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# A variational approach to the local character of $G$-closure: the convex case

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**Abstract**

This article is devoted to characterize all possible effective behaviors of composite materials by means of periodic homogenization. This is known as a $G$-closure problem. Under convexity and $p$-growth conditions ($p > 1$), it is proved that all such possible effective energy densities obtained by a $\Gamma$-convergence analysis, can be locally recovered by the pointwise limit of a sequence of periodic homogenized energy densities with prescribed volume fractions. A weaker locality result is also provided without any kind of convexity assumption and the zero level set of effective energy densities is characterized in terms of Young measures. A similar result is given for cell integrands which enables to propose new counter-examples to the validity of the cell formula in the nonconvex case and to the continuity of the determinant with respect to the two-scale convergence.


**Résumé**

Cet article est consacré à la caractérisation de toutes les limites effectives possibles de matériaux composites en terme d'homogénéisation périodique. Ce problème est connu sous le nom de $G$-fermeture. Il est démontré, sous des hypothèses de convexité et de croissance $p > 1$, que toutes ces densités d'énergies effectives, obtenues lors d'une analyse par $\Gamma$-convergence, peuvent être localement vues comme la limite ponctuelle d'une suite de densités d'énergies périodiquement homogénéisées avec une fraction de volume fixée. Un résultat plus faible est obtenu sans aucune hypothèse de convexité et l'ensemble des zéros de ce type de densités d'énergies effectives est caractérisé en terme de mesures d'Young. Un résultat similaire est donné pour des intégrandes cellulaires, permettant ainsi de proposer de nouveaux contre-exemples quant à la validité de la formule de cellule dans le cas non convexe, ainsi qu'à la continuité du déterminant par rapport à la convergence à double échelle.


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## 1. Introduction

Composites are structures constituted by two or more materials which are finely mixed at microscopic length scales. Despite the high complexity of their microstructure, composites appear essentially as homogeneous at macroscopic

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length scale. It suggests to give a description of their effective properties as a kind of average made on the respective properties of the constituents. The *Homogenization Theory* renders possible to define properly such an average, by thinking of a composite as a limit (in a certain sense) of a sequence of structures whose heterogeneities become finer and finer. There is a wide literature on the subject; we refer the reader to [29] for a starting point.

Many notions of convergence have been introduced to give a precise sense to such asymptotic analysis. One of the most general is the $H$-convergence (see [33,40]), which permits to describe the asymptotic behavior of a sequence of second order elliptic operators in divergence form. This notion appears as a generalization of the $G$-convergence (see [19, Chapter 22]) introduced independently for symmetric operators. Specifically, given a bounded open set $\Omega \subseteq \mathbb{R}^n$, a sequence $\{A_k\} \subset L^\infty(\Omega; \mathbb{R}^{n\times n})$ of uniformly elliptic tensors $H$-converges to $A_\infty \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ if, for every $g \in H^{-1}(\Omega)$, the sequence of solutions $u_k$ of

$$-\operatorname{div}(A_k \nabla u) = g \quad \text{in } \mathcal{D}'(\Omega),\ u \in H_0^1(\Omega)$$

satisfies

$$\begin{cases} u_k \rightharpoonup u_\infty & \text{weakly in } H_0^1(\Omega), \\ A_k \nabla u_k \rightharpoonup A_\infty \nabla u_\infty & \text{weakly in } L^2(\Omega; \mathbb{R}^n). \end{cases}$$

In particular, $u_\infty$ is the solution of

$$-\operatorname{div}(A_\infty \nabla u) = g \quad \text{in } \mathcal{D}'(\Omega),\ u \in H_0^1(\Omega).$$

If the tensors $A_k$ describe a certain property (for instance the conductivity) of the structures approximating the composite, it is assumed that $A_\infty$ describes the effective behavior with respect to that property.

Composite materials are characterized by three main features: the different constituents (or phases), their volume fraction and their geometric arrangement. A natural question which arises is the following: given two constituents as well as their volume fraction, what are all possible effective behaviors with respect to a certain property? In the language of $H$-convergence this problem reads as follows: given two tensors $A^{(1)}$ and $A^{(2)} \in \mathbb{R}^{n\times n}$ corresponding to the conductivities of both constituents of the composite and given a volume fraction $\theta \in [0,1]$, determine all the tensors $A^* \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ such that there exists a sequence of characteristic functions $\{\chi_k\} \subset L^\infty(\Omega; \{0,1\})$ (the geometry of the finer and finer mixture) satisfying

$$\begin{cases} A_{\chi_k} & H\text{-converges to } A^*, \\ \displaystyle -\!\!\!\!\!\!\int_\Omega \chi_k \, dx = \theta & \text{for all } k \in \mathbb{N}, \end{cases}$$

where $A_{\chi_k}(x) := \chi_k(x) A^{(1)} + (1 - \chi_k(x)) A^{(2)}$. The family of these effective tensors is called the $G$-closure of $\{A^{(1)}, A^{(2)}\}$ with fixed volume fraction $\theta$. Its determination is known as a *G-closure problem* and is, in general, very difficult to solve except in some particular cases that we shall briefly discuss later. Indeed, there is, in general, no "explicit" formula to compute a $H$-limit. An exception occurs in the periodic case, i.e., when

$$A_k(x) = A\big(\langle kx \rangle\big) \quad \text{for } x \in \Omega,$$

where $\langle\cdot\rangle$ denotes the fractional part of a vector componentwise. In this very particular case it is possible to prove (see [2, Theorem 1.3.18]) that the sequence $\{A_k\}$ $H$-converges to a constant tensor $A_{\text{cell}}$ given by

$$(A_{\text{cell}})_{ij} = \int_Q A(y)\big[e_i + \nabla\phi_i(y)\big] \cdot \big[e_j + \nabla\phi_j(y)\big]\, dy \quad \text{for } i, j \in \{1, \dots, n\}, \tag{1.1}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, $Q := (0,1)^n$ and $\phi_i$ is a solution (unique up to an additive constant) of the problem

$$-\operatorname{div}\big(A[e_i + \nabla\phi]\big) = 0 \quad \text{in } \mathcal{D}'(Q),\ \phi \in H^1_{\text{per}}(Q).$$

With the aim of finding optimal bounds and an analytical description of the $G$-closure set of two isotropic conductors in dimension two, it has been proved independently in [28] and [38] a locality property which underlines the importance of periodic structures. It states that every effective tensors, obtained by mixing two materials with a volume fraction $\theta$, can be locally recovered by the pointwise limit of a sequence of effective tensors, each of them obtained by a periodic mixture with the same proportion $\theta$. In other words, in that case, periodic mixtures capture every kind of mixtures, which enables to reduce the study of such problems to periodic geometries and materials with homogeneous effective behavior, since periodic homogenization produces homogeneous limiting ones. This locality result has been subsequently generalized by Dal Maso and Kohn in an unpublished work to higher dimension and not necessarily isotropic conductors (a proof can be found e.g. in [2, Chapter 2]). Given two tensors $A^{(1)}$ and $A^{(2)} \in \mathbb{R}^{n\times n}$, for any $\theta \in [0,1]$, define

$$P_\theta := \left\{A^* \in \mathbb{R}^{n\times n}\colon \text{ there exists } \chi \in L^\infty\big(Q;\{0,1\}\big) \text{ such that } \int_Q \chi\,dx = \theta \text{ and } A^* = (A_\chi)_{\text{cell}}\right\},$$

where $A_\chi(x) := \chi(x)A^{(1)} + (1-\chi(x))A^{(2)}$ and $(A_\chi)_{\text{cell}}$ is defined as in (1.1). Denote by $G_\theta$ the closure of $P_\theta$ in $\mathbb{R}^{n\times n}$. The local representation of $G$-closure states that if $A^* \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ and $\theta \in L^\infty(\Omega;[0,1])$, then $A^*(x_0) \in G_{\theta(x_0)}$ for a.e. $x_0 \in \Omega$ if and only if there exists a sequence of characteristic functions $\{\chi_k\} \subset L^\infty(\Omega;\{0,1\})$ such that

$$\begin{cases} A_{\chi_k}\, H\text{-converges to } A^*, \\ \chi_k \rightharpoonup \theta \quad \text{weakly* in } L^\infty\big(\Omega;[0,1]\big). \end{cases}$$

Later, a further generalization has been given in [35] to the case of nonlinear elliptic and strictly monotone operators in divergence form, using an extension of the $H$-convergence to the nonlinear monotone setting which can be found in [16].

In this paper, we wish to study similar $G$-closure problems in the framework of nonlinear elasticity. Of course, this degree of generality makes it quite impossible to characterize analytically a $G$-closure set. However, it is still possible to formulate a locality result which finds a similar statement also in that case. As nonlinear elasticity rests on the study of equilibrium states, or minimizers, of a suitable energy, we fall within the framework of the Calculus of Variations and it is natural to use the notion of $\Gamma$-convergence (see [13]) to describe the effective properties of a composite. It is a variational convergence which permits to analyze the asymptotic behaviour of a sequence of minimization problems of the form

$$\mathrm{m}_k = \min\big\{F_k(u)\colon\ u \in \mathcal{U}\big\}, \tag{1.2}$$

where $\mathcal{U}$ is a suitable functional space (in the sequel $W^{1,p}(\Omega;\mathbb{R}^m)$ with $p \in (1,+\infty)$) and $F_k$ is a sequence of functionals representing in our case the energies of the structures approximating the composite. The $\Gamma$-limit $F_\infty$ of the sequence $F_k$ has the following fundamental property: if $u_k$ is a solution of the minimization problem (1.2) and $u_k \to u_\infty$ in $\mathcal{U}$, then $F_k(u_k) \to F_\infty(u_\infty)$ and $u_\infty$ is a solution of the minimization problem

$$\mathrm{m}_\infty = \min\big\{F_\infty(u)\colon\ u \in \mathcal{U}\big\}.$$

This fact supports the assumption that $F_\infty$ describes in a good way the effective energy of the composite.

Let us briefly explain our study in the special case of mixtures of two different materials. Let $W^{(1)}$ and $W^{(2)}\colon \mathbb{R}^{m\times n} \to [0,+\infty)$ be two functions satisfying standard $p$-growth and $p$-coercivity conditions. The functions $W^{(1)}$ and $W^{(2)}$ stand for the stored energy densities of two homogeneous nonlinearly elastic materials. If $\chi \in L^\infty(\Omega;\{0,1\})$ is a characteristic function, we use the notation $W_\chi(x,\xi) := \chi(x)W^{(1)}(\xi) + (1-\chi(x))W^{(2)}(\xi)$. The function $W_\chi$ can be thought of as the stored energy density of a composite material obtained as a mixture of both previous ones.

Without any kind of convexity assumption on $W^{(1)}$ and $W^{(2)}$, we prove a weaker locality property about effective energy densities (Theorem 3.5) which states that if $W^*\colon \Omega \times \mathbb{R}^{m\times n} \to [0,+\infty)$ is a Carathéodory function and $\theta \in L^\infty(\Omega;[0,1])$, then the following conditions are equivalent:

(i) there exists a sequence of characteristic functions $\{\chi_k\} \subset L^\infty(\Omega;\{0,1\})$ such that

$$\begin{cases} \displaystyle\int_\Omega W^*\big(x,\nabla u(x)\big)\,dx = \Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_{\chi_k}\big(x,\nabla u(x)\big)\,dx, \\ \chi_k \rightharpoonup \theta \quad \text{weakly* in } L^\infty\big(\Omega;[0,1]\big); \end{cases} \tag{1.3}$$

(ii) for a.e. $x_0 \in \Omega$, there exists a sequence of characteristic functions $\{\tilde\chi_k\} \subset L^\infty(Q;\{0,1\})$ such that

$$\begin{cases} \displaystyle\int_Q W^*\big(x_0, \nabla u(y)\big)\,dy = \Gamma\text{-}\lim_{k\to+\infty}\int_Q W_{\tilde\chi_k}\big(y,\nabla u(y)\big)\,dy, \\ \tilde\chi_k \rightharpoonup \theta(x_0) \quad \text{weakly* in } L^\infty\big(Q;[0,1]\big). \end{cases}$$

We interpret this result as being enough to consider homogeneous effective energies with asymptotically homogeneous microstructure. In Theorem 4.1 we perform another characterization of the effective energy densities through a fine analysis of their zero level set. It is realized by means of gradient Young measures supported in the union of the zero level sets of $W^{(1)}$ and $W^{(2)}$.

When the functions $W^{(1)}$ and $W^{(2)}$ are both convex, we can carry out a deeper locality result. As before, for any $\theta \in [0,1]$, we define the set $P_\theta$ made of all effective energy densities obtained by a periodic homogenization of a mixture of $W^{(1)}$ and $W^{(2)}$ in proportions $\theta$ and $1-\theta$:

$$P_\theta := \left\{ W^*: \mathbb{R}^{m\times n} \to [0,+\infty)\text{: there exists } \chi \in L^\infty\big(Q;\{0,1\}\big) \text{such that } \int_Q \chi\,dx = \theta \text{ and } W^* = (W_\chi)_{\mathrm{hom}} \right\},$$

where

$$(W_\chi)_{\mathrm{hom}}(\xi) := \inf_{j\in\mathbb{N}^+} \inf\left\{ ⨍_{(0,j)^n} W_\chi\big(\langle y\rangle, \xi + \nabla\phi(y)\big)\,dy\text{: } \phi \in W^{1,p}_{\mathrm{per}}\big((0,j)^n;\mathbb{R}^m\big)\right\}. \tag{1.4}$$

The function $(W_\chi)_{\mathrm{hom}}$ is called *homogenized integrand* associated to $W_\chi$ and it is possible to prove (independently of the convex assumption) that if $\chi_k(x) = \chi(\langle kx\rangle)$ for $x \in \Omega$, then

$$\int_\Omega (W_\chi)_{\mathrm{hom}}\big(\nabla u(x)\big)\,dx = \Gamma\text{-}\lim_{k\to+\infty}\int_\Omega W_{\chi_k}\big(x,\nabla u(x)\big)\,dx.$$

In general, the set $P_\theta$ is not closed for the pointwise convergence so that it is natural to consider its closure denoted as before by $G_\theta$. Indeed, there exist effective energies that are not exactly reached by a periodic mixture: thanks to Theorem 4.1, we built an example in the case of a mixture of four materials (Example 5.2). Our main result, Theorem 5.1, states that if $W^*: \Omega \times \mathbb{R}^{m\times n} \to [0,+\infty)$ is a Carathéodory function and $\theta \in L^\infty(\Omega;[0,1])$, then there exists a sequence of characteristic functions $\{\chi_k\} \subset L^\infty(\Omega;\{0,1\})$ satisfying (1.3) if and only if $W^*(x_0,\cdot) \in G_{\theta(x_0)}$ for a.e. $x_0 \in \Omega$.

This result was stated in [21, Conjecture 3.15] and it was useful to study a quasi-static evolution model for the interaction between fracture and damage. Indeed the authors used this conjecture to prove the wellposedness of an incremental problem at fixed time step.

If we assume the functions $W^{(1)}$ and $W^{(2)}$ to be differentiable, one can write the Euler–Lagrange equation associated to the minimization problem. It is a nonlinear elliptic partial differential equation in divergence form and the first order convexity condition means that the operators

$$\xi \mapsto \frac{\partial W^{(1)}}{\partial \xi}(\xi) \quad \text{and} \quad \xi \mapsto \frac{\partial W^{(2)}}{\partial \xi}(\xi)$$

are monotone. Hence our locality result reduces to a particular case of [35]. Furthermore, if $W^{(1)}(\xi) = A^{(1)}\xi\cdot\xi$ and $W^{(2)}(\xi) = A^{(2)}\xi\cdot\xi$ for some symmetric matrices $A^{(1)}$ and $A^{(2)} \in \mathbb{R}^{n\times n}$, then Theorem 5.1 is nothing but Dal Maso and Kohn's result in the special case of symmetric operators. Note that since $H$-convergence problems have, in general, no variational structure, our result do not generalize the preceding ones, but rather gives a variational version of them.

Our proof of Theorem 5.1 works only when the functions $W^{(1)}$ and $W^{(2)}$ are both convex. The nonconvex case seems to be much more delicate to address. The reason is that the homogenization formula (1.4) in the convex case is reduced to a single cell formula, i.e., $(W_\chi)_{\mathrm{hom}} = (W_\chi)_{\mathrm{cell}}$, where

$$(W_\chi)_{\mathrm{cell}}(\xi) := \inf\left\{ \int_Q W_\chi\big(y, \xi + \nabla\phi(y)\big)\,dy\text{: } \phi \in W^{1,p}_{\mathrm{per}}\big(Q;\mathbb{R}^m\big)\right\}.$$

On the contrary, it is known that both formula do not coincide even in the quasiconvex case because of the counter-example [31, Theorem 4.3]. To illustrate this phenomena, we present here another much simpler counter-example, Example 6.1, based on a rank-one connection argument and on the characterization of the zero level set of cell integrands under quasiconvex assumption, Lemma 4.4. Indeed we prove that when $m = n = 2$, there exist suitable densities $W^{(1)}$ and $W^{(2)}$ (one of them being nonconvex) and a suitable geometry $\chi$ such that

$$(W_\chi)_{\mathrm{cell}}(C) > 0 \quad \text{while} \quad (W_\chi)_{\mathrm{hom}}(C) = 0$$

for some matrix $C \in \mathbb{R}^{2\times 2}$. Moreover, exploiting the lower semicontinuity property of certain integral functionals with respect to the two-scale convergence and noticing that the functions $W^{(1)}$ and $W^{(2)}$ in the previous example can be taken polyconvex, we also show that, in general, one cannot expect the determinant to be continuous with respect to the two-scale convergence (Example 6.5).

The paper is organized as follows: in Section 2, we introduce notations used in the sequel and we recall basic facts about $\Gamma$-convergence, periodic homogenization, Young measures and two-scale convergence. In Section 3, we prove a locality property, Theorem 3.5, enjoyed by effective energy densities obtained by mixing $N$ different materials. Section 4 is devoted to a fine analysis of the zero level set of such effective energy densities by means of Young measures: this is achieved in Theorem 4.1. We also perform a similar analysis for cell kind integrands. In Section 5, we give a proof of our main result, Theorem 5.1, which states the local property with respect to the set $G_\theta$ in the case of convex stored energy densities. Moreover, thanks to the previous Young measure analysis, we show by an example based on the four matrices configuration (Example 5.2) that in general $P_\theta \subsetneq G_\theta$. Finally, we present in Section 6 a new counter-example to the validity of the cell formula in the nonconvex case (Example 6.1) and to the continuity of the determinant with respect to the two-scale convergence (Example 6.5).

## 2. Notations and preliminaries

Throughout the paper, we employ the following notations:

- $\Omega$ is a bounded open subset of $\mathbb{R}^n$;
- $Q$ is the unit cell $(0,1)^n$;
- if $a \in \mathbb{R}^n$ and $\rho > 0$, $B_\rho(a)$ stands for the open ball in $\mathbb{R}^n$ of center $a$ and radius $\rho$ while $Q_\rho(a) := a + (0,\rho)^n$ is the open cube in $\mathbb{R}^n$ with corner $a$ and edge length $\rho$;
- $\mathcal{O}(\Omega)$ denotes the family of all open subsets of $\Omega$;
- $\mathcal{L}^n$ denotes the $n$-dimensional Lebesgue measure;
- the symbol $\fint_E$ stands for the average $\mathcal{L}^n(E)^{-1}\int_E$;
- if $\mu$ is a Borel measure in $\mathbb{R}^s$ and $E \subseteq \mathbb{R}^s$ is a Borel set, the measure $\mu \llcorner E$ stands for the restriction of $\mu$ to $E$, i.e. for every Borel set $B \subseteq \mathbb{R}^s$, $(\mu \llcorner E)(B) = \mu(E \cap B)$;
- $W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^m)$ is the space of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ functions which are $Q$-periodic;
- given $x \in \mathbb{R}^n$, $\langle x\rangle$ stands for the fractional part of $x$ componentwise;
- $\rightharpoonup$ (resp. $\overset{*}{\rightharpoonup}$) always denotes weak (resp. weak*) convergence.

### 2.1. $\Gamma$-convergence

Let $\alpha \in \mathbb{R}^3$ with $\alpha_1, \alpha_2, \alpha_3 > 0$ and $p \in (1, +\infty)$. We denote by $\mathcal{F}(\alpha, p)$ the set of all continuous functions $f : \mathbb{R}^{m\times n} \to [0,+\infty)$ satisfying the following coercivity and growth conditions:

$$\alpha_1|\xi|^p - \alpha_2 \leqslant f(\xi) \leqslant \alpha_3\big(1 + |\xi|^p\big) \quad \text{for any } \xi \in \mathbb{R}^{m\times n}. \tag{2.1}$$

Moreover, we denote by $\mathcal{F}(\alpha, p, \Omega)$ the set of all Carathéodory functions $f : \Omega \times \mathbb{R}^{m\times n} \to [0,+\infty)$ such that $f(x,\cdot) \in \mathcal{F}(\alpha, p)$ for a.e. $x \in \Omega$.

We recall the definition of $\Gamma$-convergence, referring to [12,13,19] for a comprehensive treatment on the subject.

**Definition 2.1.** Let $f$, $f_k \in \mathcal{F}(\alpha, p, \Omega)$ and define the functionals $F$, $F_k : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ by

$$F(u) := \int_\Omega f(x, \nabla u)\,dx \quad \text{and} \quad F_k(u) := \int_\Omega f_k(x, \nabla u)\,dx.$$

We say that the sequence $\{F_k\}$ $\Gamma$-converges to $F$ (with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m)$) if for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ the following conditions are satisfied:

(i) *liminf inequality*: for every sequence $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$,

$$F(u) \leqslant \liminf_{k\to+\infty} F_k(u_k);$$

(ii) *recovery sequence*: there exists a sequence $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and

$$F(u) = \lim_{k\to+\infty} F_k(u_k).$$

In the following remark, we state without proof standard results on $\Gamma$-convergence of integral functionals which can be found in the references given.

**Remark 2.2.**

(i) Thanks to the coercivity condition (2.1), Definition 2.1 coincides with the standard definition given in a topological space (see [19, Proposition 8.10]).
(ii) If $\{f_k\}$ is a sequence in $\mathcal{F}(\alpha,p,\Omega)$, then there exist a subsequence $\{f_{k_j}\}$ and a function $f \in \mathcal{F}(\alpha,p,\Omega)$ such that $\{F_{k_j}\}$ $\Gamma$-converges to $F$ (see [12, Theorem 12.5]).
(iii) If $\{F_k\}$ $\Gamma$-converges to $F$, then $F$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^m)$ and therefore $f$ is quasiconvex with respect to the second variable (see [1, Statement II-5]). Moreover, for every $U \in \mathcal{O}(\Omega)$ and every $u \in W^{1,p}(U;\mathbb{R}^m)$, we have

$$\int_U f(x,\nabla u)\,dx = \Gamma\text{-}\lim_{k\to+\infty} \int_U f_k(x,\nabla u)\,dx.$$

(iv) If $\{F_k\}$ $\Gamma$-converges to $F$, defining $\widetilde{F}_k : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ by

$$\widetilde{F}_k(u) := \int_\Omega Qf_k(x,\nabla u)\,dx,$$

then $\{\widetilde{F}_k\}$ $\Gamma$-converges to $F$ as well, where $Qf_k(x,\cdot)$ denotes the quasiconvex envelope of $f_k(x,\cdot)$.
(v) Given $f$ and $f_k \in \mathcal{F}(\alpha,p,\Omega)$, if the functions $f_k(x,\cdot)$ are quasiconvex for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{m\times n}$

$$f_k(\cdot,\xi) \to f(\cdot,\xi) \quad \text{pointwise a.e. in } \Omega,$$

then $\{F_k\}$ $\Gamma$-converges to $F$ (see [24, Lemma 7 and Remark 15]);
(vi) If $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ is a sequence such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $F_k(u_k) \to F(u)$, then one can assume that $u_k = u$ in a neighborhood of $\partial\Omega$ (see [12, Proposition 11.7]).

The following technical result will be useful in the analysis of the local properties of mixtures in Section 3.

**Lemma 2.3.** *Let $f \in \mathcal{F}(\alpha,p,\Omega)$ be quasiconvex with respect to the second variable. There exists a set $Z \subseteq \Omega$ with $\mathcal{L}^n(Z) = 0$ such that for every $\{\rho_j\} \searrow 0^+$ and every $x_0 \in \Omega \setminus Z$,*

$$\Gamma\text{-}\lim_{j\to+\infty} \int_Q f\big(x_0 + \rho_j y, \nabla u(y)\big)\,dy = \int_Q f\big(x_0, \nabla u(y)\big)\,dy \quad \textit{for all } u \in W^{1,p}\big(Q;\mathbb{R}^m\big).$$

**Proof.** By [3, Lemma 5.38], there exists a $\mathcal{L}^n$-negligible set $Z \subseteq \Omega$ such that for any $x_0 \in \Omega \setminus Z$ and any sequence $\{\rho_j\} \searrow 0^+$, one can find a subsequence $\{\rho_{j_k}\}$ of $\{\rho_j\}$ and a set $E \subseteq Q$ with $\mathcal{L}^n(E) = 0$ such that

$$f(x_0 + \rho_{j_k} y, \xi) \to f(x_0, \xi)$$

for every $\xi \in \mathbb{R}^{m\times n}$ and every $y \in Q \setminus E$. Hence from Remark 2.2(v), one gets that

$$\Gamma\text{-}\lim_{k\to+\infty} \int_Q f\big(x_0 + \rho_{j_k} y, \nabla u(y)\big)\, dy = \int_Q f\big(x_0, \nabla u(y)\big)\, dy, \quad \text{for every } u \in W^{1,p}(Q;\mathbb{R}^m).$$

Since the $\Gamma$-limit does not depend upon the choice of the subsequence, we conclude in light of [19, Proposition 8.3] that the whole sequence $\Gamma$-converges. □

*2.2. Periodic homogenization*

We now recall standard results concerning periodic homogenization. We call *cell integrand* (resp. *homogenized integrand*) associated to $f \in \mathcal{F}(\alpha, p, Q)$, the function $f_{\mathrm{cell}} \in \mathcal{F}(\alpha, p)$ (resp. $f_{\mathrm{hom}} \in \mathcal{F}(\alpha, p)$) defined by

$$f_{\mathrm{cell}}(\xi) := \inf\left\{ \int_Q f(y, \xi + \nabla\phi)\, dy\colon\ \phi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^m)\right\}$$

$$\left(\text{resp. } f_{\mathrm{hom}}(\xi) := \inf_{j\in\mathbb{N}^+} \inf\left\{ -\!\!\!\!\!\!\int_{(0,j)^n} f\big(\langle y\rangle, \xi + \nabla\phi\big)\, dy\colon\ \phi \in W^{1,p}_{\mathrm{per}}\big((0,j)^n;\mathbb{R}^m\big)\right\}\right). \tag{2.2}$$

**Remark 2.4.** If $f$ is quasiconvex in the second variable, then from standard lower semicontinuity results

$$f_{\mathrm{cell}}(\xi) = \min\left\{ \int_Q f(y, \xi + \nabla\phi)\, dy\colon\ \phi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^m)\right\}.$$

The following theorem is well known in the theory of $\Gamma$-convergence (See [12, Section 14], [11] or [31]).

**Theorem 2.5.** *Let $f \in \mathcal{F}(\alpha, p, Q)$ and $\{\varepsilon_k\} \searrow 0^+$, then the functionals $F_k : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$ defined by*

$$F_k(u) := \int_\Omega f\left(\left\langle \frac{x}{\varepsilon_k}\right\rangle, \nabla u\right) dx$$

*$\Gamma$-converges to $F_{\mathrm{hom}} : W^{1,p}(\Omega;\mathbb{R}^m) \to [0,+\infty)$, where*

$$F_{\mathrm{hom}}(u) := \int_\Omega f_{\mathrm{hom}}(\nabla u)\, dx.$$

*In particular $f_{\mathrm{hom}}$ is a quasiconvex function. Under the additional hypothesis that $f$ is convex with respect to the second variable, then $f_{\mathrm{hom}} = f_{\mathrm{cell}}$.*

*2.3. Young measures*

Let $s \in \mathbb{N}$, we denote by $\mathcal{P}(\mathbb{R}^s)$ the space of probability measures in $\mathbb{R}^s$ and by $\mathcal{Y}(\Omega;\mathbb{R}^s)$ the space of maps $\mu : x \in \Omega \mapsto \mu_x \in \mathcal{P}(\mathbb{R}^s)$ such that the function $x \mapsto \mu_x(B)$ is Lebesgue measurable for every Borel set $B \subseteq \mathbb{R}^s$ (see [3, Definition 2.25]).

The following result is a version of the Fundamental Theorem on Young Measures (a proof can be found in [5,32]). Under suitable hypothesis, it shows that the weak limit of a sequence of the type $\{f(\cdot, w_k(\cdot))\}$ can be expressed through a suitable map $\mu \in \mathcal{Y}(\Omega;\mathbb{R}^s)$ associated to $\{w_k\}$.

**Theorem 2.6.** *Let $\{w_k\}$ be a bounded sequence in $L^1(\Omega;\mathbb{R}^s)$. There exist a subsequence $\{w_{k_j}\}$ and a map $\mu \in \mathcal{Y}(\Omega;\mathbb{R}^s)$ such that the following properties hold*:

(i) *if $f : \Omega \times \mathbb{R}^s \to [0,+\infty)$ is a Carathéodory function, then*

$$\liminf_{j\to+\infty} \int_\Omega f\big(x, w_{k_j}(x)\big)\, dx \geqslant \int_\Omega \bar f(x)\, dx$$

*where*

$$\bar{f}(x) := \int_{\mathbb{R}^s} f(x,\xi)\,d\mu_x(\xi);$$

(ii) *if* $f : \Omega \times \mathbb{R}^s \to \mathbb{R}$ *is a Carathéodory function and* $\{f(\cdot, w_k(\cdot))\}$ *is equi-integrable, then* $f(x,\cdot)$ *is* $\mu_x$*-integrable for a.e.* $x \in \Omega$, $\bar{f} \in L^1(\Omega)$ *and*

$$\lim_{j\to+\infty} \int_\Omega f\big(x, w_{k_j}(x)\big)\,dx = \int_\Omega \bar{f}(x)\,dx;$$

(iii) *if* $\mathfrak{A} \subseteq \mathbb{R}^s$ *is compact, then* $\operatorname{supp}\mu_x \subseteq \mathfrak{A}$ *for a.e.* $x \in \Omega$ *if and only if* $\operatorname{dist}(w_{k_j}, \mathfrak{A}) \to 0$ *in measure.*

**Definition 2.7.** The map $\mu \in \mathcal{Y}(\Omega;\mathbb{R}^s)$ is called the *Young measure* generated by $\{w_{k_j}\}$.

**Remark 2.8.** We denote by $\bar{\mu}_x := \int_{\mathbb{R}^s} \xi\,d\mu_x$ the *barycenter* of $\mu$. If the sequence $\{w_k\}$ is equi-integrable and generates $\mu$, then $w_k \rightharpoonup \bar{\mu}$ in $L^1(\Omega;\mathbb{R}^s)$. To see this, it is sufficient to apply Theorem 2.6(ii) with $f(x,\xi) = \varphi(x)\xi^{(i)}$ where $\varphi \in L^\infty(\Omega)$ and $\xi^{(i)}$ is the $i$th component of $\xi$ ($i \in \{1,\dots,s\}$).

In the sequel we are interested in Young measures generated by sequences of gradients. We refer the reader to [32,34] and references therein for a more exhaustive study.

**Definition 2.9.** A map $\nu \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ is called *gradient Young measure* if there exists a bounded sequence $\{u_k\}$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\{\nabla u_k\}$ generates $\nu$.

A probability measure $\sigma \in \mathcal{P}(\mathbb{R}^{m\times n})$ is called *homogeneous gradient Young measure* if there exists a gradient Young measure $\nu$ such that $\sigma = \nu_x$ for a.e. $x \in \Omega$.

**Remark 2.10.** If $\nu$ is a gradient Young measure, then for a.e. $x \in \Omega$ the measure $\nu_x$ is a homogeneous gradient Young measure (see [34, Theorem 8.4]).

*2.4. Two-scale convergence*

We here briefly gather the definition and some of the main results about two-scale convergence.

**Definition 2.11.** Let $\{w_k\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^s)$ and $\{\varepsilon_k\} \searrow 0^+$. We say that $\{w_k\}$ two-scale converges to a function $w = w(x,y) \in L^p(\Omega \times Q;\mathbb{R}^s)$ (with respect to $\{\varepsilon_k\}$) if

$$\lim_{k\to+\infty} \int_\Omega \varphi(x)\phi\left(\frac{x}{\varepsilon_k}\right) w_k(x)\,dx = \int_{\Omega\times Q} \varphi(x)\phi(y)w(x,y)\,dx\,dy$$

for any $\varphi \in C_c^\infty(\Omega)$ and any $\phi \in C_{\mathrm{per}}^\infty(Q)$; we simply write $w_k \rightsquigarrow w$.

**Remark 2.12.**

(i) Every bounded sequence in $L^p(\Omega;\mathbb{R}^s)$ admits a two-scale converging subsequence (see [27, Theorem 7]).

(ii) Let $f : Q \times \mathbb{R}^s \to [0,+\infty)$ be a Carathéodory function convex with respect to the second variable and $\{\varepsilon_k\} \searrow 0^+$. If $w_k \rightsquigarrow w$ (with respect to $\{\varepsilon_k\}$), then

$$\liminf_{k\to+\infty} \int_\Omega f\left(\left\langle \frac{x}{\varepsilon_k}\right\rangle, w_k(x)\right)dx \geqslant \int_{\Omega\times Q} f\big(y, w(x,y)\big)\,dx\,dy.$$

This result is a direct consequence of [8, Theorem 4.14] together with Jensen's Inequality.

(iii) Let $\{u_k\}$ be a sequence weakly converging in $W^{1,p}(\Omega;\mathbb{R}^m)$ to a function $u$. Then $u_k \rightsquigarrow u$ and there exist a subsequence (not relabeled) and $v \in L^p(\Omega; W_{\mathrm{per}}^{1,p}(Q;\mathbb{R}^m))$ such that $\nabla u_k \rightsquigarrow \nabla u + \nabla_y v$ (see [27, Theorem 13]).

## 3. Local properties of mixtures

Let $W^{(1)},\dots,W^{(N)}$ belong to $\mathcal{F}(\alpha,p)$. We interpret these functions as the stored energy densities of $N$ different nonlinearly elastic materials and we are interested in mixtures between them.

**Definition 3.1.** Let $\mathcal{X}(\Omega)$ be the family of all functions $\chi=(\chi^{(1)},\dots,\chi^{(N)})\in L^\infty(\Omega;\{0,1\}^N)$ such that $\sum_{i=1}^N\chi^{(i)}(x)=1$ in $\Omega$. Equivalently, $\chi\in\mathcal{X}(\Omega)$ if there exists a measurable partition $\{P^{(i)}\}_{i=1,\dots,N}$ of $\Omega$ such that $\chi^{(i)}=\chi_{P^{(i)}}$ for $i=1,\dots,N$.

If $\chi\in\mathcal{X}(\Omega)$, we define $W_\chi\in\mathcal{F}(\alpha,p,\Omega)$ by

$$W_\chi(x,\xi):=\sum_{i=1}^N\chi^{(i)}(x)W^{(i)}(\xi),\quad \text{for every } x\in\Omega \text{ and } \xi\in\mathbb{R}^{m\times n}$$

and the functional $F_\chi:W^{1,p}(\Omega;\mathbb{R}^m)\to[0,+\infty)$ by

$$F_\chi(u):=\int_\Omega W_\chi(x,\nabla u)\,dx.$$

The function $W_\chi$ can be thought of as the stored energy density of a composite material obtained by mixing $N$ components having energy density $W^{(i)}$ and occupying the reference configuration $P^{(i)}$ in $\Omega$ at rest. The elastic energy of this composite material under a deformation $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ is given by $F_\chi(u)$.

To describe efficiently the complexity of a composite material at the level of the effective energy, it is convenient to identify it with the ($\Gamma$-)limit of a sequence of mixtures. Let $\{\chi_k\}$ be a sequence in $\mathcal{X}(\Omega)$, we say that $W^*\in\mathcal{F}(\alpha,p,\Omega)$ is the *effective energy density* associated to $\{(W^{(i)},\chi_k^{(i)})\}_{i=1,\dots,N}$ if the functional $F:W^{1,p}(\Omega;\mathbb{R}^m)\to[0,+\infty)$ defined by

$$F(u):=\int_\Omega W^*(x,\nabla u)\,dx$$

is the $\Gamma$-limit of the sequence $\{F_{\chi_k}\}$. The sequence $\{\chi_k\}$ is referred as the *microstructure* (or micro-geometry) of $W^*$. If the effective energy density does not depend on $x$ we say that it is *homogeneous* while if, for some $\theta=(\theta^{(1)},\dots,\theta^{(N)})\in[0,1]^N$ with $\sum_{i=1}^N\theta^{(i)}=1$, the equality

$$⨍_\Omega\chi_k\,dx=\theta$$

holds for all $k\in\mathbb{N}$, we say that the micro-geometry $\{\chi_k\}$ has *fixed volume fractions* $\theta^{(1)},\dots,\theta^{(N)}$.

The next lemma shows that every effective energy density can be generated by a micro-geometry with fixed volume fractions.

**Lemma 3.2.** *Let $\{\chi_k\}$ be a sequence in $\mathcal{X}(\Omega)$ and let $W^*\in\mathcal{F}(\alpha,p,\Omega)$ be the effective energy density associated to $\{(W^{(i)},\chi_k^{(i)})\}_{i=1,\dots,N}$. Suppose that*

$$\chi_k\overset{*}{\rightharpoonup}\theta\quad\textit{in } L^\infty\big(\Omega;[0,1]^N\big)$$

*and set $\bar\theta:=⨍_\Omega\theta\,dx$. Then there exists another sequence $\{\tilde\chi_k\}$ in $\mathcal{X}(\Omega)$ satisfying*

$$\tilde\chi_k\overset{*}{\rightharpoonup}\theta\quad\textit{in } L^\infty\big(\Omega;[0,1]^N\big),\qquad ⨍_\Omega\tilde\chi_k\,dx=\bar\theta\quad\textit{for all } k\in\mathbb{N}$$

*and such that $W^*$ is the effective energy density associated to $\{(W^{(i)},\tilde\chi_k^{(i)})\}_{i=1,\dots,N}$.*

**Proof.** For every $k\in\mathbb{N}$, let $\theta_k:=(\theta_k^{(1)},\dots,\theta_k^{(N)})$ where

$$\theta_k^{(i)}:=⨍_\Omega\chi_k^{(i)}\,dx,\quad \text{for every } i\in\{1,\dots,N\}.$$

Let $I_k^1$, $I_k^2$ and $I_k^3$ be three disjoint subsets of indexes in $\{1,\dots,N\}$ such that

$$\begin{cases} \bar\theta^{(i)} > \theta_k^{(i)} & \text{if } i \in I_k^1, \\ \bar\theta^{(i)} = \theta_k^{(i)} & \text{if } i \in I_k^2, \\ \bar\theta^{(i)} < \theta_k^{(i)} & \text{if } i \in I_k^3. \end{cases}$$

We denote $P_k^{(i)} := \{\chi_k^{(i)} = 1\}$ so that $\theta_k^{(i)} = \mathcal{L}^n(P_k^{(i)})/\mathcal{L}^n(\Omega)$. For every $i \in I_k^2$, we define $\tilde P_k^{(i)} := P_k^{(i)}$ and $\tilde\chi_k^{(i)} := \chi_{\tilde P_k^{(i)}}$. Let $R > 0$ be such that $\Omega \subseteq B_R(0)$. Fix $i \in I_k^3$ and consider the function $\ell_k(\rho) := \mathcal{L}^n(B_\rho(0) \cap P_k^{(i)})/\mathcal{L}^n(\Omega)$; $\ell_k$ is a continuous and nondecreasing function satisfying $\ell_k(0) = 0$ and $\ell_k(R) = \theta_k^{(i)}$. Hence one can find a radius $\rho_k \in (0, R)$ such that $\ell_k(\rho_k) = \bar\theta^{(i)}$ and, for any $i \in I_k^3$, we set $\tilde P_k^{(i)} := P_k^{(i)} \cap B_{\rho_k}(0)$ and $\tilde\chi_k^{(i)} := \chi_{\tilde P_k^{(i)}}$. It remains to treat the indexes $i \in I_k^1$. Set

$$Z_k := \bigcup_{i \in I_k^3} P_k^{(i)} \setminus \tilde P_k^{(i)};$$

Since

$$\sum_{i=1}^N \bar\theta^{(i)} = \sum_{i=1}^N \theta_k^{(i)} = 1$$

it follows that

$$\mathcal{L}^n(Z_k) = \sum_{i \in I_k^3} \mathcal{L}^n(\Omega)\big(\theta_k^{(i)} - \bar\theta^{(i)}\big) = \sum_{i \in I_k^1} \mathcal{L}^n(\Omega)\big(\bar\theta^{(i)} - \theta_k^{(i)}\big).$$

Hence the measure of the set $Z_k$ (that we have removed) is equal to the sum of the measures of the sets that we need to add to each $P_k^{(i)}$ (for $i \in I_k^1$) to get a larger set $\tilde P_k^{(i)}$ with Lebesgue measure $\bar\theta^{(i)}\mathcal{L}^n(\Omega)$. Thus we have enough room to find disjoint measurable sets $\{Z_k^{(i)}\}_{i \in I_k^1} \subseteq Z_k$ such that $\mathcal{L}^n(Z_k^{(i)}) = \mathcal{L}^n(\Omega)(\bar\theta^{(i)} - \theta_k^{(i)})$ and it suffices to define $\tilde P_k^{(i)} := P_k^{(i)} \cup Z_k^{(i)}$ and $\tilde\chi_k^{(i)} := \chi_{\tilde P_k^{(i)}}$ for all $i \in I_k^1$. As

$$\begin{aligned} \mathcal{L}^n\left(\bigcup_{i=1}^N \big(P_k^{(i)} \triangle \tilde P_k^{(i)}\big)\right) &= \sum_{i \in I_k^1} \mathcal{L}^n(\Omega)\big(\bar\theta^{(i)} - \theta_k^{(i)}\big) + \sum_{i \in I_k^3} \mathcal{L}^n(\Omega)\big(\theta_k^{(i)} - \bar\theta^{(i)}\big) \\ &= \mathcal{L}^n(\Omega) \sum_{i=1}^N \big|\bar\theta^{(i)} - \theta_k^{(i)}\big| \xrightarrow[k \to +\infty]{} 0, \end{aligned} \tag{3.1}$$

one immediately gets that $\tilde\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega; [0,1]^N)$.

It remains to show that the $\Gamma$-limit is unchanged upon replacing $\chi_k$ by $\tilde\chi_k$. Extracting a subsequence (not relabeled) if necessary, one may assume without loss of generality (see Remark 2.2(ii–iii)) that there exists a function $\tilde W \in \mathcal{F}(\alpha, p, \Omega)$ such that

$$\int_U \tilde W(x, \nabla u)\, dx = \Gamma\text{-}\lim_{k \to +\infty} \int_U W_{\tilde\chi_k}(x, \nabla u)\, dx,$$

for every $U \in \mathcal{O}(\Omega)$ and every $u \in W^{1,p}(U; \mathbb{R}^m)$. Let us show that $\tilde W(x, \xi) = W^*(x, \xi)$ for all $\xi \in \mathbb{R}^{m \times n}$ and a.e. $x \in \Omega$. Fix $\xi \in \mathbb{R}^{m \times n}$ and let $\{u_k\} \subset W^{1,p}(U; \mathbb{R}^m)$ be a sequence such that $u_k \rightharpoonup \xi\cdot$ in $W^{1,p}(U; \mathbb{R}^m)$ and

$$\int_U \tilde W(x, \xi)\, dx = \lim_{k \to +\infty} \int_U W_{\tilde\chi_k}(x, \nabla u_k)\, dx.$$

Thanks to the Decomposition Lemma (see [22, Lemma 1.2]) one can assume without loss of generality that the sequence $\{\nabla u_k\}$ is $p$-equi-integrable. By the $p$-growth condition (2.1) and (3.1), we get that

$$\int_U \left| W_{\tilde\chi_k}(x, \nabla u_k) - W_{\chi_k}(x, \nabla u_k)\right| dx \leqslant 2\alpha_3 \int_{U\cap\bigcup_{i=1}^N (P_k^{(i)} \triangle \tilde P_k^{(i)})} \left(1 + |\nabla u_k|^p\right) dx \to 0.$$

Hence

$$\int_U \tilde W(x, \xi)\, dx = \lim_{k\to+\infty} \int_U W_{\chi_k}(x, \nabla u_k)\, dx \geqslant \int_U W^*(x, \xi)\, dx$$

and the opposite inequality follows from a similar argument. Since the equality

$$\int_U \tilde W(x, \xi)\, dx = \int_U W^*(x, \xi)\, dx$$

holds for every open subset $U$ of $\Omega$, we deduce that $\tilde W(x, \xi) = W^*(x, \xi)$ for a.e. $x \in \Omega$. $\square$

In order to state the localization result for mixtures, namely Theorem 3.5, it will be more convenient to introduce the set of functions $\mathcal{G}_\theta(\Omega)$ which is made of all possible homogeneous energy densities obtained as $\Gamma$-limits of mixtures of $W^{(1)}, \ldots, W^{(N)}$ and having asymptotically homogeneous microstructure.

**Definition 3.3.** Let $\theta = (\theta^{(1)}, \ldots, \theta^{(N)}) \in [0, 1]^N$ be such that $\sum_{i=1}^N \theta^{(i)} = 1$. We define $\mathcal{G}_\theta(\Omega)$ as the set of all functions $W^* \in \mathcal{F}(\alpha, p)$ for which there exists a sequence $\{\chi_k\}$ in $\mathcal{X}(\Omega)$ with the properties that

$$\chi_k \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty\big(\Omega; [0, 1]^N\big)$$

and that $W^*$ is the effective energy density associated to $\{(W^{(i)}, \chi_k^{(i)})\}_{i=1,\ldots,N}$.

**Lemma 3.4.** *The set $\mathcal{G}_\theta(\Omega)$ is independent of the open set $\Omega \subseteq \mathbb{R}^n$.*

**Proof.** Consider $\Omega$ and $\Omega'$ two arbitrary open subsets of $\mathbb{R}^n$ and assume that $W^* \in \mathcal{G}_\theta(\Omega)$. Let us show that it belongs to $\mathcal{G}_\theta(\Omega')$ too. From the definition of $\mathcal{G}_\theta(\Omega)$, there exists a sequence of characteristic functions $\{\chi_k\}$ in $\mathcal{X}(\Omega)$ such that $\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega; [0, 1]^N)$ and

$$\Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_{\chi_k}(x, \nabla u)\, dx = \int_\Omega W^*(\nabla u)\, dx.$$

Let $a \in \Omega$ and $\rho > 0$ be such that $a+\rho\Omega' \subseteq \Omega$. It turns out that $\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(a+\rho\Omega'; [0, 1]^N)$ and, by a localization argument (see Remark 2.2(iii)),

$$\Gamma\text{-}\lim_{k\to+\infty} \int_{a+\rho\Omega'} W_{\chi_k}(x, \nabla u)\, dx = \int_{a+\rho\Omega'} W^*(\nabla u)\, dx.$$

Define $\tilde\chi_k(x) := \chi_k(a + \rho x)$ for all $x \in \Omega'$. Since $\theta$ is constant, it is easily seen that $\tilde\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega'; [0, 1]^N)$. Moreover a simple change of variables together with the fact that $W^*$ does not depend on $x \in \Omega$ implies that

$$\Gamma\text{-}\lim_{k\to+\infty} \int_{\Omega'} W_{\tilde\chi_k}(x, \nabla u)\, dx = \int_{\Omega'} W^*(\nabla u)\, dx. \qquad \square$$

Since $\mathcal{G}_\theta(\Omega)$ is independent of $\Omega$, we simply denote it from now on by $\mathcal{G}_\theta$. We next show that every effective energy density can be locally seen as homogeneous. Note that this result is independent of any kind of convexity assumption on the densities $W^{(1)}, \ldots, W^{(N)}$.

**Theorem 3.5.** *Let $W^{(1)}, \dots, W^{(N)} \in \mathcal{F}(\alpha, p)$ and $\theta \in L^\infty(\Omega; [0,1]^N)$ be such that $\sum_{i=1}^N \theta^{(i)}(x) = 1$ a.e. in $\Omega$. Given $W^* \in \mathcal{F}(\alpha, p, \Omega)$, the following conditions are equivalent*:

(i) *there exists a sequence $\{\chi_k\}$ in $\mathcal{X}(\Omega)$ such that $\chi_k \xrightarrow{*} \theta$ in $L^\infty(\Omega; [0,1]^N)$ and $W^*$ is the effective energy density associated to $\{(W^{(i)}, \chi_k^{(i)})\}_{i=1,\dots,N}$*;
(ii) *$W^*(x, \cdot) \in \mathcal{G}_{\theta(x)}$ for a.e. $x \in \Omega$.*

**Proof.** Thanks to Remark 2.2(iv) we can suppose without loss of generality that the functions $W^{(1)}, \dots, W^{(N)}$ are quasiconvex.

(i)⇒(ii). Let $\{\rho_j\} \searrow 0^+$ and fix a point $x_0 \in \Omega \setminus Z$, where $Z \subseteq \Omega$ is the set of Lebesgue measure zero given by Lemma 2.3 (with $f = W^*$), which is also a Lebesgue point of $\theta$. Define $\chi_{j,k}(y) := \chi_k(x_0 + \rho_j y)$, by Lemma 2.3 we have that

$$\begin{aligned}
\Gamma\text{-}\lim_{j\to+\infty}\left(\Gamma\text{-}\lim_{k\to+\infty}\int_Q W_{\chi_{j,k}}\big(y, \nabla u(y)\big)\,dy\right) &= \Gamma\text{-}\lim_{j\to+\infty}\left(\Gamma\text{-}\lim_{k\to+\infty}\int_Q W_{\chi_k}\big(x_0 + \rho_j y, \nabla u(y)\big)\,dy\right) \\
&= \Gamma\text{-}\lim_{j\to+\infty}\int_Q W^*\big(x_0 + \rho_j y, \nabla u(y)\big)\,dy \\
&= \int_Q W^*\big(x_0, \nabla u(y)\big)\,dy.
\end{aligned} \tag{3.2}$$

Moreover, for every $\varphi \in C_c(Q)$ and any $i \in \{1, \dots, N\}$,

$$\begin{aligned}
\lim_{j\to+\infty}\lim_{k\to+\infty}\left|\int_Q \big(\chi_{j,k}^{(i)}(y) - \theta^{(i)}(x_0)\big)\varphi(y)\,dy\right| &= \lim_{j\to+\infty}\lim_{k\to+\infty}\left|\fint_{Q_{\rho_j}(x_0)} \big(\chi_k^{(i)}(y) - \theta^{(i)}(x_0)\big)\varphi\left(\frac{y - x_0}{\rho_j}\right)dy\right| \\
&= \lim_{j\to+\infty}\left|\fint_{Q_{\rho_j}(x_0)} \big(\theta^{(i)}(y) - \theta^{(i)}(x_0)\big)\varphi\left(\frac{y - x_0}{\rho_j}\right)dy\right| \\
&\leqslant \lim_{j\to+\infty}\|\varphi\|_{L^\infty(Q)}\fint_{Q_{\rho_j}(x_0)} \big|\theta^{(i)}(y) - \theta^{(i)}(x_0)\big|\,dy = 0
\end{aligned} \tag{3.3}$$

since $x_0$ is a Lebesgue point of $\theta$. By density, (3.3) remains true for any $\varphi \in L^1(Q)$. By (3.2), (3.3), the metrizable character of $\Gamma$-convergence for lower semicontinuous and coercive functionals on $W^{1,p}(Q; \mathbb{R}^m)$ [19, Corollary 10.22(a)] and the metrizability of the weak* convergence in $L^\infty(Q; [0,1]^N)$, we deduce through a standard diagonalization argument the existence of a sequence $k_j \nearrow +\infty$ as $j \to +\infty$ (possibly depending on $x_0$) such that $\tilde\chi_j := \chi_{j,k_j} \xrightarrow{*} \theta(x_0)$ in $L^\infty(Q; [0,1]^N)$ and

$$\Gamma\text{-}\lim_{j\to+\infty}\int_Q W_{\tilde\chi_j}\big(y, \nabla u(y)\big)\,dy = \int_Q W^*\big(x_0, \nabla u(y)\big)\,dy.$$

(ii)⇒(i). By the Lusin and the Scorza–Dragoni Theorems (see [20]), for every $k \in \mathbb{N}$, there exists a compact set $K_k \subseteq \Omega$ satisfying $\mathcal{L}^n(\Omega \setminus K_k) < 1/k$ and such that $\theta^{(1)}, \dots, \theta^{(N)}$ are continuous on $K_k$ and $W^*$ is continuous on $K_k \times \mathbb{R}^{m\times n}$. Let $h \in \mathbb{N}$, split $\Omega$ into $h$ disjoint open sets $U_{r,h}$ such that $\delta_h := \max_{1\leqslant r\leqslant h} \operatorname{diam}(U_{r,h}) \to 0$ as $h \to +\infty$ and set

$$I_{h,k} := \big\{r \in \{1, \dots, h\}\colon\ \mathcal{L}^n(K_k \cap U_{r,h}) > 0\big\}.$$

Hence for each $r \in I_{h,k}$, one can find a point $x^k_{r,h} \in K_k \cap U_{r,h}$ such that $W^*(x^k_{r,h}, \cdot) \in \mathcal{G}_{\theta(x^k_{r,h})}$. As a consequence of Lemma 3.4, there exist sequences $\{\chi^{r,h,k}_j\} = \{((\chi^{r,h,k}_j)^{(1)}, \dots, (\chi^{r,h,k}_j)^{(N)})\}$ in $\mathcal{X}(U_{r,h})$ such that $\chi^{r,h,k}_j \overset{*}{\rightharpoonup} \theta(x^k_{r,h})$ in $L^\infty(U_{r,h}; [0,1]^N)$ as $j \to +\infty$ and

$$\Gamma\text{-}\lim_{j\to+\infty} \int_{U_{r,h}} W_{\chi^{r,h,k}_j}(x, \nabla u)\,dx = \int_{U_{r,h}} W^*\big(x^k_{r,h}, \nabla u\big)\,dx. \tag{3.4}$$

Define now

$$W^*_{h,k}(x,\xi) := \sum_{r\in I_{h,k}} \chi_{U_{r,h}}(x) W^*\big(x^k_{r,h}, \xi\big) + \chi_{U^c_{h,k}}(x) W^*(x,\xi)$$

and

$$\begin{cases} (\chi^{h,k}_j)^{(1)}(x) := \displaystyle\sum_{r\in I_{h,k}} (\chi^{r,h,k}_j)^{(1)}(x)\chi_{U_{r,h}}(x) + \chi_{U^c_{h,k}}(x), \\ (\chi^{h,k}_j)^{(i)}(x) := \displaystyle\sum_{r\in I_{h,k}} (\chi^{r,h,k}_j)^{(i)}(x)\chi_{U_{r,h}}(x), \quad i \in \{2,\dots,N\}, \end{cases}$$

where $U^c_{h,k} := \Omega \setminus \bigcup_{r\in I_{h,k}} U_{r,h}$. Then $\chi^{h,k}_j := ((\chi^{h,k}_j)^{(1)}, \dots, (\chi^{h,k}_j)^{(N)}) \in \mathcal{X}(\Omega)$ and for every $\varphi \in L^1(\Omega)$

$$\begin{aligned} \lim_{j\to+\infty} \int_\Omega \big(\chi^{h,k}_j\big)^{(1)}(x)\varphi(x)\,dx &= \lim_{j\to+\infty} \sum_{r\in I_{h,k}} \int_{U_{r,h}} \big(\chi^{r,h,k}_j\big)^{(1)}(x)\varphi(x)\,dx + \int_{U^c_{h,k}} \varphi(x)\,dx \\ &= \sum_{r\in I_{h,k}} \int_{U_{r,h}} \theta^{(1)}\big(x^k_{r,h}\big)\varphi(x)\,dx + \int_{U^c_{h,k}} \varphi(x)\,dx, \end{aligned} \tag{3.5}$$

while for each $i \in \{2,\dots,N\}$

$$\lim_{j\to+\infty} \int_\Omega \big(\chi^{h,k}_j\big)^{(i)}(x)\varphi(x)\,dx = \sum_{r\in I_{h,k}} \int_{U_{r,h}} \theta^{(i)}\big(x^k_{r,h}\big)\varphi(x)\,dx. \tag{3.6}$$

On the other hand, since $\|\theta^{(i)}\|_{L^\infty(\Omega)} \leqslant 1$ we get that for any $i \in \{1,\dots,N\}$

$$\begin{aligned} \sum_{r\in I_{h,k}} \int_{U_{r,h}} \big|\theta^{(i)}\big(x^k_{r,h}\big) - \theta^{(i)}(x)\big|\big|\varphi(x)\big|\,dx &\leqslant \sum_{r\in I_{h,k}} \int_{U_{r,h}\cap K_k} \big|\theta^{(i)}\big(x^k_{r,h}\big) - \theta^{(i)}(x)\big|\big|\varphi(x)\big|\,dx + 2\int_{\Omega\setminus K_k} |\varphi|\,dx \\ &\leqslant \omega^{(i)}_k(\delta_h) \int_\Omega |\varphi|\,dx + 2\int_{\Omega\setminus K_k} |\varphi|\,dx, \end{aligned} \tag{3.7}$$

where $\omega^{(i)}_k : [0,+\infty) \to [0,+\infty)$ is the modulus of continuity of $\theta^{(i)}$ on $K_k$. Since $\theta^{(i)}$ is uniformly continuous on $K_k$, it follows that $\omega^{(i)}_k$ is a continuous and nondecreasing function satisfying $\omega^{(i)}_k(0) = 0$. Hence taking first the limit as $h \to +\infty$ and then as $k \to +\infty$, we deduce from (3.5)–(3.7) and the fact that $\mathcal{L}^n(U^c_{h,k}) \leqslant \mathcal{L}^n(\Omega \setminus K_k) \to 0$ (as $k \to +\infty$ uniformly with respect to $h \in \mathbb{N}$), that

$$\lim_{k\to+\infty} \lim_{h\to+\infty} \lim_{j\to+\infty} \int_\Omega \big(\chi^{h,k}_j\big)^{(i)}(x)\varphi(x)\,dx = \int_\Omega \theta^{(i)}(x)\varphi(x)\,dx. \tag{3.8}$$

Moreover, by virtue of the uniform continuity of $W^*$ on $K_k \times B_R(0)$ for every $k \in \mathbb{N}$ and $R > 0$, we get that

$$\begin{aligned} \int_\Omega \sup_{|\xi|\leqslant R} \big|W^*_{h,k}(x,\xi) - W^*(x,\xi)\big|\,dx &\leqslant \sum_{r\in I_{h,k}} \int_{U_{r,h}\cap K_k} \sup_{|\xi|\leqslant R} \big|W^*\big(x^k_{r,h},\xi\big) - W^*(x,\xi)\big|\,dx \\ &\quad + 2\alpha_3\big(1+R^p\big)\mathcal{L}^n(\Omega\setminus K_k) \\ &\leqslant \omega_{k,R}(\delta_h)\mathcal{L}^n(\Omega) + 2\alpha_3\big(1+R^p\big)\mathcal{L}^n(\Omega\setminus K_k) \end{aligned}$$

where $\omega_{k,R} : [0,+\infty) \to [0,+\infty)$ is the modulus of continuity of $W^*$ on $K_k \times B_R(0)$ which is a continuous and nondecreasing function satisfying $\omega_{k,R}(0) = 0$. Hence taking the limit as $h \to +\infty$ and then as $k \to +\infty$ we obtain that for every $R > 0$,

$$\lim_{k\to+\infty} \lim_{h\to+\infty} \int_\Omega \sup_{|\xi|\leqslant R} \left| W^*_{h,k}(x,\xi) - W^*(x,\xi) \right| dx = 0. \tag{3.9}$$

By (3.8), (3.9) and the fact that $L^1(\Omega)$ is separable, one can apply a standard diagonalization technique to get the existence of a sequence $h_k \nearrow +\infty$ as $k \to +\infty$ such that, setting $\tilde\chi^k_j := \chi^{h_k,k}_j$ and $W^*_k := W^*_{h_k,k}$, then for any $i \in \{1,\dots,N\}$,

$$\lim_{k\to+\infty} \lim_{j\to+\infty} \int_\Omega \left(\tilde\chi^k_j\right)^{(i)}(x)\varphi(x)\,dx = \int_\Omega \theta^{(i)}(x)\varphi(x)\,dx. \tag{3.10}$$

and for every $R > 0$

$$\lim_{k\to+\infty} \int_\Omega \sup_{|\xi|\leqslant R} \left| W^*_k(x,\xi) - W^*(x,\xi) \right| dx = 0. \tag{3.11}$$

In particular, (3.11) shows that (at least for a not relabeled subsequence) for every $\xi \in \mathbb{R}^{m\times n}$, $W^*_k(\cdot,\xi) \to W^*(\cdot,\xi)$ pointwise a.e. in $\Omega$ and by Remark 2.2(v), we obtain that

$$\Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W^*_k(x,\nabla u)\,dx = \int_\Omega W^*(x,\nabla u)\,dx.$$

Furthermore, thanks to (3.4), one can show that

$$\Gamma\text{-}\lim_{j\to+\infty} \int_\Omega W_{\tilde\chi^k_j}(x,\nabla u)\,dx = \int_\Omega W^*_k(x,\nabla u)\,dx.$$

Indeed, the lower bound is immediate while the construction of a recovery sequence can be completed using the fact that, inside each $U_{r,h_k}$, there exists an optimal sequence which matches $u$ on a neighborhood of $\partial U_{r,h_k}$ (see Remark 2.2(vi)). In this way, we can glue continuously each pieces on $U_{r,h_k}$ and extend by $u$ on the whole set $\Omega$ to construct an optimal sequence. Thus we have that

$$\Gamma\text{-}\lim_{k\to+\infty} \left( \Gamma\text{-}\lim_{j\to+\infty} \int_\Omega W_{\tilde\chi^k_j}(x,\nabla u)\,dx \right) = \int_\Omega W^*(x,\nabla u)\,dx. \tag{3.12}$$

Appealing once again to the metrizability of $\Gamma$-convergence of lower semicontinuous and coercive functionals on $W^{1,p}(\Omega;\mathbb{R}^m)$ and the metrizability of the weak* convergence in $L^\infty(\Omega;[0,1]^N)$, from (3.10) and (3.12) we obtain, by a diagonalization process, the existence of a sequence $j_k \nearrow +\infty$ as $k \to +\infty$ such that, upon setting $\tilde\chi_k := \tilde\chi^k_{j_k}$, then $\tilde\chi_k \rightharpoonup \theta$ in $L^\infty(\Omega;[0,1]^N)$ and

$$\Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_{\tilde\chi_k}(x,\nabla u)\,dx = \int_\Omega W^*(x,\nabla u)\,dx,$$

which completes the proof of the theorem. □

## 4. Characterization of zero level sets

### 4.1. *Effective energy densities*

The first aim of this section is to study the zero level set of an effective energy density. This will be done by means of gradient Young measures. As an application to the $G$-closure problem in the convex case, one will show thanks to this result that they may exist effective energy densities that cannot be exactly reached by a periodic microstructure (Example 5.2).

**Theorem 4.1.** *Let $W^{(1)},\dots,W^{(N)}\in\mathcal{F}(\alpha,p)$ and suppose that the compact sets $\mathfrak{A}^{(i)}:=\{\xi\in\mathbb{R}^{m\times n}\colon W^{(i)}(\xi)=0\}$ are pairwise disjoint. If $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\theta\in L^\infty(\Omega;[0,1]^N)$ with $\sum_{i=1}^N\theta^{(i)}(x)=1$ a.e. in $\Omega$, then the following conditions are equivalent*:

(i) *there exists a gradient Young measure $\nu\in\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that*

$$\begin{cases}\operatorname{supp}\nu_x\subseteq\mathfrak{A}:=\displaystyle\bigcup_{i=1}^N\mathfrak{A}^{(i)} & \textit{for a.e. } x\in\Omega,\\ \nu_x(\mathfrak{A}^{(i)})=\theta^{(i)}(x) & \textit{for all } i\in\{1,\dots,N\} \textit{ and a.e. } x\in\Omega,\\ \bar\nu_x=\nabla u(x) & \textit{for a.e. } x\in\Omega;\end{cases}$$

(ii) *there exists a micro-geometry $\{\chi_k\}$ in $\mathcal{X}(\Omega)$ such that, denoting by $W^*\in\mathcal{F}(\alpha,p,\Omega)$ the effective energy density associated to $\{(W^{(i)},\chi_k^{(i)})\}_{i=1,\dots,N}$, then*

$$\begin{cases}\displaystyle\fint_\Omega\chi_k\,dx=\fint_\Omega\theta\,dx & \textit{for all } k\in\mathbb{N},\\ \chi_k\overset{*}{\rightharpoonup}\theta & \textit{in } L^\infty(\Omega;[0,1]^N),\\ W^*(x,\nabla u(x))=0 & \textit{for a.e. } x\in\Omega.\end{cases}$$

**Proof. (i)⇒(ii)**. For sake of clarity, we divide the proof into four steps.

**Step 1.** By using Theorem 2.6(iii) and the Decomposition Lemma [22, Lemma 1.2], we can suppose that there exists a sequence $\{u_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_k\rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, $\{\nabla u_k\}$ is $p$-equi-integrable, generates $\nu$ and $\operatorname{dist}(\nabla u_k,\mathfrak{A})\to 0$ in measure. For any $i\in\{1,\dots,N\}$, we define the Borel subsets of $\mathbb{R}^{m\times n}$

$$\mathfrak{C}^{(i)}:=\{\xi\in\mathbb{R}^{m\times n}\colon \operatorname{dist}(\xi,\mathfrak{A}^{(i)})\leqslant\operatorname{dist}(\xi,\mathfrak{A}^{(h)}) \text{ for } h\neq i\}$$

and, for every fixed $k\in\mathbb{N}$, the following measurable partitions $\{P_k^{(i)}\}_{i=1,\dots,N}$ of $\Omega$

$$\begin{aligned}P_k^{(1)}&:=\{x\in\Omega\colon \nabla u_k(x)\in\mathfrak{C}^{(1)}\},\\ P_k^{(2)}&:=\{x\in\Omega\colon \nabla u_k(x)\in\mathfrak{C}^{(2)}\setminus\mathfrak{C}^{(1)}\},\\ &\ \vdots\\ P_k^{(N)}&:=\left\{x\in\Omega\colon \nabla u_k(x)\in\mathfrak{C}^{(N)}\Big\backslash\bigcup_{i=1}^{N-1}\mathfrak{C}^{(i)}\right\}.\end{aligned}$$

Finally, we define the sequence $\{\chi_k\}$ in $\mathcal{X}(\Omega)$ by $\chi_k:=(\chi_{P_k^{(1)}},\dots,\chi_{P_k^{(N)}})$. Extracting a subsequence (not relabeled) if necessary, we may assume that there exists an effective energy density $W^*\in\mathcal{F}(\alpha,p,\Omega)$ associated to $\{(W^{(i)},\chi_k^{(i)})\}_{i=1,\dots,N}$ and that the sequence $\{w_k\}:=\{(\chi_k,\nabla u_k)\}$ generates a Young measure $\mu\in\mathcal{Y}(\Omega;\mathbb{R}^N\times\mathbb{R}^{m\times n})$.

**Step 2.** Denoting by $\{e_1,\dots,e_N\}$ the canonical basis of $\mathbb{R}^N$, we claim that for a.e. $x\in\Omega$,

$$\mu_x=\sum_{i=1}^N\delta_{e_i}\otimes(\nu_x\llcorner\mathfrak{A}^{(i)}).\tag{4.1}$$

Given $x\in\Omega$ and $k\in\mathbb{N}$, we may find some $h\in\{1,\dots,N\}$ so that $x\in P_k^{(h)}$ and thus

$$\begin{aligned}\operatorname{dist}\left(w_k(x),\bigcup_{i=1}^N(\{e_i\}\times\mathfrak{A}^{(i)})\right)&\leqslant\operatorname{dist}(w_k(x),\{e_h\}\times\mathfrak{A}^{(h)})\\ &=\operatorname{dist}(\nabla u_k(x),\mathfrak{A}^{(h)})=\operatorname{dist}(\nabla u_k(x),\mathfrak{A}).\end{aligned}$$

Then by Theorem 2.6(iii) we obtain that

$$\operatorname{supp}\mu_x\subseteq\bigcup_{i=1}^N(\{e_i\}\times\mathfrak{A}^{(i)})\quad\text{for a.e. } x\in\Omega.$$

Since the sets $\mathfrak{A}^{(1)}, \dots, \mathfrak{A}^{(N)}$ are pairwise disjoint, for any Borel set $\mathfrak{B} \subseteq \mathfrak{A}^{(i)}$, we get that

$$\mu_x\big(\{e_i\} \times \mathfrak{B}\big) = \mu_x\big(\mathbb{R}^N \times \mathfrak{B}\big) = \nu_x(\mathfrak{B})$$

and therefore

$$\mu_x \llcorner \big(\{e_i\} \times \mathfrak{A}^{(i)}\big) = \delta_{e_i} \otimes \big(\nu_x \llcorner \mathfrak{A}^{(i)}\big).$$

**Step 3.** We prove that $W^*(x, \nabla u(x)) = 0$ for a.e. $x \in \Omega$. By (4.1) and applying Theorem 2.6(ii) with $f(x, z, \xi) = \sum_{i=1}^N z^{(i)} W^{(i)}(\xi)$ (where $z^{(i)}$ is the $i$th component of $z$), we have that

$$\begin{aligned}\int_\Omega W^*\big(x, \nabla u(x)\big)\, dx &\leqslant \lim_{k\to+\infty} \int_\Omega \sum_{i=1}^N \chi_k^{(i)}(x) W^{(i)}\big(\nabla u_k(x)\big)\, dx \\ &= \int_\Omega \left( \int_{\mathbb{R}^N \times \mathbb{R}^{m\times n}} \sum_{i=1}^N z^{(i)} W^{(i)}(\xi)\, d\mu_x(z, \xi) \right) dx \\ &= \int_\Omega \sum_{i=1}^N \left( \int_{\mathfrak{A}^{(\mathrm{i})}} W^{(i)}(\xi)\, d\nu_x(\xi) \right) dx = 0,\end{aligned}$$

where we used the fact that the sequence $\{\nabla u_k\}$ is $p$-equi-integrable in the first equality.

**Step 4.** We prove that $\chi_k \xrightharpoonup{*} \theta$ in $L^\infty(\Omega; [0,1]^N)$. Fix $i \in \{1, \dots, N\}$ and $\varphi \in L^1(\Omega)$, by (4.1) and Theorem 2.6(ii) (with $f(x, z, \xi) = \varphi(x) z^{(i)}$), we have that

$$\begin{aligned}\lim_{k\to+\infty} \int_\Omega \varphi(x) \chi_k^{(i)}(x)\, dx &= \int_\Omega \left( \int_{\mathbb{R}^N \times \mathbb{R}^{m\times n}} \varphi(x) z^{(i)}\, d\mu_x(z, \xi) \right) dx \\ &= \int_\Omega \varphi(x) \nu_x\big(\mathfrak{A}^{(i)}\big)\, dx = \int_\Omega \varphi(x) \theta^{(i)}(x)\, dx.\end{aligned}$$

Through Lemma 3.2 we can now modify the sequence $\{\chi_k\}$ in order to get another one satisfying (ii).

**(ii)⇒(i)**. Let $\{u_k\}$ be a recovery sequence such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ and

$$\lim_{k\to+\infty} \int_\Omega W_{\chi_k}(x, \nabla u_k)\, dx = \int_\Omega W^*(x, \nabla u)\, dx = 0. \tag{4.2}$$

Up to a subsequence (not relabeled), we may assume that the sequences $\{\nabla u_k\}$ and $\{w_k\} := \{(\chi_k, \nabla u_k)\}$ generate, respectively, the Young measures $\nu \in \mathcal{Y}(\Omega; \mathbb{R}^{m\times n})$ and $\mu \in \mathcal{Y}(\Omega; \mathbb{R}^N \times \mathbb{R}^{m\times n})$. According to Remark 2.8, we get that $\bar\nu_x = \nabla u(x)$ for a.e. $x \in \Omega$.

From (4.2) and Theorem 2.6(i), we get that

$$\int_\Omega \left( \int_{\mathbb{R}^N \times \mathbb{R}^{m\times n}} \sum_{i=1}^N z^{(i)} W^{(i)}(\xi)\, d\mu_x(z, \xi) \right) dx \leqslant \liminf_{k\to+\infty} \int_\Omega \sum_{i=1}^N \chi_k^{(i)}(x) W^{(i)}\big(\nabla u_k(x)\big)\, dx = 0$$

and therefore by [7, Lemma 3.3]

$$\operatorname{supp} \mu_x \subseteq \bigcup_{i=1}^N \big(\{e_i\} \times \mathfrak{A}^{(i)}\big) \quad \text{for a.e. } x \in \Omega.$$

Thanks to the inequality

$$\operatorname{dist}\left( w_k(x), \bigcup_{i=1}^N \big(\{e_i\} \times \mathfrak{A}^{(i)}\big) \right) \geqslant \operatorname{dist}\big(\nabla u_k(x), \mathfrak{A}\big),$$

we obtain from Theorem 2.6(iii) that $\operatorname{supp}\nu_x \subseteq \mathfrak{A}$ for a.e. $x \in \Omega$. Moreover, since the sets $\mathfrak{A}^{(1)}, \dots, \mathfrak{A}^{(N)}$ are pairwise disjoint, as in steps 2 of the previous implication we have that

$$\mu_x = \sum_{i=1}^{N} \delta_{e_i} \otimes \big(\nu_x \llcorner \mathfrak{A}^{(i)}\big).$$

Finally, for all $i \in \{1, \dots, N\}$ and all $\varphi \in L^1(\Omega)$,

$$\begin{aligned}\int_\Omega \varphi(x)\nu_x\big(\mathfrak{A}^{(i)}\big)\,dx &= \int_\Omega \bigg(\int_{\mathbb{R}^N \times \mathbb{R}^{m\times n}} \varphi(x) z^{(i)}\,d\mu_x(z,\xi)\bigg)\,dx \\ &= \lim_{k\to+\infty} \int_\Omega \varphi(x)\chi_k^{(i)}(x)\,dx = \int_\Omega \varphi(x)\theta^{(i)}(x)\,dx,\end{aligned}$$

and thus, by the arbitrariness of $\varphi$, it follows that $\nu_x(\mathfrak{A}^{(i)}) = \theta^{(i)}(x)$ for a.e. $x \in \Omega$. $\square$

As a consequence of the localization result for effective energy densities (Theorem 3.5) and of gradient Young measures (Remark 2.10), we deduce the following homogeneous version of Theorem 4.1.

**Corollary 4.2.** *Under the same hypothesis than that of Theorem* 4.1*, if $A \in \mathbb{R}^{m\times n}$ and $\theta \in [0,1]^N$ with $\sum_{i=1}^N \theta^{(i)} = 1$, then the following conditions are equivalent*:

(i) *there exists a homogeneous gradient Young measure $\sigma \in \mathcal{P}(\mathbb{R}^{m\times n})$ such that*

$$\operatorname{supp}\sigma \subseteq \mathfrak{A}, \quad \bar{\sigma} = A \quad \textit{and} \quad \sigma\big(\mathfrak{A}^{(i)}\big) = \theta^{(i)} \quad \textit{for all } i \in \{1,\dots,N\};$$

(ii) *there exists $W^* \in \mathcal{G}_\theta$ such that $W^*(A) = 0$.*

*4.2. Cell integrands*

We now characterize the zero level set of $(W_\chi)_{\mathrm{cell}}$ and point out its dependence on $\chi$ and on the zero level sets of $W^{(1)}, \dots, W^{(N)}$. We refer to [9] for a similar characterization of the zero level set of $(W_\chi)_{\mathrm{hom}}$. The result obtained has a lot of interesting consequences because it will enable us to build new counter-examples to the validity of the cell formula even in the quasiconvex case (Example 6.1) and to the continuity of the determinant with respect to the two-scale convergence (Example 6.5).

**Definition 4.3.** Given a measurable partition $\{P^{(i)}\}_{i=1,\dots,N}$ of the unit cell $Q$ and a family of compact sets $\{\mathfrak{A}^{(i)}\}_{i=1,\dots,N}$, we define $\mathfrak{A}_{\mathrm{cell}}$ as the set of matrices $\xi \in \mathbb{R}^{m\times n}$ such that there exists $\phi \in W^{1,\infty}_{\mathrm{per}}(Q;\mathbb{R}^m)$ satisfying

$$\xi + \nabla\phi(y) \in \mathfrak{A}^{(i)} \quad \text{for a.e. } y \in P^{(i)} \text{ and all } i \in \{1,\dots,N\}.$$

We call $\mathfrak{A}_{\mathrm{cell}}$ the *cell set* associated to $\{(\mathfrak{A}^{(i)}, P^{(i)})\}_{i=1,\dots,N}$.

**Lemma 4.4.** *Let $\{P^{(i)}\}_{i=1,\dots,N}$ be a measurable partition of the unit cell $Q$ and define $\chi \in \mathcal{X}(Q)$ by $\chi^{(i)} := \chi_{P^{(i)}}$ for $i \in \{1,\dots,N\}$. Assume that the functions $W^{(1)}, \dots, W^{(N)} \in \mathcal{F}(\alpha, p)$ are quasiconvex and also that the compact sets $\mathfrak{A}^{(i)} := \{\xi \in \mathbb{R}^{m\times n}\colon W^{(i)}(\xi) = 0\}$ are not empty. Then we have*

$$\mathfrak{A}_{\mathrm{cell}} = \big\{\xi \in \mathbb{R}^{m\times n}\colon (W_\chi)_{\mathrm{cell}}(\xi) = 0\big\},$$

*i.e., the zero-level set of the cell integrand $(W_\chi)_{\mathrm{cell}}$ associated to $W_\chi$ coincides with the cell set $\mathfrak{A}_{\mathrm{cell}}$ associated to $\{(\mathfrak{A}^{(i)}, P^{(i)})\}_{i=1,\dots,N}$.*

**Proof.** We only prove the inclusion $(W_\chi)^{-1}_{\mathrm{cell}}(0) \subseteq \mathfrak{A}_{\mathrm{cell}}$, since the opposite one is immediate. If $(W_\chi)_{\mathrm{cell}}(\xi) = 0$, then by Remark 2.4 there exists $\phi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^m)$ such that

$$\int_Q W_\chi\big(y, \xi + \nabla\phi(y)\big)\,dy = 0$$

and so $W^{(i)}(\xi + \nabla\phi(y)) = 0$ for a.e. $y \in P^{(i)}$. As $\mathfrak{A}^{(1)}, \dots, \mathfrak{A}^{(N)}$ are compact sets and $\xi + \nabla\phi(y) \in \bigcup_{i=1}^N \mathfrak{A}^{(i)}$ for a.e. $y \in Q$, it follows that $\nabla\phi \in L^\infty(\mathbb{R}^n; \mathbb{R}^{m\times n})$ and thus $\phi$ is Lipschitz continuous. Consequently $\phi \in W^{1,\infty}_{\rm per}(Q; \mathbb{R}^m)$ which proves that $\xi \in \mathfrak{A}_{\rm cell}$. □

## 5. G-closure in the convex case

In this section we focus on effective energy densities with periodic microstructure. Let $\theta = (\theta^{(1)}, \dots, \theta^{(N)}) \in [0,1]^N$ satisfying $\sum_{i=1}^N \theta^{(i)} = 1$, we define $P_\theta$ to be the set of all functions $W^* \in \mathcal{F}(\alpha, p)$ for which there exists $\chi \in \mathcal{X}(Q)$ such that $\int_Q \chi\, dx = \theta$ and, defining the sequence $\{\chi_k\}$ by

$$\chi_k(y) = \chi\big(\langle ky\rangle\big) \quad \text{for } y \in Q,$$

$W^*$ is the effective energy density associated to $\{(W^{(i)}, \chi_k^{(i)})\}_{i=1,\dots,N}$. Since by the Riemann–Lebesgue Lemma $\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega; [0,1]^N)$, we always have the inclusion $P_\theta \subseteq \mathcal{G}_\theta$. In view of Theorem 2.5,

$$P_\theta = \left\{ W^* \in \mathcal{F}(\alpha, p)\text{: there exists } \chi \in \mathcal{X}(Q) \text{ such that } \int_Q \chi\, dy = \theta \text{ and } W^* = (W_\chi)_{\rm hom} \right\}.$$

Note that, in general, one cannot expect the set $P_\theta$ to be already closed with respect to the pointwise convergence (see Example 5.2 below). Hence we also define

$$G_\theta := \big\{ W^* \in \mathcal{F}(\alpha, p)\text{: there exists a sequence } \{W_k^*\} \text{ in } P_\theta \text{ such that } W_k^* \to W^* \text{ pointwise} \big\}.$$

Remark 2.2(v), the metrizability of $\Gamma$-convergence of lower semicontinuous and coercive functionals in $W^{1,p}(Q; \mathbb{R}^m)$ and a standard diagonalization argument show that the sets $G_\theta$ and $\mathcal{G}_\theta$ are closed for the pointwise convergence. Thus, since $P_\theta \subseteq \mathcal{G}_\theta$, it follows that $G_\theta \subseteq \mathcal{G}_\theta$. The following result states that, at least in the convex case, this inclusion is actually an equality and thus it makes more precise the result on locality of mixtures Theorem 3.5.

**Theorem 5.1.** *Let $W^{(1)}, \dots, W^{(N)} \in \mathcal{F}(\alpha, p)$ be $N$ convex functions and let $\theta \in L^\infty(\Omega; [0,1]^N)$ be such that $\sum_{i=1}^N \theta^{(i)}(x) = 1$ a.e. in $\Omega$. Given $W^* \in \mathcal{F}(\alpha, p, \Omega)$, the following conditions are equivalent*:

(i) *there exists a sequence $\{\chi_k\}$ in $\mathcal{X}(\Omega)$ such that $\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega; [0,1]^N)$ and $W^*$ is the effective energy associated to $\{(W^{(i)}, \chi_k^{(i)})\}_{i=1,\dots,N}$;*
(ii) *$W^*(x, \cdot) \in G_{\theta(x)}$ for a.e. $x \in \Omega$.*

**Proof.** From Theorem 3.5, it is sufficient to prove that $G_\theta = \mathcal{G}_\theta$ for any fixed $\theta \in [0,1]^N$. Let $W^* \in \mathcal{G}_\theta$. By Lemmas 3.2 and 3.4, there exists a sequence $\{\chi_k\}$ in $\mathcal{X}(Q)$ such that $\int_Q \chi_k\, dy = \theta$ for all $k \in \mathbb{N}$ and

$$\Gamma\text{-}\lim_{k\to+\infty} \int_Q W_{\chi_k}(y, \nabla u)\, dy = \int_Q W^*(\nabla u)\, dy. \tag{5.1}$$

As a consequence of the classical property of convergence of minimal values (see e.g. [12,19])

$$\begin{aligned} W^*(\xi) &= \min\left\{ \int_Q W^*(\xi + \nabla\phi)\, dy\text{: } \phi \in W^{1,p}_{\rm per}(Q; \mathbb{R}^m) \right\} \\ &= \lim_{k\to+\infty} \min\left\{ \int_Q W_{\chi_k}(y, \xi + \nabla\phi)\, dy\text{: } \phi \in W^{1,p}_{\rm per}(Q; \mathbb{R}^m) \right\} \\ &= \lim_{k\to+\infty} (W_{\chi_k})_{\rm cell}(\xi) \end{aligned} \tag{5.2}$$

and the conclusion follows from the very definition of $G_\theta$ and the convexity assumption. □

As the following example shows, even in the convex case the inclusion $P_\theta \subseteq G_\theta$ is in general strict and thus we cannot expect $P_\theta$ to be closed with respect to the pointwise convergence.

**Example 5.2.** Consider the following four matrices in $\mathbb{R}^{2\times 2}$:

$$A_1 := \operatorname{diag}(-1,-3),\quad A_2 := \operatorname{diag}(-3,1),\quad A_3 := \operatorname{diag}(1,3)\quad\text{and}\quad A_4 := \operatorname{diag}(3,-1)$$

and let $\mathfrak{A} := \{A_1,\dots,A_4\}$. This set has a peculiarity: despite the absence of rank-one connections, its quasiconvex hull is not trivial. The relevance of this kind of sets has been discovered independently by many authors and in different contexts ([4,15,30,36,39], see also [10]). We refer to [32] for a basic description and to [26,37] for an advanced analysis. By [32, Lemma 2.6 and Example d in Section 3.2], if $\theta := (8/15, 4/15, 2/15, 1/15)$, then

$$\sigma := \theta^{(1)}\delta_{A_1} + \theta^{(2)}\delta_{A_2} + \theta^{(3)}\delta_{A_3} + \theta^{(4)}\delta_{A_4}$$

is a homogeneous gradient Young measure. For $i \in \{1,\dots,4\}$, we consider the convex functions $W^{(i)}$ defined by

$$W^{(i)}(\xi) := |A_i - \xi|^p.$$

Since $\bar\sigma = -I$, where $I := \operatorname{diag}(1,1)$, from Corollary 4.2 and Theorem 5.1, there exists some $W^* \in G_\theta$ such that $W^*(-I) = 0$. On the other hand, if $W^* \in P_\theta$, then from Lemma 4.4, there would exist $\chi \in \mathcal{X}(Q)$ and $\phi \in W^{1,\infty}_{\mathrm{per}}(Q;\mathbb{R}^2)$ such that $\nabla\phi(x) - I = A_i$ for a.e. $y \in P^{(i)} := \{\chi^{(i)} = 1\}$. Defining $\varphi(y) := \phi(y) - y$, it follows that $\varphi$ is Lipschitz continuous and that $\nabla\varphi(y) = A_i$ for a.e. $y \in P^{(i)}$ which is impossible since the matrices $A_1,\dots,A_4$ are not rank-one connected (see [17] or [25]). Thus $W^* \notin P_\theta$.

**Remark 5.3.** Note that it is also possible to adapt the proofs of [2] and [35] to our setting. Let us briefly explain how to proceed. We first need to introduce a suitable metric space structure. Specifically, let $\mathcal{E}_p$ be the space made of all continuous functions $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ such that there exists the limit

$$\lim_{|\xi|\to+\infty} \frac{f(\xi)}{1+|\xi|^{p+1}} = 0.$$

The space $\mathcal{E}_p$ endowed with the norm

$$\|f\| := \sup_{\xi\in\mathbb{R}^{m\times n}} \frac{|f(\xi)|}{1+|\xi|^{p+1}}$$

is a normed space. In fact, only its metric structure will be useful for our purpose. Using the Ascoli–Arzela Theorem and the fact that all the functions we are considering here satisfy uniform $p$-growth and $p$-coercivity conditions, one can show that the sets $G_\theta$ and $\mathcal{G}_\theta$ are closed subsets of $\mathcal{E}_p$ and that $G_\theta$ is the closure of $P_\theta$ in $\mathcal{E}_p$. Moreover, if $W^* \in \mathcal{G}_\theta$ and $\{\chi_k\} \subset \mathcal{X}(Q)$ is a sequence of characteristic functions as in the definition of $\mathcal{G}_\theta$, by convergence of minimizers, we always have that $(W_{\chi_k})_{\mathrm{cell}} \to W^*$ pointwise and also in $\mathcal{E}_p$. The idea now consists in using a Hausdorff convergence argument to deduce directly from the previous property that $W^* \in G_\theta$. We recall the definition of the Hausdorff distance between two closed sets $G_1$ and $G_2$ in $\mathcal{E}_p$:

$$\mathbf{d}_{\mathcal{H}}(G_1,G_2) := \max\Big\{\sup_{g\in G_2}\inf_{f\in G_1}\|f-g\|,\ \sup_{f\in G_1}\inf_{g\in G_2}\|f-g\|\Big\}.$$

To do that, we need to prove that $\theta \mapsto G_\theta$ is continuous for the Hausdorff metric. Indeed, if so, setting $\theta_k := \int_Q \chi_k\,dx$, since $(W_{\chi_k})_{\mathrm{cell}} \in P_{\theta_k} \subseteq G_{\theta_k}$ and $\theta_k \to \theta$, it follows that $G_{\theta_k} \to G_\theta$ in the sense of Hausdorff. Hence remembering that $(W_{\chi_k})_{\mathrm{cell}} \to W^*$ in $\mathcal{E}_p$, it implies that $W^* \in G_\theta$.

To show that $G_\theta$ depends continuously on $\theta$, we prove an estimate on $\mathbf{d}_{\mathcal{H}}(G_{\theta_1}, G_{\theta_2})$ for any $\theta_1$ and $\theta_2 \in [0,1]^N$. This is done using a Meyers type regularity result to the solutions of the minimization problem (2.2). Indeed, one can adapt the proof of e.g. [23, Theorem 3.1] to get the existence of two universal constants $c > 0$ and $q > p$ (both depending only on $n$, $p$, $\alpha_1$, $\alpha_2$ and $\alpha_3$) such that any solution $\varphi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^m)$ of (2.2) has a higher integrability property, namely that

$$\left(\int_Q |\nabla\varphi|^q\,dx\right)^{1/q} \leqslant c\big(1+|\xi|\big).$$

Using this estimate together with the standard uniform $p$-Lipschitz property satisfied by functions in $G_{\theta_1}$ and $G_{\theta_2}$, one can show that

$$\mathbf{d}_{\mathcal{H}}(G_{\theta_1}, G_{\theta_2}) \leqslant c'|\theta_1 - \theta_2|^{(q-p)/q},$$

(where $c' > 0$ depends only on $n$, $p$, $\alpha_1$, $\alpha_2$ and $\alpha_3$) which proves the desired result.

This gives an alternative proof to the equality $\mathcal{G}_\theta = G_\theta$ and stresses the fact that the Meyers type regularity result can be seen as a stronger version of the Decomposition Lemma of [22].

## 6. Some counter-examples

An example obtained in [31] shows that the equality between cell and homogenized integrands does not hold, even in the quasiconvex case. Precisely, this result is obtained through a rank-one laminated structure assembled by mixing two polyconvex functions with 4-growth in the case $n = m = 2$. In this last section, we present an alternative counter-example. Namely we show that when $n = m = 2$, there exist functions $W^{(1)}$ and $W^{(2)} \in \mathcal{F}(\alpha, p)$ (with $p > 1$ arbitrary) and a measurable set $P \subseteq Q$ such that, setting $W_\chi(y, \xi) := \chi_P(y)W^{(1)}(\xi) + \chi_{Q\setminus P}(y)W^{(2)}(\xi)$, then $(W_\chi)_{\mathrm{hom}} < (W_\chi)_{\mathrm{cell}}$. We use an argument based on the characterization of the zero level set stated in Lemma 4.4 to show that $W_{\mathrm{cell}}$ is not rank-one convex.

**Example 6.1.** Consider the following matrices (see Fig. 1):

$$O := \mathrm{diag}(0, 0), \quad I := \mathrm{diag}(1, 1), \quad A := \mathrm{diag}(-1, 1), \quad B := \mathrm{diag}(0, 1) \quad \text{and} \quad C := \mathrm{diag}(0, 1/2).$$

Let $W^{(1)}$ and $W^{(2)} \in \mathcal{F}(\alpha, p)$ be two quasiconvex functions such that

$$\big(W^{(1)}\big)^{-1}(0) = \{O, A\} \quad \text{and} \quad \big(W^{(2)}\big)^{-1}(0) = \{O, I\}, \tag{6.1}$$

and define the function

$$W_\chi(y, \xi) := \chi_P(y)W^{(1)}(\xi) + \chi_{Q\setminus P}(y)W^{(2)}(\xi), \tag{6.2}$$

where $P = (0, 1/2) \times (0, 1)$. Then, the function $(W_\chi)_{\mathrm{cell}}$ is not rank-one convex. Moreover, $W_{\mathrm{cell}}(C) > 0$ while $W_{\mathrm{hom}}(C) = 0$.

**Proof.** Obviously $(W_\chi)_{\mathrm{cell}}(O) = (W_\chi)_{\mathrm{hom}}(O) = 0$. Let $\phi \in W^{1,\infty}_{\mathrm{per}}(Q; \mathbb{R}^2)$ be defined by $\phi(y) := (|y_1 - 1/2|, 0)$. It is immediate to check that $\nabla\phi(y) = \chi_P(y)\,\mathrm{diag}(-1, 0) + \chi_{Q\setminus P}(y)\,\mathrm{diag}(1, 0)$ and therefore $(W_\chi)_{\mathrm{cell}}(B) = (W_\chi)_{\mathrm{hom}}(B) = 0$. Since $(W_\chi)_{\mathrm{hom}}$ is rank-one convex (because it is quasiconvex), necessarily $(W_\chi)_{\mathrm{hom}}(C) = 0$. We shall prove that on the contrary, $(W_\chi)_{\mathrm{cell}}(C) > 0$. If not, by Lemma 4.4, there exists $\phi \in W^{1,\infty}_{\mathrm{per}}(Q; \mathbb{R}^2)$ such that

$$C + \nabla\phi(y) \in \begin{cases} \{O, A\} & \text{for a.e. } y \in P; \\ \{O, I\} & \text{for a.e. } y \in Q \setminus P. \end{cases}$$

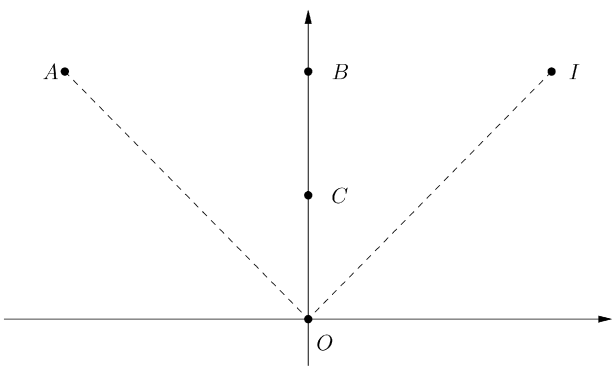


Fig. 1. A representation of the matrices $O$, $I$, $A$, $B$ and $C$ in $\mathbb{R}^2$, identified with the set of the diagonal matrices.

Since $\operatorname{rank}(I) = \operatorname{rank}(A) = 2$, it follows from a classical result on rigidity of Lipschitz functions (see e.g. [6, Proposition 1]) that $\phi$ is an affine function on $P$ and $Q \setminus P$ and that either

$$C + \nabla\phi(y) = O$$

or

$$C + \nabla\phi(y) = \chi_P(y)A + \chi_{Q\setminus P}(y)I$$

for a.e. $y \in Q$. But in both cases we get a contradiction with the fact that $\phi$ should be $Q$-periodic and thus $(W_\chi)_{\mathrm{cell}}(C) \neq 0$. $\square$

**Remark 6.2.** Since the sets $\{O, A\}$ and $\{O, I\}$ are quasiconvex, we can take as $W^{(1)}$ (resp. $W^{(2)}$) the quasiconvexification of $D^{(1)}(\xi) := \operatorname{dist}^p(\xi, \{O, A\})$ (resp. $D^{(2)}(\xi) := \operatorname{dist}^p(\xi, \{O, I\})$).

**Remark 6.3.** If a function $\phi \in W^{1,\infty}(Q; \mathbb{R}^2)$ is affine on $P$ and

$$C + \nabla\phi(y) \in \begin{cases} \{O, A\} & \text{for a.e. } y \in P; \\ \{O, I\}^{\mathrm{co}} & \text{for a.e. } y \in Q \setminus P, \end{cases}$$

then it cannot be $Q$-periodic (the superscript $^{\mathrm{co}}$ denotes the convex hull of a set). By a straightforward modification of Example 6.1, we can take $W^{(2)}(\xi) := \operatorname{dist}^p(\xi, \{O, I\}^{\mathrm{co}})$. This proves that also by mixing a quasiconvex function and a convex function, the equality $(W_\chi)_{\mathrm{cell}} = (W_\chi)_{\mathrm{hom}}$ could not occur.

**Remark 6.4.** If $p \geqslant 2$, then we can take $W^{(1)}$ and $W^{(2)}$ polyconvex. We recall that a function $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is said to be *polyconvex* if there is a convex function $V : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $W(\xi) = V(\xi, \det(\xi))$ for all $\xi \in \mathbb{R}^{2\times 2}$. We refer to [18,32] for more details.

We define $V^{(1)}$ and $V^{(2)} : \mathbb{R}^{2\times 2} \times \mathbb{R} \to [0, +\infty)$ by

$$V^{(1)}(\xi, z) := \max\left\{\operatorname{dist}^p\left(\xi, \{O, A\}^{\mathrm{co}}\right), \operatorname{dist}^{\frac{p}{2}}\left((\xi, z), \left\{(O, 0), (A, -1)\right\}^{\mathrm{co}}\right)\right\},$$
$$V^{(2)}(\xi, z) := \max\left\{\operatorname{dist}^p\left(\xi, \{O, I\}^{\mathrm{co}}\right), \operatorname{dist}^{\frac{p}{2}}\left((\xi, z), \left\{(O, 0), (I, 1)\right\}^{\mathrm{co}}\right)\right\}.$$

Setting

$$W^{(1)}(\xi) := V^{(1)}\big(\xi, \det(\xi)\big) \quad \text{and} \quad W^{(2)}(\xi) := V^{(2)}\big(\xi, \det(\xi)\big), \tag{6.3}$$

we can verify that $W^{(1)}$ and $W^{(2)} \in \mathcal{F}(\alpha, p)$ for a suitable $\alpha$ and that (6.1) holds. In particular $(W_\chi)_{\mathrm{hom}}(C) = 0$ while $(W_\chi)_{\mathrm{cell}}(C) > 0$.

This last remark enables to underline the lack of continuity of the determinant with respect to the two-scale convergence. It is known (see e.g. [18, Theorem 2.6 in Chapter 4]) that if $p > 2$ and $\{u_k\}$ is a sequence in $W^{1,p}(\Omega; \mathbb{R}^2)$, then

$$u_k \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^2) \quad \text{implies that} \quad \det(\nabla u_k) \rightharpoonup \det(\nabla u) \quad \text{in } L^{p/2}(\Omega).$$

It would be interesting to ask whether this result still holds in the two-scale convergence framework. The answer is negative as the following example shows.

**Example 6.5.** Let $u(x) := Cx$, where $C = \operatorname{diag}(0, 1/2)$. By Theorem 2.5, there exists a sequence $\{u_k\}$ in $W^{1,p}(\Omega; \mathbb{R}^2)$ such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^2)$ and

$$\lim_{k\to+\infty} \int_\Omega W_\chi\left(\left\langle \frac{x}{\varepsilon_k} \right\rangle, \nabla u_k(x)\right) dx = \mathcal{L}^n(\Omega)(W_\chi)_{\mathrm{hom}}(C) = 0,$$

where $W_\chi$ is defined by (6.2) with $W^{(1)}$ and $W^{(2)}$ given by (6.3) for some $p > 2$. By Remark 2.12(i)–(iii), there exist a subsequence (not relabeled), $v \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Q; \mathbb{R}^2))$ and $w \in L^{p/2}(\Omega \times Q)$ such that $\nabla u_k \rightsquigarrow C + \nabla_y v$ and $\det(\nabla u_k) \rightsquigarrow w$. Suppose now that $w = \det(C + \nabla_y v)$, then by Remark 2.12(ii),

$$\begin{aligned}
\mathcal{L}^n(\Omega)(W_\chi)_{\mathrm{cell}}(C) &\leqslant \int\limits_{\Omega\times Q} W_\chi\big(y, C+\nabla_y v(x,y)\big)\,dx\,dy \\
&= \int\limits_{\Omega\times Q} \big[\chi_P(y)V^{(1)}\big(C+\nabla_y v(x,y), w(x,y)\big) \\
&\quad + \chi_{Q\setminus P}(y)V^{(2)}\big(C+\nabla_y v(x,y), w(x,y)\big)\big]\,dx\,dy \\
&\leqslant \liminf_{k\to+\infty}\int\limits_{\Omega}\left[\chi_P\left(\left\langle\frac{x}{\varepsilon_k}\right\rangle\right)V^{(1)}\big(\nabla u_k, \det(\nabla u_k)\big)+\chi_{Q\setminus P}\left(\left\langle\frac{x}{\varepsilon_k}\right\rangle\right)V^{(2)}\big(\nabla u_k, \det(\nabla u_k)\big)\right]dx \\
&= \lim_{k\to+\infty}\int\limits_{\Omega} W_\chi\left(\left\langle\frac{x}{\varepsilon_k}\right\rangle, \nabla u_k(x)\right)dx = 0,
\end{aligned}$$

which is against the fact that $(W_\chi)_{\mathrm{cell}}(C) > 0$. Hence $w \neq \det(C+\nabla_y v)$. This example can be connected to the recent investigation of the validity of the div-curl lemma in two-scale convergence (see [14,41]).

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