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Density of hyperbolicity and tangencies in sectional dissipative regions

Enrique R. Pujals [a,*], Martin Sambarino [b]

[a] *Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina 110, CEP: 22460-320, Rio de Janeiro, RJ, Brazil*
[b] *Centro de Matemática, Universidad de la República, Iguá 4225 esq. Mataojo, CP:11400, Montevideo, Uruguay*

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Abstract

In this paper we extend the notion of sectionally dissipative periodic points to arbitrarily compact invariant sets. We show that given a sectionally dissipative and attracting region for a diffeomorphisms f, there is a neighborhood of f and a dense subset of it such that any diffeomorphism g in this dense subset either exhibits a sectional dissipative homoclinic tangency or the part of the limit set of g in this attracting region is a hyperbolic compact set. The proof goes extending some results on dominated splitting obtained for compact surfaces maps.

Résumé

Dans cet article nous étendons la notion de points périodiques sectionnellement dissipatifs à des ensembles compacts invariants quelconques. Nous montrons qu'ayant une région sectionnellement dissipative et attrayante pour un difféomorphisme f, il y a un voisinage de f et un sous-ensemble dense de celui-ci tels que tout difféomorphisme g dans ce sous-ensemble a une tangence homoclinique sectionnellement dissipative oú la partie de l'ensemble limite de g dans la région attrayante est un ensembe compact hyperbolique. La preuve est une géneralisation des résultats obtenus pour des difféomorphismes de surfaces.

Keywords: Generic dynamics; Partial hyperbolicity; Dominated splitting; Homoclinic bifurcations; Homoclinic tangencies

1. Introduction

During the early times of non-conservative dynamics was a common sense that "non-pathological" systems behave in a very simple form such as the nonwandering set consisting of finitely many periodic elements. The achievement of Peixoto that an open and dense subset of C^1 vector fields on surfaces consist of the now-called Morse–Smale systems is paradigmatic of this view. However, in the early sixties (by Anosov and Smale following Birkhoff, Cartwright and Littlewood, and others) it was shown that "chaotic behavior" may exist within stable systems and this was the starting point of the hyperbolic theory and the modern non-conservative dynamical systems theory. A major result

* Corresponding author.
E-mail addresses: enrique@impa.br (E.R. Pujals), samba@cmat.edu.uy (M. Sambarino).

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in hyperbolic theory is the so-called Ω spectral decomposition theorem for Axiom A systems. This means that for these systems, the nonwandering set can be decomposed into finitely many compact, disjoint and transitive pieces. Although this pieces could exhibit a chaotic behavior (and nowadays well understood) there are just finitely many of them and this recovers the old vision by replacing finitely many periodic elements by these finitely many "dynamically irreducible" pieces.

It was soon realized that hyperbolic systems were not as universal as initially thought: there were given examples of open sets of diffeomorphism were none of them are hyperbolic. Nevertheless in all these new examples the nonwandering set still decomposes into finitely many compact, disjoint and transitive pieces. It was through the seminal work of Newhouse (see [10–12]) where a new phenomena was shown: the existence of infinitely many periodic attractors (today called Newhouse's phenomena) for residual subsets in the space of C^r diffeomorphisms ($r \geqslant 2$) of compact surfaces. The underlying mechanism here was the presence of a *homoclinic tangency*: non-transversal intersection of the stable and unstable manifold of a periodic point.

In the late eighties, Palis conjectured (see [14,15,21]) that for surface diffeomorphisms, homoclinic tangencies are the solely mechanisms that leads to the explosion of the limit set into an infinite number of transitive isolated sets: *Any C^r-diffeomorphism on a surface can be C^r-approximated by one which is hyperbolic or by one exhibiting a homoclinic tangency.*

The above conjecture was proved to be true for the case of surfaces and the C^1 topology (see [17]). Moreover, in [20], it was proved that any C^2-diffeomorphisms having infinitely many periodic attracting points with unbounded period, can be C^1-approximated by another diffeomorphisms exhibiting a homoclinic tangency.

One may think that in higher dimensions the unfolding of a homoclinic tangency may lead to the breakdown of a finite decomposition of the nonwandering set. However, there are examples of robust transitive diffeomorphisms that coexist with the presences of a homoclinic tangency (see for instance [2]).

Nevertheless, it was shown in [22] that for smooth diffeomorphisms on manifold with dimension larger than two, the unfold of *tangencies associated to sectional dissipative periodic points* (tangencies associated to a periodic point such that the modulus of the product of any pair of eigenvalues is smaller than one) leads to the same phenomena that holds in dimension two: residual subsets of diffeomorphisms exhibiting infinitely many periodic attractors.

Regarding the previous comments and following the conjecture formulated by Palis, it is naturally to ask if is true that *any diffeomorphisms on a finite-dimensional manifold can be either C^r-approximated by another one such its dynamic is hyperbolic restricted to a "sectionally dissipative regions of the limit set", or it is C^r-approximated by a system exhibiting a sectional dissipative homoclinic tangency.* In few words, any result in this direction, would be a converse to the one proved in [22] and mentioned above. This is one of the aims of this paper (see Corollary 1.1).

1.1. Definitions and statements

Let $f : M \to M$ be a diffeomorphism of a compact Riemannian manifold without boundary. We denote by $\Omega(f)$ the nonwandering set of f and by $L(f)$ its limit set which is defined as the closure of the forward and backward accumulation points of all orbits, i.e.

$$L(f) = \overline{\bigcup_{x\in M} \omega(x) \cup \alpha(x)}.$$

A set Λ is called hyperbolic for f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\left\|Df^n_{/E^s(x)}\right\| \leqslant C\lambda^n \quad \text{and} \quad \left\|Df^{-n}_{/E^u(x)}\right\| \leqslant C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n.

We say that f is a hyperbolic diffeomorphism if $L(f)$ is hyperbolic.

We recall that the stable and unstable sets

$$W^s(p) = \left\{y \in M\colon \operatorname{dist}\left(f^n(y), f^n(p)\right) \to 0 \text{ as } n \to \infty\right\},$$
$$W^u(p) = \left\{y \in M\colon \operatorname{dist}\left(f^n(y), f^n(p)\right) \to 0 \text{ as } n \to -\infty\right\}$$

are C^r-injectively immersed submanifold when p is a hyperbolic periodic point of f. The *index* of p is the dimension of $W^s(p)$. If $W^s(p)$ and $W^u(p)$ has a nontransverse intersection we say that p has a *homoclinic tangency*.

A periodic point p of period m is called *sectionally dissipative* if the modulus of the product of any two distinct eigenvalues of Df_p^m is smaller than one. A homoclinic tangency associated to a sectionally dissipative periodic point is called a *sectionally dissipative tangency*.

We wish to "extend" the notion of sectional dissipativeness to non-periodic points. This is done as follows.

Definition 1.1 *(Two-dimensional determinant).* Let $\langle\, ,\rangle$ be the Riemannian metric of M. Let $G_2(M)$ be the Grassmannian space of all two-dimensional subspaces in TM. Observe that given $(x, L) \in G_2(M)$ we can consider the metric $\langle\, ,\rangle_x$ restricted to L and in particular it induces a two dimension volume form w_L on L. The derivative of f acts naturally over $G_2(M)$, i.e.: $Df(x, L) = (f(x), Df(L))$. The determinant of Df at (x, L) is defined as the unique real number $\det(Df_{x|L})$ such that

$$f^*(w_{Df(L)}) = \det(Df_{x|L})w_L,$$

where f^* is the pull back associated to f.

Definition 1.2 *(Sectionally dissipative compact sets).* Let $f : M \to M$ be a C^1-diffeomorphism and Λ a compact invariant set. We say that f is sectionally dissipative on Λ (or Λ is a sectionally dissipative set for f) if for any point $x \in \Lambda$ and for any two-dimensional subspace L holds that

$$\left|\det(Df_{x|L})\right| < 1.$$

We remark that if p is a periodic point and the orbit $\mathcal{O}(p)$ is a sectionally dissipative set then p is a sectionally dissipative periodic point, i.e, the modulus of the product of any two eigenvalues of Df_p^m is less than one. The converse is not true even if p if fixed.

More generally, given $\lambda > 0$, we denote with $SD_f(\lambda)$ the set

$$SD_f(\lambda) := \left\{x \in M\colon\ \left|\det(Df_{x|L})\right| < \lambda \text{ for any two-dimensional subspace } L \subset T_xM\right\}.$$

We define

$$\mathcal{SD}_f(\lambda) := \overline{\left\{x\colon\ \mathcal{O}(x, f) \subset SD_f(\lambda)\right\}},$$

where $\mathcal{O}(x, f)$ is the orbit of x by f. Notice that if Λ is a sectionally dissipative set then $\Lambda \subset \mathcal{SD}_f(1)$.

We denote by

$$L(f, 1) := L(f) \cap \mathcal{SD}_f(1).$$

Finally, given two compact invariant sets $A \subset B$ we say that A is isolated within B if there is a neighborhood U of A such that $B \cap U = A$. We say that U is an attracting region if $\overline{f(U)} \subset U$ and observe that $L(f) \cap U$ is isolated within $L(f)$.

Now we can formulate our main theorem that relates tangencies and hyperbolicity in the sectionally dissipative regions of the limit set:

Theorem A. *Let $f : M \to M$ be a C^2-diffeomorphism of a finite-dimensional compact Riemannian manifold M. Let $\Lambda \subset L(f, 1)$ be a compact invariant isolated set in $L(f)$ and such that the periodic points in Λ are hyperbolic. Then, one of the following statements holds*:

1. *For any neighborhood $\mathcal{U}(f)$ and a neighborhood U of Λ there exits $g \in \mathcal{U}(f)$ exhibiting a sectionally dissipative tangency associated to a (sectionally dissipative) periodic point p such that $\mathcal{O}(p, g) \subset U$.*
2. *$\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is a hyperbolic set and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, normally hyperbolic and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

The following corollary is an immediate consequence of Theorem A and represents a weak converse of the main result in [22]. Before to state it, we introduce the set

$$\Lambda_0 = \overline{P_0(f)}$$

the closure of the periodic attractors. Observe that Λ_0 is a compact invariant set in $L(f)$.

Corollary 1.1. *Let $f \in \mathit{Diff}^2(M)$ be a diffeomorphism exhibiting infinitely many attracting periodic points and let us assume that Λ_0 is an isolated compact set in $L(f, 1)$ such that all the periodic points are hyperbolic. Then, f can be C^1-approximated by a diffeomorphism g having a sectionally dissipative tangency in a neighborhood of Λ_0.*

The previous result follows immediately since Λ_0 cannot verify the second option of Theorem A. Another version of this corollary is the following:

Corollary 1.2. *Let $f \in \mathit{Diff}^2(M)$ be a Kupka–Smale diffeomorphism and let U be an attracting region such that $U \subset SD_f(\mu)$, $\mu < 1$. Assume that f has infinitely many periodic attractors in U. Then, f can be C^1-approximated by a diffeomorphism g having a sectionally dissipative tangency in U.*

Indeed, observe that $\Lambda = L(f) \cap U$ is isolated within $L(f)$ since U is an attracting region and $\Lambda \subset \mathcal{SD}_f(1)$. Since the second option of Theorem A cannot happen since f has infinitely many periodic attractors then the first one must occur.

An important consequence of Theorem A is also the following result which extends in some sense a bidimensional result in [9]:

Corollary 1.3. *Let $f \in \mathit{Diff}^1(M)$ and let U be an attracting region such that $U \subset SD_f(\mu)$, $\mu < 1$. Then, there exist a neighborhood $\mathcal{U}(f)$ and a residual subset $\mathcal{R} \subset \mathcal{U}$ such that for any $g \in \mathcal{R}$ one of the following statements holds*:

1. *g has infinitely many periodic attractors in U.*
2. *$L(g) \cap U$ is hyperbolic.*

Another straightforward important consequence is also the following result which extends in some sense a bidimensional result in [17]:

Corollary 1.4. *Let $f \in \mathit{Diff}^1(M)$ and let U be an attracting region such that $U \subset \mathcal{SD}_f(1)$. Then, there exists a neighborhood $\mathcal{U}(f)$ such that any $g \in \mathcal{U}(f)$ can be C^1-approximated by a diffeomorphism g either exhibiting a sectionally dissipative tangency in U or such that $L(g) \cap U$ is hyperbolic.*

The proof of these two last corollaries are given in the next section.

In the direction to prove Theorem A, we shall extend some results on dominated splitting that we have obtained for compact surfaces. Let $f : M \to M$ be a C^1 diffeomorphism of a compact Riemannian manifold M. An f-invariant set Λ is said to have dominated splitting if we can decompose its tangent bundle in two invariant subbundle $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{/E(x)}\| \|Df^{-n}_{/F(f^n(x))}\| \leqslant C\lambda^n, \quad \text{for all } x \in \Lambda,\ n \geqslant 0,$$

with $C > 0$ and $0 < \lambda < 1$.

We say that the dominated splitting is a *codimension one dominated splitting* if dimension$(F) = 1$. We say that a codimension one dominated splitting is a *contractive codimension one dominated splitting* if the direction E is a contractive direction, i.e.: there exist $C > 0$ and $0 < \lambda_1 < 1$ such that for any x and any positive integer n holds that

$$|Df^n_{/E(x)}| < C\lambda_1^n.$$

In this case we denote the direction E as E^s.

The next result establishes the relation between contractive codimension one dominated splitting and not being approximated by sectionally dissipative tangency. Let us state first a definition.

Definition 1.3. Given a compact invariant set Λ we say that $f_{/\Lambda}$ is C^1-far from sectionally dissipative homoclinic tangencies if there is a neighborhood $\mathcal{U} \subset Diff^1(M)$ of f and a neighborhood U of Λ such that any $g \in \mathcal{U}$ does not exhibit a sectionally dissipative tangency associated to a periodic point p of g with $\mathcal{O}(p, g) \subset U$.

Theorem B. *Let Λ be a compact invariant set in $L(f, 1)$ and isolated within $L(f)$. Let us assume that $f_{/\Lambda}$ is C^1-far from sectionally dissipative tangencies and all the periodic points in Λ are hyperbolic. Then, $\Lambda \setminus P_0(f_{/\Lambda})$ (where $P_0(f_{/\Lambda})$ is the set of periodic attractors of f in Λ) has a contractive codimension one dominated splitting.*

Now we prove, that under certain conditions, contractive codimension one dominated splitting are actually hyperbolic.

Theorem C. *Let $f : M \to M$ be a C^2-diffeomorphism. Let Λ be a compact invariant set contained in $L(f)$ and exhibiting a contractive codimension one dominated splitting. Let also assume that Λ is isolated within $L(f)$ and all the periodic points in Λ are hyperbolic. Then,*

$$\Lambda = \Lambda_1 \cup \Lambda_2$$

where Λ_1 is a hyperbolic set and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \dots \mathcal{C}_n$, normally hyperbolic and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugated to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).

Remark 1.1. Observe that in Theorem C we are not assuming that the set Λ is contained in $L(f, 1)$.

We will prove also the next corollary from Theorem C.

Corollary 1.5. *Let $f : M \to M$ be a C^2-diffeomorphism. Assume that M has a contractive codimension one dominated splitting and all the hyperbolic periodic points are of saddle type. Then f is an Anosov diffeomorphism and $M = T^n$.*

The paper is organized as follows: In Section 2 we give the proofs of Theorem A, Corollaries 1.3, 1.4 and 1.5 assuming that Theorems B and C hold. In Section 3 we give the proof of Theorem B. The proof of Theorem C is given in Section 5. To perform the proof, we need a series of results about the dynamical geometry of sets having a contractive codimension one dominated splitting. In this direction, in Section 4, under the hypothesis of contractive codimension one dominated splitting, we show the existence of Markov partition for a general class of sets that include the homoclinic classes. This result is a fundamental tool in the proof of the rest of Theorem C.

2. Proof of Theorem A and Corollaries 1.3, 1.4 and 1.5

Through this section we assume that Theorems B and C hold.

Proof of Theorem A. Let f and Λ be as in the statement and assume that the first option does not happen, i.e. $f_{/\Lambda}$ is C^1-far from sectionally dissipative tangencies. By Theorem B follows that $f_{/\Lambda \setminus P_0(f_{/\Lambda})}$ exhibits a contractive codimension one dominated splitting. Given a neighborhood V of $\Lambda \setminus P_0(f_{/\Lambda})$ we have that $\sharp P_0(f_{/\Lambda}) \cap V^c < \infty$ and set $P_0(f_{/\Lambda}) \cap V^c = \{p_1, \dots, p_n\}$. If the neighborhood V has been appropriately chosen, we have that any compact invariant set in V has contractive codimension one dominated splitting. Therefore $\tilde{\Lambda} = \Lambda \setminus \{\mathcal{O}(p_1), \dots, \mathcal{O}(p_n)\}$ has contractive codimension one dominated splitting. On the other hand, if Λ is isolated within $L(f)$ so it is $\tilde{\Lambda}$. Now applying Theorem C to $\tilde{\Lambda}$ we have the desired decomposition of it as a union of a hyperbolic set and finitely many periodic curves "supporting an irrational rotation". Since $\Lambda = \tilde{\Lambda} \cup \{\mathcal{O}(p_1), \dots, \mathcal{O}(p_n)\}$ and $p_1, \dots, p_n$ are hyperbolic (periodic attractors) we have the desired decomposition of Λ as required in the second option of Theorem A. □

Proof of Corollary 1.3. Let $\mathcal{U}$ be such that for any $g \in \mathcal{U}$ we have that $\overline{g(U)} \subset U$ and $L(g) \cap U \subset \mathcal{SD}_g(1)$. For $g \in \mathcal{U}$ consider the map Γ that $\Gamma(g) = \overline{P_0(g, U)}$ where $P_0(g, U)$ is the set of attracting periodic point of g in U. This map is lower semicontinuous and there is a residual subset $R_1 \subset \mathcal{U}(f)$ of continuity points of Γ. Let $R_2 =$

$\{g \in R_1\colon \sharp P_0(g, U) = \infty\}$ and consider $\mathcal{V} = \mathcal{U} \setminus \overline{R_2}$. It follows that if $g \in R_1 \cap \mathcal{V}$ then g has finitely many periodic attractors and since R_1 is formed by continuity points of Γ we have that there is $\mathcal{V}_1$ open and dense in $\mathcal{V}$ such that any $g \in \mathcal{V}_1$ has finitely many periodic attractors. The set of C^2 Kupka–Smale diffeomorphism g in $\mathcal{V}_1$ is dense in $\mathcal{V}_1$ and by Theorem A all of them satisfy $L(g) \cap U$ is hyperbolic. Since there cannot exist a cycle among the basic pieces of $L(g)$ (since g is Kupka–Smale and the basic pieces have index $n-1$ or are periodic attractors) it follows by a straightforward adaptation of the Ω-stability theorem that there is $\mathcal{V}_2$ open and dense in $\mathcal{V}_1$ such that for any $g \in \mathcal{V}_2$ holds that $L(g) \cap U$ is hyperbolic. Hence

$$\mathcal{R} = \mathcal{V}_2 \cup R_2$$

satisfies the conclusion of Corollary 1.3. $\square$

Proof of Corollary 1.4. In the same way as in the proof of Corollary 1.3, let $\mathcal{U}(f)$ be such that any $g \in \mathcal{U}(f)$ satisfies $\overline{g(U)} \subset U$ and $L(g) \cap U \subset L(g, 1)$. Recalling again that the set of C^2 Kupka–Smale diffeomorphism in $\mathcal{U}(f)$ are dense and arguing again as in Corollary 1.3 the result follows by a direct application of Theorem A. $\square$

Proof of Corollary 1.5. By Theorem B holds that $L(f)$ is the union of a hyperbolic set and a finite number of periodic simple closed curves normally hyperbolic (attracting) "supporting an irrational rotation". It follows that there must be a hyperbolic repeller in $L(f)$. In other words, there exists $\Lambda \subset L(f)$ such that Λ is maximal invariant with local product structure and it is a repeller. Moreover it has stable index $n-1$. On the other hand, by [19] follows that F is uniquely integrable. Now, the exact same proof of the main theorem in [13] where it is proved that a repeller in a codimension one Anosov diffeomorphism is also an open set applies here to Λ. Thus $M = \Lambda$ and hence f is Anosov. By a result in [5] follows that $M = \mathbb{T}^n$. $\square$

3. Proof of Theorem B: Dominated splitting for systems far from sectionally dissipative homoclinic tangencies

Let Λ be as in Theorem B, that is, $\Lambda \subset L(f, 1)$ and it is isolated within $L(f)$. Recall that $f_{/\Lambda}$ is far from sectionally dissipative homoclinic tangencies and hence there exist a neighborhood U of Λ and $\mathcal{U}(f)$ such that any $g \in \mathcal{U}(f)$ does not exhibit a homoclinic tangency associated to a sectionally dissipative periodic point of g whose orbit lies entirely in U. From now on and through this section, U and $\mathcal{U}(f)$ will be as above.

We denote by

$$Per^{SD}_{n-1}(g, U)$$

the set of sectionally dissipative periodic points (i.e the product of any two distinct eigenvalues is less than one) of g having index $n-1$ and whose orbit lies entirely in U.

We shall split the proof of Theorem B in the following sequence of propositions.

Proposition 3.1. *Let Λ be as in Theorem B and fix any $\eta > 0$. Let $x \in \Lambda$ and assume that x is not a periodic attractor. Then, there exist sequences of diffeomorphisms $\{g_n\}$ and periodic points $\{q_n\}$ such that $g_n \to f$, $q_n \to x$ such that $q_n \in Per^{SD}_{n-1}(g, U) \cap \mathcal{SD}_{g_n}(1+\eta)$.*

For the next proposition we need the definition of angle between subspaces. Let E and F be two subspaces of finite-dimensional vector space V with an inner product and assume that $E \oplus F = V$. Hence $\dim(F) = \dim(E^{\perp})$ and F is the graph of the linear map $L : E^{\perp} \to E$ defined as follows: given $w \in F$ there exists a unique pair of vectors $v \in E$, $u \in E^{\perp}$, such that $v + u = w$. Define $L(v) = u$ obtaining that graph$(L) = F$. We define, as it is done in [9], the angle $\angle(E, F)$ between E and F as $\|L\|^{-1}$ In particular $\angle(E, E^{\perp}) = +\infty$.

Remark 3.1. If E and F are subspaces of a vector space W with an inner product and such that $E \cap F = \{0\}$ then we can define the angle between them as before just setting $V = E \oplus F$ with the inner product inherited from W.

Proposition 3.2. *Let Λ be as in Theorem B. There exist a neighborhood $\mathcal{V}(f)$ of f, a neighborhood U of Λ, $\gamma > 0$ and $\eta > 0$ such that for any $g \in \mathcal{V}(f)$ and any $q \in Per^{SD}_{n-1}(g, U) \cap \mathcal{SD}_g(1+\eta)$ it holds that*

$$\angle\big(E^s_q, E^u_q\big) > \gamma.$$

With the above two propositions we prove Theorem B and this is the content of our last proposition in this section.

Proposition 3.3. *Under the assumptions of Theorem B, Propositions* 3.1 *and* 3.2 *imply that the set $\Lambda \setminus P_0(f_{/\Lambda})$ has contractive codimension one dominated splitting.*

The proofs of these propositions are done in the next subsections. Before given the proof, we state now a classical C^1 perturbation technique known as Franks' lemma.

Lemma 3.0.1. *(See [4, Lemma 1.1].) Let M be a closed n-manifold and $f : M \to M$ be a C^1 diffeomorphism, and let $\mathcal{U}(f)$ a neighborhood of f. Then, there exist $\mathcal{U}_1(f) \subset \mathcal{U}(f)$ and $\epsilon > 0$ such that if $g \in \mathcal{U}_1(f)$, $S \subset M$ is a finite set, $S = \{p_1, p_2, \dots, p_m\}$ and L_i, $i = 1, \dots, m$, are linear maps $L_i : T_{p_i}M \to T_{f(p_i)}M$ satisfying $\|L_i - D_{p_i}g\| \leqslant \epsilon$, $i = 1, \dots, m$, then there exists $\tilde{g} \in \mathcal{U}(f)$ satisfying $\tilde{g}(p_i) = g(p_i)$ and $D_{p_i}\tilde{g} = L_i$, $i = 1, \dots, m$. Moreover, if U is any neighborhood of S then we may chose $\tilde{g}$ so that $\tilde{g}(x) = g(x)$ for all $x \in \{p_1, p_2, \dots, p_m\} \cup (M \setminus U)$.*

3.1. Proof of Proposition 3.1

We first recall a version of the closing lemma (see [9, Lemma I.2]).

Lemma 3.1.1. *Given $f \in \mathit{Diff}^1(M)$, $x \in M$, $\epsilon > 0$ and a neighborhood $\mathcal{U}(f)$ there exist $r > 0$, $\rho > 1$ such that if $w \in B_{\bar{r}}(x)$ with $0 < \bar{r} \leqslant r$ and $f^m(w) \in B_{\bar{r}}(x)$ for some $m > 0$ then there exist $0 \leqslant m_1 < m_2 \leqslant m$ and $g \in \mathcal{U}(f)$ such that $f^{m_i}(w) \in B_{\rho\bar{r}}$, $i = 1, 2$, $g^{m_2 - m_1}(f^{m_2}(w)) = f^{m_2}(w)$ and $d(g^j(f^{m_2}(w)), f^j(f^{m_1}(w))) \leqslant \epsilon$ for $0 \leqslant j \leqslant m_2 - m_1$.*

Corollary 3.1. *Let Λ be as in Theorem B and let $x \in \Lambda$. Then, there exist sequences $g_n \to f$, $q_n \to x$ and $\eta_n \to 0$ such that $q_n \in \mathit{Per}(g_n, U) \cap SD_{g_n}(1 + \eta_n)$.*

Proof. Let η_n be a sequence of positive real numbers decreasing to 0. Then there exist $\epsilon_n \searrow 0$ and $\mathcal{U}_n(f)$ such that if $g \in \mathcal{U}_n(f)$ and z satisfies that $d(z, \Lambda) \leqslant 2\epsilon$ then $z \in SD_g(1 + \eta_n)$. This follows by a standard continuity argument since $\Lambda \subset SD_f(1)$. We may assume that $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ and $\bigcap_n \mathcal{U}_n = f$.

Now, let $x \in \Lambda$. If x is periodic then there is nothing to prove. Assume that x is not periodic. Since Λ is isolated within $L(f)$, we may assume that for all ϵ_n it holds that $L(f) \cap \{z\colon d(z, \Lambda) \leqslant \epsilon_n\} = \Lambda$. It follows that if $\omega(y)$ (or $\alpha(y)$) intersects $\{z\colon d(z, \Lambda) \leqslant \epsilon_n\}$ then it is contained in Λ and hence $d(f^j(y), \Lambda) \leqslant \epsilon_n$ for all $j \geqslant j_0$ (or $j \leqslant j_0$). Now fix n. Since $x \in L(f)$ there exist y_k and $z_k \in \omega(y_k) \cup \alpha(y_k)$ such that $z_k \to x$. Then, the result follows by direct application of the preceding lemma by setting w as an appropriate iterate of y_k for k large enough. □

In order to finish the proof of the proposition we have to prove that if $x \in \Lambda$ is not a periodic attractor, then the sequences given by the above corollary can be chosen so that q_n is also sectionally dissipative periodic point of g of index $n - 1$.

We may assume without loss of generality that the periodic points q_n of g_n are hyperbolic with simple spectrum and denote by m_n the period of q_n.

Assume first that q_n are saddles for infinitely many n's (and we may assume without loss of generality that this holds for any n), and let $\lambda_u = \max\{|\lambda|\colon \lambda \text{ eigenvalue of } Dg_n^{m_n}(q_n)\}$. Since $q_n \in SD_{g_n}(1 + \eta_n)$ it follows that the product of any two eigenvalues is less than $(1 + \eta_n)^{m_n}$. We now choose a number ρ:

- If $\lambda_u > (1 + \eta_n)^{m_n}$ then $\rho = (1 + \eta_n)^{m_n}$.
- Otherwise choose $1 < \rho < \lambda_u$ that ρ^2 is bigger than the product of any two distinct eigenvalues.

Let $\mu = \rho^{\frac{1}{m_n}}$. Apply Franks' lemma to $L_i = \mu^{-1} Dg_n(g_n^i(q_n))$. Thus we obtain $\tilde{g}_n$ such that q_n is a periodic point of $\tilde{g}_n$ and $\mathcal{O}(q_n, g_n) = \mathcal{O}(q_n, \tilde{g}_n)$. Notice that $\tilde{g}_n \to f$. The largest eigenvalue of q_n will now have modulus equals to λ_u/ρ and hence q_n is a saddle. On the other hand, either $q_n \in SD_{\tilde{g}_n}(1)$ or the product of any two distinct eigenvalues is equal to the product of any two distinct eigenvalues of $Dg_n^{m_n}$ divided by ρ^2. In any case the periodic point is sectionally dissipative. Moreover it is always true that $q_n \in SD_{\tilde{g}_n}(1 + \eta_n)$.

It remains to prove the proposition in case the periodic points given by Corollary 3.1 are periodic attractors for every large n.

The periods of the periodic points q_n must be unbounded. Otherwise, the point $x \in \Lambda$ is also periodic and cannot be an attractor by our hypothesis. It cannot be a saddle because otherwise the points q_n are also saddle for g_n. Thus, x is nonhyperbolic periodic point contradicting the assumption that any periodic point of f in Λ is hyperbolic.

It is left to prove the proposition in the case the period of the periodic attractors q_n are unbounded. For this we need a result which essentially due to Pliss [16].

Theorem 3.1. *Let g_n be a sequence of diffeomorphisms converging to $f \in \mathit{Diff}^1(M)$. Assume that there is a sequence q_n such that q_n is a periodic attractor of g_n and whose periods k_n are unbounded. Then, for every sequence $\epsilon_m \searrow 0$ there exist a subsequence n_m and a sequence $\tilde{g}_m$ such that*:

1. $g_{n_m}^j(q_{n_m}) = \tilde{g}_m^j(q_{n_m})$ *for* $0 \leqslant j \leqslant k_{n_m}$.
2. $\|Dg_{n_m}(g_{n_m}^j(q_{n_m})) - D\tilde{g}_m(g_m^j(q_{n_m}))\| < \epsilon_m$ *for* $0 \leqslant j \leqslant k_{n_m}$.
3. q_{n_m} *is a saddle hyperbolic periodic point of* g_m.
4. $g_m \to f$.

Before giving the (outline of) the proof of this theorem let us remark that with it, the proof of our Proposition 3.1 can be finished since we fall again in the case where the sequence of points q_n in Corollary 3.1 can be chosen as hyperbolic saddles and then we finish the proof as it was done before.

Proof. As we said this is essentially due to Pliss. We will give an outline of the proof so that the reader could complete it by itself. Fix the sequence ϵ_m. For ever m we have to find q_{n_m} and $\tilde{g}_m$. Fix m and set $\epsilon = \epsilon_m$. It is enough, by a direct application of Franks' lemma, to show that for some $n \geqslant m$ it holds that there are linear maps $L_i : T_{g_n^i(q_n)}M \to T_{g_n^{i+1}(q_n)}M$, $i = 0, \dots, k_n - 1$, such that $\|L_i - Dg_n(g_n^i(q_n))\| < \epsilon$ and $\prod_{i=0}^{k_n-1} L_i$ has an eigenvalue of modulus equal to one. Arguing by contradiction, assume that this does not hold. This means (following [9]) that the family of sequence of periodic matrices induced by $\{Dg_n(g_n^i(q_n))\colon\ i \in \mathbb{Z},\ n \geqslant m\}$ is uniformly attracting. It follows by Lemma II.5 of [9] that there exit K_0, $0 < \lambda < 1$ and m_0 such that

$$\prod_{j=0}^{k-1} \left\| Dg_n^{m_0}\left(g_n^{m_0 j}(q_n)\right)\right\| \leqslant K_0 \lambda^k$$

where $k = [k_n/m_0]$.

To continue we need a lemma known as Pliss' lemma [16] (see also [8]):

Lemma 3.1.2. *Let $H > 0$ and $0 < \lambda_2 < \lambda_1 < 1$ be given. Then there exist a positive integer N and $0 < c < 1$ such that given positive real numbers a_j, $j = 0, \dots, k-1$, with $k - 1 \geqslant N$ such that $a_j \leqslant H$ for $j = 0, \dots, k$ and satisfying*

$$\prod_{j=0}^{k-1} a_j \leqslant \lambda_2^k$$

then there exist $0 \leqslant j_1 < j_2 < \cdots < j_l \leqslant k-1$ such that

$$\prod_{j=0}^{p} a_{j+j_i} \leqslant \lambda_1^p \quad \text{for any } 1 \leqslant p \leqslant k-1-j_i \text{ and } i = 1, 2, \dots, l.$$

Moreover, $l \geqslant ck$.

Continuing with the proof of the theorem we observe that given m_0 there is a constant H such that $\|Dg_n^{m_0}(x)\| \leqslant H$ for any $x \in M$ since $g_n \to f$. Since the periods k_n of the periodic points q_n are unbounded (and we may assume that $k_n \to \infty$) we may choose $0 < \lambda_2 < \lambda_1 < 1$ such that $K_0\lambda^k < \lambda_2^k$ where $k = [k_n/m_0]$ for every n large enough.

Applying Pliss' lemma, we have that for every large n that there exist $0 \leqslant j_1(n) < j_2(n) < \cdots < j_{l_n}(n) \leqslant [k_n/m_0] - 1$ such that

$$\prod_{j=0}^{p} \left\| Dg_n^{m_0}\left(g_n^{(j+j_i)m_0}(q_n)\right)\right\| \leqslant \lambda_1^p \quad \forall 0 \leqslant p \leqslant [k_n/m_0] - 1 - j_i.$$

By this uniform contraction of rate λ_1 we have that there exist $\gamma > 0$ and $\lambda_1 < \rho < 1$ such that for any $x, y \in B_\gamma(g_n^{j_i}(q_n))$ it holds that

$$d\left(g_n^{pm_0}(x), g_n^{pm_0}(y)\right) \leqslant \rho^p d(x, y) \quad \forall 0 \leqslant p \leqslant [k_n/m_0] - 1 - j_i.$$

Let p_0 be such that $\rho^p < \gamma/4$ for all $p \geqslant p_0$. Now, since the number of "times" $0 \leqslant j_1(n) < j_2(n) < \cdots < j_{l_n}(n) \leqslant [k_n/m_0] - 1$ goes to infinity as n grows, we may find n large enough and $0 \leqslant i < t < l_n$ such that

$$j_t(n) - j_i(n) \geqslant p_0 \quad \text{and} \quad d\left(g_n^{j_i m_0}(q_n), g_n^{j_t m_0}(q_n)\right) < \gamma/4.$$

Therefore, setting $p = j_t - j_i$ we have that

$$g_n^{pm_0} : B_\gamma\left(g_n^{j_i m_0}(q_n)\right) \to B_{\gamma/4}\left(g_n^{(p+j_i)m_0}(q_n)\right) \subset B_\gamma\left(g_n^{j_i m_0}(q_n)\right)$$

is a contraction and hence every point in $B_\gamma(g_n^{j_i m_0}(q_n))$ under iteration of $g_n^{pm_0}$ converge to the unique fixed point of this contraction. This is not possible because the point $g_n^{j_i m_0}(q_n)$ is periodic of g_n of period k_n and cannot be fixed by $g_n^{pm_0}$. This is a contradiction and the proof is completed. □

3.2. Proof of Proposition 3.2

Recall that $f_{/\Lambda}$ is far from sectionally dissipative tangencies and so there are neighborhoods $\mathcal{U}(f)$ and $U(\Lambda)$ such that there are no homoclinic tangencies associated to points in $Per_{n-1}^{SD}(g, U)$ for any $g \in \mathcal{U}(f)$.

Proposition 3.2 asserts that there exist $\mathcal{V}(f)$, $\gamma > 0$ and $\eta > 0$ such that for any $g \in \mathcal{V}(f)$ and $q \in Per_{n-1}^{SD}(g, U) \cap \mathcal{SD}_g(1+\eta)$ then

$$\angle\left(E_q^s, E_q^u\right) > \gamma.$$

We state first the key lemma of this section which establishes the relationship between small angle of stable and unstable subspaces and homoclinic tangencies. It is a straightforward adaptation of Lemma 2.2.1 of [17]. Compare also with Lemma 4.2 of [24] where an explicit proof can be found.

Lemma 3.2.1. *Let $\epsilon > 0$ and let $g \in \mathit{Diff}^1(M)$. Assume that $p \in M$ is a hyperbolic periodic point of g with period m. Assume that there are $E_1^u \subset E_p^u$ and E_p^s invariant under $Dg^m(p)$ and such that $\|Dg^m_{/E_1^s}\| < \lambda < 1$ and $\|Dg^{-m}_{/E_1^u}\|^{-1} > \sigma > 1$. Assume that $\lambda\sigma < 1$ and let $\gamma = \angle(E_1^s, E_1^u)$. If*

$$\gamma < \frac{\sigma - 1}{\sigma + 1}\frac{\epsilon}{2}$$

then there is $\tilde{g}$ ϵ-C^1 close to g such that p as a hyperbolic periodic point of $\tilde{g}$, $g^i(p) = \tilde{g}^i(p)$ for $i = 0, 1, \ldots, m$ and $\tilde{g}$ exhibits a homoclinic tangency associated to p. Furthermore $Dg^m(p) = D\tilde{g}^m(p)$.

Although the last part is not included in the original bidimensional statement it follows from the proof since the support of the perturbation is disjoint of the orbit of p. Hence the orbit remains the same and if for instance p is sectionally dissipative still it is after the perturbation.

Now let $\mathcal{U}_0(f)$ be a neighborhood of f and ϵ_1 such that if $g \in \mathcal{U}_0(f)$ and $\tilde{g}$ is $\epsilon_1 - C^1$ close to g then $\tilde{g} \in \mathcal{U}(f)$. Let $\mathcal{U}_1(f)$ and ϵ be from Lemma 3.0.1 applied to $\mathcal{U}_0(f)$.

Lemma 3.2.2. *There exists m_0 such that if $g \in \mathcal{U}_1(f)$ and p is a periodic point of g whose orbit is in U of period $m_p \geqslant m_0$ such that*:

1. *All eigenvalues of* $Dg^{m_p}: T_pM \to T_pM$ *has modulus* $\leqslant 1$.
2. *There is a bidimensional subspace* $P \subset T_pM$ *such that* $Dg^{m_p}_{/P} = Id$.

Then, there exists $\tilde g \in \mathcal{U}(f)$ *such that* $p \in Per^{SD}_{n-1}(\tilde g, U)$ *and has a homoclinic tangency associated to* p.

Proof. Let $K > 1$ and choose $\gamma > 0$ such that $\gamma < \frac{K-1}{K+1}\frac{\epsilon_1}{2}$. Consider also $C = \sup\{\|Dg\|: g \in \mathcal{U}_0(f)\}$ and set $\epsilon' = \epsilon/C$. Finally, let m_0 be such that

$$m_0 \frac{\epsilon'\epsilon}{4(K+1)} > 1.$$

We will show that m_0 as above satisfies the lemma. So, let g and p satisfying the conditions of the lemma. By performing a very small perturbation we may assume that $Dg^{m_p}(p)$ is diagonalizable and that $Ker(Dg^{m_p}(p) - Id) = P$ and also $g \in \mathcal{U}_1(f)$. Let E and F be two one-dimensional subspaces of P such that $\angle(E, F) < \gamma$. After performing a very small perturbation, we obtain g_1 still in $\mathcal{U}_1(f)$ such that:

- $g^i(p) = g_1^i(p)$ for $i = 0, \dots, m_p - 1$.
- $p \in Per^{SD}_{n-1}(g_1, U)$.
- E and F are invariant by $Dg_1^{m_p}(p)$.
- $\|Dg_1^{m_p}{}_{/E}\| = \lambda < 1$; $\|Dg_1^{m_p}{}_{/F}\| = \sigma > 1$ and $\lambda\sigma < 1$.

Notice that $E^u_p = F$ and $E \subset E^s_p$. Let $\gamma_1 = \angle(E, F)$ and we may assume without loss of generality that $\angle(E, F) = \min\{\angle(Dg_1^i(E), Dg_1^i(F)),\ i = 0, \dots, m_p - 1\}$. Moreover we may assume that $\gamma_1 \leqslant \angle(E^s(g_1^i(p)), E^u(g_1^1(p)))$, $i = 0, \dots, m_p - 1$.

If

$$\gamma_1 < \frac{(\sigma - 1)\epsilon_1}{(\sigma + 1)2}$$

then, by Lemma 3.2.1 we are done. Otherwise we set $\delta = \gamma_1\epsilon'/2$ and for $0 \leqslant i \leqslant m_p - 1$ consider $T_i : T_{g_1^i(p)}M \to T_{g_1^{i+1}(p)}M$ such that

$$T_{i/E^s_{g_1^i(p)}} = (1 - \delta)Id \quad \text{and} \quad T_{i/E^u_{g_1^i(p)}} = (1 + \delta)Id.$$

It follows that $\|T_i - Id\| < \epsilon'$. For $0 \leqslant i \leqslant m_p - 2$ let $L_i = T_{i+1} \circ Dg_1(g_1^i(p))$ and $L_{m_p-1} = T_0 \circ Dg_1(g_1^{m_p-1}(p))$. It holds that $\|L_i - Dg_1(g_1^i(p))\| < \epsilon$ and by applying Lemma 3.0.1 we obtain $\tilde g \in \mathcal{U}_0(f)$ such that $g_1^i(p) = \tilde g^i(p)$ and $D\tilde g(\tilde g^i(p)) = L_i$. It is straightforward to check that:

- $p \in Per^{SD}_{n-1}(\tilde g, U)$.
- $E \subset E^s_p$, $F \subset E^u_p$.
- $\tilde\lambda = \|D\tilde g^{m_p}_{/E}\| = \lambda(1 - \delta)^{m_p}$ and $\tilde\sigma\|D\tilde g^{m_p}_{/F}\| = \sigma(1 + \delta)^{m_p}$.
- $\angle(E, F) = \gamma_1$.

If $\tilde\sigma \geqslant K$ then it follows that $\gamma_1 \leqslant \frac{(\tilde\sigma-1)\epsilon_1}{(\tilde\sigma+1)2}$ and by Lemma 3.2.1 we are done. On the other hand, if $\tilde\sigma \leqslant K$ by the way we choose m_0 we have that:

$$\frac{(\tilde\sigma - 1)\epsilon_1}{(\tilde\sigma + 1)2} = \frac{((1+\delta)^{m_p}\sigma - 1)\epsilon_1}{((1+\delta)^{m_p}\sigma + 1)2} \geqslant \frac{((1 + m_p\delta)\sigma - 1)\epsilon_1}{(K + 1)2} \geqslant \frac{(m_p\delta\sigma)\epsilon_1}{(K + 1)2} \geqslant m_0 \frac{\epsilon'\epsilon_1}{(K + 1)4}\gamma_1 > \gamma_1$$

and again by Lemma 3.2.1 the proof is now complete. □

Corollary 3.2. *Under the assumptions of Theorem B, there exist* $\delta > 0$ *and a neighborhood* $\mathcal{U}_2(f)$ *such that for any* $g \in \mathcal{U}_2(f)$ *and* $p \in Per^{SD}_{n-1}(g, U)$ *then there exists only one eigenvalue of* $Dg^m(p)$ (*where* m *is period of* p) *of modulus* $> (1 - \delta)^m$ *and it is real and simple.*

Proof. Let $\mathcal{U}_1 \subset \mathcal{U}_0 \subset \mathcal{U}$ be as before where any $g \in \mathcal{U}$ does not exhibit a homoclinic tangency associated to a periodic point in $Per^{SD}_{n-1}(g, U)$. Hence by the above lemma no $g \in \mathcal{U}_1$ has a periodic point of period $\geqslant m_0$ satisfying the conditions in the statement. Let $\mathcal{U}_2(f)$ and ϵ_0 from Lemma 3.0.1, that is, any ϵ_0-perturbation of the linear maps along a periodic orbit of $g \in \mathcal{U}_2$ can be realized as the linear maps on the same orbit of $\tilde{g} \in \mathcal{U}_1$. Let $C = \sup\{\|Dg\|\colon g \in \mathcal{U}_2\}$ and set $\delta_0 = \epsilon_0/C$ and choose δ such that $(1-\delta)^{-1} < 1 + \delta_0$. We shall prove the corollary for this $\mathcal{U}_2$ and δ. So, let $g \in \mathcal{U}_2$ and $p \in Per^{SD}_{n-1}(g, U)$ and assume by contradiction that $Dg^m(p)$ where m is the period of p has two eigenvalues of modulus $> 1-\delta$. Let λ_u be the largest eigenvalue (in modulus) of $Dg^m(p)$. Notice that λ_u is real and simple and $|\lambda_u| > 1$ and any other eigenvalue is smaller than 1. Let λ_s be the largest (in modulus) of the eigenvalues with modulus smaller than 1. Thus $|\lambda_s| > (1-\delta)^m$ and $|\lambda_s \lambda_u| < 1$. We shall pursue a similar argument as in Lemma 3.5 of [24] and split the proof into several cases. First assume that λ_s is real and $(1-\delta)^m < \lambda_s < 1 < \lambda_u$. Let $\delta_u = \lambda_u^{1/m}$ and $\delta_s = \lambda_s^{1/m}$. Let $T_i : T_{g^i(p)}M \to T_{g^i(p)}M$ such that $T_{i/E^u} = \delta_u^{-1} Id$ and $T_{i/E^{u\perp}} = \delta_s^{-1} Id$. Consider $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ defined by $L_i = T_{i+1} \circ Dg(g^i(p))$, $i = 0, \dots, m-1$. By applying Lemma 3.0.1 we find $\tilde{g} \in \mathcal{U}_1$ such that 1 is a double root of $D\tilde{g}^m(p)$. By an arbitrarily small perturbation (if necessary) we may assume that there is a bidimensional subspace $P \subset T_pM$ such that $D\tilde{g}^m_{/P} = Id$. If $m \geqslant m_0$ we get a contradiction with the previous lemma. On the other hand, if $m < m_0$ then after a small perturbation we may assume that $D\tilde{g}^m_{/P} = R_\phi$ where R_ϕ is a rotation with a very small rational angle ϕ such that $l = \min\{n \geqslant 0\colon n\phi = 1\} \geqslant m_0$. On the other hand, we also may assume that g^m coincides with Dg^m in a neighborhood of p. Thus, take $q \in P$ near p but different from p. It follows that q is a periodic point of period $l \geqslant m_0$ and $Dg^l_{q/P} = Id$ and by the previous lemma we get a contradiction.

Any other case with λ_s real can be treated similarly. In case λ_s is complex we perturb first to have one real eigenvalue of modulus one and a complex eigenvalue of modulus one. The complex eigenvalue can be interpreted in an appropriate basis as a rotation with rational angle. And a similar argument as before can be done. $\square$

Lemma 3.2.3. *There exist $\eta > 0$, $K > 0$ and $0 < \lambda < 1$ and a neighborhood $\mathcal{U}_3(f)$ such that for any $g \in \mathcal{U}_3(f)$ and $p \in Per^{SD}_{n-1}(g, U) \cap SD_g(1+\eta)$ then*

$$\left\|Dg^{m_p}_{/E^s(p)}\right\| \leqslant K\lambda^{m_p}$$

where m_p is the period of p.

Proof. Let $\mathcal{U}_2$ and δ be as in Corollary 3.2. Let η, $0 < \eta < \delta/2$. Let ϵ and $\mathcal{U}_3$ be the constant and neighborhood obtained from Lemma 3.0.1 when it is applied to $\mathcal{U}_2$. We may assume that ϵ is so small that if $g \in \mathcal{U}_3$ and $p \in Per(g) \cap \mathcal{SD}_g(1+\eta)$ and we perform an ϵ perturbation of Dg along the orbit of p then we obtain $\tilde{g} \in \mathcal{U}_2$ and such that $p \in Per(\tilde{g}) \cap \mathcal{SD}_{\tilde{g}}(1+2\eta)$. For this $\mathcal{U}_3$ and η we will find K and λ as in the lemma. For this it is enough following Lemma II.4 of [9] to show that the family of periodic sequence of linear isomorphism of $\mathbb{R}^{n-1}$ induced by $\{Dg_{/E^s_p}\colon p \in Per^{SD}_{n-1}(g, U) \cap \mathcal{SD}_g(1+\eta);\ g \in \mathcal{U}_3\}$ is a uniformly contracting family.

Notice that if $p \in Per_{n-1}(g)$ and $L_i^s : E^s_{g^i(p)} \to E^s_{g^{i+1}(p)}$ satisfy $\|L_i^s - Dg_{/E^s_{g^i(p)}}\| < \epsilon$ then it induced $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ such that $\|L_i - Dg(g^i(p))\| < \epsilon$ just by declaring

$$L_{i/E^s_{g^i(p)}} = L_i^s \quad \text{and} \quad L_{i/E^{s\perp}_{g^i(p)}} = Id.$$

So, let us show that if $g \in \mathcal{U}_3$, $p \in Per^{SD}_{n-1}(g, U) \cap \mathcal{SD}_g(1+\eta)$ and for any $L_i^s : E^s_{g^i(p)} \to E^s_{g^{i+1}(p)}$ satisfying $\|L_i^s - Dg_{/E^s_{g^i(p)}}\| < \epsilon$ for $i = 0, \dots, m_p - 1$ where m_p is the period of p then $\prod_0^{m_p-1} L_i^s : E^s_p \to E^s_p$ has no eigenvalue of modulus one. Arguing by contradiction, assume that there exist p and g and L_i^s as before such that $\prod L_i^s$ has an eigenvalue of modulus one. We may take, for each i and any $0 \leqslant t \leqslant 1$ a map $L_i^s(t) : E^s_{g^i(p)} \to E^s_{g^{i+1}(p)}$ satisfy $\|L_i^s(t) - Dg_{/E^s_{g^i(p)}}\| < \epsilon$ depending continuously on t and such that $L_i^s(0) = Dg_{/E^s_{g^i(p)}}$ and $L_i^s(1) = L_i^s$. Moreover, we may assume that $\prod_0^{m_p-1} L_i^s(t)$ has all the eigenvalues with modulus less than one for $0 \leqslant t < 1$.

Consider the induced maps $L_i(t) : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ and by applying Frank's lemma 3.0.1 we obtain $g_t \in \mathcal{U}_2$ such that $g_t^i(p) = g^i(p)$ and $Dg_t(g^i(p)) = L_i(t)$. Let $\lambda_u(g)$ be the largest (in modulus) eigenvalue of $Dg^{m_p}(p)$. By the way the induce map are defined we have that $\lambda_u(g_t) = \lambda_u(g)$ for any $0 \leqslant t \leqslant 1$.

Now, if $|\lambda_u(g)| \geqslant (1+\delta)^{m_p}$ we arrive to a contradiction since $p \in \mathcal{SD}_{g_1}(1+2\eta)$ but the product of $\lambda_u(g_1)$ with the eigenvalue of modulus one is larger than $(1+\delta)^{m_p}$ which is bigger than $(1+2\eta)^{m_p}$.

On the other hand, if $|\lambda_u(g)| < (1+\delta)^{m_p}$ then there exists $0 \leqslant t_0 < 1$ such that all eigenvalues of $Dg_{t_0}^{m_p}$ in E_p^s have modulus $< (1+\delta)^{-m_p}$ but has at least one eigenvalue with modulus larger than $(1-\delta)^{m_p}$. This contradicts Corollary 3.2 for g_{t_0}. $\square$

Now we are ready to finish the proof of Proposition 3.2. Recall that we have found $\mathcal{U}_3 \subset \mathcal{U}_2 \subset \mathcal{U}_1 \subset \mathcal{U}_0 \subset \mathcal{U}$ with some desired properties stated in the previous results. We shall prove Proposition 3.2 for $\mathcal{V}(f) = \mathcal{U}_3$ and η as in Lemma 3.2.3. So, we have to find $\gamma > 0$ such that for any $g \in \mathcal{V}$ and $p \in Per_{n-1}^{SD}(g,U) \cap \mathcal{SD}_g(1+\eta)$ then

$$\angle\big(E_p^s, E_p^u\big) > \gamma.$$

First we prove that for any m there is $\gamma > 0$ such that the above holds provided the period of p is smaller than m. Assume by contradiction that this does not hold. Thus, there is a sequence g_n and $p_n \in Per_{n-1}^{SD}(g_n,U) \cap \mathcal{SD}_{g_n}(1+\eta)$ of period $\leqslant m$ such that $\angle(E_{p_n}^s, E_{p_n}^u) \to 0$. We may assume without loss of generality that the periods of p_n are all the same, say m. Let $C = \sup\{\|Dg^m\|\colon g \in \mathcal{V}\} < \infty$. Write $Dg_n^m : T_{p_n}M \to T_{p_n}M$ with respect to $E_{p_n}^s \oplus E_{p_n}^{s\perp}$ as

$$Dg_n^m = \begin{pmatrix} A_n & v_n \\ 0 & \lambda_u(n) \end{pmatrix}$$

where λ_u is the eigenvalue of modulus larger than 1. Let $L_n : E_{p_n}^{s\perp} \to E_{p_n}^s$ be such that $\|L_n\|^{-1} = \angle(E_{p_n}^s, E_{p_n}^u) \to 0$. This means that there exists w_n such that $L_n(0,\dots,0,1) = w_n$ and $\|L_n\| = \|w_n\|$. Moreover, w_n satisfies

$$\big(A_n - \lambda_u(n)I\big)w_n = v_n.$$

By Corollary 3.2 the spectrum of A_n is contained in $B(0,(1-\delta)^m)$. Since $\lambda_u(n) \geqslant 1$ it follows that $\|(A_n - \lambda_u(n)I)^{-1}\|$ is uniformly bounded. Since $\|v_n\| \leqslant C$ for every n we have a contradiction since $\|w_n\| \to \infty$ and $\|w_n\| \leqslant \|(A_n - \lambda_u(n)I)^{-1}\|\|v_n\|$.

Finally, to finish the proof of our proposition we must show that the angle between the subspaces are bounded away of zero for any periodic point of arbitrarily large period. Arguing again by contradiction let $g_n \in \mathcal{V}$ and $p_n \in Per_{n-1}^{SD}(g_n,U)$ of period $m_n \to \infty$ such that $\angle(E_{p_n}^s, E_{p_n}^u) = \gamma_n \to 0$. By Lemma 3.2.3 we have that $\lambda_n = \|Dg_n^{m_n}/E_{p_n}^s\| \leqslant K\lambda^{m_n} < 1$ for n large enough. Let $\sigma_n = \|Dg_n^{m_n}/E_{p_n}^u\| > 1$. Since $p_n \in Per_{n-1}^{SD}(g_n,U)$ we may assume that $\lambda_n\sigma_n < 1$. Thus, if $\gamma_n < \frac{(\sigma_n-1)\epsilon_1}{(\sigma_n+1)2}$ for some n then by Lemma 3.2.1 we get a contradiction. Otherwise we argue exactly as in the last part of the proof of Lemma 3.2.2 and we also get a contradiction with Lemma 3.2.1. This completes the proof of Proposition 3.2.

3.3. Proof of Proposition 3.3

The proof of this proposition is based on the following lemma.

Lemma 3.3.1. *There exist $\mathcal{V}_1$, $\eta_1 > 0$ and a positive integer m_0 such that for any $g \in \mathcal{V}_1$ and $p \in Per_{n-1}^{SD}(g,U) \cap \mathcal{SD}_g(1+\eta_1)$ it holds for some $1 \leqslant m(p) \leqslant m_0$ that*

$$\big\|Dg^m_{/E_p^s}\big\|.\big\|Dg^{-m}_{/E^u_{g^m(p)}}\big\| < \frac{1}{2}.$$

Proof. The proof is strongly based on the strategy developed by Mañé in [9] and we shall follow [17].

Let $\mathcal{V}(f)$, γ and η be as in Proposition 3.2. Let $\mathcal{V}_1(f)$ and ϵ be as in Lemma 3.0.1 applied to $\mathcal{V}(f)$. Let also $0 < \eta_1 < \eta$ and we may assume that ϵ is so small that any ϵ perturbation of the linear map of Dg along a periodic orbit $p \in \mathcal{SD}_g(1+\eta_1)$ then Franks' lemma gives $\tilde{g} \in \mathcal{V}$ such that $p \in \mathcal{SD}_{\tilde{g}}(1+\eta)$.

Now, arguing by contradiction, for this $\mathcal{V}_1$ and η_1 assume that such m_0 does not exist. Thus, for every n we may find $g_n \in \mathcal{V}_1$ and $p_n \in Per_{n-1}^{SD}(g_n,U) \cap \mathcal{SD}_{g_n}(1+\eta_1)$ such that

$$\big\|Dg^m_{n/E_p^s}\big\|.\big\|Dg^{-m}_{n/E^u_{g^m(p)}}\big\| \geqslant \frac{1}{2}$$

for any $1 \leqslant m \leqslant n$. The period of the periodic points p_n must be unbounded. Otherwise, we may assume that all the periods are equal, say m_0 and by taking a subsequence (and identifying $\mathbb{R}^n$ with $T_{p_n}M$ we may assume that

$Dg_n^{m_0}$ converge to a linear isomorphism A of $\mathbb{R}^n$, recall that $\|Dg^{m_0}\|$ is uniformly bounded). Furthermore, we may assume that $E^s_{p_n}$ and $E^u_{p_n}$ converge to subspaces E^s of dimension $n-1$ and E^u of dimension 1 (which are different since $\angle(E^s_{p_n}, E^u_{p_n}) > \gamma$). These subspaces are invariant by A and moreover, the spectrum of A/E^s is contained in $\{z\colon |z| \leqslant (1-\delta)^{m_0}\}$ by Lemma 3.2 and $\|A_{/E^u}\| \geqslant 1$. On the other hand

$$\|A^m_{/E^s}\|.\|A^{-m}_{/E^u}\| \geqslant \frac{1}{2}$$

for every $m \geqslant 1$. However $\|A^m_{/E^s}\| \to 0$ (by the spectrum) and $\|A^{-m}_{/E^u}\| \leqslant 1$. This is a contradiction.

So, let us assume that the periods of the periodic points p_n are unbounded. Let $C = \sup\{\|Dg\|\colon g \in \mathcal{V}_1\}$ and take ϵ_0 satisfying $(2\epsilon_0 + \epsilon_0^2)C \leqslant \epsilon, \epsilon_1$ and m such that

$$\epsilon_1 \leqslant \frac{\gamma}{1+\gamma}\epsilon_0 \quad \text{and} \quad (1+\epsilon_1)^m \geqslant 4 + \frac{2}{\gamma}.$$

Since the periods of p_n are unbounded, we can choose p_n such that its period $m_n > 2m$ and such that $\|Dg^{m_n}_{n/E^s_{p_n}}\| \leqslant K\lambda^{m_n} < 1$ (see Lemma 3.2.3). For the sake of simplicity in notation, set $p = p_n$, $g = g_n$ and $n_0 = m_n$. Take $v \in E^u_p$ and $w \in E^s_p$ with $\|v\| = \|w\| = 1$ and observe that

$$\frac{1}{2}\|Dg^m v\| \leqslant \|Dg^m w\|.$$

Take a linear map $L : E^u_p \to E^s_p$ satisfying $Lv = \epsilon_1 w$ and $\|L\| = \epsilon_1$. Define $\tilde{L} = Dg^{n_0}_{/E^s_p} \circ L \circ Dg^{-n_0}_{/E^u_p}$ and observe that $\|\tilde{L}\| \leqslant \epsilon_1$. Define

$$G = \{u + Lu\colon u \in E^u_p\} \quad \text{and} \quad \tilde{G} = \{u + \tilde{L}u\colon u \in E^u_p\}$$

and take linear maps P, S from T_pM to itself such that

$$P/E^s_p = 0; \qquad (Id + P).E^u_p = G; \qquad S_{/E^s_p} = 0; \qquad (Id + S).\tilde{G} = E^u_p.$$

By Lemma II.10 of [9] it follows that $\|P\| \leqslant \epsilon_0$ and $\|S\| \leqslant \epsilon_0$. Now, for $1 \leqslant j \leqslant m$ define $T_j : T_{g^j(p)}M \to T_{g^j(p)}M$ such that

$$T_{j/E^s_{g^j(p)}} = (1+\epsilon_1)Id \quad \text{and} \quad T_{j/E^u_{g^j(p)}} = Id$$

and for $m+1 \leqslant j \leqslant 2m$ define $T_j : T_{g^j(p)}M \to T_{g^j(p)}M$ such that

$$T_{j/E^s_{g^j(p)}} = (1+\epsilon_1)^{-1}Id \quad \text{and} \quad T_{j/E^u_{g^j(p)}} = Id.$$

It follows also that $\|T_j\| \leqslant \epsilon_0$. Finally, let $L_0 : T_pM \to T_{g(p)}M$ by $L_0 = T_1 \circ Dg \circ (Id + P)$; for $1 \leqslant j \leqslant 2m-1$ let $L_j = T_{j+1} \circ Dg$; for $2m \leqslant j \leqslant n_0 - 2$ let $L_j = Dg$ and $L_{n_0-1} = (Id + S) \circ Dg$. It follows that $\|L_j - Dg_{g^j(p)}\| \leqslant \epsilon$ for $0 \leqslant j \leqslant n_0 - 1$. Therefore, by Lemma 3.0.1, there exists $\tilde{g} \in \mathcal{V}$ such that $\tilde{g}^j(p) = g^j(p)$ and $D\tilde{g}_{g^j(p)} = L_j$. Notice that $p \in \mathcal{SD}_{\tilde{g}}(1+\eta)$. It also holds that $D\tilde{g}(E^s_{g^j}) = E^s_{g^{j+1}(p)}$ and $D\tilde{g}^{n_0}_{/E^s_p} = Dg^{n_0}_{/E^s_p}$. On the other hand

$$D\tilde{g}^{n_0}.E^u_p = L_{n_0-1} \circ \cdots \circ L_0.E^u_p = E^u_p$$

and

$$\|D\tilde{g}^{n_0}_{/E^u_p}\| = \|Dg^{n_0}_{E^u_p}\| > 1$$

and so $p \in Per^{SD}_{n-1}(\tilde{g}, U)$ and

$$E^s_p(\tilde{g}) = E^s_p, \qquad E^u_p(\tilde{g}) = E^u_p.$$

The estimate of the angle between $E^s_m = D\tilde{g}^m(E^s_p)$ and $E^u_m = D\tilde{g}(E^u_p)$ are exactly the same as in [9] or [17] and yields

$$\angle(E^s_m, E^u_m) \leqslant \gamma$$

a contradiction with Proposition 3.2. $\square$

The next corollary is straightforward.

Corollary 3.3. *Let $\mathcal{V}_1$, η_1 and m_0 be as in Lemma* 3.3.1. *Then, there exist $K_0 > 0$ and $0 < \mu < 1$ such that for any $g \in \mathcal{V}_1$ and $p \in Per^{SD}_{n-1}(g, U) \cap \mathcal{SD}_g(1 + \eta_1)$ it holds that*

$$\left\| Dg^n_{/E^s_p} \right\| . \left\| Dg^{-n}_{/E^u_{g^n(p)}} \right\| \leqslant K_0 \mu^n$$

for any $n \geqslant 0$.

Now we will finish the proof of Proposition 3.3. Hence, let $x \in \Lambda$ which is not a periodic attractor. By Proposition 3.1 there exist $g_n \to f$ and $p_n \in Per^{SD}_{n-1} \cap \mathcal{SD}_g(1 + \eta_1)$ such that $p_n \to x$. We may assume that $g_n \in \mathcal{V}_1$ for all n. Taking a subsequence if necessary we may assume that the subspaces $E^s_{p_n}$ converge to a subspace $E_p \subset T_p M$ (of dimension $n - 1$) and the subspaces $E^u_{p_n}$ converge to a one-dimensional subspace $F_p \subset T_p M$ and $T_p M = E \oplus F$. It follows that for any fixed j that $E^s_{g^j_n(p_n)}$ converge to the subspace $Df^j(E_p) := E_{f^j(p)} \subset T_{f^j(p)} M$ and the sequence $E^u_{g^j_n(p_n)}$ converges to $Df^j(F_p) := F_{f^j(p)} \subset T_{f^j(p)} M$. From Corollary 3.3 it holds that

$$\left\| Df^n_{/E^s_z} \right\| . \left\| Df^{-n}_{/E^u_{f^n(z)}} \right\| \leqslant K_0 \mu^n$$

for any $n \geqslant 0$ and for any $z \in \mathcal{O}(x)$ This implies that the subspaces E_x and F_x are unique and does not depends on the sequence of g_n and p_n. This proves that $\Lambda \setminus P_0(f/\Lambda)$ has codimension one dominated splitting $T_{\Lambda \setminus P_0(f/\Lambda)} M = E \oplus F$.

It is just left to prove that the subbundle E is contractive. This is done as follows. First notice that, since $\angle(E, F)$ is bounded away from zero, there exists $c > 0$ such that if $v \in E_x$ and $w \in E^u_x$ with $\|v\| = \|w\| = 1$ and L is the bidimensional subspace spanned by $\{v, w\}$ then for any $n \geqslant 0$ it holds that

$$\det\left(Df^n_{/L}\right) \geqslant c \left\| Df^n v \right\| . \left\| Df^n w \right\|.$$

Now, for any x and n let v_n be such that $\|Df^n_{/E_x}\| = \|Df^n v_n\|$ and let L_n be the subspace spanned by v_n and F. Since $\Lambda \subset L(f, 1)$ we have that $\det(Df^n_{/L_n}) \leqslant 1$ for any n. Now, since F is one-dimensional, we have

$$\left\| Df^n_{/E_x} \right\|^2 = \left\| Df^n_{/E_x} \right\| . \left\| Df^{-n}_{F_{f^n(x)}} \right\| . \left\| Df^n v_n \right\| \left\| Df^n_{F_x} \right\| \leqslant \frac{K_0}{c} \mu^n$$

and therefore

$$\left\| Df^n_{/E_x} \right\| \leqslant \sqrt{\frac{K_0}{c}} \left(\mu^{1/2}\right)^n.$$

This completes the proof of Proposition 3.3 and Theorem B.

4. Markov partitions for contractive codimension one dominated splitting

In this section, we show the existence of Markov partition for "basic sets" (see Definition 4.6) exhibiting a contractive codimension one dominated splitting (see Section 4.2 for the definitions). First, in the next subsection we show some dynamical properties that hold for the center unstable manifold. More precisely, we prove that it has dynamical meaning and we use this to prove in Section 4.2 that for some special sets it is possible to exhibit a Markov partition.

4.1. Some dynamical properties

Let $I_1 = (-1, 1)$ and $I_\epsilon = (-\epsilon, \epsilon)$, and denote by $Emb^r(I_1, M)$ the set of C^r-embedding of I_1 on M, and denote by $Emb^r(I_1^{n-1}, M)$ the set of C^r-embedding of I_1^{n-1} on M, where n is the dimension of M.

Following the classical results of stable manifold theorems (see [7]) we get that for a contractive codimension one dominated splitting holds the next:

Lemma 4.1.1. *There exist two continuous functions $\phi^s : \Lambda \to Emb^1(I_1^{n-1}, M)$ and $\phi^{cu} : \Lambda \to Emb^1(I_1, M)$ such that if define $W^s_\epsilon(x) = \phi^s(x) I_\epsilon^{n-1}$ and $W^{cu}_\epsilon(x) = \phi^{cu}(x) I_\epsilon$ the following properties hold*:

(a) $T_x W^s_\epsilon(x) = E(x)$ *and* $T_x W^{cu}_\epsilon(x) = F(x)$.
(b) *There is* $\lambda < 1$ *such that*

$$f\big(W^s_\epsilon(x)\big) \subset W^s_{\lambda\epsilon}\big(f(x)\big).$$

(c) *For all* $0 < \epsilon_1 < 1$ *there exists* ϵ_2 *such that and*

$$f^{-1}\big(W^{cu}_{\epsilon_2}(x)\big) \subset W^{cu}_{\epsilon_1}\big(f^{-1}(x)\big).$$

Sometimes, one needs the central manifold to be of class C^2. This is guaranteed, for C^2 diffeomorphisms, by the so-called 2-domination: the splitting $E \oplus F$ is 2-dominated if there exists $0 < \sigma < 1$ such that

$$\big\|Df^n_{/E(x)}\big\|\big\|Df^{-n}_{/F(f^n(x))}\big\|^2 \leqslant C\sigma^n, \quad n \geqslant 0.$$

Remark 4.1. It follows that if f is a C^2 diffeomorphisms and Λ is a compact invariant manifold exhibiting a codimension one dominated splitting which is also 2-dominated then the map ϕ^{cu} in Lemma 4.1.1 is indeed a map $\phi^{cu} : \Lambda \to Emb^2(I_1, M)$ (see [7] for details).

The following result in [18] guarantees that a codimension one dominated splitting is 2-dominated. In the mentioned paper the result is only proved for surfaces map's, but the adaptation is straightforward:

Lemma 4.1.2. *Let f be a C^2 diffeomorphisms and let Λ be a compact invariant manifold exhibiting a codimension one dominated splitting. Then, there exists at most finitely many periodic attractors* (*sinks*) *in Λ such that any compact invariant set $\Lambda_0 \subset \Lambda$ and disjoint from those periodic attractors is* 2*-dominated.*

We conclude some dynamical properties for the center unstable manifold tangent to the F direction. First, we appeal to some results and definitions proved in [19] for "codimension one dominated splitting". It what follows with $\ell(I)$ it is denoted the usual length of an arc I.

Definition 4.1. Let $f : M \to M$ be a C^2 diffeomorphism and let Λ be a compact invariant set having dominated splitting $E \oplus F$ with $dim(F) = 1$. Let U be an open set containing Λ where is possible to extend the previous dominated splitting. We say that a C^2-arc I in M (i.e, a C^2-embedding of the interval $(-1, 1)$) is a δ–E-arc provided the next two conditions holds:

1. $f^n(I) \subset U$, and $\ell(f^n(I)) \leqslant \delta$ for all $n \geqslant 0$.
2. $f^n(I)$ is always transverse to the E-subbundle.

Related to this kind of arcs it is proved in [19] the following result (see Theorem 3.2 in [19]).

Theorem 4.1 *(Denjoy Theorem). Let f be a C^2 diffeomorphisms, and let Λ be a compact invariant set exhibiting a codimension one dominated splitting. There exists δ_0 such that if I is a δ–E-arc with $\delta \leqslant \delta_0$, then one of the following properties holds*:

1. $\omega(I) = \bigcup_{\{x\in I\}} \omega(x)$ *is a periodic simple closed curve and* $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (*where m is the period of $\mathcal{C}$*) *is conjugated to an irrational rotation,*
2. $\omega(I) \subset J$ *where J is a periodic arc.*

As a consequence of the Denjoy Theorem, we can conclude the following lemma related to the center unstable manifolds. The proof is a straightforward version of Lemma 3.3.2 of [17] for codimension one dominated splitting.

Lemma 4.1.3. *Let f be a C^2 diffeomorphisms, and let Λ be a compact invariant set exhibiting a codimension one dominated splitting such that all the periodic points are hyperbolic. There exists $\epsilon > 0$ such that for all $\gamma < \epsilon$ there exists $r = r(\gamma)$ such that*:

1. *For any positive integer n follows that* $f^{-n}(W_r^{cu}(x)) \subset W_\gamma^{cu}(f^{-n}(x))$.
2. *For every* $r \leqslant r(\gamma)$*, either*:
   (a) $\ell(f^{-n}(W_r^{cu}(x))) \to 0$ *as* $n \to +\infty$,
   (b) *or* $x \in W_r^u(p)$ *for some* $p \in Per(f_{/\Lambda})$ *such that* $p \in W_r^{cu}(x)$ *and there exists in another periodic points* $q \in \overline{W_r^u(p)}$ *which is a sink or a nonhyperbolic periodic point,*
   (c) $x \in \mathcal{C}$ *such that* $\mathcal{C}$ *is a periodic simple closed curve and* $f^m_{/\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ *(where m is the period of* $\mathcal{C}$*) is conjugated to an irrational rotation.*

4.2. Markov partitions

In what follows we assume that Λ exhibits a codimension one contractive dominated splitting and Λ is not a periodic simple closed curve. First we give a series of definitions inspired in similar definitions introduced for hyperbolic sets.

Definition 4.2. We say that Λ has local product structure if exists $\alpha > 0$ such that if for any $x, y \in \Lambda$ with $d(x, y) < \alpha$ holds that $W_\epsilon^s(x) \cap W_\epsilon^{cu}(y) \in \Lambda$. We denote with

$$[x, y] := W_\epsilon^s(x) \cap W_\epsilon^{cu}(y).$$

Definition 4.3. A subset $B \subset \Lambda$ is called a box if

1. $[x, y] \in B$ whenever $x, y \in B$,
2. $B = \overline{int(B)}$, where $int(B)$ denotes the interior of B in Λ.

We also define the diameter of B as the maximum distance between points in B.

Definition 4.4. Let Λ be a compact and invariant set having contractive codimension one dominated splitting. A Markov partition of Λ is a collection of boxes $\mathcal{P} = \{B_1, \dots, B_n\}$ such that:

1. $\Lambda \subset \bigcup_{1 \leqslant i \leqslant n} B_i$,
2. $int(B_i) \cap int(B_j) = \emptyset$ if $i \neq j$,
3. for any $x \in \Lambda$, if $x \in B_i$ for some $B_i \in \mathcal{P}$ follow that:
   (a) there exists $B_j \in \mathcal{P}$ such that $f^{-1}(W_\epsilon^{cu}(x) \cap B_i) \subset B_j$,
   (b) there exists $B_k \in \mathcal{P}$ such that $f(W_\epsilon^s(x) \cap B_i) \subset B_k$.

Moreover, we define the size of the Markov partition as the maximum of the diameters of B_i.

Definition 4.5. We say that a point x in the limit set $L(f)$ is isolated if there exists a neighborhood U_x of x such that $U_x \cap L(f) \subset Per(f)$. Let $\tilde{L}(f) \subset L(f)$ be the sets of the non-isolated points.

Definition 4.6. We say that a compact and invariant set Λ with contractive codimension one dominated splitting is a basic piece if it is transitive and has local product structure.

The next theorem is the main one in the present subsection.

Theorem 4.2. *Let* Λ *be a basic piece of* $\tilde{L}(f)$ *such that all the periodic points are hyperbolic. Then, there exists a Markov partition of* Λ *of arbitrarily small size.*

The proof is different for the case that for any points the local center unstable manifold of any point is contained in the local unstable manifold, and for the case that this does not hold, that is:

Case A. There exists $r > 0$ such that for any $x \in \Lambda$ holds that $W_r^{cu}(x) \subset W_{loc}^u(x)$.

Case B. For any r small there exists a point $x \in \Lambda$ satisfying the item $2(b)$ of Lemma 4.1.3.

4.2.1. Proof of Theorem 4.2 in Case A

In this case, we just follows the strategy developed by Fathi in [6]. Observe that in this case, $f_{/\Lambda}$ is expansive, and since Λ has local product structure it follows that Λ is a maximal invariant set. In fact in [6] it is proved that for expansive homeomorphisms, it is possible to obtain a hyperbolic adapted metric, not necessarily coherent with a Riemannian structure but defining the same topology. Using the hyperbolic metric for f, and the fact that we are dealing with a maximal invariant set, the proof of the shadowing lemma for hyperbolic sets with local product structure can be pushed in the present case, and after that it is possible to repeat the classic construction of a Markov partition done for a maximal invariant hyperbolic set (see [1]).

4.2.2. Proof of Theorem 4.2 in Case B

The proof of Theorem 4.2 goes through different steps:

- *Step* I: First we study the boundary points in Λ.
- *Step* II: We induce an expansive quotient map and following [1] we get a Markov partition for the quotient map.
- *Step* III: We refine the Markov partition obtained in Step II, using the periodic boundary points. From the that, we construct a Markov partition for f of arbitrarily small size.

Definition 4.7 *(Boundary points).* Let Λ be a basic piece of $\tilde{L}(f)$. Let ϵ be the positive constant given by Lemma 4.1.3. We say that x is a boundary point, if there exists $\epsilon_1 < \epsilon$ such that one of the connected components of $W_{\epsilon_1}^{cu}(x) \setminus \{x\}$ does not contain points in Λ.

Let $\gamma < \epsilon$; we say that x is a γ-boundary point if one of the connected components of $W_\gamma^{cu}(x) \setminus \{x\}$ does not contain points in Λ but both end points of this connected component are in Λ.

Remark 4.2. Observe that any $x \in \Lambda$ is at least accumulated by points in Λ contained in one of the connected components of $W_{\epsilon_1}^{cu}(x) \setminus \{x\}$.

Remark 4.3. Observe that in the present case, always exist boundary points.

Lemma 4.2.1. *Let Λ be a basic piece of $\tilde{L}(f)$. The following hold*:

1. *if x is a boundary points then it belongs to the stable manifold of a periodic point p in Λ*;
2. *there exists $\gamma > 0$ such that for any $\gamma_2 < \gamma_1 < \gamma$ follows that*

$$\mathit{Cardinal}\big(\{\gamma'\text{-boundary periodic points},\ \gamma_2 < \gamma' < \gamma_1\}\big) < \infty.$$

Proof. Let x be a boundary point. Then, there is $\epsilon_1 < \epsilon$ such that one of the connected components of $W_{\epsilon_1}^{cu}(x) \setminus \{x\}$ do not contain points in Λ.

To see the first item, let us start observing that we can assume that there exists $r = r(\epsilon_1)$ such that

$$\ell\big(f^{-n}\big(W_r^{cu}\big(f^n(x)\big)\big)\big) \to 0 \tag{1}$$

otherwise, we have that there exists r' such that $\ell(f^n(W_{r'}^{cu}(x)))$ does not converge to zero and therefore, it follows from Lemma 4.1.3 that belongs to the stable manifold of a periodic points and so the lemma is proved in this case. Now, to conclude, it is enough to show that there are positive integers $m < n$ such that $f^n(x) \in W_\epsilon^s(f^m(x))$. If this does not hold, we would have positive integers n_1, n_2, n_3 such that $[W_r^{cu}(f^{n_2}(x)) \setminus \{f^{n_2}(x)\}] \cap W_\epsilon^s(f^{n_1}(x)) \neq \emptyset$ and $[W_r^{cu}(f^{n_2}(x)) \setminus \{f^{n_2}(x)\}] \cap W_\epsilon^s(f^{n_3}(x)) \neq \emptyset$, such that these intersections hold at both side of $f^{n_2}(x)$. But this implies from (1) that $W_{\epsilon_1}^{cu}(x)$ has points in Λ in both sides of x which is a contradiction with the assumption that x is a boundary point.

The second item is immediate. $\square$

Lemma 4.2.2. *For any periodic point p in Λ it follows that $W^s(p)$ is dense. Moreover, for any $\epsilon > 0$ and any periodic point p there is a compact disk D contained in the stable manifold of p such that for any $x \in \Lambda$ follows that $D \cap W^{cu}_\epsilon(x) \neq \emptyset$.*

Proof. Let z such that $\omega(z) = \alpha(z) = \Lambda$. It follows that for this point the central unstable manifold is dynamically defined: otherwise, by Lemma 4.1.3 follows that $z \in W^s(q)$ for some periodic point q and therefore, $\omega(z) = \mathcal{O}(q)$; a contradiction.

Then, given any periodic point p, there exists $n > 0$ such that $\mathrm{dist}(f^n(z), p) < \frac{\epsilon}{2}$ and therefore, $W^s_\epsilon(p) \cap W^{cu}_\epsilon(f^n(z)) \neq \emptyset$. Noting with z' the point of intersection, it follows that

$$\mathrm{dist}\big(f^{-m}\big(f^n(z)\big), f^{-m}(z')\big) \to 0, \quad n \to +\infty,$$

and since $\alpha(z) = \Lambda$ it follows that $\alpha(z') = \Lambda$.

The second part, follows from compactness and transversality between the local center unstable and local stable manifolds. □

Lemma 4.2.3. *Let Λ be a basic piece of $\tilde{L}(f)$. Given β, there are a finite number of periodic points $p_1, \dots, p_r$ and $D_1, \dots, D_r$ compact disks contained in $\bigcup_{1 \leqslant i \leqslant r} W^s(p_i)$ such that*

$$f\Big(\bigcup_i D_i\Big) \subset \bigcup_i D_i \tag{2}$$

and if $x \notin \Lambda \cap D = \Lambda \cap [\bigcup_{1 \leqslant i \leqslant r} D_i]$ then:

(1) *$W^{cu}_\epsilon(x)$ has intersection with D at both sides of x (i.e.: D intersects both connected components of $W^{cu}_\epsilon(x) \setminus \{x\}$).*
(2) *The connected component of $W^{cu}_\epsilon(x) \setminus D$ containing x has length smaller than $\beta/2$.*

Proof. We take $\epsilon_1 < \beta/2$, $\epsilon_2 < \beta/2$ and such that $\ell(f^{-n}(W^{cu}_{\epsilon_2}(x))) < \epsilon_1$. Take $\gamma < \epsilon_1, \epsilon_2$ and take all the γ-boundary periodic points $p_1, \dots, p_r$. Let us assume that the lemma is not true. Then, there exists a sequence x_n of points in Λ and compacts disks $D_n = \bigcup_i D_{i,n}$ such that the conclusion (1) of the lemma does not holds for any x_n and D_n. Take x and accumulation point of $\{x_n\}$. If x is in the stable manifold of some p_i, from the fact that p_i is a boundary point, then all points x_n are converging either from one side of the stable compact disk D_x of $W^s(p_i)$ or are contained in D_x. Using Lemma 4.2.2 we get that there are compact disks $\hat{D}_n$ contained in the stable manifold of p_i converging to D_x, and so the points x_n are enclosed by compact disks of the stables manifolds of the points p_i getting a contradiction. If x does not belong to any of the stables manifolds of the points p_i, we get two alternatives; either x is a boundary point, or it is not a boundary point. In the first case, x belong to the stable manifold of some δ-boundary periodic point q with $\delta < \gamma$. This implies that on one of the connected components of $W^{cu}_\epsilon(q) \setminus \{q\}$ we get points of Λ converging to q and on the other components there are points of Λ also contained in $W^{cu}_\gamma(q)$. Taking n large enough such that $f^n(x)$ is close to q we get that there are points of Λ contained in both side of $W^{cu}_\gamma(f^n(x))$, and this implies that there are points of Λ on both sides of $W^{cu}_\gamma(x)$. Again, using that the stables manifolds of the periodic points are dense, we conclude the points x_n are closed by compact disks of the stables manifolds of the points p_i getting a contradiction. In the case that x is not a boundary point, there are points of Λ on both sides of $W^{cu}_\gamma(x)$, and again we get a contradiction. □

Notation. If $y \in W^{cu}_\gamma(x)$ we denote by $W^{cu}(x, y)$ the (open) arc in the central unstable manifolds $W^{cu}_\gamma(x)$ whose endpoints are x an y.

Definition 4.8. Let β_1 be a small positive number. We define a relation in Λ as follows: we say that $x \sim_{\beta_1} y$ if:

1. $x = y$ or
2. (a) $y \in W^{cu}_\gamma(x)$,
   (b) $W^{cu}(x, y) \cap \Lambda = \emptyset$ (in particular x and y are boundary points),
   (c) there exist periodic points p_x, p_y in Λ such that $x \in W^s(p_x)$ and $y \in W^s(p_y)$ and p_x, p_y are η-boundary points with $\eta \leqslant \beta_1$. Notice that $p_y \in W^{cu}_\gamma(p_x)$ and $W^{cu}_\gamma(p_x, p_y) \cap \Lambda = \emptyset$.

Remark 4.4. The relation $\sim_{\beta_1}$ is an equivalence relation. It is obvious by definition that is identical and reflexive. The transitivity follows because if $x \sim_{\beta_1} y$ and $y \sim_{\beta_1} z$ then $z = x$ or $z = y$. Otherwise some of these points is a boundary points at both sides of the central unstable manifold and this is impossible due to the transitivity of Λ (unless Λ is just a periodic orbit).

Lemma 4.2.4. *Let $x \sim_{\beta_1} y$ then $f^m(x) \sim_{\beta_1} f^m(y)$ for all $m \in \mathbb{Z}$.*

Proof. Let r be such that if $w \in W_r^{cu}(z)$ then $f^{-n}(w) \in W_\gamma^{cu}(f^{-n}(z))$ for all $n \geqslant 0$. Choose $\beta_1 \leqslant r/2$. Take n large enough so that $f^n(x)$ is close to p_x. It follows that $W_r^{cu}(f^n(x)) \cap W_\epsilon^s(p_y)$ is nonempty, and let w be this intersection. It suffices to prove the lemma that $w = f^n(y)$. Otherwise, notice that $W_r^{cu}(f^n(x), w) \cap \Lambda = \emptyset$ since $f^n(x)$ is a boundary point. It follows that $f^{-n}(w) \in W_\gamma^{cu}(x)$. Since x cannot be a boundary point at both sides of $W_\gamma^{cu}(x) \setminus \{x\}$ then $y \in W_\gamma^{cu}(x, f^{-n}(w))$ or $f^{-n}(w) \in W_\gamma^{cu}(x, y)$. In any case we arrive to a contradiction, because in these open arcs there are no point of Λ. $\square$

Lemma 4.2.5. *Given γ_0 there exists η such that if $\beta_1 \leqslant \eta$ and $x \sim_{\beta_1} y$ then $d(f^n(x), f^n(y)) \leqslant \gamma_0$ for all $n \in \mathbb{Z}$.*

Proof. Let r be such that if $w \in W_r^{cu}(z)$ then $f^{-n}(w) \in W_{\gamma_0}^{cu}(f^{-n}(z))$ for all $n \geqslant 0$. Let $\eta \leqslant r/2$. Now let $x \sim_{\beta_1} y$ and $m \in \mathbb{Z}$. It follows that for n large enough (we may assume that $n > |m|$) $f^n(x)$ is arbitrarily close to p_x and as in the previous lemma, $f^n(x) \in W_r^{cu}(f^n(x))$ then $f^m(y) \in W_{\gamma_0}^{cu}(f^m(x))$ and hence $d(f^m(x), f^m(y)) \leqslant \gamma_0$. $\square$

Lemma 4.2.6. *If γ_1 is small then if two points satisfy $d(f^n(x), f^n(y)) \leqslant \gamma_1$ for all $n \in \mathbb{Z}$ it holds that $x \sim_\eta y$ for some η.*

Proof. Let r be such that $f^{-n}(W_r^{cu}(w))$ has length less than $\epsilon/2$ for all $n \geqslant 0$ and any $w \in \Lambda$. Choose $\gamma_1 < \epsilon/2$ be such that if $d(x, y) < \gamma_1$ then $z = W_\gamma^{cu}(y) \cap W_\epsilon^s(x) \in W_r^{cu}(y)$. Now assume that x and y are as in the lemma. First we prove that $y \in W_\gamma^{cu}(x)$. If $x \notin W_\gamma^{cu}(y)$, we take $z = W_\gamma^{cu}(y) \cap W_\epsilon^s(x)$ and observe that for some positive integer n follows that $\operatorname{dist}(f^{-n}(z), f^{-n}(x)) > \epsilon$ and for any positive integer m, $\operatorname{dist}(f^{-m}(z), f^{-m}(y)) < \gamma_1$. Therefore we get that

$$\gamma_1 > \operatorname{dist}\bigl(f^{-n}(y), f^{-n}(x)\bigr) > \operatorname{dist}\bigl(f^{-n}(z), f^{-n}(x)\bigr) - \operatorname{dist}\bigl(f^{-n}(z), f^{-n}(y)\bigr) > \epsilon - \epsilon/2 > \gamma_1,$$

a contradiction.

On the other hand, if $W^{cu}(x, y) \cap \Lambda \neq \emptyset$ it follows that $W^s(q) \cap W^{cu}(x, y) \neq \emptyset$ for some q with unbounded unstable manifold. Hence the arc length of $W^{cu}(x, y)$ growths by positive iteration and this contradicts that $d(f^n(x), f^n(y)) < \gamma_1$ for all n.

Finally, from above it follows that x and y are boundary points. Therefore they belongs to the stable manifold of some periodic points p_x and p_y, respectively. Since $f^n(x) \to p_x$ and $f^n(y) \to p_y$ and $W^{cu}(x, y)$ does not grow in length by future iteration we conclude the proof of the lemma. $\square$

Now we fix γ_1 in such a way that the above lemma applies and we choose $\gamma_0 < \gamma_1/2$ and we take β_1 from Lemma 4.2.5 corresponding to γ_0. To avoid notation we denote the equivalence relation $\sim_{\beta_1}$ with $\sim$.

Let

$$\tilde{\Lambda} = \Lambda/\sim \quad \text{and} \quad p : \Lambda \to \tilde{\Lambda}$$

the canonical projection and endow $\tilde{\Lambda}$ with the quotient topology. Denote by $[x] = p(x)$. Moreover, denote by

$$\tilde{f} : \tilde{\Lambda} \to \tilde{\Lambda}$$

the induced homeomorphism (recall Lemma 4.2.4).

Lemma 4.2.7. *With the notations above the following hold*:

1. *p is closed and $\tilde{\Lambda}$ is a compact Hausdorff metrizable space*;
2. *$\tilde{f}$ is expansive.*

Proof. If $x \sim y$ we may find $z_1, z_2 \in W^{cu}_\gamma(x)$ which are not boundary points and that $W^{cu}(x, y) \subset W^{cu}(z_1, z_2)$. Now, for any w in a neighborhood of x consider the set of points in Λ which lies in $W^{cu}_\gamma(w)$ between the local stable manifolds of z_1 and z_2. These points form an open set U such that if $z \in U$ and $w \sim z$ then $w \in U$. In other words U is a saturated open set. Now for different equivalent classes it is not difficult to find disjoint open sets as above. This implies that $\tilde{\Lambda}$ is Hausdorff. Since p is continuous, Λ is compact and $\tilde{\Lambda}$ is Hausdorff it follows that p is closed. Finally, if $\{U_n\}$ is a countable basis (and closed under finite unions) for the topology of Λ it is not difficult to see that $\{p(U_n^c)^c\}$ is a basis for $\tilde{\Lambda}$. Therefore $\tilde{\Lambda}$ is metrizable. Let $\tilde{d}$ be a metric in $\tilde{\Lambda}$ compatible with the topology.

Let us show that $\tilde{f}$ is expansive. Let $W = \{(x, y) \in \Lambda \times \Lambda\colon d(x, y) \geqslant \gamma_1\}$. Then $p \times p(W)$ is a compact subset of $\tilde{\Lambda} \times \tilde{\Lambda}$ which does not contain the diagonal $\tilde{\Delta}$ of $\tilde{\Lambda} \times \tilde{\Lambda}$. Let α_1 be such that the $B(\tilde{\Delta}, \alpha_1) \cap p \times p(W) = \emptyset$ where $B(\tilde{\Delta}, \alpha_1)$ denote the α_1 neighborhood of $\tilde{\Delta}$.

Let P denote the set of η-boundary periodic points of Λ with $\eta > \beta_1$. It is clear that P is a finite set. Let α_2 be such that if $q_1 \neq q_2$ are two different points in P then $\tilde{d}(p(q_1), p(q_2)) > \alpha_2$.

Let

$$\alpha_0 = \min\{\alpha_1, \alpha_2\}.$$

We are going to show that α_0 is the constant of expansivity. Let $[x]$ and $[y]$ be such that their orbits by $\tilde{f}$ remain to a distance smaller than α_0. Therefore, it follows that $d(f^n(x), f^n(y)) < \gamma_1$ for all n (otherwise, $(f^n(x), f^n(y)) \in W$ and hence $\tilde{d}([f^n(x)], [f^n(y)]) > \alpha_0$). From Lemma 4.2.6 follows that $x \in W^s(p_x)$ and $y \in W^s(p_y)$ and such that $p_x \in W^{cu}_{loc}(p_y)$ with $W^{cu}(p_x, p_y) \cap \Lambda = \emptyset$. Since, by continuity $\tilde{d}([p_x], [p_y]) \leqslant \alpha_0$ it follows that p_x, p_y are not η boundary points with $\eta > \beta_1$. Therefore they are η boundary points with $\eta \leqslant \beta_1$. Hence, $[x] = [y]$. □

Lemma 4.2.8. *For all α, there exists a Markov partition of $\tilde{\Lambda}$ of size smaller than α.*

Proof. For any η we define $W^s_\eta([x]) = \bigcup_{z\sim x} p(W^s_\eta(z))$. We also define $W^{cu}_\eta([x]) = \bigcup_{z\sim x} p(W^{cu}_\eta(z))$. It is straightforward to verify that $W^s_\eta([x])$ and $W^{cu}_\eta([x])$ are true local stable and unstable sets and that $\tilde{\Lambda}$ has local product structure.

Using the notion of adapted metric for expansive maps introduced by Fathi in [6] it follows the shadowing property. And arguing exactly in the same way as in [1] we can construct a Markov partition $\tilde{\mathcal{P}} = \{\tilde{B}_1, \dots, \tilde{B}_n\}$ on $\tilde{\Lambda}$ of size less than β_1 for $\tilde{f}$. □

Lemma 4.2.9. *Given $\beta > 0$ there exists $\alpha > 0$ such that if $z \in W^s_\epsilon(x)$ and $\tilde{d}([z], [x]) < \alpha$ then $d(x, z) < \beta/2$.*

Proof. Otherwise, there are sequences z_n, x_n such that $z_n \in W^s_\epsilon(x_n)$, $d(x_n, z_n) \geqslant \beta/2$ and such that $\tilde{d}([x_n], [z_n]) < 1/n$. Taking limit points x and z of x_n and y_n we have $z \in W^s_\epsilon(x)$ and $x \sim z$ which is not possible. □

End of the proof of Theorem 4.2. We have to prove that given $\beta > 0$ Λ has a Markov partition of size smaller than β. Choose α so that the previous lemma applies and take a Markov partition $\tilde{\mathcal{P}} = \{\tilde{B}_1, \dots, \tilde{B}_n\}$ on $\tilde{\Lambda}$ of size less than α for $\tilde{f}$. Define $B_i = p^{-1}(\tilde{B}_i)$. It is straightforward to verify that $\mathcal{P} = \{B_1, \dots, B_n\}$ is a Markov partition of Λ.

It remains the question if it has size less than β. By the previous lemma the "stable" size of this boxes is smaller than $\beta/2$. Nevertheless, a priori we have no much control on the "unstable" size. So we argue as follows. From Lemma 4.2.3 we have that there are a finite number of periodic points $p_1, \dots, p_r$ and $D_1, \dots, D_r$ compact disks contained in $\bigcup_{1\leqslant i\leqslant r} W^s(p_i)$ such that

$$f\Bigl(\bigcup_i D_i\Bigr) \subset \bigcup_i D_i \tag{3}$$

and if $x \notin \Lambda \cap D = \Lambda \cap [\bigcup_{1\leqslant i\leqslant r} D_i]$ then:

1. $W^{cu}_\epsilon(x)$ has intersection with D at both sides of x;
2. the connected component of $W^{cu}_\epsilon(x) \setminus D$ containing x has length smaller than $\beta/2$.

We refine the Markov partition in the following way: Fix B_i any one of the previous Markov box. There are finitely many points $x_1, \dots, x_m$ in B_i such that $D \cap int(B_i) \subset \bigcup_j W^s_\epsilon(x_j)$. Now we define a relation in $B_i - \bigcup_j W^s_\epsilon(x_j)$: $z \sim w$ if

1. $z \in W^s_\epsilon(w)$ or
2. denoting by $u = W^s_\epsilon(w) \cap W^{cu}_\gamma(z)$ it holds that $W^{cu}(z,u) \cap (\bigcup_j W^s_\epsilon(x_j)) = \emptyset$.

The above relation is an equivalence relation. Let $C_{i,j},\ j = 1, \dots, k_i$, be the set of equivalent classes. The refinement of B_i is the collection $B_i^j = \overline{C_{i,j}},\ j = 1, \dots, k_i$.

Finally, the collection $\mathcal{B} = \{B_i^j\colon\ B_i \in \mathcal{P},\ j = 1, \dots, k_i\}$ is a Markov partition of Λ of size smaller than β. □

Remark 4.5. In the proof of Theorem C (see next section) we shall use also Markov boxes that consists on a collection of central unstable arcs. If B_i is a Markov box and $x \in B_i$ we take x^-, x^+ points in $W^{cu}_\gamma(x) \cap B_i$ such that $W^{cu}_\gamma(x) \cap B_i$ is contained in the closed arc in $W^{cu}_\gamma(x)$ whose endpoints are x^- and x^+. We denote this arc by $W^{cu}[x^-, x^+]$.

Now, if $\mathcal{B} = \{B_1, \dots, B_k\}$ is a Markov partition as before the collection $\mathcal{P} = \{\hat{B}_1, \dots, \hat{B}_k\}$ defined by:

$$\hat{B}_i = \bigcup_{x \in B_i} W^{cu}\big[x^-, x^+\big]$$

is a Markov partition consisting of central unstable arcs.

In the sequel, we consider the especial case of homoclinic class, and we show that they exhibit Markov partition.

Definition 4.9. We define the homoclinic class of a saddle hyperbolic periodic point as the closure of intersection of the stable and unstable manifold of p and it is denoted with $H(p) = \overline{W^s(p) \cap W^u(p)}$.

Proposition 4.1. *If a homoclinic class has codimension one contractive dominated splitting, then it is a basic piece. In particular, it has Markov partition.*

Proof. It is well known that a homoclinic class is transitive. So, to finish the proof we need to prove that has local product structure. If one of the components $W^u(p) \setminus \{p\}$ has finite length, we take γ smaller than ϵ and the length of the connected component of $W^u(p) \setminus \{p\}$ that does not intersect the class. If x belongs to the intersection of the stable and unstable manifold of p we have that $W^{cu}_\gamma(x) \subset W^u(p)$. And also $W^s_\gamma(x) \subset W^s(p)$. Thus, if $x, y \in W^s(p) \cap W^u(p)$ and $\mathrm{dist}(x,y)$ is small then $W^{cu}_\gamma(x) \cap W^s_\gamma(y) \in W^s(p) \cap W^u(p)$, i.e., $[x,y] \in \Lambda = H(p)$. Since $H(p)$ is the closure of the intersection of the stable and unstable manifold of p we conclude, by continuity, the local product structure on $H(p)$. □

5. Proof of Theorem C

Theorem C is an extension of Theorem B in [17]. Although the proofs have strong similarity there are nontrivial difficulties to overcome in our context. This is the main reason why we assume contractive codimension one dominated splitting and that the set Λ is isolated in $L(f)$ (in order to obtain a Markov partition).

First, Theorem C follows from the next theorem.

Theorem 5.1. *Let $f : M \to M$ be a C^2 diffeomorphism. Let $\Lambda \subset L(f)$ be a compact invariant set such that it is isolated in $L(f)$, all the periodic points are hyperbolic, and has contractive codimension one dominated splitting. Then, one of the following statements holds*:

1. *Λ is a hyperbolic set*;
2. *there exists a simple closed curve $\mathcal{C} \subset \Lambda$ which is invariant under f^m for some m and it is normally hyperbolic. Moreover $f^m : \mathcal{C} \to \mathcal{C}$ is conjugated to an irrational rotation.*

Assuming that this last theorem is true we show that in this case, the number of periodic simple closed curves normally hyperbolic and conjugated to an irrational rotation contained in Λ is finite. This implies Theorem C. For more details see [17, p. 977].

The first step in the proof of Theorem 5.1 is the following elementary lemma.

Lemma 5.0.10. *Let Λ_0 be a compact invariant set having a contractive codimension one dominated splitting $T_{/\Lambda}M = E^s \oplus F$. If for any $x \in \Lambda_0$ holds that $\|Df^{-n}_{/F(x)}\| \to 0$ as $n \to \infty$ then Λ_0 is a hyperbolic set.*

Now, using the previous lemma, we prove Theorem 5.1 based on the next lemma.

Main Lemma. *Let $f : M \to M$ and Λ be as in Theorem* 5.1 *and assume that Λ does not contain a periodic simple closed curve normally hyperbolic conjugated to an irrational rotation. Let $\Lambda_0 \subset \Lambda$ be a nontrivial compact invariant set such that every properly compact invariant subset of Λ_0 is hyperbolic. Then, Λ_0 is a hyperbolic set.*

To show how the Main Lemma implies Theorem 5.1 we argue as follows: assume that statement 2 in Theorem 5.1 does not hold and we have to prove then that Λ is hyperbolic. If this is not the case, we take a compact invariant subset $\Lambda_0 \subset \Lambda$ which is the minimal set, in the Zorn's lemma sense, such that Λ_0 is not hyperbolic. To prove the existence of this set, it is enough to show that given a sequences of nonhyperbolic compacts invariant sets $\{\Lambda_\alpha\}_{\alpha\in\mathcal{A}}$ ordered by inclusion follows that $\bigcap_{\alpha\in\mathcal{A}} \Lambda_\alpha$ is a nonhyperbolic compact invariant set. By election of Λ_0 it follows that every properly compact invariant subset of Λ_0 is hyperbolic. By the Main Lemma it follows that Λ_0 is hyperbolic, a contradiction. More details can be found in [17].

The proof of the Main Lemma is given in the next subsection. Nevertheless we give here the basics steps of it proof:

1. The central unstable manifolds (which are of class C^2, recall Lemma 4.1.2) have dynamics properties. In fact for every $x \in \Lambda_0$ there exists $\epsilon(x)$ such that $W^{cu}_{\epsilon(x)}(x)$ is an unstable manifold of x, meaning that $\ell(f^{-n}(W^{cu}_{\epsilon(x)}(x))) \to 0$ as $n \to \infty$.
2. For point x in an open set B in Λ_0 we have
$$\sum_{n\geqslant 0} \ell\big(f^{-n}\big(W^{cu}_{\epsilon(x)}(x)\big)\big) < \infty.$$
3. For every point $x \in \Lambda_0$ we have
$$\big\|Df^{-n}_{/F(x)}\big\| \to 0$$
when $n \to \infty$.

5.1. Proof of the Main Lemma

In this section we shall assume that Λ_0 is in the hypothesis of the Main Lemma, i.e., Λ_0 is a nontrivial compact invariant transitive set, such that every proper compact invariant subset is hyperbolic and $\Lambda_0 \subset \Lambda$ where Λ has contractive codimension one dominated splitting with all its periodic points hyperbolic and has no periodic simple curve normally hyperbolic conjugated to an irrational rotation. Under this conditions, we will prove that for every $x \in \Lambda_0$, $\|Df^{-n}_{/F(x)}\| \to 0$. Then next lemma show a sufficient condition. The rest of the paper will consist on showing that this condition holds.

Lemma 5.1.1. *Assume that there exists a set B containing an open set of Λ_0 such that for every $y \in B \cap \Lambda_0$ we have $\|Df^{-n}_{/F(y)}\| \to_{n\to\infty} 0$. Then for every $z \in \Lambda_0$, $\|Df^{-n}_{/F(z)}\| \to 0$.*

Proof. Let z be any point in Λ. There are two possibilities:

- $\alpha(z)$ (the α-limit set of z) is properly contained in Λ_0. Then, $\alpha(z)$ is a hyperbolic set, thus
$$\big\|Df^{-n}_{/F(z)}\big\| \to_{n\to\infty} 0.$$

- $\alpha(z) = \Lambda_0$. Then, there exists m_0 such that $f^{-m_0}(z) \in B_0$, implying that

  $$\|Df^{-n}_{/F(f^{-m_0}(z))}\| \to_{n\to\infty} 0$$

  and so

  $$\|Df^{-n}_{/F(z)}\| \to_{n\to\infty} 0. \qquad \square$$

The next lemma of this subsection is classical in one-dimensional dynamics (see for example [3]) and the proof is left to the reader. We only have to remark, since the diffeomorphism f is of class C^2, the center unstable manifolds are also C^2, and the center unstable manifolds varies continuously in the C^2 topology, we have a uniform Lipschitz constant K_0 of $\log(Df)$ restricted to the center unstable manifolds.

Lemma 5.1.2. *There exists K_0 such that for all $x \in \Lambda_0$ and $J \subset W^{cu}_\gamma(x)$ we have for all $z, y \in J$ and $n \geqslant 0$:*

1. $\frac{\|Df^{-n}_{/\tilde F(y)}\|}{\|Df^{-n}_{/\tilde F(z)}\|} \leqslant \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$,
2. $\|Df^{-n}_{/\tilde F(x)}\| \leqslant \frac{\ell(f^{-n}(J))}{\ell(J)} \exp(K_0 \sum_{i=0}^{n-1} \ell(f^{-i}(J)))$

where $\tilde F(z) = T_z W^{cu}_\epsilon(x)$.

In order to prove the existence of the open set B as in Lemma 5.1.1 we need a Markov partition. For this reason we show that Λ_0 is contained in a homoclinic class.

Lemma 5.1.3. *There exists a periodic point p such that Λ_0 is contained in the homoclinic class or p.*

Proof. Let $x \in \Lambda_0$ be such that $\Lambda_0 = \alpha(x)$. So, there is a subsequence m_i of positive integers such that $f^{-m_i}(x) \to x$. We can assume that x does not belong to the unstable manifold of a periodic point (in other case, Λ would be a periodic point) and so by Lemma 4.1.3 we get that there is γ such that $\ell(f^{-n}(W^{cu}_\gamma(x))) \to 0$. Then, for m_{i_0} large enough, we get that for any $y \in f^{-m_{i_0}}(W^{cu}_\gamma(x))$ follows that $W^s_\epsilon(y) \cap W^{cu}_{\gamma/3}(x) \neq \emptyset$. Then, from standard arguments, we get a periodic point p_1 with orbit in a neighborhood of Λ_0 (and hence it is in Λ since Λ is isolated within $L(f)$) and such that $W^{cu}_\gamma(p_1) \cap W^s_\epsilon(x) \neq \emptyset$. If $W^{cu}_\gamma(p_1) \subset W^u(p_1)$ we set $p = p_1$. Otherwise, there must exists $p \in W^{cu}_\gamma(p_1)$ such that for one connected component of $W^{cu}_\gamma(p) \setminus \{p\}$, say $W^{cu,+}_\gamma(p)$ we have that

$$W^{cu,+}_\gamma(p) \subset W^u(p) \quad \text{and} \quad W^{cu,+}_\gamma(p) \cap W^s_\epsilon(x) \neq \emptyset.$$

Moreover, we get that for any $y \in \Lambda_0$ in a small ball centered at x we get that $W^s_\epsilon(p) \cap W^{cu}_\gamma(y) \neq \emptyset$ and $W^{cu,+}_\gamma(p) \cap W^s_\epsilon(y) \neq \emptyset$. In particular we get that $W^s_\epsilon(p) \cap W^{cu}_\gamma(f^{-m_i}(x)) \neq \emptyset$ and $W^{cu,+}_\gamma(p) \cap W^s_\epsilon(f^{-m_i}(x)) \neq \emptyset$ for any m_i large enough. Notice that the orbits of the points in these intersection remains in a neighborhood of Λ_0.

From the fact that $W^{cu,+}_\gamma(p) \cap W^s_\epsilon(f^{-m_i}(x)) \neq \emptyset$ for any m_i and that $W^{cu,+}_\gamma(p) \subset W^u(p)$, we conclude that there are compact disks of $W^u(p)$ converging to the central unstable manifold of x. On the other hand, since $W^s_\epsilon(p) \cap W^{cu}_\gamma(f^{-m_i}(x)) \neq \emptyset$, $f^{-m_i}(x) \to x$ and the dynamical properties of the central unstable manifold, we get that there are compact disks of $W^s(p)$ converging to the local stable manifold of x. These two fact, together, imply that there are homoclinic points of p converging to x. Thus, $x \in H(p)$, the homoclinic class of p. Therefore, since $\alpha(x) = \Lambda_0$ it follows that $\Lambda_0 \subset H(p)$ $\square$

Our next goal is to show that if p is as above, then $H(p) \subset \Lambda$.

Lemma 5.1.4. *Let $\Lambda \subset L(f)$ be a compact invariant set isolated in $L(f)$. Then, there is a neighborhood U of Λ such that if $\Lambda \cap \omega(x) \neq \emptyset$ for some $x \in M$ it follows that there exists a positive integer n_0 such that $f^n(x) \in U$ for any $n > n_0$.*

Proof. Let us assume that the lemma is false. Then for any closed neighborhood U of Λ such that $\Lambda \subset interior(\Lambda)$ and $L(f) \cap U = \Lambda$, there exist x and $y \in \Lambda \cap \omega(x)$ such that $\mathcal{O}_n^+(x) = \{f^k(x): k > n\}$ is not contained in U for any positive integer n. Let $n_i \to +\infty$ be such that $f^{n_i}(x) \to y$. Let for each n_i the first positive k_i such that $f^{n_i+k_i}(x) \notin U$ and let z be an accumulation point of $\{f^{n_i+k_i-1}(x)\}_{i>0}$. We can assume that $f^{n_i+k_i-1}(x) \to z$. It follows that $z \in U$ and $z \in \omega(x)$. Therefore, $z \in \Lambda$ and so $f(z) \in \Lambda$. However, $f^{n_i+k_i}(x) \to f(z)$ and so $f(z) \in interior(U)^c$. A contradiction. □

It follows immediately the following corollary.

Corollary 5.1. *Let Λ and Λ_0 be as in the Main Lemma and let p be as in Lemma* 5.1.3. *Then $\Lambda_0 \subset H(p) \subset \Lambda$.*

Now, since $\Lambda_0 \subset H(p) \subset \Lambda$ for some hyperbolic periodic point p and since Λ has contractive codimension one dominated splitting and all its hyperbolic points are hyperbolic the same holds for $H(p)$. Finally, by Proposition 4.1, there is a Markov partition $\mathcal{P} = \{B_1, \ldots, B_n\}$ associated for $H(p)$ of arbitrarily small size (to be fixed later) and we will use it to conclude the Main Lemma. Recall also that we may define the Markov partition as consisting of central unstable manifolds (see Remark 4.5).

Definition 5.1. Given a Markov partition $\mathcal{P} = \{B_1, \ldots, B_n\}$ we say that a set B is a Markov subbox if there exist $k \geqslant 0$ and two boxes B_i and B_j of $\mathcal{P}$ such that

1. $B \subset f^{-k}(B_i) \cap B_j$.
2. If $x \in B \cap H(p)$ then the connected component of $W_\gamma^{cu}(x) \cap (f^{-k}(B_i) \cap B_j)$ that contains x is $W_\gamma^{cu}(x) \cap B$.

Now, given a Markov subbox B, for any $y \in B \cap H(p)$ we define:

$$J_B(y) = W_\gamma^{cu}(y) \cap B.$$

Notice that $J_B(y) = f^{-k}(J_{B_i}(f^k(y)))$. Moreover, since $\mathcal{P}$ is a Markov partition we get that for any $y \in B$ and any $k \geqslant 0$ either,

1. $f^{-k}(J_B(y)) \cap B = \emptyset$ or
2. $f^{-k}(J_B(y)) \subset B$.

In many occasions, we need to estimate the length between different central unstable arcs in a Markov subbox. In this direction, we introduce the following definitions.

Definition 5.2. Let $\mathcal{P} = \{B_1, \ldots, B_n\}$ be a Markov partition in $H(p)$ and let $B \subset B_j$ be a Markov subbox B. We say that B has distortion (or cu-distortion) C if for any two arcs J_1, J_2 where $J_i = J_B(y_i)$ for some $y_i \in B \cap H(p)$, $i = 1, 2$, the following holds:

$$\frac{1}{C} \leqslant \frac{\ell(J_1)}{\ell(J_2)} \leqslant C.$$

Lemma 5.1.5. *Let Λ be a compact invariant set having a contractive codimension one dominated splitting. It follows that the local stable foliation defined on Λ is a C^1-foliation.*

The proof follows from the classical C^r-section theorem (for details, see Theorem 5.18 in [7, p. 58] and also [23, p. 44]). To conclude that a stable lamination associated to a dominated splitting $E^s \oplus F$ is C^1, it is necessary to show that

$$\frac{\|Df_{/E^s}\|}{m(Df_{/F})}\|Df_{/F}\| < \lambda < 1$$

where $m(.)$ is the minimum norm. In particular, if the subbundle F is one-dimensional, this condition translates into the condition $\|Df_{/E^s}\| < \lambda < 1$.

Lemma 5.1.6. *Let $\mathcal{P} = \{B_1, \ldots, B_n\}$ be a Markov partition of sufficiently small size. Then, there exists $C = C(\mathcal{P})$ such that any Markov subbox B has distortion C.*

Proof number one. Let B_i be an element of the Markov partition. Notice that there exist $x_i^+, x_i^- \in B_i \cap H(p)$ such that $\partial_B^{cu} \subset W_\epsilon^s(x_i^+) \cup W_\epsilon^s(x_i^-)$. It is not difficult to show that there exists a C^1 codimension one foliation $\mathcal{F}_i^s$ in an open set (in M) that contains B_i whose tangent spaces are close to the E^s subbundle and such that $\mathcal{F}_i^s(x_i^\pm)$ (the leaf of this foliation through the points $x_i^\pm$) contains $W_\epsilon^s(x_i^\pm)$, respectively. There exists C_i such that for any two arcs $J_j = J_{B_i}(y_j)$, $j = 1, 2$, for some points $y_j \in B_i \cap H(p)$ and setting $\Pi_i = \Pi_i(J_1, J_2)$ the projection from J_1 onto J_2 along the foliation $\mathcal{F}_i^s$ then

$$\frac{1}{C_i} \leqslant \|\Pi_i'\| \leqslant C_i.$$

For each box B_i we fix the foliation $\mathcal{F}_i^s$. Now, let B be any Markov subbox. Let $k \geqslant 0$ and B_i, B_j element of the Markov partition as in Definition 5.1 and let any two arcs J_1, J_2 where $J_i = J_B(y_i)$ for some $y_i \in B \cap H(p)$, $i = 1, 2$. We define $\Pi_B : J_1 \to J_2$ as follows:

$$\Pi_B = f^{-k} \circ \Pi_i\big(f^k(J_1), f^k(J_2)\big).$$

If we show that there exists C such that for any Markov subbox B and any J_1, J_2 as before we have that

$$\frac{1}{C} \leqslant \|\Pi_B'\| \leqslant C$$

we are done. From standard arguments about foliations and contractive direction (see for instance [23] and the proof of Lemma 3.4.1 of [17]) it follows that there exists D such that

$$\frac{1}{DC_i} \leqslant \|\Pi_B'\| \leqslant DC_i.$$

Thus, taking $C = \max\{DC_1, \ldots, DC_n\}$ the proof is finished. □

Proof number two. We could also argue as follows. The stable foliation in $H(p)$ is of class C^1. Extends this foliation to a C^1 foliation in a neighborhood of $H(p)$. Choose a Markov partition with such a small size that any element of this partition is contained in that neighborhood. For any B_i of the Markov partition consider Π_i the (local) projection along this foliation. Now for any Markov subbox $B \subset B_j$ project along the foliation in B_j (it is well defined since the stable foliation in $H(p)$ is invariant and ∂_B^{cu} is contained in the stable foliation of Λ). Therefore there if $C = \max\{C_1, \ldots, C_n\}$ where C_i is such that

$$\frac{1}{C_i} \leqslant \|\Pi_i'\| \leqslant C_i$$

we are done. □

The previous lemma, help us to prove the following.

Lemma 5.1.7. *Let $\mathcal{P} = \{B_1, \ldots, B_n\}$ be a Markov partition. Then, exists $K = K(\mathcal{P})$ such that for any Markov subbox B and any $z \in B \cap \Lambda_0$ holds that*

$$\sum_{i=0}^{n} \ell\big(f^{-i}\big(J_B(z)\big)\big) \leqslant K$$

provided $f^{-i}(z) \notin B$, $1 \leqslant i \leqslant n$.

Proof. Let $\mathcal{P} = \{B_1, \ldots, B_n\}$ be a Markov partition of $H(p)$. For each box B_j we choose a point $x_j \in B_j \cap H(p)$ and we take $J_j = J_{B_j}(x_j)$.

Let B be a Markov subbox and let $z \in B \cap \Lambda_0$ such that $f^{-i}(z) \notin B$ for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$ let B_{k_i} the element of the partition $\mathcal{P}$ that contains $f^{-i}(z)$. Let $B(i)$ the Markov subbox contained in $f^{-i}(B) \cap B_{k_i}$ that contains $f^{-i}(J_B(z))$. Now, we consider $J_{i,k_i}(z) = J_{k_i} \cap B(i)$. Observe that:

1. Since each $B \in \mathcal{P}$ is Markovian in the stable direction follows that for $i \neq j$, $B(i) \cap B(j) = \emptyset$, and in particular $J_{i,k_i}(z) \cap J_{j,k_j}(z) = \emptyset$ (no matter if $k_i = k_j$).
2. For any i,

$$\frac{1}{C} \leqslant \frac{\ell(J_{i,k_i}(z))}{\ell(f^{-i}(J_B(z)))} \leqslant C.$$

Then,

$$\sum_{i=0}^{n} \ell\bigl(f^{-i}\bigl(J_B(z)\bigr)\bigr) < C \sum_{i=0}^{n} \ell\bigl(J_{i,k_i}(z)\bigr) \leqslant C \sum_{j} \ell(J_j) := K. \qquad \square$$

Now, we proceed to conclude the proof of the Main Lemma. We will split the rest of the proof in two cases: either there exists a point $x \in \Lambda_o$ such that $x \notin \omega(x)$ or no such a point exists.

5.1.1. Proof of Main Lemma when $\exists x \in \Lambda_0$ with $x \notin \omega(x)$

Let U be a neighborhood of x such that $f^n(x) \notin U$ for any $n \geqslant 1$. Such a neighborhood of x exists since $x \notin \omega(x)$. Now fix a Markov partition $\mathcal{P} = \{B_1, \ldots, B_m\}$ such that if $x \in B_i$ then $B_i \subset U$.

Definition 5.3. Given an element B_i of the Markov partition $\mathcal{P}$, we say that it has infinitely many returns (associated to Λ_0) if there are points $x_n \in B_i \cap \Lambda_0$ such that $f^{-k_n}(x_n) \in B_i$, $f^{-j}(x_n) \notin B$ for $j = 1, \ldots, k_n - 1$ and $k_n \to \infty$. For the point x_n we call the integer k_n, the return time of x_n.

Notice that by the way we choose the Markov partition there is B_i such that $x \in B_i$ and B_i has infinitely many returns associated to Λ_0 since Λ_0 is transitive (and so there is a point whose forward orbit is dense and as it goes very near x for a long time does not return to B_i).

Lemma 5.1.8. *Let B be a Markov subbox (see Definition 5.1) and assume that there is $\xi < 1$ such that for every $y \in B \cap \Lambda_0$ we have $\|Df^{-k}_{/\tilde{F}(z)}\| < \xi$ for all $z \in J_B(y)$ where k is such that $f^{-k}(y) \in B$ and $f^{-i}(y) \notin B$ for $1 \leqslant i < k$. Then for all $y \in B \cap \Lambda_0$ the following holds*:

$$\sum_{n \geqslant 0} \ell\bigl(f^{-n}\bigl(J_B(y)\bigr)\bigr) < \infty.$$

In particular this implies that

$$\bigl\|Df^{-n}_{/F(y)}\bigr\| \to_{n \to \infty} 0.$$

The proof of the previous lemma is the same as the proof of Lemma 3.7.2 in [17].

Now, we show that it is possible to find a Markov subbox verifying the condition of Lemma 5.1.8.

Lemma 5.1.9. *Let B_i be an element of the Markov Partition with infinitely many returns. Then there exists Markov subbox B contained in B_i such that satisfies the conditions of Lemma 5.1.8.*

From this, it follows from Lemmas 5.1.8 and 5.1.1 the proof of the Main Lemma in the case we are dealing with.

Now we give the proof of Lemma 5.1.9.

Proof. Let B_i be an element of the Markov partition as in the hypothesis of the lemma, and let K, K_0, C be as in Lemmas 5.1.7, 5.1.2 and 5.1.6, respectively. Consider also $L = \min\{\ell(J_B(z))\colon z \in B \cap \Lambda_0\}$.

Let $r > 0$ such that

$$r\frac{C_1}{L}\exp(2K_0K) < \frac{1}{2}.$$

Let r_1 be such that $\ell(f^{-n}(W^{cu}_{r_1}(z))) \leqslant r$ for any $z \in \Lambda_0$ (recall Lemma 4.1.3). Since B_i has infinite returns, there exists $w \in B_i \cap \Lambda_0$ such that if we take B the Markov subbox in $f^{-k_0}(B_i) \cap B_i$ that contains $f^{-k_0}(w)$, where k_0 is the return time of w, then follows that

$$J_B(z) \subset W^{cu}_{r_1}(z), \quad \forall z \in B \cap \Lambda_0,$$

and therefore

$$\ell\bigl(f^{-j}\bigl(J_B(z)\bigr)\bigr) < r, \quad \forall j \geqslant 0,\ \forall z \in B \cap \Lambda_0.$$

Let us prove that the box B satisfied the thesis of the lemma. Let $y \in B \cap \Lambda_0$ and let k be the return time of y to B. Notice that $k \geqslant k_0$. Observe that if $w \in f^{k_0}(B) \cap \Lambda_0$, then for $z \in J(w) = J_{B_i}(w)$,

$$\bigl\| Df^{-k_0}_{/\tilde F(z)} \bigr\| \leqslant \frac{\ell(f^{-k_0}(J(w)))}{J(w)} \exp(K_0 K)$$

and moreover $J_B(f^{-k_0}(w)) = f^{-k_0}(J(w))$.

Now, set $n_0 = k - k_0$ ($k \geqslant k_0$) and we have $f^{-n_0}(y) \in f^{k_0}(B)$.

Then, for $z \in J_B(y)$,

$$\begin{aligned}
\bigl\| Df^{-k}_{/\tilde F(z)} \bigr\| &\leqslant \bigl\| Df^{-k_0}_{/\tilde F(f^{-n_0}(z))} \bigr\| \bigl\| Df^{-n_0}_{/\tilde F(z)} \bigr\| \\
&\leqslant \frac{\ell(f^{-k_0}(J(f^{-n_0}(y))))}{\ell(J(f^{-n_0}(y)))} \exp(K_0 K) \frac{\ell(f^{-n_0}(J_B(y)))}{\ell(J_B(y))} \exp(K_0 K) \\
&= \ell\bigl(f^{-n_0}\bigl(J_B(y)\bigr)\bigr) \frac{\ell(J_B(f^{-k}(y)))}{\ell(J_B(y))} \frac{1}{\ell(J(f^{-n_0}(y)))} \exp(2K_0 K) \\
&\leqslant r C_1 \frac{1}{L} \exp(2K_0 K) < \frac{1}{2}.
\end{aligned}$$

So, the proof is finished. □

5.1.2. Proof of Main Lemma when $x \in \omega(x)$ for all $x \in \Lambda_0$

We begin remarking that we cannot expect to do the same argument here as in the preceding case since by our assumption then, for every Markov box, the set of returns associated to Λ_0 of this box is always finite. Nevertheless we shall exploit the fact that in the case Λ_0 the central unstable manifold is in fact an unstable together with the existence of "boundary points of Λ_0". We begin by showing that in the present case there are no periodic points in Λ_0.

Lemma 5.1.10. *Assume that $x \in \omega(x)$ for all $x \in \Lambda_0$. Then there are no periodic points in Λ_0.*

Proof. Assume that there is a periodic point $q \in \Lambda_0$. Since Λ_0 is not just a periodic orbit and it is transitive, then there is $x \in \Lambda_0$ such that $x \in W^s(q)$. Hence $x \notin \omega(x)$. □

Notice in particular that any x in Λ_0 is not a boundary point of $H(p)$. The next corollary follows immediately from the above lemma and Lemma 4.1.3.

Corollary 5.2. *Let $x \in \Lambda_0$. Then $\ell(W^{cu}_\gamma(x)) \to 0$.*

We introduce some notations. Given a Markov partition (consisting of central unstable manifolds) and let B be an element of it. Let $x \in int(B)$ and let $J_B(x) = W^{cu}_\gamma(x) \cap B$. We order $J = J_B(x)$ in some way and we denote $J^+ = \{y \in J\colon\ y > x\}$, $J^- = \{y \in J\colon\ y < x\}$. Notice that there are points z^+, z^- in $H(p) \cap B$ such that $W^s_\epsilon(z^+) \cap J^+ \neq \emptyset$ and $W^s_\epsilon(z^-) \cap J^- \neq \emptyset$ and $W^s_\epsilon(z^+) \cap J^- = \emptyset = W^s_\epsilon(z^-) \cap J^+$ (since x is not a boundary point of $H(p)$).

For any other point $y \in B \cap H(p)$ we denote by $J^+(y)$ the connected component of $J_B(y) \setminus W^s_\epsilon(x)$ that contains a point of $W^s_\epsilon(z^+)$ and by J^- the other one (and so contains a point of $W^s_\epsilon(z^-)$).

We shall denote by B^+ the collection of $J^+(y)$ for $y \in B \cap H(p)$ and by B^- the collection of $J^-(y)$ for $y \in B \cap H(p)$.

Lemma 5.1.11. *There exist a Markov partition arbitrary small and B and element of it such that there is $x \in int(B) \cap \Lambda_0$ such that (with the notations above) that $B^+ \cap \Lambda_0 = \emptyset$ or $B^- \cap \Lambda_0 = \emptyset$.*

Proof. Let us start with any Markov partition of $H(p)$ (with small size) and let B_1 be an element of this partition such that contains a point of $z \in \Lambda_0$. Let A be the subset of $J_B(z)$ that consist of points of $W^s_\epsilon(x)$ with $x \in B \cap \Lambda_0$. It follows that A is compact and has empty interior (as a subset of $J_B(z)$) since there are no periodic points in Λ_0. If there is a connected component L of the complement of A in $J_B(z)$ with one endpoint in common with an endpoint of $J_B(z)$ we are done since the other point must belong to $W^s_\epsilon(x)$ for some point $x \in B \cap \Lambda_0$. Otherwise, take L any component and there exists $x_1, x_2 \in B \cap \Lambda_0$ such that the endpoints of L are in $W^s_\epsilon(x_1)$ and $W^s_\epsilon(x_2)$. Now we consider a Markov partition $\mathcal{P}_k$ consisting of the subboxes of $f^{-k}(B_i) \cap B_j$ where $B_i, B_j \in \mathcal{P}$ and k is large enough so that the central unstable size of $\mathcal{P}_k$ is smaller that the distance between $W^s_\epsilon(x_1)$ and $W^s_\epsilon(x_2)$ in B. Let $\hat{B}$ be the element of $\mathcal{P}_k$ in B that contains x_1. We claim that $\hat{B}$ satisfy the thesis of our lemma. It is enough to show that x_1 in not in $\partial^{cu}(\hat{B})$. Assume that x_1 is in $\partial^{cu}(\hat{B})$. Since $x_1 \in \omega(x_1)$ there are points of the forward orbit of x_1 in $\hat{B}$ and arbitrarily close to x_1. There cannot be in the $\partial^{cu}(\hat{B})$ since otherwise $\omega(x_1)$ is a periodic orbit. Therefore, for some m, $f^m(x_1) \in int(\hat{B})$. Then $f^{-m}(J_{\hat{B}}(f^m(x_1))) \cap \hat{B} \neq \emptyset$ but $f^{-m}(J_{\hat{B}}(f^m(x_1))) \not\subseteq \hat{B}$, a contradiction. □

Lemma 5.1.12. *Let B be a Markov box such that $B^+ \cap \Lambda_0 = \emptyset$. Then there exists K such that for every $y \in B \cap \Lambda_0$,*

$$\sum_{j \geqslant 0} \ell\big(f^{-j}\big(J^+(y)\big)\big) < K.$$

In particular there exist $J_1(y)$, $J^+(y) \subset J_1(y) \subset J(y)$ such that the length of $J_1(y) - J^+(y)$ is bounded away from zero (independently of y) and such that

$$\sum_{n=0}^{\infty} \ell\big(f^{-n}\big(J_1(y)\big)\big) < \infty.$$

Assuming this lemma, we can prove the Main Lemma in the present case. Using the notation of the preceding lemma, take

$$B = \bigcup_{y \in B \cap \Lambda_0} J_1(y).$$

Notice that B is an open set of Λ_0, and for every $y \in B \cap \Lambda_0$ (i.e. $y \in J_1(y)$), we have

$$\sum_{n=0}^{\infty} \ell\big(f^{-n}\big(J_1(y)\big)\big) < \infty$$

and so

$$\big\|Df^{-n}_{/F(y)}\big\| \to_{n\to\infty} 0.$$

Thus, Lemma 5.1.1 provides the end of the proof of the Main Lemma in the present case if Lemma 5.1.12 holds. To prove this lemma we show first that the box introduced in Lemma 5.1.11, has some similar property as the Markov box.

Lemma 5.1.13. *Let B be a Markov box such that $B^+ \cap \Lambda_0 = \emptyset$. Then B^+ verifies that for all $y \in B \cap \Lambda_0$ and $n \geqslant 0$,*

$$f^{-n}\big(J^+(y)\big) \cap B^+ = \emptyset \quad \text{or} \quad f^{-n}\big(J^+(y)\big) \subset B^+.$$

Moreover, there exists K_1 such that if $y \in B \cap \Lambda_0$ and $f^{-j}(J^+(y)) \cap B^+ = \emptyset$, $1 \leqslant j < n$, then

$$\sum_{j=0}^{n} \ell\big(f^{-j}\big(J^+(y)\big)\big) < K_1.$$

Proof. Assume that for some $y \in B \cap \Lambda_0$ and $n > 0$ $f^{-n}(J^+(y)) \cap B^+ \neq \emptyset$ holds. As B is a Markov box we conclude that $f^{-n}(J(y)) \subset B$. If $f^{-n}(J^+(y))$ is not contained in B^+, then $f^{-n}(J^+(y)) \cap W^s_\epsilon(x) \neq \emptyset$. However, this implies that $f^n(x) \in B^+$. Since $x \in \Lambda_0$ this is a contradiction, and completes the proof of the first part.

The existence of K_1 it can be proved with the same arguments as in Lemma 5.1.7. □

Definition 5.4. Let B be as in the previous lemma and let $y \in B \cap \Lambda_0$ The set of positive integers m such that $f^{-m}(J^+(y)) \subset B^+$ will be called the set of return times of y.

Now, we conclude the proof of Lemma 5.1.12.

Proof of Lemma 5.1.12. First, if we have that there are not points in B^+ that for some negative iterates are also in B^+, by the preceding lemma, we conclude the thesis.

Take $r > 0$ such that

$$\frac{r}{L}\exp(K_0K_1) < \frac{1}{2}$$

where $L = \min\{\ell(J^+(y))\colon\ y \in B \cap \Lambda_0\}$. Let $N > 0$ be such that if $n > N$ then $\ell(f^{-n}(J(y))) < r$.

It follows that if for some $y \in B \cap \Lambda_0$ we have that $f^{-k}(J^+(y)) \cap B^+ = \emptyset$ for $1 \leqslant k \leqslant m$ with $m \geqslant N$ then, by item 2 of Lemma 5.1.2 that $\|Df^{-m}_{\tilde{F}(z)}\| < \frac{1}{2}$ for all $z \in J^+(y)$.

On the other hand we claim that there exists r_1 $(\leqslant r)$ such that if $y \in B \cap \Lambda_0$ and $f^{-k}(J^+(y)) \cap B^+ = \emptyset$ for $1 \leqslant k < k(y) \leqslant N$ and $f^{-k(y)}(J^+(y)) \subset B^+$ then

$$\operatorname{dist}\bigl(W^s_\epsilon(x), f^{-k_0}\bigl(J^+(y)\bigr)\bigr) > r_1.$$

Otherwise, since $k(y)$ is bounded by N we get a point $y \in B \cap \Lambda_0$ such that $f^{-k(y)}(J^+(y)) \cap W^s_\epsilon(x) \neq \emptyset$; then, using that B is a Markov box, and that the extremal points of $J^+(y)$ and $f^{-k(y)}(J^+(y))$ are $W^s_\epsilon(x)$, we conclude that $\omega(x)$ is a periodic point, which is impossible.

Now, let N_1 be such that if $n > N_1$ then $\ell(f^{-n}(J(y))) < r_1$. Let $y \in B \cap \Lambda_0$ and consider $k_1(y), k_2(y), \ldots, k_n(y), \ldots$ be the successive return times of $J^+(y)$, i.e, $f^{-j}(J^+(f^{-k_1(y)-\cdots-k_i(y)}(y))) \cap B^+ = \emptyset$ if $1 \leqslant j < k_{i+1}$ and $f^{-k_{i+1}(y)}(J^+(f^{-k_1(y)-\cdots-k_i(y)}(y))) \subset B^+$. We claim that if $i \geqslant N_1$ then $k_i > N$. Otherwise, since $m = k_1(y) + \cdots + k_i(y) \geqslant N_1$ then $\ell(f^{-m}(J(y))) < r_1$ and $\operatorname{dist}(W^s_\epsilon(x), f^{-m}(J^+(y))) > r_1$ and so $f^{-m}(y) \in B^+$, a contradiction.

Finally, for any point $y \in B \cap \Lambda_0$ we have three possibilities:

1. the set of return times is empty,
2. the set of return times is finite,
3. the set of return times is infinite.

In any case we have that, taking into account the previous estimations that for every $y \in B \cap \Lambda_0$ we have

$$\sum_{j\geqslant 0} \ell\bigl(f^{-j}\bigl(J_1^+(y)\bigr)\bigr) \leqslant N_1K_1 + \sum_{j\geqslant 0} \operatorname{diam}(M)\exp(2K_0K_1)\left(\frac{1}{2}\right)^j + K_1 = K < \infty.$$

In particular, as in the Schwarz's proof of the Denjoy Theorem, we conclude that $\forall y \in B \cap \Lambda_0$ there exist $J_1(y)$, $J^+(y) \subset J_1(y) \subset J(y)$ such that the length of $J_1(y) - J^+(y)$ is bounded away from zero (independently of y) and such that

$$\sum_{n=0}^{\infty} \ell\bigl(f^{-n}\bigl(J_1(y)\bigr)\bigr) < \infty$$

and the proof of the lemma is finished. □

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