled with a 1, with the condition that no 2 occurs directly below a 1. As an illustration, the partition $(10, 10, 8, 7, 7, 4, 2, 2, 1)$ has 2-modular diagram

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2	2	2	2	2
2	2	2	2	2
2	2	2	2	
2	2	2	1	
2	2	2	1	
2	2			
2				
2				
1				

If we add the condition that a 1 may only occur in the last entry of a column, then these 2-modular diagrams correspond to partitions whose odd parts may not be repeated. Partitions without repeated odd parts and their 2-modular diagrams have long played a role in combinatorial studies of q-series identities (see [1, 2, 8, 11], for example). Most recently, Berkovich and Garvan [8] introduced what they called the M_2-rank of such partitions. The M_2-rank of a partition λ without repeated odd parts is defined to be the number of columns minus the number of rows of its 2-modular diagram, or equivalently,

$$M_2\text{-rank}\,(\lambda) = \left\lceil \frac{l(\lambda)}{2} \right\rceil - \nu(\lambda),$$

where $l(\lambda)$ is the largest part of λ and $\nu(\lambda)$ is the number of parts of λ.

The two-variable generating function for the M_2-rank has a particularly nice form. Namely, using the fact that partitions with distinct odd parts correspond to overpartitions in which the odd parts are all overlined, it may be deduced from [16, Theorem 1.2] that if $N_2(m,n)$ denotes the number of partitions of n without repeated odd parts whose M_2-rank is m, then

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} N_2(m,n) z^m q^n = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(zq^2, q^2/z; q^2)_n}. \tag{1.1}$$

Here we have introduced the standard q-series notation [13],

$$(a_1, a_2, \ldots, a_j; q)_n = \prod_{k=0}^{n-1} (1-a_1q^k)(1-a_2q^k)\cdots(1-a_jq^k),$$

following the custom of dropping the "$;q$" unless the base is something other than q.

The generating function for the M_2-rank in (1.1) appears numerous times in Ramanujan's "lost" notebook [3], [5, Ch. 12]. When $z=-1$, we have the mock theta function $\mu(-q)$ of McIntosh [19] and when $z=i$, this

is the mock theta function $U_0(q)$ of Gordon and McIntosh [14, 19]. More generally, Bringmann, Ono, and Rhoades [9] have shown that one obtains the holomorphic part of a weak Maass form when z is replaced by certain roots of unity. There are many nice consequences of this number-theoretic structure, including the fact that the generating function for $N_2(s,\ell,n) - N_2(t,\ell,n)$ will often be a classical modular form when n is restricted to arithmetic progressions. Here $N_2(s,\ell,n)$ denotes the number of partitions of n without repeated odd parts whose M_2-rank is congruent to s modulo ℓ. In this paper we obtain formulas for all of the generating functions $N_2(s,\ell,\ell n+d) - N_2(t,\ell,\ell n+d)$, when $\ell = 3$ or 5, in terms of modular forms and generalized Lambert series. We shall indeed see that many of these functions are simply modular forms.

Using the notation

$$R_{st}(d) = \sum_{n\geq 0} \left(N_2(s,\ell,\ell n+d) - N_2(t,\ell,\ell n+d)\right) q^n, \tag{1.2}$$

where the prime ℓ will always be clear, the main results are summarized in Theorems 1.1 and 1.2 below.

Theorem 1.1. *For $\ell = 3$, we have*

$$\begin{aligned} R_{01}(0) = -1 - 3q^3 \frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{6n^2+9n}}{1-q^{6n+4}} \\ + \frac{(q^6;q^6)_\infty^4 (-q^3;q^3)_\infty^4 (q;q^2)_\infty}{(q^4;q^4)_\infty (q^2,q^{10},q^{12};q^{12})_\infty^2}, \end{aligned} \tag{1.3}$$

$$R_{01}(1) = \frac{(-q^3,q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}, \tag{1.4}$$

$$R_{01}(2) = \frac{(q^3;q^3)_\infty (-q^6;q^6)_\infty}{(q,q^5;q^6)_\infty (q^4,q^8;q^{12})_\infty}. \tag{1.5}$$

Theorem 1.2. *For $\ell = 5$, we have*

$$\begin{aligned} R_{12}(0) = -1 - q^2 \frac{(-q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{10n^2+15n}}{1-q^{10n+2}} \\ + \frac{(q,q^9;q^{10})_\infty^2 (q^6,q^8,q^{12},q^{14};q^{20})_\infty (q^{10};q^{20})_\infty^3 (q^{20};q^{20})_\infty^2}{(q;q)_\infty}, \end{aligned} \tag{1.6}$$

$$R_{12}(1) = 0, \tag{1.7}$$

$$R_{12}(2) = \frac{q(q^2,q^{18};q^{20})_\infty (q^5;q^5)_\infty (-q^{10};q^{10})_\infty}{(q,q^4;q^5)_\infty}, \tag{1.8}$$

$$R_{12}(3) = \frac{(-q^5,q^{10};q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}, \tag{1.9}$$

$$\begin{aligned} R_{12}(4) &= 2q^3 \frac{(-q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{10n^2+15n}}{1-q^{10n+4}} \\ &\quad + \frac{(q^3,q^7,q^{10};q^{10})_\infty^2}{(q;q^2)_\infty (q^6,q^8,q^{12},q^{14},q^{20};q^{20})_\infty}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} R_{02}(0) &= 1 + 2q^2 \frac{(-q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{10n^2+15n}}{1-q^{10n+2}} \\ &\quad - \frac{(q,q^9;q^{10})_\infty^2 (q^{10};q^{10})_\infty^3 (q^6,q^8,q^{12},q^{14};q^{20})_\infty}{(q;q)_\infty (q^{20};q^{20})_\infty}, \end{aligned} \tag{1.11}$$

$$R_{02}(1) = \frac{(-q^5,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty}, \tag{1.12}$$

$$R_{02}(2) = \frac{(q^5;q^5)_\infty (-q^{10};q^{10})_\infty (q^6,q^{14};q^{20})_\infty}{(q^2,q^3;q^5)_\infty}, \tag{1.13}$$

$$R_{02}(3) = 0, \tag{1.14}$$

$$\begin{aligned} R_{02}(4) &= q^3 \frac{(-q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{10n^2+15n}}{1-q^{10n+4}} \\ &\quad + \frac{(q^3,q^7,q^{10};q^{10})_\infty^2}{(q;q^2)_\infty (q^6,q^8,q^{12},q^{14},q^{20};q^{20})_\infty}. \end{aligned} \tag{1.15}$$

To prove Theorems 1.1 and 1.2, we shall roughly follow the method developed by Atkin and Swinnerton-Dyer [6] in their study of Dyson's rank for partitions. This method may be generally described as regarding groups of identities as equalities between polynomials of degree $\ell - 1$ in q whose coefficients are power series in q^ℓ. Specifically, we first consider the expression

$$\sum_{n=0}^{\infty} \Big\{ N_2(s,\ell,n) - N_2(t,\ell,n) \Big\} q^n \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{1.16}$$

By (2.4), (2.5), and (5.3), we write (1.16) as a polynomial in q whose coefficients are power series in q^ℓ. We then alternatively express (1.16) in the same manner using Theorems 1.1 and 1.2 and Lemma 3.1. Finally, we use various q-series identities to show that these two resulting polynomials are the same for each pair of values of s and t.

The paper is organized as follows. In Section 2 we collect some basic definitions, notations and generating functions. In Section 3 we record a number of equalities between an infinite product and a sum of infinite products. These are ultimately required for the simplification of identities that end up being more complex than we would like, principally because there are only two 0's in Theorems 1.1 and 1.2. In Section 4 we prove two key q-series identities relating generalized Lambert series to infinite products, and in Section 5 we give the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

We begin by introducing some notation and definitions, essentially following [6]. With $y = q^\ell$, let

$$r_s(d) := \sum_{n=0}^{\infty} N_2(s, \ell, \ell n + d) y^n$$

and

$$r_{st}(d) := r_s(d) - r_t(d).$$

Thus we have

$$\sum_{n=0}^{\infty} N_2(s, \ell, n) q^n = \sum_{d=0}^{\ell-1} r_s(d) q^d.$$

To abbreviate the sums appearing in Theorems 1.1 and 1.2, we define

$$\Sigma(z, \zeta, q) := \sum_{n \in \mathbb{Z}} \frac{(-1)^n \zeta^{4n} q^{2n^2+3n}}{1 - z^2 q^{2n}}.$$

Henceforth we assume that a is not a multiple of ℓ. We write

$$\Sigma(a, b) := \Sigma(y^a, y^b, y^\ell) = \sum_{n \in \mathbb{Z}} \frac{(-1)^n y^{4bn+\ell n(2n+3)}}{1 - y^{2\ell n+2a}}$$

and

$$\Sigma(0, b) := \sum_{n \in \mathbb{Z}}{}' \frac{(-1)^n y^{4bn+\ell n(2n+3)}}{1 - y^{2\ell n}},$$

where the prime means that the term corresponding to $n = 0$ is omitted.

To abbreviate the products occurring in Theorems 1.1 and 1.2, we define

$$P(z,q) := \prod_{r=1}^{\infty}(1-zq^{r-1})(1-z^{-1}q^{r})$$

and

$$P(0) := \prod_{r=1}^{\infty}(1-y^{2\ell r}).$$

We have the relations

$$P(z^{-1}q,q) = P(z,q) \tag{2.1}$$

and

$$P(zq,q) = -z^{-1}P(z,q). \tag{2.2}$$

Now, for any integer m we have (see Section 5 of [8] or [16])

$$\sum_{n\geq 0} N_2(m,n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 1}(-1)^{n+1}q^{2n^2-n+2|m|n}(1-q^{2n}). \tag{2.3}$$

It is then a simple matter to deduce that

$$\sum_{n=0}^{\infty} N_2(s,m,n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\in\mathbb{Z}}{}' \frac{(-1)^n q^{2n^2+n}(q^{2sn}+q^{2(m-s)n})}{1-q^{2mn}}. \tag{2.4}$$

Unfortunately, it does not appear that one can go directly from differences of (2.4) to the formulas in Theorems 1.1 and 1.2. Hence it will be beneficial to consider sums of the form

$$S_2(b) := \sum_{n\in\mathbb{Z}}{}' \frac{(-1)^n q^{2n^2+bn}}{1-q^{2\ell n}}. \tag{2.5}$$

We will require the relation

$$S_2(b) = -S_2(2\ell - b), \tag{2.6}$$

which follows from the substitution $n \to -n$ in (2.5). We shall also require the fact that

$$S_2(b) - S_2(2\ell+b) = \sum_{n\in\mathbb{Z}}(-1)^n q^{2n^2+bn} - 1 = (q^{2+b}, q^{2-b}, q^4; q^4)_\infty - 1 \tag{2.7}$$

if b is odd. This follows by applying the case $z = -q^b$ and replacing q with q^2 in Jacobi's triple product identity

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty. \tag{2.8}$$

3. Infinite product identities

In this section we record some identities involving infinite products. These will be needed later on for simplification and verification of certain identities. First, we have a result which is the analogue of Lemma 6 in [6] and Lemma 3.1 in [17]. The proof, which just amounts to an application of (2.8), is similar to that of Lemma 3.1 in [17] and thus is omitted.

Lemma 3.1. *We have*

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = (q^3, -q^6, -q^9, -q^{12}, q^{15}, q^{18}; q^{18})_\infty - q(q^9, q^{27}, q^{36}; q^{36})_\infty \tag{3.1}$$

and

$$\begin{aligned}\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} &= (-q^{10}, q^{15}, -q^{25}, q^{35}, -q^{40}, q^{50}; q^{50})_\infty \\ &\quad - q(q^5, -q^{20}, -q^{25}, -q^{30}, q^{45}, q^{50}; q^{50})_\infty \\ &\quad - q^3(q^{25}, q^{75}, q^{100}; q^{100})_\infty.\end{aligned} \tag{3.2}$$

Next, we quote a result of Hickerson [15, Theorem 1.1] along with some of its corollaries.

Lemma 3.2.

$$\begin{aligned}P(x,q)P(z,q)(q)_\infty^2 &= P(-xz,q^2)P(-qz/x,q^2)(q^2;q^2)_\infty^2 \\ &\quad - xP(-xzq,q^2)P(-z/x,q^2)(q^2;q^2)_\infty^2.\end{aligned}$$

The first corollary was recorded by Hickerson [15, Theorem 1.2]. It follows by applying Lemma 3.2 twice, once with x replaced by $-x$ and once with z replaced by $-z$, and then subtracting.

Lemma 3.3.

$$\begin{aligned}P(-x,q)P(z,q)(q)_\infty^2 - P(x,q)P(-z,q)(q)_\infty^2 &= 2xP(z/x,q^2) \\ &\quad \times P(xzq,q^2)(q^2;q^2)_\infty^2.\end{aligned}$$

The second corollary follows just as the first, except we add instead of subtract in the final step.

Lemma 3.4.

$$\begin{aligned}P(-x,q)P(z,q)(q)_\infty^2 + P(x,q)P(-z,q)(q)_\infty^2 &= 2P(xz,q^2) \\ &\quad \times P(qz/x,q^2)(q^2;q^2)_\infty^2.\end{aligned}$$

4. Two lemmas

Theorems 1.1 and 1.2 will follow from identities which relate the sums $\Sigma(a,b)$ to products $P(z,q)$. The key steps are the two Lemmas below. These results are similar in nature to Lemmas 7 and 8 in [6] and Lemmas 4.1 and 4.2 in [17].

Lemma 4.1. *We have*

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+3n}\Big[\frac{\zeta^{-4n}}{1-z^2\zeta^{-2}q^{2n}}+\frac{\zeta^{4n+6}}{1-z^2\zeta^2q^{2n}}\Big]\\
&=\frac{\zeta^2(-q,-q,\zeta^4,q^2\zeta^{-4};q^2)_\infty}{(\zeta^2,q^2\zeta^{-2},-q\zeta^2,-q\zeta^{-2};q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{2n^2+3n}}{1-z^2q^{2n}}\\
&+\frac{(-z^2q,-qz^{-2},\zeta^4,q^2\zeta^{-4},\zeta^2,q^2\zeta^{-2};q^2)_\infty(q^2;q^2)_\infty^2}{(z^2\zeta^{-2},q^2\zeta^2z^{-2},z^2\zeta^2,q^2z^{-2}\zeta^{-2},z^2,q^2z^{-2},-q\zeta^2,-q\zeta^{-2};q^2)_\infty}.
\end{aligned}\tag{4.1}$$

Proof. This is just the case $r=1$, $s=3$, $q=q^2$, $a_1=-z^2q$, $b_1=z^2/\zeta^2$, $b_2=z^2\zeta^2$, and $b_3=z^2$ of [10, Theorem 2.1],

$$\begin{aligned}
&\frac{P(a_1,q)\cdots P(a_r,q)(q)_\infty^2}{P(b_1,q)\cdots P(b_s,q)}\\
&=\frac{P(a_1/b_1,q)\cdots P(a_r/b_1,q)}{P(b_2/b_1,q)\cdots P(b_s/b_1,q)}\\
&\times\sum_{n\in\mathbb{Z}}\frac{(-1)^{(s-r)n}q^{(s-r)n(n+1)/2}}{1-b_1q^n}\left(\frac{a_1\cdots a_rb_1^{s-r-1}}{b_2\cdots b_s}\right)^n\\
&+\operatorname{idem}(b_1;b_2,\dots,b_s).
\end{aligned}\tag{4.2}$$

Here we use the usual notation

$$\begin{aligned}
F(b_1,b_2,\dots,b_m)+\operatorname{idem}(b_1;b_2,\dots,b_m)&\\
:=F(b_1,b_2,\dots,b_m)+F(b_2,b_1,b_3,\dots,b_m)&\\
+\cdots+F(b_m,b_2,\dots,b_{m-1},b_1).&
\end{aligned}$$

□

We now specialize Lemma 4.1 to the case $\zeta=y^a$, $z=y^b$, and $q=y^\ell$:

$$\begin{aligned}
&y^{6a}\Sigma(b+a,a)+\Sigma(b-a,-a)\\
&-y^{2a}\frac{P(-y^\ell,y^{2\ell})P(y^{4a},y^{2\ell})}{P(y^{2a},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}\Sigma(b,0)\\
&-\frac{P(-y^{2b+\ell},y^{2\ell})P(y^{4a},y^{2\ell})P(y^{2a},y^{2\ell})P(0)^2}{P(y^{2b-2a},y^{2\ell})P(y^{2b+2a},y^{2\ell})P(y^{2b},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}=0.
\end{aligned}\tag{4.3}$$

We now define

$$g(z,q) := z^2\frac{P(-q,q^2)P(z^4,q^2)}{P(z^2,q^2)P(-qz^2,q^2)}\Sigma(z,1,q) - z^6\Sigma(z^2,z,q) - {\sum_{n=-\infty}^{\infty}}'\frac{(-1)^n z^{-4n}q^{n(2n+3)}}{1-q^{2n}}$$

and

$$g(a) := g(y^a,y^\ell) = y^{2a}\frac{P(-y^\ell,y^{2\ell})P(y^{4a},y^{2\ell})}{P(y^{2a},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}\Sigma(a,0) - y^{6a}\Sigma(2a,a) - \Sigma(0,-a). \tag{4.4}$$

The second key lemma is the following.

Lemma 4.2. *We have*

$$2g(z,q) - g(z^2,q) + 1 = \frac{P(qz^2,q^2)P(-z^2,q^2)P(0)^2P(-1,q)^2}{P(-qz^2,q^2)P(z^2,q^2)P(-1,q^2)^2} + \frac{z^4}{2}\frac{P(q^2z^{16},q^4)P(-1,q^2)(q)_\infty^2}{P(z^8,q)} \tag{4.5}$$

and

$$g(z,q) + g(z^{-1}q,q) = 0. \tag{4.6}$$

Proof. We first require a short computation involving $\Sigma(z,\zeta,q)$. Note that

$$\begin{aligned} z^4\Sigma(z,\zeta,q) + q\zeta^4\Sigma(zq,\zeta,q) &= \sum_{n=-\infty}^{\infty}(-1)^n\frac{z^4\zeta^{4n}q^{n(2n+3)}}{1-z^2q^{2n}} \\ &\quad + \sum_{n=-\infty}^{\infty}(-1)^n\frac{\zeta^{4n+4}q^{n(2n+3)+1}}{1-z^2q^{2n+2}} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n\zeta^{4n}q^{n(2n-1)}\Big(\frac{z^4q^{4n}-1}{1-z^2q^{2n}}\Big) \\ &= -\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{4n}q^{n(2n-1)}(1+z^2q^{2n}) \end{aligned} \tag{4.7}$$

upon writing $n-1$ for n in the second sum of the first equation. Taking $\zeta = 1$ yields

$$z^4\Sigma(z,1,q)+q\Sigma(zq,1,q)=-\sum_{n=-\infty}^{\infty}(-1)^n q^{n(2n-1)}(1+z^2q^{2n}). \tag{4.8}$$

Now write $g(z,q)$ in the form

$$g(z,q)=f_1(z)-f_2(z)-f_3(z)$$

where

$$f_1(z):=z^2\frac{P(-q,q^2)P(z^4,q^2)}{P(z^2,q^2)P(-z^2q,q^2)}\Sigma(z,1,q),$$

$$f_2(z):=z^6\Sigma(z^2,z,q),$$

and

$$f_3(z):=\sum_{n=-\infty}^{\infty}{}'\ \frac{(-1)^n z^{-4n}q^{n(2n+3)}}{1-q^{2n}}.$$

By (2.1) and (2.2) (replacing the base q with q^2), and (4.8),

$$f_1(zq)-f_1(z)=(z^{-2}+1)\sum_{n=-\infty}^{\infty}(-1)^n q^{n(2n+1)}\frac{P(z^4,q^2)P(-q,q^2)}{P(z^2,q^2)P(-z^2q,q^2)}. \tag{4.9}$$

A similar argument as in (4.7) yields

$$\begin{aligned} f_2(zq)-f_2(z)=&\sum_{n=-\infty}^{\infty}(-1)^n z^{4n-2}q^{n(2n-1)}\\ &+\sum_{n=-\infty}^{\infty}(-1)^n z^{4n+2}q^{n(2n+1)} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} f_3(zq)-f_3(z)=&-2+\sum_{n=-\infty}^{\infty}(-1)^n z^{-4n}q^{n(2n-1)}\\ &+\sum_{n=-\infty}^{\infty}(-1)^n z^{-4n}q^{n(2n+1)}. \end{aligned} \tag{4.11}$$

Adding (4.10) and (4.11), then subtracting from (4.9) gives

$$g(z,q)-g(zq,q)=-2. \tag{4.12}$$

Here we have used the identity

$$
\begin{aligned}
&(z^{-2}+1)\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}\frac{P(z^4,q^2)P(-q,q^2)}{P(z^2,q^2)P(-z^2q,q^2)}\\
&=\sum_{n=-\infty}^{\infty}(-1)^n z^{4n-2}q^{n(2n-1)}+\sum_{n=-\infty}^{\infty}(-1)^n z^{4n+2}q^{n(2n+1)}\\
&+\sum_{n=-\infty}^{\infty}(-1)^n z^{-4n}q^{n(2n-1)}+\sum_{n=-\infty}^{\infty}(-1)^n z^{-4n}q^{n(2n+1)}
\end{aligned}
\tag{4.13}
$$

which follows from [13, Ex. 5.5, p.134], the triple product identity (2.8), and a little simplification. If we now define

$$
\begin{aligned}
f(z) :=\ & 2g(z,q)-g(z^2,q)+1-\frac{P(qz^2,q^2)P(-z^2,q^2)P(0)^2P(-1,q)^2}{P(-qz^2,q^2)P(z^2,q^2)P(-1,q^2)^2}\\
&-\frac{z^4}{2}\frac{P(q^2z^{16},q^4)P(-1,q^2)(q)_\infty^2}{P(z^8,q)},
\end{aligned}
$$

then from (2.1), (2.2), and (4.12), one can verify that

$$
f(zq)-f(z)=0. \tag{4.14}
$$

Now, it follows from a routine complex analytic argument similar to the proof of Lemma 4.2 in [17] (see also Lemma 2 in [6]) that $f(z)=0$. This proves (4.5).

To prove (4.6), it suffices to show, after (4.12),

$$
g(z^{-1},q)+g(z,q)=-2. \tag{4.15}
$$

Note that

$$
\begin{aligned}
\Sigma(z,1,q)+z^{-6}\Sigma(z^{-1},1,q) &= \sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(2n+3)}}{1-z^2q^{2n}}\\
&-z^{-4}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{n(2n-1)}}{1-z^2q^{2n}}\\
&=-z^{-4}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(2n-1)}(1+z^2q^{2n})
\end{aligned}
\tag{4.16}
$$

where we have written $-n$ for n in the second sum in the first equation. Thus, by (2.1), (2.2), and (4.16), we have

$$\begin{aligned} &f_1(z)+f_1(z^{-1}) \\ &=-z^{-2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(2n-1)}(1+z^2q^{2n})\frac{P(-q,q^2)P(z^4,q^2)}{P(z^2,q^2)P(-z^2q,q^2)}. \end{aligned} \tag{4.17}$$

Again, a similar argument as in (4.16) gives

$$f_2(z)+f_2(z^{-1})=-z^{-2}\sum_{n=-\infty}^{\infty}(-1)^n z^{4n}q^{n(2n-1)}(1+z^4q^{2n}) \tag{4.18}$$

and

$$f_3(z)+f_3(z^{-1})=2-\sum_{n=-\infty}^{\infty}(-1)^n z^{4n}q^{n(2n-1)}(1+q^{2n}). \tag{4.19}$$

Adding (4.18) and (4.19), then subtracting from (4.17) yields (4.15). Here we have used the identity

$$\begin{aligned} &z^{-2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(2n-1)}(1+z^2q^{2n})\frac{P(-q,q^2)P(z^4,q^2)}{P(z^2,q^2)P(-z^2q,q^2)} \\ &=z^{-2}\sum_{n=-\infty}^{\infty}(-1)^n z^{4n}q^{n(2n-1)}(1+z^4q^{2n}) \\ &+\sum_{n=-\infty}^{\infty}(-1)^n z^{4n}q^{n(2n-1)}(1+q^{2n}) \end{aligned} \tag{4.20}$$

which is easily seen to be equivalent to (4.13).

□

Letting $z=y^a$ and $q=y^\ell$ in Lemma 4.2, we get

$$\begin{aligned} &2g(a)-g(2a)+1 \\ &=\frac{P(y^{\ell+2a},y^{2\ell})P(-y^{2a},y^{2\ell})P(0)^2P(-1,y^\ell)^2}{P(-y^{\ell+2a},y^{2\ell})P(y^{2a},y^{2\ell})P(-1,y^{2\ell})^2} \\ &+\frac{y^{4a}}{2}\frac{P(y^{2\ell+16a},y^{4\ell})P(-1,y^{2\ell})(y^\ell;y^\ell)_\infty^2}{P(y^{8a},y^\ell)} \end{aligned} \tag{4.21}$$

and

$$g(a)+g(\ell-a)=0. \tag{4.22}$$

These two identities will be of key importance in the proofs of Theorems 1.1 and 1.2.

5. Proofs of Theorems 1.1 and 1.2

We now compute the sums $S_2(3\ell - 4m)$. The reason for this choice is two-fold. First, we would like to obtain as simple an expression as possible in the final formulation (5.3). Secondly, to prove Theorem 1.1, we only need to compute $S_2(1)$ whereas to prove Theorem 1.2, we need $S_2(11)$ and $S_2(7)$. The latter in turn yields $S_2(1)$ and $S_2(3)$ via (2.6) and (2.7). For $\ell = 3$, we can choose $m = 2$ and for $\ell = 5$, $m = 1$ and $m = 2$ respectively. As this point, we follow the idea of Section 6 in [6]. Namely, we write

$$n = \ell r + m + b, \tag{5.1}$$

where $-\infty < r < \infty$. The idea is to simplify the exponent of q in $S_2(3\ell - 4m)$. Thus

$$(3\ell - 4m)n + 2n^2 = \ell^2 r(2r + 3) + 2(b + m)(b - m + \ell) + \ell(m + b) + 4b\ell r.$$

We now substitute (5.1) into (2.5) and let b take the values 0, $\pm a$, and $\pm m$. Here a runs through 1, 2, ... , $\frac{\ell-1}{2}$ where the value $a \equiv \pm m \bmod \ell$ is omitted. As in [6], we use the notation $\sum_a''$ to denote the sum over these values of a. We thus obtain

$$\begin{aligned} S_2(3\ell - 4m) &= \sum_{n=-\infty}^{\infty}{}' (-1)^n \frac{q^{(3\ell-4m)n+2n^2}}{1 - y^{2n}} \\ &= \sum_b \sum_{r=-\infty}^{\infty}{}' (-1)^{r+m+b} y^{m+b} q^{2(b+m)(b-m+\ell)} \frac{y^{\ell r(2r+3)+4br}}{1 - y^{2(\ell r+m+b)}}, \end{aligned}$$

where b takes values 0, $\pm a$, and $\pm m$ and the term corresponding to $r = 0$ and $b = -m$ is omitted. Thus

$$\begin{aligned} &S_2(3\ell - 4m) \\ &= (-1)^m y^m q^{2m(\ell-m)} \Sigma(m, 0) + \Sigma(0, -m) + y^{6m} \Sigma(2m, m) \\ &+ \sum_a'' \Big\{ (-1)^{m+a} y^{m+a} q^{2(a+m)(a-m+\ell)} \\ &\times \Big\{ \Sigma(m + a, a) + y^{-6a} \Sigma(m - a, -a) \Big\} \Big\}. \end{aligned} \tag{5.2}$$

Here the first three terms arise from taking $b = 0$, $-m$, and m respectively. We now can use (4.3) to simplify this expression. By taking $b = m$ and

dividing by y^{6a} in (4.3), the sum of the two terms inside the curly brackets becomes

$$y^{-4a}\frac{P(-y^{\ell},y^{2\ell})P(y^{4a},y^{2\ell})}{P(y^{2a},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}\Sigma(m,0)$$
$$+\,y^{-6a}\frac{P(-y^{2m+\ell},y^{2\ell})P(y^{4a},y^{2\ell})P(y^{2a},y^{2\ell})P(0)^2}{P(y^{2m-2a},y^{2\ell})P(y^{2m+2a},y^{2\ell})P(y^{2m},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}.$$

Similarly, upon taking $a=m$ in (4.4), then the sum of the second and third terms in (5.2) is

$$y^{2m}\frac{P(-y^{\ell},y^{2\ell})P(y^{4m},y^{2\ell})}{P(y^{2m},y^{2\ell})P(-y^{2m+\ell},y^{2\ell})}\Sigma(m,0)-g(m).$$

In total, we have

$$\begin{aligned}
&S_2(3\ell-4m)=\\
&-g(m)\\
&+\sum_a{}''\Bigg\{(-1)^{m+a}y^{m-5a}q^{2(a+m)(a-m+\ell)}\\
&\times\frac{P(-y^{2m+\ell},y^{2\ell})P(y^{4a},y^{2\ell})P(y^{2a},y^{2\ell})P(0)^2}{P(y^{2m-2a},y^{2\ell})P(y^{2m+2a},y^{2\ell})P(y^{2m},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}\Bigg\}\\
&+\Sigma(m,0)\Bigg\{(-1)^m y^m q^{2m(\ell-m)}+y^{2m}\frac{P(-y^{\ell},y^{2\ell})P(y^{4m},y^{2\ell})}{P(y^{2m},y^{2\ell})P(-y^{2m+\ell},y^{2\ell})}\\
&+\sum_a{}''(-1)^{m+a}y^{m-3a}q^{2(a+m)(a-m+\ell)}\frac{P(-y^{\ell},y^{2\ell})P(y^{4a},y^{2\ell})}{P(y^{2a},y^{2\ell})P(-y^{2a+\ell},y^{2\ell})}\Bigg\}.
\end{aligned}\tag{5.3}$$

We can simplify some of the terms appearing in (5.3) as we are interested in certain values of ℓ, m, and a. To this end, we prove the following result. Let $\{\quad\}$ denote the coefficient of $\Sigma(m,0)$ in (5.3).

Proposition 5.1. *If $\ell=3$ and $m=2$, then*

$$\{\quad\}=-q^9\frac{(q^2;q^2)_\infty(-q^9;q^{18})_\infty}{(-q;q^2)_\infty(q^{18};q^{18})_\infty}.$$

If $\ell=5$, $m=2$, and $a=1$, then

$$\{\quad\}=-q^{19}\frac{(q^2;q^2)_\infty(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty(q^{50};q^{50})_\infty}.$$

If $\ell=5$, $m=1$, $a=2$, then

$$\{\quad\} = q^{10}\frac{(q^2;q^2)_\infty(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty(q^{50};q^{50})_\infty}.$$

Proof. These are easily deduced from Lemma 3.1. □

We are now in a position to prove Theorems 1.1 and 1.2. We begin with Theorem 1.1.

Proof. By (2.4), (2.5), and (2.6), we have

$$\sum_{n=0}^{\infty}\Big\{N_2(0,3,n) - N_2(1,3,n)\Big\}q^n\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = 2S_2(1) + S_2(7). \tag{5.4}$$

By (2.1), (2.2), (5.3), and Proposition 5.1 we have

$$S_2(1) = -g(2) - y^3\frac{(q^2;q^2)_\infty(-q^9;q^{18})_\infty}{(-q;q^2)_\infty(q^{18};q^{18})_\infty}\Sigma(2,0). \tag{5.5}$$

Taking $b = 1$ in (2.7) yields

$$S_2(1) - S_2(7) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} - 1. \tag{5.6}$$

By (5.4), (5.5), and (5.6), we have that

$$\begin{aligned}&- 3g(2) - 3y^2\frac{(q^2;q^2)_\infty(-q^9;q^{18})_\infty}{(-q;q^2)_\infty(q^{18};q^{18})_\infty}\Sigma(2,0) - \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} + 1\\ &= \Big\{r_{01}(0)q^0 + r_{01}(1)q + r_{01}(2)q^2\Big\}\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.\end{aligned}$$

We now multiply the right hand side of the above expression using Lemma 3.1 and the $r_{01}(d)$ from Theorem 1.1 (recall that $r_{01}(d)$ is just $R_{01}(d)$ with q replaced by q^3). We then equate coefficients of powers of q and verify the resulting identities. The only power of q for which the resulting equation does not follow easily upon cancelling factors in infinite products is the constant term. We obtain

$$\begin{aligned}&- 3g(2) + 1\\ &= \frac{(q^{18};q^{18})_\infty^4(-q^9;q^9)_\infty^4(q^3;q^6)_\infty(q^3,-q^6,-q^9,-q^{12},q^{15},q^{18};q^{18})_\infty}{(q^{12};q^{12})_\infty(q^6,q^{30},q^{36};q^{36})_\infty^2}\\ &- y\frac{(q^9;q^9)_\infty(-q^{18};q^{18})_\infty(q^9,q^{27},q^{36};q^{36})_\infty}{(q^3,q^{15};q^{18})_\infty(q^{12},q^{24};q^{36})_\infty}.\end{aligned}$$

The first term (resp. second term) above is easily seen to be identical to the first term (resp. second term) in (4.21) with $a = 1$. Applying (4.22), this

then establishes the above identity and completes the proof of Theorem 1.1. □

We now turn to Theorem 1.2.

Proof. We begin with the rank differences $R_{12}(d)$. By (2.4), (2.5), and (2.6), we have

$$\sum_{n=0}^{\infty}\Big\{N_2(1,5,n)-N_2(2,5,n)\Big\}q^n\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}=2S_2(3)-S_2(1). \tag{5.7}$$

By (2.1), (2.2), (2.7), (5.3), and Proposition 5.1,

$$\begin{aligned}S_2(1)=&-g(1)+q\frac{P(0)^2P(-y^7,y^{10})}{P(y^2,y^{10})P(-y^9,y^{10})}\\&+y^2\Sigma(1,0)\frac{(q^2;q^2)_\infty(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty(q^{50};q^{50})_\infty}+\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}-1\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}S_2(3)=&\,g(2)+yq^4\frac{P(0)^2P(-y^9,y^{10})}{P(y^4,y^{10})P(-y^7,y^{10})}\\&+y^3q^4\Sigma(2,0)\frac{(q^2;q^2)_\infty(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty(q^{50};q^{50})_\infty}.\end{aligned} \tag{5.9}$$

By (5.7), (5.8), and (5.9), we have

$$\begin{aligned}&2g(2)+2yq^4\frac{P(0)^2P(-y^9,y^{10})}{P(y^4,y^{10})P(-y^7,y^{10})}+2y^3q^4\Sigma(2,0)\frac{(q^2;q^2)_\infty(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty(q^{50};q^{50})_\infty}\\&+g(1)-q\frac{P(0)^2P(-y^7,y^{10})}{P(y^2,y^{10})P(-y^9,y^{10})}-y^2\Sigma(1,0)\frac{(q^2;q^2)_\infty(-q^{25};q^{50})_\infty}{(-q;q^2)_\infty(q^{50};q^{50})_\infty}\\&-\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}+1\\&=\Big\{r_{12}(0)q^0+r_{12}(1)q+r_{12}(2)q^2+r_{12}(3)q^3+r_{12}(4)q^4\Big\}\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.\end{aligned}$$

We now multiply the right hand side of the above expression using Lemma 3.1 and the $R_{12}(d)$ from Theorem 1.2, equating coefficients of powers of q. The coefficients of q^0, q^1, q^2, q^3, q^4 give us, respectively,

$$
\begin{aligned}
&2g(2)+g(1)+1\\
&=(q^5,q^{45};q^{50})_\infty^2(q^{30},q^{40},q^{60},q^{70};q^{100})_\infty(q^{50};q^{100})_\infty^3\\
&\times\frac{(q^{100};q^{100})_\infty^2(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty}\\
&-y\frac{(q^{15},q^{35},q^{50};q^{50})_\infty^2(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty(q^{30},q^{40},q^{60},q^{70},q^{100};q^{100})_\infty}\\
&-y^2\frac{(q^{10},q^{90};q^{100})_\infty(q^{25};q^{25})_\infty(-q^{50};q^{50})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^5,q^{20};q^{25})_\infty},
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
&\frac{(q^{50};q^{50})_\infty^2(-q^{15},-q^{35};q^{50})_\infty}{(q^{10},q^{40};q^{50})_\infty(-q^5,-q^{45};q^{50})_\infty}\\
&=(q^5,q^{45};q^{50})_\infty^2(q^{30},q^{40},q^{60},q^{70};q^{100})_\infty(q^{50};q^{100})_\infty^3\\
&\times\frac{(q^{100};q^{100})_\infty^2(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty}\\
&+y\frac{(-q^{25},q^{50};q^{50})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^{20},q^{30};q^{50})_\infty},
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
&(q^{10},q^{90};q^{100})_\infty(q^{25};q^{25})_\infty(-q^{50};q^{50})_\infty\\
&\times\frac{(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^5,q^{20};q^{25})_\infty}\\
&=\frac{(q^{15},q^{35},q^{50};q^{50})_\infty^2(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^5;q^{10})_\infty(q^{30},q^{40},q^{60},q^{70},q^{100};q^{100})_\infty},
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
&\frac{(-q^{25},q^{50};q^{50})_\infty(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^{20},q^{30};q^{50})_\infty}\\
&=y(q^{10},q^{90};q^{100})_\infty(q^{25};q^{25})_\infty(-q^{50};q^{50})_\infty\\
&\times\frac{(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^5,q^{20};q^{25})_\infty}\\
&+(q^5,q^{45};q^{50})_\infty^2(q^{30},q^{40},q^{60},q^{70};q^{100})_\infty(q^{50};q^{100})_\infty^3\\
&\times\frac{(q^{100};q^{100})_\infty^2(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^5;q^5)_\infty},
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
&2y\frac{(q^{50};q^{50})_\infty^2(-q^5,-q^{45};q^{50})_\infty}{(q^{20},q^{30};q^{50})_\infty(-q^{15},-q^{35};q^{50})_\infty}\\
&=\frac{(q^{15},q^{35},q^{50};q^{50})_\infty^2(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty(q^{30},q^{40},q^{60},q^{70},q^{100};q^{100})_\infty}\\
&-\frac{(-q^{25},q^{50};q^{50})_\infty(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^{20},q^{30};q^{50})_\infty}.
\end{aligned}
\tag{5.14}
$$

Equation (5.12) is immediate after some simplification. The other identities follow from routine (though tedious) reduction and application of one of Lemmas 3.2 - 3.4. Specifically, upon clearing denominators in (5.11) and simplifying, we have

$$
\begin{aligned}
&(-q^{15},q^{20},q^{30},-q^{35};q^{50})_\infty\\
&=(q^5,q^{10},-q^{15},-q^{20},-q^{25},-q^{25},-q^{30},-q^{35},q^{40},q^{45};q^{50})_\infty\\
&+y(-q^5,q^{10},q^{40},-q^{45};q^{50})_\infty.
\end{aligned}
\tag{5.15}
$$

Now replacing q by $-q$, this may be verified using the case $(x,z,q)=(-q^5,q^{10},q^{25})$ of Lemma 3.2. After clearing denominators and simplifying, (5.13) may be reduced to

$$
\begin{aligned}
&(-q^{25};q^{50})_\infty^2(q^{20},q^{80};q^{100})_\infty(q^{15},q^{35};q^{50})_\infty\\
&=y(q^{10},q^{90};q^{100})_\infty(q^{10},-q^{20},-q^{30},q^{40};q^{50})\\
&+(q^{40},q^{60};q^{100})_\infty(q^5,-q^{15},-q^{35},q^{45};q^{50})_\infty.
\end{aligned}
\tag{5.16}
$$

Factoring out $(q^5,-q^{20},-q^{30},q^{45};q^{50})_\infty$ from the right hand side, replacing q by $-q$ and applying the case $(x,z,q)=(q^5,-q^{10},q^{25})$ of Lemma 3.2 verifies (5.16). For (5.14), we clear denominators and simplify to get

$$
\begin{aligned}
&2y(q^{10},q^{25},q^{40};q^{50})_\infty(q^{100};q^{100})_\infty\\
&=(q^{10},-q^{10},-q^{10},q^{15},q^{15},-q^{15},-q^{25},q^{35},\\
&\quad q^{35},-q^{35},q^{40},-q^{40},-q^{40},q^{50};q^{50})_\infty\\
&-(q^5,q^5,-q^{15},q^{20},-q^{20},-q^{20},-q^{25},q^{30},\\
&\quad -q^{30},-q^{30},-q^{35},q^{45},q^{45},q^{50};q^{50})_\infty.
\end{aligned}
\tag{5.17}
$$

Factoring out $(-q^{15},-q^{25},-q^{35},q^{50};q^{50})_\infty$ from both terms on the right hand side, replacing q by $-q$ and writing the right hand side in base q^{25} yields an expression to which the case $(x,z,q)=(-q^5,-q^{10},q^{25})$ of Lemma 3.3 may applied, confirming (5.17).

As for (5.10), taking $a = 2$ in (4.21) and applying (4.22) gives

$$\begin{aligned} &2g(2) + g(1) + 1 \\ &= \frac{(q^5, q^{45}; q^{50})_\infty (-q^{20}, -q^{30}; q^{50})_\infty (-q^{25}; q^{50})_\infty^4 (q^{50}; q^{50})_\infty^2}{(-q^5, -q^{45}; q^{50})_\infty (q^{20}, q^{30}; q^{50})_\infty} \\ &- y^2 \frac{(q^{10}, q^{90}; q^{100})_\infty (q^{25}; q^{25})_\infty^2 (-q^{50}; q^{50})_\infty^2}{(q^5, q^{20}; q^{25})_\infty}. \end{aligned} \tag{5.18}$$

The final term above is identical to the final term in (5.10). After some simplification, the fact the the first term above is equal to the first two terms in (5.10) is equivalent to the identity

$$\begin{aligned} &(q^{30}, q^{70}; q^{100})_\infty (-q^{10}, -q^{15}, -q^{35}, -q^{40}; q^{50})_\infty \\ &- y(-q^5, -q^{45}; q^{50})_\infty \\ &= (q^5, -q^{15}, -q^{25}, -q^{25}, -q^{30}, -q^{35}, q^{45}; q^{50})_\infty. \end{aligned} \tag{5.19}$$

Equation (5.19) is seen to be true after multiplying both sides by $(q^{10}, q^{40}; q^{50})_\infty$, replacing q by $-q$, and applying the case $(x, z, q) = (-q^5, q^{10}, q^{25})$ of Lemma 3.2.

We now turn to the rank differences $R_{02}(d)$, proceeding as above. Again by (2.4), (2.5), and (2.6), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \Big\{ N_2(0,5,n) - N_2(2,5,n) \Big\} q^n \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \\ &= 2S_2(1) + S_2(3) - \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} + 1. \end{aligned} \tag{5.20}$$

By (5.20), (5.8), and (5.9), we have

$$\begin{aligned} &-2g(1) + 2q \frac{P(0)^2 P(-y^7, y^{10})}{P(y^2, y^{10}) P(-y^9, y^{10})} + 2y^2 \Sigma(1,0) \frac{(-q^{25}; q^{50})_\infty}{(-q; q^2)_\infty (q^{50}; q^{50})_\infty} \\ &+ g(2) + yq^4 \frac{P(0)^2 P(-y^9, y^{10})}{P(y^4, y^{10}) P(-y^7, y^{10})} + y^3 q^4 \Sigma(2,0) \frac{(-q^{25}; q^{50})_\infty}{(-q; q^2)_\infty (q^{50}; q^{50})_\infty} \\ &+ \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} - 1. \\ &= \Big\{ r_{02}(0) q^0 + r_{02}(1) q + r_{02}(2) q^2 + r_{02}(3) q^3 + r_{02}(4) q^4 \Big\} \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty}. \end{aligned}$$

Again, substituting for $r_{02}(d)$ from Theorem 1.2 and equating coefficients of powers of q yields the following identities to be verified.

$$
\begin{aligned}
&2g(1)-g(2)+1\\
&=(q^5,q^{45};q^{50})_\infty^2(q^{50};q^{50})_\infty^3(q^{30},q^{40},q^{60},q^{70};q^{100})_\infty\\
&\times\frac{(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty(q^{100};q^{100})_\infty}\\
&+y\frac{(q^{15},q^{35},q^{50};q^{50})_\infty^2(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty(q^{30},q^{40},q^{60},q^{70},q^{100};q^{100})_\infty}\\
&+y\frac{(q^{30},q^{70};q^{100})_\infty(q^{25};q^{25})_\infty(-q^{50};q^{50})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^{10},q^{15};q^{25})_\infty},
\end{aligned}
\tag{5.21}
$$

$$
\begin{aligned}
&2\frac{(q^{50};q^{50})_\infty^2(-q^{15},-q^{35};q^{50})_\infty}{(q^{10},q^{40};q^{50})_\infty(-q^5,-q^{45};q^{50})_\infty}\\
&=\frac{(-q^{25};q^{50})_\infty(q^{50};q^{50})_\infty(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^{10},q^{40};q^{50})_\infty}\\
&+(q^5,q^{45};q^{50})_\infty^2(q^{50};q^{50})_\infty^3(q^{30},q^{40},q^{60},q^{70};q^{100})_\infty\\
&\times\frac{(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty(q^{100};q^{100})_\infty},
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
&(q^{30},q^{70};q^{100})_\infty(q^{25};q^{25})_\infty(-q^{50};q^{50})_\infty\\
&\times\frac{(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^{10},q^{15};q^{25})_\infty}\\
&=\frac{(-q^{25};q^{50})_\infty(q^{50};q^{50})_\infty(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^{10},q^{40};q^{50})_\infty}\\
&+y\frac{(q^{15},q^{35},q^{50};q^{50})_\infty^2(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^5;q^{10})_\infty(q^{30},q^{40},q^{60},q^{70},q^{100};q^{100})_\infty},
\end{aligned}
\tag{5.23}
$$

$$
\begin{aligned}
&(q^{30},q^{70};q^{100})_\infty(q^{25};q^{25})_\infty(-q^{50};q^{50})_\infty\\
&\times\frac{(q^5,-q^{20},-q^{25},-q^{30},q^{45},q^{50};q^{50})_\infty}{(q^{10},q^{15};q^{25})_\infty}\\
&=(q^5,q^{45};q^{50})_\infty^2(q^{50};q^{50})_\infty^3(q^{30},q^{40},q^{60},q^{70};q^{100})_\infty\\
&\times\frac{(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^5;q^5)_\infty(q^{100};q^{100})_\infty},
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
& y\frac{(q^{50};q^{50})_\infty^2(-q^5,-q^{45};q^{50})_\infty}{(q^{20},q^{30};q^{50})_\infty(-q^{15},-q^{35};q^{50})_\infty} \\
&= \frac{(q^{15},q^{35},q^{50};q^{50})_\infty^2(-q^{10},q^{15},-q^{25},q^{35},-q^{40},q^{50};q^{50})_\infty}{(q^5;q^{10})_\infty(q^{30},q^{40},q^{60},q^{70},q^{100};q^{100})_\infty} \\
&- \frac{(-q^{25};q^{50})_\infty(q^{50};q^{50})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty}{(q^{10},q^{40};q^{50})_\infty}.
\end{aligned} \tag{5.25}
$$

These follow in the same way as equations (5.10) - (5.14). The arduous details are left to the interested reader. The point is to simplify and reduce in order to arrive at an expression that can be verified using an appropriate instance of one of the Lemmas 3.2 - 3.4.

□

6. Concluding remarks

With the present paper and previous work on rank differences for overpartitions [17], we have seen the effectiveness of the approach developed by Atkin and Swinnerton-Dyer [6] for proving formulas for rank differences in arithmetic progressions in terms of modular forms and generalized Lambert series. We should stress that two major difficulties in this method are the requirement that all of the formulas be ascertained beforehand and the apparent need for a new set of key q-series identities for each application. Nevertheless, the ideas should in principle be reliable in other instances where there is a two-variable generating function like (2.3). For example, one might consider the $M2$-rank for overpartitions [16], ranks arising in Andrews' study of Durfee symbols [4], or the generalized ranks of Garvan [12]. Finally, as evidenced by work of Atkin and Hussain [7] on the partition rank, the formulas for rank differences quickly become more complicated as ℓ grows. It would be interesting to try to extend the method used for $\ell = 3$ and 5 here and in [17] to the case $\ell = 7$.

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