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An optimal partial regularity result for minimizers of an intrinsically defined second-order functional

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Abstract

For a compact Riemannian manifold N and a domain $\Omega \subset \mathbb{R}^m$, we consider the intrinsic bi-energy

$$\mathcal{E}_2(u) := \int_\Omega |\nabla Du|^2 \, dx$$

for maps $u : \Omega \to N$. We prove that the minimizers of $\mathcal{E}_2$ constructed by R. Moser satisfy $u \in W^{2,2}_{\mathrm{loc}}(\Omega, N)$. Furthermore, we apply a dimension reduction argument in order to show $\mathcal{H}\text{-}\dim(\mathrm{sing}(u)) \leqslant m-5$ for all minimizers $u \in W^{2,2}(\Omega, N)$ of the functional $\mathcal{E}_2$. This result is optimal since we show that the map $u_0 : B^m \to S^{m-1}$, $x \mapsto \frac{x}{|x|}$ minimizes $\mathcal{E}_2$ in its Dirichlet class for $m \geqslant 5$.

Résumé

Pour une variété riemanienne compacte et un domaine $\Omega \subset \mathbb{R}^m$, nous considérons la bi-énergie intrinsèque

$$\mathcal{E}_2(u) := \int_\Omega |\nabla Du|^2 \, dx$$

pour les applications $u : \Omega \to N$. Nous démontrons que les minimiseurs de $\mathcal{E}_2$ construée par R. Moser satisfont $u \in W^{2,2}_{\mathrm{loc}}(\Omega, N)$. En outre, nous utilisons une méthode de réduction de la dimension pour prouver $\mathcal{H}\text{-}\dim(\mathrm{sing}(u)) \leqslant m-5$ pour tout minimiseur $u \in W^{2,2}(\Omega, N)$ de la fonctionnelle $\mathcal{E}_2$. Ce résultat est optimal parce que nous démontrons que l'application $u_0 : B^m \to S^{m-1}$, $x \mapsto \frac{x}{|x|}$ minimise $\mathcal{E}_2$ dans sa classe de Dirichlet pour $m \geqslant 5$.

Keywords: Biharmonic maps; Defect measure; Dimension reduction; Intrinsic bi-energy; Partial regularity

1. Introduction and statement of the results

Throughout this article, we assume that N is a smooth, compact Riemannian manifold without boundary, and $\Omega \subset \mathbb{R}^m$ a bounded flat domain with smooth boundary, where $m \geqslant 4$. For simplicity, we may assume that $N \subset \mathbb{R}^K$ is embedded isometrically into some Euclidean space $\mathbb{R}^K$.

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For maps $u:\Omega\to N$, several second-order functionals have been considered as analogues of the classical Dirichlet energy

$$E_1(u):=\frac{1}{2}\int_\Omega |Du|^2\,dx,$$

which is well known from the setting of harmonic maps. From the analytic point of view, the so-called extrinsic bi-energy

$$E_2(u):=\frac{1}{4}\int_\Omega |\Delta u|^2\,dx$$

seems to be the most natural one. For example, E_2 is a coercive functional on the space $W^{2,2}(\Omega,N)$ and therefore, the existence of minimizers follows easily by the direct method. Critical points $u\in W^{2,2}(\Omega,N)$ with an additional stationarity assumption enjoy the partial regularity property $\mathcal{H}^{m-4}(\mathrm{sing}(u))=0$, where $\mathcal{H}^{m-4}$ denotes the $(m-4)$-dimensional Hausdorff measure, compare [3,14,17–19] and [15]. In a preceding paper [13], the author reduced the dimension of the singular set of minimizers $u\in W^{2,2}(\Omega,N)$ to $\mathcal{H}\text{-}\dim(\mathrm{sing}(u))\leqslant m-5$. Nevertheless, the functional E_2 is not a completely satisfactory choice because its definition depends on the embedding $N\hookrightarrow\mathbb{R}^K$ and not only on the intrinsic geometrical properties of N. It would be preferable, from the geometric point of view, to consider an intrinsically defined functional with similar properties as the extrinsic bi-energy.

A more geometrical functional that has been studied in the literature is

$$F_2(u):=\frac{1}{4}\int_\Omega \big|\mathrm{trace}(\nabla Du)\big|^2\,dx,$$

where ∇ denotes the Levi-Civita connection on N. We refer to [8] for a survey on the differential geometric aspects of this functional. From the analytic point of view, it has some unfavorable properties, for example it is not coercive on $W^{2,2}(\Omega,N)$. This is the reason why the existence of minimizers is to our knowledge still an open problem, apart from harmonic maps in $W^{2,2}(\Omega,N)$, which trivially minimize F_2. Moreover, as pointed out by Moser [10, Section 7], critical points of F_2 cannot be expected to satisfy a monotonicity property unless the target manifold has nonpositive sectional curvature. Consequently, the question of regularity for critical points of F_2 remains open for general targets.

In order to overcome these difficulties, Moser introduced the functional

$$\mathcal{E}_2(u)=\frac{1}{4}\int_\Omega |\nabla Du|^2\,dx.$$

Here, $|\cdot|$ denotes the Hilbert–Schmidt norm on the space of bilinear functions, i.e. $|\nabla Du|^2=\sum_{\alpha,\beta=1}^m|\nabla_{e_\alpha}\partial_\beta u|^2$ with the standard basis $\{e_1,\dots,e_m\}\subset\mathbb{R}^m$. This functional is intrinsically defined, and we will show below that its critical points satisfy similar regularity properties as the critical points of the extrinsic bi-energy E_2.

The existence of minimizers of $\mathcal{E}_2$ is rather intricate, since $\mathcal{E}_2$ is not coercive in $W^{2,2}(\Omega,N)$. Nevertheless, Moser [10] was able to construct minimizers of $\mathcal{E}_2$ in the space

$$H_N^2(\Omega):=\big\{u\in W^{1,2}(\Omega,N)\colon\ \nabla Du\in L^2\big(\Omega,\big(\mathbb{R}^m\big)^*\otimes\big(\mathbb{R}^m\big)^*\otimes\mathbb{R}^K\big)\big\}$$

with respect to Dirichlet boundary conditions

$$u=u_0\quad\text{and}\quad Du=Du_0\quad\text{on }\partial\Omega \tag{1.1}$$

for a given map $u_0\in H_N^2(\Omega)$. This boundary condition has to be understood in the sense of traces, which is possible since $u_0\in H_N^2(\Omega)$ implies $u_0\in W^{2,1}(\Omega,\mathbb{R}^K)$.

In the sequel, we will employ the notion that a map $u\in H_N^2(\Omega)$ is an *H_N^2-minimizer of $\mathcal{E}_2$* iff $\mathcal{E}_2(u)\leqslant\mathcal{E}_2(v)$ for all $v\in H_N^2(\Omega)$ with the same boundary data in the sense of (1.1). For H_N^2-minimizers u of $\mathcal{E}_2$, Moser was able to prove $u\in W^{2,2}_{\mathrm{loc}}(\Omega\setminus\Sigma,N)$ for a singular set $\Sigma\subset\Omega$ with $\mathcal{H}^{m-2^*}(\Sigma)=0$, where $2^*=\frac{2m}{m-2}$ [10, Corollary 4.1]. Here, we improve this result as follows.

Theorem 1.1. *Every H_N^2-minimizer $u\in H_N^2(\Omega)$ of $\mathcal{E}_2$ satisfies $u\in W^{2,2}_{\mathrm{loc}}\cap W^{1,4}_{\mathrm{loc}}(\Omega,N)$.*

This theorem holds more generally for maps satisfying a certain stationarity condition, see Theorem 3.4. We point out that for $m = 4$, the theorem follows from the result by Moser.

In the proof of the above theorem, the key tool with which we compensate the lack of coercivity is the following Morrey–Sobolev embedding theorem involving only covariant derivatives. We use the notation $M^p_\mu(\Omega)$ for the Morrey spaces as defined in (2.1).

Theorem 1.2. *Every map $u \in H^2_N(\Omega)$ with $|\nabla Du| \in M^2_4(\Omega)$ and $|Du| \in M^2_4(\Omega)$ satisfies*

$$\|Du\|_{M^4_4(\Omega)} \leqslant C\big(\|\nabla Du\|_{M^2_4(\Omega)} + \|Du\|_{M^2_4(\Omega)}\big).$$

For the proof we refer to Theorem 2.1.

Because of Theorem 1.1, it is legitimate to consider H^2_N-minimizers u for $\mathcal{E}_2$ with $u \in W^{2,2} \cap W^{1,4}(\Omega, N)$. Such minimizers are solutions of the Euler equation for $\mathcal{E}_2$, see Section 4. Since the leading part of the Euler equation is the Bilaplacian, we will call smooth solutions of the Euler equation (*intrinsically*) *biharmonic maps*. We say that $u \in W^{2,2}(\Omega, N)$ is a *minimizing* (*intrinsically*) *biharmonic map* if $\mathcal{E}_2(u) \leqslant \mathcal{E}_2(v)$ holds for all $v \in W^{2,2}(\Omega, N)$ with $u - v \in W^{2,2}_0(\Omega, \mathbb{R}^K)$.

The fact $u \in W^{2,2}(\Omega, N)$, combined with an ε-regularity result by Moser implies immediately $\mathcal{H}^{m-4}(\mathrm{sing}(u)) = 0$ for all minimizing intrinsically biharmonic maps $u \in W^{2,2}(\Omega, N)$, see Corollary 5.2. By a dimension reduction argument similar to that in [13], we are able to improve this result by the following theorem.

Theorem 1.3. *Let N be a smooth, compact Riemannian manifold without boundary and $\Omega \subset \mathbb{R}^m$ an open domain. Then, every minimizing intrinsically biharmonic map $u \in W^{2,2}(\Omega, N)$ satisfies*

$$\mathcal{H}\text{-}\dim\big(\mathrm{sing}(u)\big) \leqslant m - 5. \tag{1.2}$$

Here, $\mathcal{H}$-dim denotes the Hausdorff dimension and $\mathrm{sing}(u)$ is the complement of the set $\{x \in \Omega\colon u \text{ is } C^\infty \text{ in a neighborhood of } x\}$.

Moreover, if for all dimensions $4 \leqslant k \leqslant k_0$ up to some $k_0 \geqslant 4$, there are no nonconstant minimizing intrinsically biharmonic maps $v \in C^\infty(\mathbb{R}^{k+1} \setminus \{0\}, N)$ that are homogeneous of degree zero, then $\mathcal{H}\text{-}\dim(\mathrm{sing}(u)) \leqslant m - k_0 - 2$. If this assumption holds with $k_0 = m - 2$, then $\mathrm{sing}(u) \cap A$ is finite for every compact subset $A \subset \Omega$.

For the proof of the last theorem, we employ Federer's dimension reduction principle. The crucial step that enables us to use Federer's argument is the following compactness theorem, which we prove by analyzing the defect measures with tools from geometric measure theory, see Section 6.

Theorem 1.4. *Let $M(\Omega) \subset W^{2,2}_{\mathrm{loc}}(\Omega, N)$ be the closure of the set of minimizing intrinsically biharmonic maps with respect to the $W^{2,2}_{\mathrm{loc}}$-topology. Assume that $\{u_i\}_{i\in\mathbb{N}} \subset M(\Omega)$ is a sequence with*

$$\sup_{i\in\mathbb{N}} \int_\Omega \big(|\nabla Du_i|^2 + |Du_i|^2\big)\,dx < \infty.$$

Then there is a subsequence $\{i_j\} \subset \mathbb{N}$ and a limit map $u \in W^{2,2}_{\mathrm{loc}} \cap W^{1,4}_{\mathrm{loc}}(\Omega, \mathbb{R}^K)$ with $u_{i_j} \to u$ strongly in $W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^K)$ and in $W^{1,4}_{\mathrm{loc}}(\Omega, \mathbb{R}^K)$, as $j \to \infty$.

The estimate (1.2) for the dimension of the singular set is optimal since the map $B^5 \ni x \mapsto \frac{x}{|x|} \in S^4$ is minimizing biharmonic. More generally, we prove the following proposition in Section 8.

Proposition 1.5. *The map $u_0(x) := \frac{x}{|x|}$, where $x \in B^m$, minimizes $\mathcal{E}_2$ in the Dirichlet class $W^{2,2}(B^m, S^{m-1}) \cap (u_0 + W^{2,2}_0(B^m, \mathbb{R}^m))$ for all $m \geqslant 5$.*

2. Sobolev–Morrey embedding for intrinsically defined Sobolev spaces

As noted by Moser [10], the function space naturally associated with $\mathcal{E}_2$ is

$$H^2_N(\Omega) := \{u \in W^{1,2}(\Omega, N):\ \nabla Du \in L^2(\Omega, (\mathbb{R}^m)^* \otimes (\mathbb{R}^m)^* \otimes \mathbb{R}^K)\},$$

where the covariant second derivative of a map $u : \Omega \to N \subset \mathbb{R}^K$ is given by

$$\nabla Du := D^2 u - (\mathbb{I} \circ u)(Du \otimes Du)$$

in the distributional sense. Here, $\mathbb{I}$ denotes the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^K$, defined by $\mathbb{I}(y)(v, w) := (\partial_v W(y))^\perp$ for $y \in N$ and $v, w \in \mathrm{T}_y N$, where $W \in C^1(N, \mathbb{R}^K)$ is any vector field that is tangential along N and satisfies $W(y) = w$, and $^\perp$ indicates the orthogonal projection onto $\mathrm{T}^\perp_y N$, see e.g. [6]. We note that $u \in H^2_N(\Omega)$ implies in particular $u \in W^{2,1}(\Omega, \mathbb{R}^K)$ by the above definition of the covariant derivative and the compactness of N.

The analysis of minimizers of $\mathcal{E}_2$ is delicate because the functional is not coercive with respect to the $W^{2,2}$-norm. In order to deal with this problem, we need the following Morrey–Sobolev type embedding theorem adapted to the function space $H^2_N(\Omega)$. For this we employ the notation $M^p_\mu(\Omega)$ for the Morrey spaces of functions $f \in L^p(\Omega)$ with

$$\|f\|_{M^p_\mu(\Omega)} := \sup_{y\in\Omega,\ 0<\rho<1} \left(\rho^{\mu-m} \int\limits_{\Omega\cap B_\rho(y)} |f|^p\, dx \right)^{1/p} < \infty, \tag{2.1}$$

where $1 \leqslant p < \infty$ and $0 < \mu \leqslant m$.

Theorem 2.1. *Assume that $u \in H^2_N(\Omega)$ with $|Du| \in M^2_4(\Omega)$ and $|\nabla Du| \in M^2_4(\Omega)$. Then, $|Du| \in M^4_4(\Omega)$ and*

$$\|Du\|_{M^4_4(\Omega)} \leqslant C\big(\|\nabla Du\|_{M^2_4(\Omega)} + \|Du\|_{M^2_4(\Omega)}\big) \tag{2.2}$$

for an appropriate constant C, depending only on $\Omega \subset \mathbb{R}^m$.

Proof. For the function $g_\varepsilon := \sqrt{\varepsilon^2 + |Du|^2}$, where $\varepsilon > 0$, we calculate

$$Dg_\varepsilon = \frac{\nabla Du \cdot Du}{\sqrt{\varepsilon^2 + |Du|^2}}$$

and conclude

$$\|Dg_\varepsilon\|_{M^2_4(\Omega)} \leqslant \|\nabla Du\|_{M^2_4(\Omega)}. \tag{2.3}$$

The above calculations can be justified by observing that since $u \in H^2_N(\Omega) \subset W^{2,1}(\Omega, \mathbb{R}^K)$, the components of Du are absolutely continuous along the coordinate lines $t \mapsto x + te_i$ for almost every $x \in \Omega$ and $1 \leqslant i \leqslant m$, compare also [10, Lemma 2.1]. Because of (2.3), we can apply the Sobolev–Morrey embedding theorem to the function g_ε, see e.g. [1, Theorem 3.2] or [16, Corollary 3.4]. We deduce $g_\varepsilon \in M^4_4(\Omega)$ with

$$\|Du\|_{M^4_4(\Omega)} \leqslant \|g_\varepsilon\|_{M^4_4(\Omega)} \leqslant C\|Dg_\varepsilon\|_{M^2_4(\Omega)} + C\|g_\varepsilon\|_{M^2_4(\Omega)} \leqslant C\|\nabla Du\|_{M^2_4(\Omega)} + C\|g_\varepsilon\|_{M^2_4(\Omega)}.$$

Letting $\varepsilon \searrow 0$, we obtain the claim (2.2). □

3. $W^{2,2}$-regularity for H^2_N-minimizers

In [10], Moser proved the existence of minimizers $u \in H^2_N(\Omega)$ of $\mathcal{E}_2$ with respect to boundary conditions of the type (1.1). In this section we study the $W^{2,2}$-regularity of these minimizers and more generally of maps $u \in H^2_N(\Omega)$ that satisfy a stationarity property.

For this we adapt the following definition from [10].

Definition 3.1. A map $u \in H^2_N(\Omega)$ is called stationary for $\mathcal{E}_2$ if and only if

$$\int_\Omega \left(\nabla_{e_\alpha} Du \cdot \nabla_{e_\beta} Du \partial_\beta \psi^\alpha + \frac{1}{2} \nabla_{e_\alpha} \partial_\beta u \cdot \partial_\gamma u \partial_\alpha \partial_\beta \psi^\gamma - \frac{1}{4} |\nabla Du|^2 \operatorname{div} \psi \right) dx = 0 \tag{3.1}$$

holds for all $\psi = (\psi^1, \dots, \psi^m) \in C_0^\infty(\Omega, \mathbb{R}^m)$. Here, $e_1, \dots, e_m$ denotes the standard basis of $\mathbb{R}^m$.

As was shown in [10], the differential equation (3.1) is satisfied by all maps $u \in H^2_N(\Omega)$ with the property $\frac{\partial}{\partial t} \mathcal{E}_2(u_t)|_{t=0} = 0$ for all inner variations $u_t(x) := u(x + t\psi(x))$ with $\psi \in C_0^\infty(\Omega, \mathbb{R}^m)$. In particular, H^2_N-minimizers of $\mathcal{E}_2$ are stationary for $\mathcal{E}_2$. The differential equation (3.1) implies the following monotonicity property.

Theorem 3.2. *(See [10, Theorem 3.1].) Assume that $u \in H^2_N(\Omega)$ is stationary for $\mathcal{E}_2$ and $B_R(a) \subset \Omega$. With the vector field $X(x) := x - a$, the expression*

$$\begin{aligned} \Phi_u(a, r) := {} & \frac{1}{4} r^{4-m} \int_{B_r(a)} |\nabla Du|^2 \, dx + \frac{3}{4} r^{3-m} \int_{\partial B_r(a)} |Du|^2 \, d\mathcal{H}^{m-1} \\ & + \frac{1}{4} r^{1-m} \int_{\partial B_r(a)} \big[(m-2) |\partial_X u|^2 - 2 \nabla_X \partial_X u \cdot \partial_X u \big] \, d\mathcal{H}^{m-1} \end{aligned}$$

is well defined and monotonously nondecreasing for all $r \in (0, R]$ outside a zero set. More precisely, for almost all $0 < r_1 \leqslant r_2 \leqslant R$, there holds

$$\Phi_u(a, r_2) - \Phi_u(a, r_1) = \int_{B_{r_2}(a) \setminus B_{r_1}(a)} \left(\frac{|\nabla \partial_X u|^2}{|x-a|^{m-2}} + (m-2) \frac{|\partial_X u|^2}{|x-a|^m} \right) dx.$$

The most important consequence of this property is the following

Corollary 3.3. *(See [10, Corollary 3.2].) There is a constant C, depending only on m, such that every map $u \in H^2_N(\Omega)$ that is stationary for $\mathcal{E}_2$ satisfies*

$$\rho^{4-m} \int_{B_\rho(a)} \big(|\nabla Du|^2 + \rho^{-2} |Du|^2 \big) \, dx \leqslant C R^{4-m} \int_{B_R(a)} \big(|\nabla Du|^2 + R^{-2} |Du|^2 \big) \, dx$$

whenever $B_R(a) \subset \Omega$ and $0 < \rho \leqslant R$.

The preceding results by Moser enable us to prove the following regularity result in the scale of Morrey spaces.

Theorem 3.4. *Every map $u \in H^2_N(B_2)$ that is stationary for $\mathcal{E}_2$ satisfies $u \in W^{2,2} \cap W^{1,4}(B_1, N)$. Furthermore, $|D^2 u| \in M^2_4(B_1, N)$ and $|Du| \in M^4_4(B_1, N)$ with the estimates*

$$\|Du\|_{M^4_4(B_1)} \leqslant C \big(\|\nabla Du\|_{L^2(B_2)} + \|Du\|_{L^2(B_2)} \big) \quad \text{and}$$
$$\|D^2 u\|_{M^2_4(B_1)} \leqslant C \big(\|\nabla Du\|_{L^2(B_2)} + \|Du\|_{L^2(B_2)} + \|\nabla Du\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)} \big).$$

Here, the constant C depends only on m and N.

Proof. According to Corollary 3.3, the stationary map u satisfies

$$\|\nabla Du\|^2_{M^2_4(B_1)} + \|Du\|^2_{M^2_4(B_1)} \leqslant C \int_{B_2} \big(|\nabla Du|^2 + |Du|^2 \big) \, dx. \tag{3.2}$$

With the help of Theorem 2.1, we deduce furthermore $|Du| \in M^4_4(B_1)$ with

$$\|Du\|_{M^4_4(B_1)} \leqslant C \big(\|\nabla Du\|_{L^2(B_2)} + \|Du\|_{L^2(B_2)} \big). \tag{3.3}$$

Since $D^2u = \nabla Du + (\mathbb{I}\circ u)(Du\otimes Du)$, we can estimate

$$\|D^2u\|_{M_4^2(B_1)} \leqslant \|\nabla Du\|_{M_4^2(B_1)} + C\|Du\|^2_{M_4^4(B_1)} \leqslant C\big(\|\nabla Du\|_{L^2(B_2)} + \|Du\|_{L^2(B_2)} + \|\nabla Du\|^2_{L^2(B_2)} + \|Du\|^2_{L^2(B_2)}\big)$$

by (3.2) and (3.3). This completes the proof. □

4. The Euler equation

Having established the $W^{2,2}$-regularity, we are now able to derive the Euler equation for biharmonic maps. We consider variations $u_t := \pi_N\circ(u+tV)$ for $V\in W_0^{2,2}\cap L^\infty(\Omega,\mathbb{R}^K)$ with $V(x)\in \mathrm{T}_{u(x)}N$ for almost every $x\in\Omega$. Here, π_N denotes the nearest-point retraction onto N, defined on a suitable neighborhood of N. We calculate

$$\frac{\partial}{\partial t}\bigg|_{t=0}\mathcal{E}_2(u_t) = \frac12\int_\Omega \nabla_{e_\alpha}\partial_\beta u\cdot\nabla_t\nabla_{e_\alpha}\partial_\beta u_t|_{t=0}\,dx = \frac12\int_\Omega\big[\nabla_{e_\alpha}\partial_\beta u\cdot\nabla_{e_\alpha}\nabla_{e_\beta}V + \nabla_{e_\alpha}\partial_\beta u\cdot(R\circ u)(V,\partial_\alpha u)\partial_\beta u\big]\,dx$$

by the definition of the Riemannian curvature tensor R associated with N and since $\frac{\partial}{\partial t}u_t|_{t=0} = V$. The last equation motivates the following

Definition 4.1. A map $u\in W^{2,2}(\Omega,N)$ is called (intrinsically) weakly biharmonic if and only if it satisfies the Euler equation

$$\int_\Omega\big(\nabla_{e_\alpha}\partial_\beta u\cdot\nabla_{e_\alpha}\nabla_{e_\beta}V + \nabla_{e_\alpha}\partial_\beta u\cdot(R\circ u)(V,\partial_\alpha u)\partial_\beta u\big)\,dx = 0 \tag{4.1}$$

for all vector fields $V\in W_0^{2,2}\cap L^\infty(\Omega,\mathbb{R}^K)$ that are tangential along u in the sense $V(x)\in\mathrm{T}_{u(x)}N$ for almost every $x\in\Omega$.

We say that a map $u\in W^{2,2}(\Omega,N)$ is (intrinsically) stationary biharmonic if it is intrinsically weakly biharmonic and additionally stationary for $\mathcal{E}_2$.

Clearly, minimizing intrinsically biharmonic maps are intrinsically stationary biharmonic. In the sequel, we will often omit the adverb 'intrinsically' for the sake of brevity since in this article, we are always considering the intrinsic case.

Next, we rewrite the Euler equation in a form that is more useful for some purposes.

Lemma 4.2. *There are functions $F[u]:\Omega\to(\mathbb{R}^m)^*\otimes\mathbb{R}^K$ and $G[u]:\Omega\to\mathbb{R}^K$ so that the Euler equation* (4.1) *for $u\in W^{2,2}(\Omega,N)$ is equivalent to the differential equation*

$$\int_\Omega \Delta u\cdot\Delta W\,dx = \int_\Omega\big((\mathbb{I}\circ u)(Du\otimes Du)\cdot D^2W + F[u]\cdot DW + G[u]\cdot W\big)\,dx \tag{4.2}$$

for all vector fields $W\in W_0^{2,2}\cap L^\infty(\Omega,\mathbb{R}^K)$. More precisely, the coefficient functions have the form

$$F[u] = f(u;\nabla Du, Du)\quad\text{and}\quad G[u] = g_1(u;\nabla Du,\nabla Du) + g_2(u;\nabla Du, Du, Du)$$

for functions f, g_1 and g_2 that depend only on m and N, are smooth in the first argument and linear in all the others.

Proof. We assume $N\subset\mathbb{R}^K$ and let $\Pi(y):\mathbb{R}^K\to\mathrm{T}_yN$ be the orthogonal projection for $y\in N$. Suppose that $u\in W^{2,2}(\Omega,N)$ satisfies the Euler equation (4.1). For an arbitrary vector field $W\in W_0^{2,2}\cap L^\infty(\Omega,\mathbb{R}^K)$ we let $V := (\Pi\circ u)W$ and compute

$$\nabla_{e_\beta}V = (\Pi\circ u)\partial_\beta\big((\Pi\circ u)W\big) = (\Pi\circ u)\partial_\beta W + (\Pi\circ u)D\Pi\circ u(W,\partial_\beta u)$$

and consequently,

$$\begin{aligned}\nabla_{e_\alpha}\nabla_{e_\beta}V &= (\Pi\circ u)\partial_\alpha\partial_\beta W + (\Pi\circ u)D\Pi\circ u(\partial_\beta W,\partial_\alpha u) + (\Pi\circ u)D\Pi\circ u(\partial_\alpha W,\partial_\beta u)\\ &\quad + (\Pi\circ u)D\Pi\circ u\big(D\Pi\circ u(W,\partial_\beta u),\partial_\alpha u\big) + (\Pi\circ u)D^2\Pi\circ u(W,\partial_\alpha u,\partial_\beta u)\\ &\quad + (\Pi\circ u)D\Pi\circ u(W,\nabla_{e_\alpha}\partial_\beta u)\\ &= (\Pi\circ u)\partial_\alpha\partial_\beta W + a_{\alpha\beta}[u]\cdot DW + b_{\alpha\beta}[u]\cdot W \end{aligned}\tag{4.3}$$

for suitable functions $a_{\alpha\beta}[u]:\Omega\to(\mathbb{R}^m)^*\otimes\mathbb{R}^K$ and $b_{\alpha\beta}[u]:\Omega\to\mathbb{R}^K$ of the form

$$a_{\alpha\beta}[u] = a'_{\alpha\beta}(u;Du) \quad\text{and}\quad b_{\alpha\beta}[u] = b'_{\alpha\beta}(u;\nabla Du) + b''_{\alpha\beta}(u;Du,Du)$$

with functions a', b' and b'' that are C^∞ in the first and linear in the remaining components. Furthermore, we use the identity

$$\nabla_{e_\alpha}\partial_\beta u = \partial_\alpha\partial_\beta u - (\mathbb{I}\circ u)(\partial_\alpha u,\partial_\beta u).\tag{4.4}$$

Plugging (4.3) and (4.4) into the Euler equation (4.1), we arrive at

$$\begin{aligned}\int_\Omega \partial_\alpha\partial_\beta u\cdot\partial_\alpha\partial_\beta W\,dx &= \int_\Omega (\mathbb{I}\circ u)(\partial_\alpha u,\partial_\beta u)\cdot\partial_\alpha\partial_\beta W\,dx\\ &\quad - \int_\Omega \nabla_{e_\alpha}\partial_\beta u\cdot\big(a_{\alpha\beta}[u]\cdot DW + b_{\alpha\beta}[u]\cdot W + (R\circ u)\big((\Pi\circ u)W,\partial_\alpha u\big)\partial_\beta u\big)\,dx\\ &=: \int_\Omega \big((\mathbb{I}\circ u)(Du\otimes Du)\cdot D^2W + F[u]\cdot DW + G[u]\cdot W\big)\,dx.\end{aligned}$$

From this Eq. (4.2) follows by two integrations by parts on the left-hand side.

On the other hand, suppose that u satisfies (4.2) for all vector fields $W\in W_0^{2,2}\cap L^\infty(\Omega,N)$. For an arbitrary $V\in W_0^{2,2}\cap L^\infty(\Omega,N)$ that is tangential along u, consider Eq. (4.2) with $W=V$. From the derivation of this equation it is clear that (4.2) is equivalent to (4.1) in this case. This proves the reverse implication. □

5. Partial regularity

The following ε-regularity result was established in [10, Corollary 6.1]. It also follows more directly from [10, Theorem 6.1], combined with our Theorem 3.4. For the proof of the higher regularity, we refer to [4].

Theorem 5.1. *There exists a constant $\varepsilon_1>0$ such that every intrinsically stationary biharmonic map $u\in W^{2,2}(B_\rho(x_0),N)$ with*

$$\rho^{4-m}\int_{B_\rho(x_0)}\big(|\nabla Du|^2+\rho^{-2}|Du|^2\big)\,dx<\varepsilon_1$$

satisfies $u\in C^\infty(B_{\rho/2}(x_0),N)$.

Combined with our results from Section 3, the above theorem implies

Corollary 5.2. *Let $u\in W^{2,2}(\Omega,N)$ be intrinsically stationary biharmonic. Then there is a closed subset $\Sigma\subset\Omega$ with $\mathcal{H}^{m-4}(\Sigma)=0$ so that $u\in C^\infty(\Omega\setminus\Sigma,N)$.*

Proof. With the constant ε_1 from Theorem 5.1, we define

$$\Sigma := \left\{y\in\Omega:\ \liminf_{\rho\searrow 0}\rho^{4-m}\int_{B_\rho(y)}\big(|\nabla Du|^2+\rho^{-2}|Du|^2\big)\,dx\geqslant\varepsilon_1\right\}.$$

From Theorem 5.1 we infer that Σ is a closed subset and $u \in C^\infty(\Omega \setminus \Sigma, N)$. Since $u \in W^{1,4}(\Omega, N)$ by Theorem 3.4, we may apply the Cauchy–Schwarz inequality with the result

$$\liminf_{\rho \searrow 0} \rho^{4-m} \int_{B_\rho(y)} \left(|\nabla Du|^2 + \rho^{-2}|Du|^2\right) dx \leqslant \liminf_{\rho \searrow 0} \left[\rho^{4-m} \int_{B_\rho(y)} |\nabla Du|^2\, dx + C\left(\rho^{4-m} \int_{B_\rho(y)} |Du|^4\, dx\right)^{1/2}\right],$$

and the right-hand side vanishes for $\mathcal{H}^{m-4}$-almost every $y \in \Omega$ by classical density theorems, see e.g. [20, Lemma 3.2.2]. We conclude $\mathcal{H}^{m-4}(\Sigma) = 0$, as desired. □

We also have the following uniform higher-order estimates, which follow from a well-known scaling technique due to [12], see also [9, Lemma 5.3] for a version adapted to higher-order equations. We include a proof for the convenience of the reader.

Lemma 5.3. *For every $\delta > 0$, there is a constant $\varepsilon(\delta) > 0$ such that every intrinsically biharmonic map $u \in C^\infty(B_1, N)$ with*

$$\sup_{B_\rho(a) \subset B_1} \rho^{2-m} \int_{B_\rho(a)} |Du|^2\, dx < \varepsilon(\delta) \tag{5.1}$$

satisfies

$$\left|D^3 u(x)\right|^{1/3} + \left|D^2 u(x)\right|^{1/2} + \left|Du(x)\right| \leqslant \frac{\delta}{1 - |x|} \quad \textit{for all } x \in B_1.$$

Proof. We abbreviate $[u]_{C^3}(x) := \sum_{k=1}^3 |D^k u(x)|^{1/k}$ for $x \in B_1$. Suppose that the lemma was not true, then we could find a sequence of biharmonic maps $u_i \in C^\infty(B_1, N)$ with

$$\sup_{B_\rho(a) \subset B_1} \rho^{2-m} \int_{B_\rho(a)} |Du_i|^2\, dx \to 0 \quad \text{as } i \to \infty, \tag{5.2}$$

but

$$\sup_{0<r<1} (1-r) \sup_{B_r(0)} [u_i]_{C^3} > \delta \quad \text{for all } i \in \mathbb{N}. \tag{5.3}$$

For every $i \in \mathbb{N}$, we choose $r_i \in [0, 1)$ with

$$(1 - r_i) \sup_{B_{r_i}(0)} [u_i]_{C^3} = \sup_{0<r<1} (1-r) \sup_{B_r(0)} [u_i]_{C^3}$$

and $x_i \in \overline{B_{r_i}(0)}$ with

$$[u_i]_{C^3}(x_i) = \sup_{B_{r_i}(0)} [u_i]_{C^3},$$

which we interpret as $x_i = 0$ in the case $r_i = 0$. Moreover, we define

$$\lambda_i := \frac{\delta}{2\,[u_i]_{C^3}(x_i)}.$$

As a consequence of (5.3) and the choice of x_i, we have $\lambda_i < (1 - r_i)/2$. Therefore, we may define rescaled maps

$$v_i(x) := u(x_i + \lambda_i x) \quad \text{for } x \in B_1,$$

which are again stationary biharmonic by the scaling invariance of $\mathcal{E}_2$, and satisfy

$$[v_i]_{C^3}(0) = \lambda_i [u_i]_{C^3}(x_i) = \frac{\delta}{2} \tag{5.4}$$

as well as

$$\sup_{B_1(0)} [v_i]_{C^3} \leqslant \lambda_i \sup_{B_{(1+r_i)/2}(0)} [u_i]_{C^3} \leqslant \lambda_i \left(1 - \frac{1+r_i}{2}\right)^{-1} (1-r_i)\,[u_i]_{C^3}(x_i) = \delta$$

by the definition of λ_i. The biharmonic maps v_i satisfy the Euler equation for $\mathcal{E}_2$ in the classical sense, which, according to Lemma 4.2, has the form

$$\Delta^2 v_i = f[v_i] = \tilde{f}\big(v_i, Dv_i, D^2 v_i, D^3 v_i\big), \quad \text{where } \big|f[v_i]\big| \leqslant C(m,N)[v_i]_{C^3}^4.$$

The last estimate follows by Young's inequality. We conclude $|\Delta^2 v_i| \leqslant C\delta^4$ on B_1 for all $i \in \mathbb{N}$, which, combined with $[v_i]_{C^3} \leqslant \delta$, yields

$$\sup_{i\in\mathbb{N}} \|v_i\|_{C^{3,\alpha}(B_{1/2})} < \infty \quad \text{for every } \alpha \in (0,1)$$

by classical Schauder estimates for the Laplace operator. Hence, the Theorem of Arzéla–Ascoli implies that after passing to a subsequence, we can find a map $v \in C^\infty(B_{1/2}, N)$ with $v_i \to v$ in $C^3(B_{1/2}, N)$ as $i \to \infty$, which implies in particular

$$[v]_{C^3}(0) = \lim_{i\to\infty} [v_i]_{C^3}(0) = \frac{\delta}{2} > 0 \tag{5.5}$$

by (5.4). On the other hand, the property (5.2) implies

$$\int_{B_{1/2}} |Dv|^2\,dx = \lim_{i\to\infty} \int_{B_{1/2}} |Dv_i|^2\,dx = 0,$$

so that v must be constant, in contradiction to (5.5). This completes the proof. □

Corollary 5.4. *There is a constant $\varepsilon_0 > 0$ such that every intrinsically stationary biharmonic map $u \in W^{2,2}(B_r(y), N)$ with*

$$r^{4-m} \int_{B_r(y)} \big(|\nabla Du|^2 + r^{-2}|Du|^2\big)\,dx < \varepsilon_0 \tag{5.6}$$

satisfies $u \in C^\infty(B_{r/2}(y), N)$ with

$$r|Du| + r^2\big|D^2u\big| + r^3\big|D^3u\big| \leqslant 1 \quad \text{on } B_{r/4}(y). \tag{5.7}$$

Proof. If we choose the constant $\varepsilon_0 > 0$ not larger than the constant ε_1 from Theorem 5.1, the mentioned theorem yields $u \in C^\infty(B_{r/2}(y), N)$. As a consequence of the monotonicity property of u, see Corollary 3.3, the assumption (5.6) implies

$$\sup_{B_\rho(a)\subset B_{r/2}(y)} \rho^{2-m} \int_{B_\rho(a)} |Du|^2\,dx \leqslant C\varepsilon_0. \tag{5.8}$$

By rescaling $B_{r/2}(y)$ to the unit ball, we infer from Lemma 5.3 that we can achieve (5.7) by choosing $\varepsilon_0 > 0$ small enough. □

6. Compactness for sequences of minimizing maps

6.1. The defect measure

In this section we are concerned with sequences of stationary biharmonic maps $u_i \in W^{2,2}(B_4, N)$ with $\sup_{i\in\mathbb{N}}(\|\nabla Du_i\|_{L^2(B_4)} + \|Du_i\|_{L^2(B_4)}) < \infty$. From Theorem 3.4 we infer $u_i \in W^{2,2} \cap W^{1,4}(B_2, N)$ with $\|D^2u_i\|^2_{M_4^2(B_2)} + \|Du_i\|^4_{M_4^4(B_2)} \leqslant \Lambda$ for an appropriate constant $\Lambda > 0$ and all $i \in \mathbb{N}$. We may thus assume, after passing to a subsequence if necessary, that there is a map $u \in W^{2,2} \cap W^{1,4}(B_2, \mathbb{R}^K)$ with $u_i \rightharpoonup u$ weakly in $W^{2,2}(B_2, \mathbb{R}^K)$

and in $W^{1,4}(B_2,\mathbb{R}^K)$ as well as $u_i \to u$ strongly in $W^{1,2}(B_2,\mathbb{R}^K)$ and almost everywhere, as $i\to\infty$. Furthermore, we can assume

$$\mathcal{L}^m \llcorner |\nabla Du_i|^2 \rightharpoonup \mathcal{L}^m \llcorner |\nabla Du|^2 + \nu \quad \text{in the sense of measures}$$

for some Radon measure ν on $\overline{B}_2$, see e.g. [7, Theorem 1.23]. From the lower semicontinuity of $\mathcal{E}_2$ with respect to weak convergence in $W^{2,2}$ we conclude $\nu \geqslant 0$. We call this measure the *defect measure* of the sequence u_i, a notion that will be justified in Lemma 6.2 below. We consider the pair (u,ν) as the limit configuration of the sequence u_i. This motivates the following

Definition 6.1. For sequences of maps $u_i \in W^{2,2}(B_2,N)$ and nonnegative Radon measures ν_i on $\overline{B}_2$, where $i\in\mathbb{N}_0$, we write $(u_i,\nu_i) \rightrightarrows (u_0,\nu_0)$ as $i\to\infty$ if and only if convergence holds in the following sense.

$$\begin{aligned}
&u_i \rightharpoonup u_0 \quad \text{weakly in } W^{2,2}\big(B_2,\mathbb{R}^K\big) \text{ and in } W^{1,4}\big(B_2,\mathbb{R}^K\big),\\
&u_i \to u_0 \quad \text{strongly in } W^{1,2}(B_2,N) \text{ and almost everywhere},\\
&\mathcal{L}^m \llcorner |\nabla Du_i|^2 + \nu_i \rightharpoonup \mathcal{L}^m \llcorner |\nabla Du_0|^2 + \nu_0 \quad \text{in the sense of measures.}
\end{aligned}$$

For the set of all limit configurations of stationary biharmonic maps, we write

$$\mathcal{B}_\Lambda := \left\{ (u_0,\nu_0) \;\middle|\; \begin{array}{l} (u_i,0) \rightrightarrows (u_0,\nu_0), \text{ where } u_i \in W^{2,2}\cap W^{1,4}(B_2,N) \text{ are} \\ \text{stationary biharmonic with } \|D^2u_i\|^2_{M^2_4} + \|Du_i\|^4_{M^4_4} \leqslant \Lambda \end{array} \right\}$$

for a given constant $\Lambda > 0$. Here, '0' denotes the zero measure. Similarly, we write $\mathcal{M}_\Lambda$ for the set that is defined as above, but with minimizing biharmonic maps instead of stationary biharmonic maps.

For a given pair $\mu = (u,\nu) \in \mathcal{B}_\Lambda$, we define the *energy concentration set* Σ_μ as the set of points $a \in \overline{B}_1$ with the property

$$\liminf_{\rho\searrow 0}\bigg(\rho^{4-m}\int_{B_\rho(a)} \big(|\nabla Du|^2 + \rho^{-2}|Du|^2\big)\,dx + \rho^{4-m}\nu\big(B_\rho(a)\big)\bigg) \geqslant \varepsilon_0,$$

where the constant $\varepsilon_0 > 0$ is chosen according to Corollary 5.4.

The following lemma clarifies the meaning of the defect measure and the energy concentration set.

Lemma 6.2. *Assume that $\{u_i\}_{i\in\mathbb{N}} \subset W^{2,2}(B_2,N)$ is a sequence of stationary biharmonic maps with $\sup_i(\|D^2u_i\|^2_{M^2_4(B_2)} + \|Du_i\|^4_{M^4_4(B_2)}) \leqslant \Lambda$ and $(u_i,0) \rightrightarrows \mu$ as $i\to\infty$, for some $\mu = (u,\nu) \in \mathcal{B}_\Lambda$. Then there holds*

(i) *$u_i \to u$ in $C^2_{\mathrm{loc}}(\overline{B}_1 \setminus \Sigma_\mu, N)$ as $i\to\infty$.*
(ii) *If the defect measure ν vanishes, then we have strong convergence $u_i \to u$ in $W^{2,2}(B_1,\mathbb{R}^K)$ and in $W^{1,4}(B_1,\mathbb{R}^K)$, as $i\to\infty$.*

Proof. In order to prove (i), we choose an arbitrary point $a \in \overline{B}_1 \setminus \Sigma_\mu$. By the definition of Σ_μ, we may choose a $\rho \in (0,1)$ with

$$\rho^{4-m}\int_{B_\rho(a)} \big(|\nabla Du|^2 + \rho^{-2}|Du|^2\big)\,dx + \rho^{4-m}\nu\big(B_\rho(a)\big) < \varepsilon_0.$$

We may also assume that $\nu(\partial B_\rho(a)) = 0$, because $\nu(\partial B_\rho(a)) > 0$ can hold at most for countably many values of $\rho \in (0,1)$. Using the convergence $(u_i,0) \rightrightarrows (u,\nu)$, we conclude

$$\lim_{i\to\infty}\rho^{4-m}\int_{B_\rho(a)} \big(|\nabla Du_i|^2 + \rho^{-2}|Du_i|^2\big)\,dx < \varepsilon_0.$$

Corollary 5.4 thus yields $\sup_i \|u_i\|_{C^3(B_{\rho/4}(a))} \leqslant C(N) + \rho^{-3}$, from which we infer by Arzéla–Ascoli's theorem $u_i \to u$ in $C^2(B_{\rho/4}(a), N)$ as $i \to \infty$. Here it is not necessary to extract a subsequence since the assumptions include convergence $u_i \to u$ almost everywhere. Since $a \in \overline{B}_1 \setminus \Sigma_\mu$ was arbitrary, we conclude (i).

For the proof of the second claim, we note that in the case of $\nu = 0$, the definition of Σ_μ becomes

$$\Sigma_\mu = \left\{ y \in \Omega\colon \liminf_{\rho \searrow 0} \rho^{4-m} \int_{B_\rho(y)} \left(|\nabla Du|^2 + \rho^{-2}|Du|^2\right) dx \geqslant \varepsilon_0 \right\}.$$

Since $u \in W^{1,4}(B_2, N)$, the Cauchy–Schwarz inequality and [20, Lemma 3.2.2] imply $\mathcal{H}^{m-4}(\Sigma_\mu) = 0$. Thus, for any given $\varepsilon > 0$, we can choose a cover $\bigcup_{k\in\mathbb{N}} B_{\rho_k}(a_k) \supset \Sigma_\mu$ of open balls with radii $\rho_k \in (0,1)$ such that $\sum_{k\in\mathbb{N}} \rho_k^{m-4} < \varepsilon$. With the abbreviation $A := \overline{B}_1 \setminus \bigcup_k B_{\rho_k}(a_k)$, we conclude for all $i, j \in \mathbb{N}$

$$\int_{B_1\setminus A} \left|D^2u_i - D^2u_j\right|^2 dx \leqslant 2\sum_{k\in\mathbb{N}} \int_{B_{\rho_k}(a_k)} \left(\left|D^2u_i\right|^2 + \left|D^2u_j\right|^2\right) dx \leqslant 4\Lambda \sum_{k\in\mathbb{N}} \rho_k^{m-4} \leqslant 4\Lambda\varepsilon.$$

On the compact set $A \subset \overline{B}_1 \setminus \Sigma_\mu$, the conclusion (i) implies $u_i \to u$ strongly in $W^{2,2}(A, \mathbb{R}^K)$. Hence,

$$\lim_{i,j\to\infty} \int_{B_1} \left|D^2u_i - D^2u_j\right|^2 dx \leqslant 4\Lambda\varepsilon + \lim_{i,j\to\infty} \int_A \left|D^2u_i - D^2u_j\right|^2 dx = 4\Lambda\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\{u_i\}$ is a Cauchy sequence in $W^{2,2}(B_1, \mathbb{R}^K)$. In the same way, one checks that u_i is strongly convergent in $W^{1,4}(B_1, \mathbb{R}^K)$. □

Next we will analyze the relation between the defect measure and the energy concentration set.

Lemma 6.3. *There are positive constants c and C, depending only on m, such that every pair $\mu = (u, \nu) \in \mathcal{B}_\Lambda$ satisfies*

$$c\varepsilon_0 \mathcal{H}^{m-4} \llcorner \Sigma_\mu \leqslant \nu \llcorner \overline{B}_1 \leqslant C\Lambda \mathcal{H}^{m-4} \llcorner \Sigma_\mu. \tag{6.1}$$

Furthermore, Σ_μ is a closed set and $\Sigma_\mu = \operatorname{sing}(u) \cup \operatorname{spt}(\nu)$.

Proof. The inclusion $\operatorname{sing}(u) \cup \operatorname{spt}(\nu) \subset \Sigma_\mu$ holds by Lemma 6.2(i). For the converse inclusion, we assume that there is some point $a \in \Sigma_\mu \setminus \operatorname{sing}(u)$. By the choice of a, the functions $|\nabla Du|$ and $|Du|$ are bounded on a neighborhood of a. Therefore, the definition of Σ_μ implies

$$\liminf_{\rho\searrow 0} \rho^{4-m} \nu\big(B_\rho(a)\big) \geqslant \varepsilon_0,$$

from which we infer $a \in \operatorname{spt}(\nu)$. We have thus proven $\Sigma_\mu = \operatorname{sing}(u) \cup \operatorname{spt}(\nu)$, which implies in particular that Σ_μ is a closed set.

Now we turn our attention to the proof of (6.1). For a Borel set $A \subset \Sigma_\mu$, we choose an arbitrary cover $\bigcup_{j\in\mathbb{N}} B_{\rho_j}(a_j) \supset A$ of balls with radii $\rho_j \in (0,1)$ and centers $a_j \in \overline{B}_1$. Since $(u, \nu) \in \mathcal{B}_\Lambda$, there holds

$$\nu(A) \leqslant \sum_{j\in\mathbb{N}} \nu\big(B_{\rho_j}(a_j)\big) \leqslant \Lambda \sum_{j\in\mathbb{N}} \rho_j^{m-4}.$$

Since the cover of A was arbitrary, we conclude $\nu(A) \leqslant C\Lambda\mathcal{H}^{m-4}(A)$. More precisely, $C = \alpha(m-4)^{-1}$, where $\alpha(m-4)$ denotes the volume of the $(m-4)$-dimensional unit ball. For the proof of the first estimate in (6.1), we choose a set $B \subset \Sigma_\mu$ with $\mathcal{H}^{m-4}(B) = 0$ and

$$\lim_{\rho\searrow 0} \rho^{4-m} \int_{B_\rho(a)} \left(|\nabla Du|^2 + |Du|^4\right) dx = 0 \quad \text{for all } a \in \Sigma_\mu \setminus B.$$

A set with this property exists by [20, Lemma 3.2.2]. From the choice of B and the definition of Σ_μ, we know

$$\liminf_{\rho\searrow 0} \rho^{4-m} \nu\big(B_\rho(a)\big) \geqslant \varepsilon_0 \quad \text{for all } a \in \Sigma_\mu \setminus B. \tag{6.2}$$

Let $\varepsilon > 0$ be given. By the definition of the Hausdorff measure, we may choose $\delta > 0$ small enough to ensure

$$\mathcal{H}^{m-4}(A \setminus B) \leqslant \varepsilon + \inf\left\{\alpha(m-4)\sum_{j\in\mathbb{N}} r_j^{m-4} \;\middle|\; A \setminus B \subset \bigcup_{j\in\mathbb{N}} B_{r_j}(a_j),\ 0 < r_j \leqslant \delta\right\}.$$

We recall that ν is a Radon measure. We may thus choose an open set $O_\varepsilon \supset A$ with $\nu(O_\varepsilon) \leqslant \nu(A) + \varepsilon$. We consider the family of balls $B_{\rho_j}(a_j) \subset O_\varepsilon$ with $0 < \rho_j \leqslant \delta/5$, centers $a_j \in A \setminus B$ and the property

$$\rho_j^{4-m}\nu\big(B_{\rho_j}(a_j)\big) \geqslant \frac{\varepsilon_0}{2}. \tag{6.3}$$

By (6.2), the union of all balls with these properties covers $A \setminus B$. A standard covering argument, see for example [11, Theorem 3.3], yields the existence of a countable disjoint family $\{B_{\rho_j}(a_j)\}_{j\in\mathbb{N}}$ of balls with the property (6.3), $B_{\rho_j}(a_j) \subset O_\varepsilon$ and $A \setminus B \subset \cup_j B_{5\rho_j}(a_j)$. We conclude

$$\mathcal{H}^{m-4}(A) = \mathcal{H}^{m-4}(A \setminus B) \leqslant \varepsilon + \alpha(m-4)\sum_{j\in\mathbb{N}}(5\rho_j)^{m-4}$$

by the choice of δ. On the other hand, since the balls are pairwise disjoint and satisfy (6.3), we can estimate

$$\sum_{j\in\mathbb{N}} \rho_j^{m-4} \leqslant \frac{2}{\varepsilon_0}\sum_{j\in\mathbb{N}} \nu\big(B_{\rho_j}(a_j)\big) \leqslant \frac{2}{\varepsilon_0}\nu(O_\varepsilon) \leqslant \frac{2}{\varepsilon_0}\big(\nu(A) + \varepsilon\big).$$

Putting together the last two estimates, we arrive at

$$c\varepsilon_0\mathcal{H}^{m-4}(A) \leqslant \nu(A) + (1 + c\varepsilon_0)\varepsilon$$

with a constant $c = c(m)$. This completes the proof of the lemma, since $\varepsilon > 0$ can be chosen arbitrarily small. □

In the special case that the weak limit map is constant, the defect measure inherits a monotonicity property from the sequence of biharmonic maps.

Lemma 6.4. *Assume $(c, \nu) \in \mathcal{B}_\Lambda$, where $c \in N$ is constant. Then, the functions*

$$(0, 1] \ni r \mapsto r^{4-m}\nu\big(B_r(a)\big)$$

are monotonously nondecreasing for every $a \in \overline{B}_1$.

Proof. By the definition of $\mathcal{B}_\Lambda$, there is a sequence $u_i \in W^{2,2}(B_2, N)$ of stationary biharmonic maps with $\|D^2u_i\|^2_{M^2_4} + \|Du_i\|^4_{M^4_4} \leqslant \Lambda$ and $(u_i, 0) \rightrightarrows (c, \nu)$ as $i \to \infty$. For a fixed $a \in \overline{B}_1$, radii $r \in (0, 1)$ and $X(x) := x - a$, let

$$f_i(r) := \frac{r^{3-m}}{4}\int_{\partial B_r(a)} \left(3|Du_i|^2 + \frac{m-2}{r^2}|\partial_X u_i|^2 - \frac{2}{r^2}\nabla_X\partial_X u_i \cdot \partial_X u_i\right) d\mathcal{H}^{m-1}.$$

Since $u_i \to c$ strongly in $W^{1,2}(B_2, \mathbb{R}^K)$ and $\sup_i \|D^2u_i\|^2_{L^2(B_1(a))} \leqslant \Lambda$, one checks that $f_i \to 0$ in $L^1_{\mathrm{loc}}[0, 1]$, as $i \to \infty$. Therefore, there is a subsequence $\{i_j\} \subset \mathbb{N}$ with $f_{i_j} \to 0$ almost everywhere, as $j \to \infty$. We conclude that for almost every $r \in (0, 1]$,

$$\Phi_{u_{i_j}}(a, r) = \frac{1}{4}r^{4-m}\int_{B_r(a)} |\nabla Du_{i_j}|^2\,dx + f_{i_j}(r) \underset{j\to\infty}{\longrightarrow} \frac{1}{4}r^{4-m}\nu\big(B_r(a)\big),$$

if we choose in particular $r \in (0, 1]$ in such a way that $\nu(\partial B_r(a)) = 0$. Thus, the monotonicity property of $\Phi_{u_{i_j}}$ from Theorem 3.2 implies the claim. □

6.2. Blow-up analysis of the defect measure

In this section we consider tangent pairs for a given pair $\mu = (u, \nu)$ in the following sense. We define the rescaled pair $\mu_{a,r} = (u_{a,r}, \nu_{a,r})$ at a given point $a \in \overline{B}_1$ with a scaling factor $r \in (0, 1)$ by

$$\begin{aligned} u_{a,r}(x) &:= u(a + rx) && \text{for } x \in B_{1/r}(0), \\ \nu_{a,r}(A) &:= r^{4-m}\nu(a + rA) && \text{for every Borel set } A \subset B_{1/r}(0). \end{aligned} \tag{6.4}$$

If for some sequence $r_i \searrow 0$ we have convergence $\mu_{a,r_i} \rightrightarrows (u_*, \nu_*) =: \mu_*$ for some map $u_* \in W^{2,2} \cap W^{1,4}(B_2, N)$ and a Radon measure ν_* on $\overline{B}_2$, then we call μ_* a *tangent pair* of the pair $\mu \in \mathcal{B}_\Lambda$ at the point $a \in \overline{B}_1$. For the family of all tangent pairs of a given $\mu \in \mathcal{B}_\Lambda$, we write

$$\mathrm{T}(\mu) := \{\mu_* \mid \mu_{a,r_i} \rightrightarrows \mu_* \text{ for a sequence } r_i \searrow 0 \text{ and some } a \in \overline{B}_1\}$$

and similarly, $\mathrm{T}(\mathcal{A}) := \{\mathrm{T}(\mu) \mid \mu \in \mathcal{A}\}$ for subsets $\mathcal{A} \subset \mathcal{B}_\Lambda$. At first we prove that the sets $\mathcal{B}_\Lambda$ and $\mathcal{M}_\Lambda$ are closed under blow-up.

Lemma 6.5. *There holds* $\mathrm{T}(\mathcal{B}_\Lambda) \subset \mathcal{B}_\Lambda$ *and* $\mathrm{T}(\mathcal{M}_\Lambda) \subset \mathcal{M}_\Lambda$.

Proof. For any pair $\mu = (u, \nu) \in \mathcal{B}_\Lambda$, respectively $\mu = (u, \nu) \in \mathcal{M}_\Lambda$, we choose a sequence $u_i \in W^{2,2}(B_2, N)$ of stationary, respectively minimizing biharmonic maps with $(u_i, 0) \rightrightarrows \mu$ as $i \to \infty$ and $\|D^2u_i\|^2_{M^2_4} + \|Du_i\|^4_{M^4_4} \leqslant \Lambda$ for all $i \in \mathbb{N}$. Consider a tangent pair $\mu_* \in \mathrm{T}(\mu)$, which means

$$\mu_{a,r_n} \rightrightarrows \mu_*, \quad \text{as } n \to \infty$$

for some sequence $r_n \searrow 0$ and $a \in \overline{B}_1$. Since $\mathcal{E}_2$ is scaling invariant, the rescaled maps $(u_i)_{a,r_n}$ are stationary biharmonic for any sequence $r_n \searrow 0$ and in the case $\mu \in \mathcal{M}_\Lambda$, they are additionally minimizing biharmonic. Moreover, there holds

$$\big((u_i)_{a,r_n}, 0\big) \rightrightarrows \mu_{a,r_n}, \quad \text{for all } n \in \mathbb{N}, \text{ as } i \to \infty.$$

Thus, a diagonal sequence argument yields a sequence $\{v_i\} \subset \{(u_i)_{a,r_n}\}$ with

$$(v_i, 0) \rightrightarrows \mu_*, \quad \text{as } i \to \infty.$$

By the definition of the Morrey spaces, it is clear that $\|D^2v_i\|^2_{M^2_4} + \|Dv_i\|^4_{M^4_4} \leqslant \Lambda$. Thus, we have $\mu_* \in \mathcal{B}_\Lambda$, respectively $\mu_* \in \mathcal{M}_\Lambda$, as desired. We point out that the choice of the diagonal sequence as above is possible since the weak convergence of measures is metrizable, cf. [7, Lemma 14.13], and the weak topology in $W^{2,2}(B_2, N)$ and $W^{1,4}(B_2, N)$ is metrizable on bounded subsets of these spaces. □

In the following lemma, we perform a double blow-up in order to simplify the situation to the case of a flat defect measure.

Lemma 6.6. *Assume that there is a pair* $\mu = (u, \nu) \in \mathcal{B}_\Lambda$ *with* $\nu \llcorner \overline{B}_1 \neq 0$. *Then there exists a pair* $(c, \bar{\nu}) \in \mathcal{B}_\Lambda$, *where* $c \in N$ *denotes a constant map and*

$$\bar{\nu} \llcorner \overline{B}_1 = C\mathcal{H}^{m-4} \llcorner (V \cap \overline{B}_1) \tag{6.5}$$

for a constant $C > 0$ *and an* $(m-4)$*-dimensional linear subspace* $V \subset \mathbb{R}^m$.

Moreover, if $\mu \in \mathcal{A}$ *for a subset* $\mathcal{A} \subset \mathcal{B}_\Lambda$ *that is closed under blow-up in the sense* $\mathrm{T}(\mathcal{A}) \subset \mathcal{A}$, *then the above pair can be chosen with* $(c, \bar{\nu}) \in \mathcal{A}$.

Proof. Since $u \in W^{2,2} \cap W^{1,4}(B_2, N)$, we know that for $\mathcal{H}^{m-4}$-a.e. $a \in \overline{B}_1$,

$$\lim_{\rho \searrow 0} \rho^{4-m} \int_{B_\rho(a)} \big(|D^2u|^2 + |Du|^4\big)\,dx = 0, \tag{6.6}$$

cf. [20, Lemma 3.2.2]. Since by Lemma 6.3, the fact $\nu \llcorner \overline{B}_1 \neq 0$ is equivalent to $\mathcal{H}^{m-4}(\Sigma_\mu) > 0$, we may choose a point $a \in \Sigma_\mu$ with the property (6.6). This property implies clearly that for any sequence $r_i \searrow 0$, the rescaled maps converge to some constant $c \in N$ in the sense

$$u_{a,r_i} \to c \quad \text{strongly in } W^{2,2}\big(B_2, \mathbb{R}^K\big) \text{ and in } W^{1,4}\big(B_2, \mathbb{R}^K\big), \text{ as } i \to \infty.$$

By extracting a subsequence if necessary, we can thus assume that $\mu_{a,r_i} \rightrightarrows (c, \nu_*)$ for some Radon measure ν_* on $\overline{B}_2$ as $i \to \infty$, which means $\mu_* := (c, \nu_*) \in \mathrm{T}(\mu)$. After this first blow-up, we have achieved that $r^{4-m}\nu_*(B_r(y))$ is monotonously nondecreasing in $r \in (0, 1]$ for every $y \in \overline{B}_1$, according to Lemma 6.4. Hence, the $(m-4)$-dimensional density

$$\Theta^{m-4}(\nu_*, y) := \lim_{r \searrow 0} r^{4-m}\nu_*\big(B_r(y)\big)$$

exists for every $y \in \overline{B}_1$. Consequently, we can rewrite the definition of Σ_{μ_*} in the way

$$\Sigma_{\mu_*} = \big\{y \in \overline{B}_1\colon\ \Theta^{m-4}(\nu_*, y) \geqslant \varepsilon_0\big\}.$$

Since we know $\nu_*(\overline{B}_1 \setminus \Sigma_{\mu_*}) = 0$ by Lemma 6.3, we conclude

$$0 < \varepsilon_0 \leqslant \Theta^{m-4}(\nu_*, y) \leqslant \Lambda \quad \text{for } \nu_*\text{-a.e. } y \in \overline{B}_1.$$

Here, the bound from above follows from the fact $(c, \nu_*) \in \mathcal{B}_\Lambda$. The above property implies the existence of an $(m-4)$-flat tangent measure of ν_*, see [7, Theorem 14.18]. To be more precise, for ν_*-a.e. $y \in \overline{B}_1$, there is a sequence $\rho_j \searrow 0$ with $(\nu_*)_{y,\rho_j} \rightharpoonup \bar{\nu}$ in the sense of measures, where $\bar{\nu} = C\mathcal{H}^{m-4} \llcorner V$ for an $(m-4)$-dimensional subspace $V \subset \mathbb{R}^m$ and a constant $C > 0$. This yields the desired pair $(c, \bar{\nu}) \in \mathrm{T}(\mathrm{T}(\mu)) \subset \mathcal{B}_\Lambda$. □

6.3. Proof of Theorem 1.4

For the proof of the compactness theorem we will employ the following lemma, which provides an analogue of radially constant extension in the setting of $W^{2,2}$-maps. It was proven in [13, Lemma 3.8].

Lemma 6.7. *For $0 \leqslant \sigma < \frac{1}{2}$ and $k, m \in \mathbb{N}$ with $4 \leqslant k < m$, we define tori $T^k_\sigma := \{x \in \mathbb{R}^m\colon\ [x] \leqslant \sigma\}$, where*

$$[x] := \left[\left(|x'| - \frac{1}{2}\right)^2 + |x''|^2\right]^{1/2} \quad \textit{for } x = (x', x'') \in \mathbb{R}^{m-k} \times \mathbb{R}^k.$$

For every $\sigma \in (0, \frac{1}{8})$, there is a retraction $\Psi \in C^\infty(T^k_{2\sigma} \setminus T^k_0, T^k_{2\sigma} \setminus T^k_\sigma)$ with $\Psi = \mathrm{id}$ and $D\Psi \equiv \mathrm{Id}$ on $\partial T^k_{2\sigma}$ and the estimates

$$\big|D\Psi(x)\big| \leqslant \frac{C\sigma}{[x]}, \qquad \big|D^2\Psi(x)\big| \leqslant \frac{C\sigma}{[x]^2} \quad \textit{and} \quad \det D\Psi(x) \geqslant \frac{c\sigma^k}{[x]^k} \tag{6.7}$$

for all $x \in T^k_{2\sigma} \setminus T^k_0$ and constants $c, C > 0$ depending only on k and m. Here, id *denotes the identity map on $\partial T^k_{2\sigma}$ and* Id *the identity map on $\mathbb{R}^m$.*

Proof of Theorem 1.4. By rescaling, it suffices to prove that for any sequence $w_i = W^{2,2}\text{-}\lim_{j\to\infty} w_{ij} \in W^{2,2}(B_4, N)$, where $w_{ij} \in W^{2,2}(B_4, N)$ are minimizing biharmonic maps, the property $\sup_{i\in\mathbb{N}}(\|\nabla D w_i\|_{L^2} + \|D w_i\|_{L^2}) < \infty$ implies that the sequence $\{w_i\}$ is strongly convergent in $W^{2,2}(B_{1/2}, \mathbb{R}^K)$ and in $W^{1,4}(B_{1/2}, \mathbb{R}^K)$, after passing to a subsequence. The maps w_i are stationary for $\mathcal{E}_2$ since the differential equation (3.1) remains valid under strong $W^{2,2}$-convergence. Hence, Theorem 3.4 yields $\|D^2 w_i\|^2_{M^2_4(B_2)} + \|D w_i\|^4_{M^4_4(B_2)} < \Lambda$ for a suitable constant $\Lambda > 0$ and all $i \in \mathbb{N}$. This implies $(w_i, 0) \in \mathcal{M}_\Lambda$ for all $i \in \mathbb{N}$, and we may assume that $(w_i, 0) \rightrightarrows (u, \nu)$ as $i \to \infty$, where $(u, \nu) \in \mathcal{M}_\Lambda$ by a diagonal sequence argument. In order to show strong convergence for the sequence w_i, it is thus sufficient to show $\nu \llcorner \overline{B}_1 = 0$ for all pairs $(u, \nu) \in \mathcal{M}_\Lambda$, compare Lemma 6.2.

We assume for contradiction that there is a pair $(u, \nu) \in \mathcal{M}_\Lambda$ with $\nu \llcorner \overline{B}_1 \neq 0$. Lemma 6.6, combined with Lemma 6.5, yields a pair $\bar{\mu} := (c, \bar{\nu}) \in \mathcal{M}_\Lambda$, where $c \in N$ is constant and $\bar{\nu}$ is a measure with the property (6.5). After a rotation, we can assume furthermore

$$\bar{\nu} \llcorner \overline{B}_1 = C\mathcal{H}^{m-4} \llcorner \big(\overline{B}^{m-4}_1 \times \{0\}\big)$$

with a positive constant C. By the definition of $\mathcal{M}_\Lambda$, there is a sequence of minimizing biharmonic maps $u_i \in W^{2,2} \cap W^{1,4}(B_2, N)$ with $(u_i, 0) \rightrightarrows (c, \bar{\nu})$ as $i \to \infty$. Since $\Sigma_{\bar{\mu}} = \overline{B}_1^{m-4} \times \{0\}$, Lemma 6.2 yields the convergence

$$u_i \to c \quad \text{in } C^2_{\mathrm{loc}}\big(B_1^m \setminus \big(B_1^{m-4} \times \{0\}\big), N\big) \text{ as } i \to \infty. \tag{6.8}$$

For an arbitrary $\sigma \in (0, \frac{1}{8})$ we choose a cut-off function $\varphi \in C^\infty(B_1, [0,1])$ with $\varphi \equiv 0$ on $B_{(1-\sigma)/2}$ and $\varphi \equiv 1$ outside of $B_{(1+\sigma)/2}$. Moreover, we can choose φ in such a way that $|D\varphi| \leqslant \frac{C}{\sigma}$ and $|D^2\varphi| \leqslant \frac{C}{\sigma^2}$ on B_1. We apply the notation of Lemma 6.7. Since the set $\{0 < \varphi < 1\} \setminus T^4_\sigma$ has positive distance from the energy concentration set $\overline{B}_1^{m-4} \times \{0\}$, we infer from (6.8) that $u_i \to c$ in C^2-norm on the former set, as $i \to \infty$. Hence, we may define

$$\tilde{v}_i(x) := \pi_N\big(c + \varphi(x)\big(u_i(x) - c\big)\big) \quad \text{for } x \in B_1 \setminus T^4_\sigma$$

if $i \in \mathbb{N}$ is chosen sufficiently large. Here, we write π_N for the nearest-point retraction from a tubular neighborhood of N onto itself. In the dimension $m = 4$, the above definition is even possible on all of B_1, so that in this case the construction is completed. In the case $m > 4$ however, we employ the retraction $\Psi \in C^\infty(T^k_{2\sigma} \setminus T^4_0, T^4_{2\sigma} \setminus T^4_\sigma)$ from Lemma 6.7 for the definition

$$v_i(x) := \begin{cases} \tilde{v}_i(x) & \text{for } x \in B_1 \setminus T^4_{2\sigma}, \\ \tilde{v}_i(\Psi(x)) & \text{for } x \in T^4_{2\sigma}. \end{cases}$$

Note that the construction of Ψ ensures $v_i \in W^{2,2}(B_1, N)$. In order to estimate $\mathcal{E}_2(v_i)$, we calculate, using $\tilde{v}_i = u_i$ on $B_1 \setminus B_{(1+\sigma)/2}$ and $\tilde{v}_i \equiv c$ on $B_{(1-\sigma)/2}$,

$$\int\limits_{B_1 \setminus T^4_\sigma} |\nabla D\tilde{v}_i|^2\,dx \leqslant \int\limits_{B_1 \setminus B_{1/2}} |\nabla Du_i|^2\,dx + C(N) \int\limits_{\{0<\varphi<1\} \setminus T^4_\sigma} \big[\big|D^2\big(\varphi(u_i - c)\big)\big|^2 + \big|D\big(\varphi(u_i - c)\big)\big|^4\big]\,dx \tag{6.9}$$

and since $u_i \to c$ in C^2-norm on $\{0 < \varphi < 1\} \setminus T^4_\sigma$ and $\mathcal{L}^m \llcorner |\nabla Du_i|^2 \rightharpoonup \bar{\nu}$ in the sense of measures, we conclude

$$\limsup_{i\to\infty} \int\limits_{B_1 \setminus T^4_\sigma} |\nabla D\tilde{v}_i|^2\,dx \leqslant \bar{\nu}(B_1 \setminus B_{1/2}), \tag{6.10}$$

where we used $\bar{\nu}(\partial(B_1 \setminus B_{1/2})) = 0$. Furthermore we estimate, using the properties (6.7) of Ψ,

$$\begin{aligned} \int\limits_{T^4_{2\sigma}} |\nabla Dv_i|^2\,dx &\leqslant C \int\limits_{T^4_{2\sigma}} \big(|\nabla D\tilde{v}_i \circ \Psi|^2 |D\Psi|^4 + |D\tilde{v}_i \circ \Psi|^2 \big|D^2\Psi\big|^2\big)\,dx \\ &\leqslant C \int\limits_{T^4_{2\sigma}} \big(|\nabla D\tilde{v}_i \circ \Psi|^2 + \sigma^{-2} |D\tilde{v}_i \circ \Psi|^2\big) \det D\Psi\,dx \\ &= C \int\limits_{T^4_{2\sigma} \setminus T^4_\sigma} \big(|\nabla D\tilde{v}_i|^2 + \sigma^{-2}|D\tilde{v}_i|^2\big)\,dx. \end{aligned} \tag{6.11}$$

Using $\tilde{v}_i = u_i$ on $B_1 \setminus B_{(1+\sigma)/2}$ and $\tilde{v}_i \equiv c$ on $B_{(1-\sigma)/2}$ as well as $u_i \to c$ in C^2-norm on $\{0 < \varphi < 1\} \setminus T^4_\sigma$, we estimate similarly as in (6.9) above

$$\limsup_{i\to\infty} \int\limits_{T^4_{2\sigma}} |\nabla Dv_i|^2\,dx \leqslant C \lim_{i\to\infty} \int\limits_{T^4_{2\sigma}} \big(|\nabla Du_i|^2 + \sigma^{-2}|Du_i|^2\big)\,dx = C\bar{\nu}\big(T^4_{2\sigma}\big) \tag{6.12}$$

by the convergence $u_i \to c$ strongly in $W^{1,2}(B_1, \mathbb{R}^K)$. Putting together (6.10) and (6.12) and using the minimizing property of the maps u_i, we arrive at

$$\lim_{i\to\infty} \int\limits_{B_1} |\nabla Du_i|^2\,dx \leqslant \limsup_{i\to\infty} \int\limits_{B_1} |\nabla Dv_i|^2\,dx \leqslant \bar{\nu}(B_1 \setminus B_{1/2}) + C\bar{\nu}\big(T^4_{2\sigma}\big) < \bar{\nu}(B_1)$$

for sufficiently small values of $\sigma > 0$. Since $\mathcal{L}^m \llcorner |\nabla Du_i|^2 \rightharpoonup \bar{\nu}$ in the sense of measures, we reached the desired contradiction. □

7. Dimension reduction for the singular set

We begin by introducing the notion of a tangent map for a given map $u \in W^{2,2}(B_2, N)$. We say that a map $v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ is a *tangent map* of u in a point $a \in B_1$ if there are scaling factors $r_i \searrow 0$ such that strong convergence $u_{a,r_i} \to v$ holds in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ and in $W^{1,4}_{\mathrm{loc}}(\mathbb{R}^m, N)$, as $i \to \infty$. Here, the rescaled maps u_{a,r_i} are defined according to (6.4). We have the following theorem about the structure of tangent maps of minimizing biharmonic maps.

Theorem 7.1. *Let N be a smooth, compact Riemannian manifold without boundary and $m \geqslant 4$. Suppose that $u \in M(B_2) \subset W^{2,2}(B_2^m, N)$, where $M(B_2)$ denotes the closure of the set of minimizing biharmonic maps with respect to the $W^{2,2}$-norm. Then for every $a \in B_1$, there exists a stationary biharmonic tangent map $v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ of u in the point a and every tangent map of u is homogeneous of degree zero. Moreover, for any given $s \geqslant 0$, at $\mathcal{H}^s$-a.e. point $a \in \mathrm{sing}(u)$, there is a tangent map $v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ of u with $\mathcal{H}^s(\mathrm{sing}(v)) > 0$.*

Proof. For every $a \in B_1$ and any sequence $r_i \searrow 0$, the rescaled maps u_{a,r_i} satisfy

$$\sup_{i\in\mathbb{N}} \int_{B_1} \big(|\nabla D u_{a,r_i}|^2 + |D u_{a,r_i}|^2\big)\, dx = \sup_{i\in\mathbb{N}} r_i^{4-m} \int_{B_{r_i}(a)} \big(|\nabla D u|^2 + r_i^{-2}|Du|^2\big)\, dx < \infty$$

by Corollary 3.3. Therefore, the existence of a tangent map $v = \lim_{r_i\searrow 0} u_{a,r_i}$ follows from the compactness Theorem 1.4. Here, the convergence holds in $W^{2,2}_{\mathrm{loc}}$ and $W^{1,4}_{\mathrm{loc}}$, which implies that the tangent map is stationary biharmonic.

In order to show that v is homogeneous of degree zero, we recall the monotonicity formula from Theorem 3.2,

$$\int_{B_R\setminus B_r} \left(\frac{|\nabla \partial_X v|^2}{|x|^{m-2}} + (m-2)\frac{|\partial_X v|^2}{|x|^m}\right) dx = \Phi_v(0,R) - \Phi_v(0,r) \tag{7.1}$$

for almost all $0 < r < R$, where $X(x) := x$. From the strong convergence $u_{a,r_i} \to v$, we conclude

$$\Phi_{u_{a,r_i}}(0,\cdot) \to \Phi_v(0,\cdot) \quad \text{in } L^1[0,1], \text{ as } i \to \infty.$$

After passing to a subsequence, the convergence holds also almost everywhere, from which we deduce for a.e. $R \in (0,1]$

$$\Phi_v(0,R) = \lim_{i\to\infty} \Phi_{u_{a,r_i}}(0,R) = \lim_{i\to\infty} \Phi_u(a, r_i R) = \lim_{\rho\searrow 0} \Phi_u(a,\rho).$$

Here, the latter limit exists by the monotonicity of Φ_u. We conclude that $\Phi_v(0,R)$ does not depend on R, which implies $\partial_X v \equiv 0$ by (7.1).

For the proof of the last claim, we employ [11, Theorem 3.6] to infer that for $\mathcal{H}^s$-a.e. point $a \in \mathrm{sing}(u)$, we can achieve

$$\lim_{i\to\infty} \mathcal{H}^s_\infty\big(\mathrm{sing}(u_{a,r_i}) \cap B_1(0)\big) = \lim_{i\to\infty} r_i^{-s} \mathcal{H}^s_\infty\big(\mathrm{sing}(u) \cap B_{r_i}(a)\big) \geqslant 2^{-s}\alpha(s) \tag{7.2}$$

for a suitable sequence $r_i \searrow 0$ and $\alpha(s) := \Gamma(\frac{1}{2})^s / \Gamma(\frac{s}{2}+1)$. Here,

$$\mathcal{H}^s_\infty(A) := \inf\left\{\alpha(s)\sum_{j\in\mathbb{N}} \rho_j^s \;\Big|\; A \subset \bigcup_{j\in\mathbb{N}} A_j,\ \mathrm{diam}(A_j) \leqslant 2\rho_j\right\}$$

for every Borel set $A \subset \mathbb{R}^m$. As shown above, we can assume $u_{a,r_i} \to v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ in $W^{2,2}_{\mathrm{loc}}$-topology, as $i \to \infty$. We choose an arbitrary cover $U := \bigcup_{j\in\mathbb{N}} B_j \supset \mathrm{sing}(v) \cap \overline{B_1(0)}$ of open balls B_j with radii ρ_j and claim that $\mathrm{sing}(u_{a,r_i}) \cap B_1(0) \subset U$ for sufficiently large $i \in \mathbb{N}$. Assume that this is not the case, then after extracting a subsequence there are points $p_i \in \mathrm{sing}(u_{a,r_i}) \cap B_1(0)$ for all $i \in \mathbb{N}$ with $p_i \to p \in \overline{B_1(0)} \setminus \mathrm{sing}(v)$ as $i \to \infty$. Because p is a regular point of v, we have for all sufficiently small $\rho > 0$

$$\rho^{4-m} \int_{B_\rho(p)} \big(|\nabla D v|^2 + \rho^{-2}|Dv|^2\big)\, dx < 2^{2-m}\varepsilon_1 \tag{7.3}$$

with the constant $\varepsilon_1 > 0$ from Theorem 5.1. For such a radius ρ and $i > i_0(\rho)$ large enough, there holds

$$\left(\frac{\rho}{2}\right)^{4-m} \int_{B_{\rho/2}(p_i)} \left(|\nabla Du_{a,r_i}|^2 + 4\rho^{-2}|Du_{a,r_i}|^2\right) dx < \varepsilon_1.$$

Theorem 5.1 yields $p_i \notin \operatorname{sing}(u_{a,r_i})$ in contradiction to the choice of p_i. Thus, we have established the claim $\operatorname{sing}(u_{a,r_i}) \cap B_1(0) \subset U = \bigcup_j B_j$ for all but finitely many values of $i \in \mathbb{N}$. If we assume for contradiction that $\mathcal{H}^s(\operatorname{sing}(v)) = 0$, then we can choose the cover $\bigcup_j B_j \supset \operatorname{sing}(v) \cap \overline{B_1(0)}$ in such a way that $\sum_j \rho_j^s < 2^{-s}$. As a consequence,

$$\lim_{i\to\infty} \mathcal{H}^s_\infty\left(\operatorname{sing}(u_{a,r_i}) \cap B_1(0)\right) < 2^{-s}\alpha(s)$$

in contradiction to (7.2). We conclude $\mathcal{H}^s(\operatorname{sing}(v)) > 0$, as claimed. □

In the following lemma we prove that the tangent maps arising in Federer's dimension reduction argument are in fact minimizing. For this we apply a comparison argument similar to the one in Section 6.3.

Lemma 7.2. *Suppose that $\hat{v} \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ is a tangent map of a minimizing biharmonic map with $\operatorname{sing}(\hat{v}) = \mathbb{R}^{m-k} \times \{0\}$ for some $5 \leqslant k \leqslant m$ and $\partial_i \hat{v} \equiv 0$ for $1 \leqslant i \leqslant m-k$. Then the restriction $v := \hat{v}|_{\{0\}\times\mathbb{R}^k} \in C^\infty(\mathbb{R}^k \setminus \{0\}, N)$ is minimizing biharmonic and homogeneous of degree zero.*

Proof. The homogeneity of v is a consequence of Theorem 7.1. We abbreviate $\mathcal{Z}_1 := B_1^{m-k} \times B_1^k$ and observe that, since v is a tangent map of the minimizing map u, there are minimizing biharmonic maps $v_i \in W^{2,2}(\mathcal{Z}_1, N)$ with $v_i \to \hat{v}$ strongly in $W^{2,2}(\mathcal{Z}_1, \mathbb{R}^K)$ and in $W^{1,4}(\mathcal{Z}_1, \mathbb{R}^K)$, as $i \to \infty$. Additionally, we can assume $v_i \to \hat{v}$ in $C^2_{\mathrm{loc}}(\mathcal{Z}_1 \setminus (B_1^{m-k} \times \{0\}), N)$, according to Theorem 6.2 and since $\operatorname{sing}(\hat{v}) = \mathbb{R}^{m-k} \times \{0\}$. The homogeneity of v makes it sufficient to consider comparison maps $w \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^k, N)$ with $w = v$ outside of B_1^k. As in Lemma 6.7, we consider tori $T_\sigma^k := \{x \in \mathbb{R}^m\colon [x] \leqslant \sigma\}$ for $\sigma \in [0, \frac{1}{2})$, where

$$[x] := \left[\left(|x'| - \frac{1}{2}\right)^2 + |x''|^2\right]^{1/2} \quad \text{for } x = (x', x'') \in \mathbb{R}^{m-k} \times \mathbb{R}^k.$$

For an arbitrary $\sigma \in (0, \frac{1}{16})$, we choose a cut-off function $\varphi \in C^\infty(\mathcal{Z}_1, [0,1])$ with $\varphi \equiv 0$ on $B_{1/2-\sigma}$, $\varphi \equiv 1$ on $\mathcal{Z}_1 \setminus B_{1/2+\sigma}$, as well as $|D\varphi| \leqslant \frac{C}{\sigma}$ and $|D^2\varphi| \leqslant \frac{C}{\sigma^2}$. Then the map $w_\sigma(x', x'') := w(\sigma^{-1}x'')$, defined for $(x', x'') \in \mathbb{R}^{m-k} \times \mathbb{R}^k$, satisfies $w_\sigma = \hat{v}$ on $\mathbb{R}^m \setminus (\mathbb{R}^{m-k} \times B_\sigma^k)$, in particular $w_\sigma = \hat{v}$ on $\{0 < \varphi < 1\} \setminus T_{2\sigma}^k$. Because $v_i \to \hat{v}$ in $C^2(\{0 < \varphi < 1\} \setminus T_{2\sigma}^k, N)$, we may define

$$\tilde{w}_i := \pi_N\left(w_\sigma + \varphi(v_i - w_\sigma)\right) \quad \text{on } \mathcal{Z}_1 \setminus T_{2\sigma}^k$$

for large values of $i \in \mathbb{N}$, where π_N denotes the nearest-point retraction onto N. For $k = m$, the above definition is actually possible on all of $\mathcal{Z}_1$ and yields an admissible comparison map for v_i. In the case $k < m$, we choose a retraction $\Psi \in C^\infty(T_{4\sigma}^k \setminus T_0^k, T_{4\sigma}^k \setminus T_{2\sigma}^k)$ as in Lemma 6.7 and define

$$w_i := \begin{cases} \tilde{w}_i & \text{on } B_1 \setminus T_{4\sigma}^k, \\ \tilde{w}_i \circ \Psi & \text{on } T_{4\sigma}^k. \end{cases}$$

By the choice of Ψ, this defines a map $w_i \in W^{2,2}(\mathcal{Z}_1, N)$, compare Lemma 6.7. Since $\varphi \equiv 0$ on $B_{1/2-\sigma}$, we have

$$\tilde{w}_i = w_\sigma \quad \text{on } \left(B_{1/2}^{m-k} \times B_\sigma^k\right) \setminus T_{2\sigma}^k. \tag{7.4}$$

Using $v_i \to \hat{v}$ in $W^{2,2}(\mathcal{Z}_1, \mathbb{R}^K)$ and in $W^{1,4}(\mathcal{Z}_1, \mathbb{R}^K)$ as $i \to \infty$, together with $w_\sigma = \hat{v}$ on $\mathbb{R}^m \setminus (\mathbb{R}^{m-k} \times B_\sigma^k)$, we deduce furthermore

$$\tilde{w}_i \to \hat{v} \quad \text{on } \left(\mathcal{Z}_1 \setminus \left(B_{1/2}^{m-k} \times B_\sigma^k\right)\right) \setminus T_{2\sigma}^k \tag{7.5}$$

with respect to the $W^{2,2}$-norm, as $i \to \infty$. From this we infer the following estimate, where we write $\alpha(m-k)$ for the volume of B_1^{m-k}.

$$\limsup_{i\to\infty}\int_{\mathcal{Z}_1\setminus T^k_{2\sigma}}|\nabla D\tilde{w}_i|^2\,dx\leqslant\int_{\mathcal{Z}_1\setminus(B^{m-k}_{1/2}\times B^k_\sigma)}|\nabla D\hat{v}|^2\,dx+\int_{B^{m-k}_{1/2}\times B^k_\sigma}|\nabla Dw_\sigma|^2\,dx$$
$$=\alpha(m-k)\int_{B^k_1}|\nabla Dv|^2\,dx+\frac{\alpha(m-k)}{2^{m-k}}\int_{B^k_\sigma}\bigl(|\nabla Dw_\sigma|^2-|\nabla Dv|^2\bigr)\,dx. \tag{7.6}$$

Using the properties (6.7) of Ψ, we can estimate similarly as in (6.11)

$$\int_{T^k_{4\sigma}}|\nabla Dw_i|^2\,dx\leqslant C\int_{T^k_{4\sigma}}\bigl(|\nabla D\tilde{w}_i\circ\Psi|^2+\sigma^{-2}|D\tilde{w}_i\circ\Psi|^2\bigr)\left(\frac{[x]}{\sigma}\right)^{k-4}\det D\Psi\,dx$$
$$\leqslant C\int_{T^k_{4\sigma}\setminus T^k_{2\sigma}}\bigl(|\nabla D\tilde{w}_i|^2+\sigma^{-2}|D\tilde{w}_i|^2\bigr)\,dx.$$

Keeping in mind (7.4) and (7.5), we arrive at

$$\limsup_{i\to\infty}\int_{T^k_{4\sigma}}|\nabla Dw_i|^2\,dx\leqslant C\int_{T^k_{4\sigma}}\bigl(|\nabla D\hat{v}|^2+\sigma^{-2}|D\hat{v}|^2+|\nabla Dw_\sigma|^2+\sigma^{-2}|Dw_\sigma|^2\bigr)\,dx.$$

By the homogeneity of $\hat{v}$, the definition of w_σ and by the inclusion $T^k_{4\sigma}\subset(B^{m-k}_{1/2+4\sigma}\setminus B^{m-k}_{1/2-4\sigma})\times B^k_{4\sigma}$, we can thus estimate

$$\limsup_{i\to\infty}\int_{T^k_{4\sigma}}|\nabla Dw_i|^2\,dx\leqslant C\sigma\int_{B^k_{4\sigma}}\bigl(|\nabla Dv|^2+\sigma^{-2}|Dv|^2+|\nabla Dw_\sigma|^2+\sigma^{-2}|Dw_\sigma|^2\bigr)\,dx$$
$$=C\sigma^{k-3}\int_{B^k_4}\bigl(|\nabla Dv|^2+|Dv|^2+|\nabla Dw|^2+|Dw|^2\bigr)\,dx=:I. \tag{7.7}$$

By the minimizing property of the v_i, we conclude

$$\alpha(m-k)\int_{B^k_1}|\nabla Dv|^2\,dx=\int_{\mathcal{Z}_1}|\nabla D\hat{v}|^2\,dx\leqslant\limsup_{i\to\infty}\int_{\mathcal{Z}_1}|\nabla Dw_i|^2\,dx$$
$$\leqslant\alpha(m-k)\int_{B^k_1}|\nabla Dv|^2\,dx+\frac{\alpha(m-k)}{2^{m-k}}\int_{B^k_\sigma}\bigl(|\nabla Dw_\sigma|^2-|\nabla Dv|^2\bigr)\,dx+I,$$

where we used (7.6) and (7.7) in the last step. As a result of the above estimate, we get

$$\int_{B^k_1}\bigl(|\nabla Dw|^2-|\nabla Dv|^2\bigr)\,dx=\sigma^{4-k}\int_{B^k_\sigma}\bigl(|\nabla Dw_\sigma|^2-|\nabla Dv|^2\bigr)\,dx\geqslant-C\sigma^{4-k}I$$
$$=-C\sigma\int_{B^k_4}\bigl(|\nabla Dv|^2+|Dv|^2+|\nabla Dw|^2+|Dw|^2\bigr)\,dx.$$

Since the last term can be made arbitrarily small by choosing $\sigma>0$ small enough, we conclude

$$\int_{B^k_1}|\nabla Dv|^2\,dx\leqslant\int_{B^k_1}|\nabla Dw|^2\,dx,$$

which is the stated minimality of v. □

Having established Theorem 7.1 and Lemma 7.2, the remainder of the proof of Theorem 1.3 is a standard application of Federer's dimension reduction argument (see e.g. [11, Theorem A.4]). We do no repeat it here in detail, but give only the

Sketch of the proof of Theorem 1.3. Assume for contradiction that there is a minimizing biharmonic map $u \in W^{2,2}(\Omega, N)$ with $\mathcal{H}\text{-}\dim(\operatorname{sing}(u)) > m-5$. Then, Theorem 7.1 enables us to linearize the singular set by taking tangent maps repeatedly. More precisely, we can construct a stationary biharmonic tangent map $v \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ with $\mathcal{H}\text{-}\dim(\operatorname{sing}(v)) > m-5$, for which $\operatorname{sing}(v)$ is a linear subspace of $\mathbb{R}^m$. This clearly contradicts $\mathcal{H}^{m-4}(\operatorname{sing}(v)) = 0$ by Corollary 5.2.

Furthermore, if there is a minimizing biharmonic map $u \in W^{2,2}(\Omega, N)$ with $m-k-2 < \mathcal{H}\text{-}\dim(\operatorname{sing}(u)) \leqslant m-k-1$, then the same blow-up construction as above yields a tangent map $\hat{v} \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^m, N)$ that is homogeneous of degree zero, satisfies $\partial_i \hat{v} \equiv 0$ for $1 \leqslant i \leqslant m-k-1$ and $\operatorname{sing}(\hat{v}) = \mathbb{R}^{m-k-1} \times \{0\}$. Applying Lemma 7.2, we infer that $v := \hat{v}|_{\{0\}\times\mathbb{R}^{k+1}} \in C^\infty(\mathbb{R}^{k+1} \setminus \{0\}, N)$ is minimizing biharmonic. Furthermore, v is homogeneous of degree zero and $\operatorname{sing}(v) = \{0\}$, which implies in particular that v is not constant. Thus, we can reduce the dimension of the singular set further if tangent maps with the above properties do not exist. □

8. The minimality of the map $\frac{x}{|x|}$

In this section we prove, as stated in Proposition 1.5, that the map $u_0 : B^m \to S^{m-1}$, $x \mapsto \frac{x}{|x|}$ is minimizing intrinsically biharmonic for $m \geqslant 5$. We point out that $u_0 \in W^{2,2}(B^m, S^{m-1})$ if and only if $m \geqslant 5$. For the proof, we employ the Null-Lagrangians introduced in [2] for the proof that u_0 minimizes the functional $\int_{B^m} |Du|^4\,dx$. In contrast to the mentioned work, we will estimate these Null-Lagrangian with Maclaurin's inequality between the fourth and second elementary symmetric polynomial, which reads as follows (see e.g. [5, Theorem 52]). For all numbers $\lambda_1, \dots, \lambda_{m-1} \geqslant 0$, there holds

$$\binom{m-1}{4}^{-1} \sum_{1\leqslant i_1<\dots<i_4\leqslant m-1} \lambda_{i_1}\dots\lambda_{i_4} \leqslant \binom{m-1}{2}^{-2} \Bigg(\sum_{1\leqslant i<j\leqslant m-1} \lambda_i\lambda_j \Bigg)^2. \tag{8.1}$$

Proof of Proposition 1.5. Let $u \in W^{2,2}(B^m, S^{m-1})$ be an arbitrary map with $u \in u_0 + W^{2,2}_0(B^m, \mathbb{R}^m)$. We point out that Nirenberg interpolation implies $u \in W^{1,4}(B^m, S^{m-1})$. Denoting the Levi-Civita connection of S^{m-1} by ∇, we observe

$$|\nabla Du|^2 = \big|D^2u\big|^2 - |Du\otimes Du|^2 \quad\text{and}\quad \big|\operatorname{trace}(\nabla Du)\big|^2 = |\Delta u|^2 - |Du|^4.$$

Therefore, one checks by two integrations by part that $\mathcal{E}_2(u)$ differs from

$$\int_{B^m} \big(\big|\operatorname{trace}(\nabla Du)\big|^2 + |Du|^4 - |Du\otimes Du|^2\big)\,dx$$

only by boundary terms, which are determined by the fixed boundary data. Since $u_0 \in W^{2,2}(B^m, S^{m-1})$ is a harmonic map, there holds $|\operatorname{trace}(\nabla Du_0)| \equiv 0$. Thus, it suffices to show that u_0 minimizes the functional

$$H(u) := \int_{B^m} \big(|Du|^4 - |Du\otimes Du|^2\big)\,dx$$

in the class $W^{2,2}(B^m, S^{m-1}) \cap (u_0 + W^{2,2}_0(B^m, \mathbb{R}^m))$. We calculate

$$\begin{aligned} H(u) &= \int_{B^m} \Bigg[|Du|^4 - \sum_{i=1}^m \sum_{\alpha,\beta=1}^m \big(\partial_\alpha u^i\,\partial_\beta u^i\big)^2 \Bigg]\,dx = \int_{B^m} \Bigg[\Bigg(\sum_{i=1}^m \big|Du^i\big|^2 \Bigg)^2 - \sum_{i=1}^m \big|Du^i\big|^4 \Bigg]\,dx \\ &= 2 \sum_{1\leqslant i<j\leqslant m} \int_{B^m} \big|Du^i\big|^2 \big|Du^j\big|^2\,dx =: \int_{B^m} h(u)\,dx. \end{aligned}$$

Following [2], we define for every subset $I \subset \{1, \dots, m\}$

$$\omega_I(u) := d\alpha_1 \wedge \cdots \wedge d\alpha_m, \quad \text{where } \alpha_i := \begin{cases} u^i & \text{if } i \in I, \\ x^i & \text{if } i \notin I, \end{cases}$$

and

$$\Lambda(u) := \sum_{|I|=4} \omega_I(u).$$

Since $\omega_I(u) = d(\alpha_1 d\alpha_2 \wedge \cdots \wedge d\alpha_m)$, the theorem of Stokes yields

$$\int_{B^m} \Lambda(u) = \int_{B^m} \Lambda(u_0) \quad \text{for all } u \in u_0 + W_0^{1,4}(B^m, \mathbb{R}^m).$$

We claim that for all $u \in W^{2,2}(B^m, S^{m-1})$, there holds

$$\Lambda(u) \leqslant \frac{(m-3)(m-4)}{24} h(u) \quad \text{on } B^m. \tag{8.2}$$

Applying a rotation, it suffices to prove this inequality for the case $u(x) = \mathrm{e} := (0, \dots, 0, 1) \in \mathbb{R}^m$. Since $\partial_\alpha u(x) \in T_{\mathrm{e}} S^{m-1}$ for all $1 \leqslant \alpha \leqslant m$, we conclude that $Du^m(x) \equiv 0$. Using the inequality of Hadamard and (8.1), we estimate in the point x

$$\begin{aligned} \Lambda(u) &\leqslant \sum_{1 \leqslant i_1 < \cdots < i_4 \leqslant m-1} |Du^{i_1}||Du^{i_2}||Du^{i_3}||Du^{i_4}| \\ &\leqslant \binom{m-1}{4}\binom{m-1}{2}^{-2}\Big(\sum_{1 \leqslant i < j \leqslant m-1} |Du^i||Du^j|\Big)^2 \\ &\leqslant \binom{m-1}{4}\binom{m-1}{2}^{-1} \sum_{1 \leqslant i < j \leqslant m-1} |Du^i|^2 |Du^j|^2, \end{aligned}$$

where we applied the Cauchy–Schwarz inequality in the last step. This implies the claim (8.2). On the other hand, one checks that $h(u_0)(x) = \frac{2}{|x|^4}\binom{m-1}{2}$ and $\Lambda(u_0)(x) = \frac{1}{|x|^4}\binom{m-1}{4}$ for all $x \in B^m$. Hence, for $u = u_0$ we have equality in (8.2). This completes the proof since

$$H(u_0) = \frac{24}{(m-3)(m-4)} \int_{B^m} \Lambda(u_0) = \frac{24}{(m-3)(m-4)} \int_{B^m} \Lambda(u) \leqslant H(u). \qquad \square$$

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