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Ann. I. H. Poincaré – AN 25 (2008) 471–503

ANNALES DE L'INSTITUT HENRI POINCARÉ
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Partial continuity for elliptic problems

Continuité partielle pour des problèmes elliptiques

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Received 11 December 2006; accepted 14 February 2007

Available online 17 July 2007

Abstract

We prove the partial Hölder continuity for solutions to elliptic systems and for minimizers of quasi-convex integrals, under the assumption of continuous coefficients. The proof relies upon an iteration scheme of a decay estimate for a new type of excess functional measuring the oscillations in the solution and its gradient. To establish the decay estimate, we use the technique of $\mathcal{A}$-harmonic approximation, based on Duzaar and Steffen's $\mathcal{A}$-harmonic approximation lemma [F. Duzaar, K. Steffen, Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals, J. Reine Angew. Math. (Crelles J.) 546 (2002) 73–138].

Résumé

Nous prouvons la continuité Hölderienne partielle pour des solutions de systèmes elliptiques ou pour des minimiseurs d'intégrales quasi-convexes, sous réserve que les coefficients soient continus. La preuve consiste en un schéma itératif d'estimations de décroissance pour un nouveau type de fonctionnelles d'excès servant à mesurer les oscillations de la solution et de son gradient. Afin d'établir les estimations de décroissance, nous utilisons la technique de l'approximation $\mathcal{A}$-harmonique, basée sur le lemme d'approximation $\mathcal{A}$-harmonique de Duzaar et Steffen [F. Duzaar, K. Steffen, Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals, J. Reine Angew. Math. (Crelles J.) 546 (2002) 73–138].

MSC: 35B65; 49N60

Keywords: Partial regularity; Elliptic systems; Quasiconvexity

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.02.003

1. Introduction and results

The purpose of this paper is to settle a rather longstanding issue in the regularity theory of vectorial elliptic and variational problems. Let us first consider the following non-linear elliptic system in divergence form:

$$\operatorname{div} a\big(x, u(x), Du(x)\big) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is a continuous vector field such that $z \mapsto a(\cdot,\cdot,z)$ is of class C^1, and satisfying the following standard ellipticity, growth and continuity assumptions

$$\begin{cases} \nu(1+|z|)^{p-2}|\lambda|^2 \leqslant \langle a_z(x,u,z)\lambda,\lambda\rangle \leqslant L(1+|z|)^{p-2}|\lambda|^2, \\ |a(x,u,z) - a(x_0,v,z)| \leqslant L\omega(|x-x_0|^2 + |u-v|^2)(1+|z|)^{p-1}, \\ |a_z(x,u,z_2) - a_z(x,u,z_1)| \leqslant L\mu(\frac{|z_2-z_1|}{1+|z_1|+|z_2|})(1+|z_1|+|z_2|)^{p-2}, \end{cases} \tag{1.2}$$

for all $x, x_0 \in \Omega$, $u, v \in \mathbb{R}^N$ and $z, z_1, z_2, \lambda \in \mathbb{R}^{N\times n}$. Here $n, N \geqslant 2$, $p \geqslant 2$, $0 < \nu \leqslant L$, and $\mu, \omega : \mathbb{R}^+ \to \mathbb{R}^+$ are two moduli of continuity i.e. two bounded, concave, and non-decreasing functions such that $\mu(0) = \omega(0) = 0$. By a_z we are denoting the partial derivatives of a with respect to the z-variable.

The emphasis in this paper is on the role played by the function $\omega(\cdot)$, which describes the degree of continuity of the partial map $(x,u) \mapsto (1+|z|)^{1-p}a(x,u,z)$; that is, roughly speaking, the continuity of a with respect to the "coefficients" (x,u). For the moment, let us concentrate on the scalar case $N = 1$, when the solution u is a scalar valued function, and assume that $\omega(\varrho) \leqslant \varrho^{\beta/2}$, for some $\beta \in (0,1)$. This means that $(x,u) \mapsto (1+|z|)^{1-p}a(x,u,z)$ is a Hölder continuous function with exponent β. Under this additional assumption it turns out that $Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, whenever $u \in W^{1,p}(\Omega)$ is a weak solution to the equation (1.1). If instead we only assume continuity with respect to (x,u); i.e.

$$\lim_{\varrho \searrow 0} \omega(\varrho) = 0; \tag{1.3}$$

then we can no longer expect Du to be continuous, but we still find that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$. See for instance [8,25–27] with the references therein. Turning to the vectorial case $N > 1$, such full interior Hölder continuity results for Du and u generally do not hold anymore. Indeed, it is known that singularities may appear (see for instance the recent examples in [31], or [27] for a discussion of these examples). Although everywhere Hölder continuity no longer expected in the vectorial case, it is often possible to obtain partial Hölder continuity: regularity of solutions outside a negligible closed subset of Ω, referred to as *the singular set*. In fact, if we assume $\omega(\varrho) \leqslant \varrho^{\beta/2}$, then it is known that Du is locally Hölder continuous with exponent β in an open subset $\Omega_u \subseteq \Omega$, with $|\Omega \setminus \Omega_u| = 0$ (see for instance [11] and the related references). The point of this paper is to examine what happens in the vectorial case $N > 1$ while just assuming the continuity (1.3) of a with respect to its coefficients.

In view of the established results in the scalar case $N = 1$, the natural expectation is that u is Hölder continuous – with every exponent – in an open subset $\Omega_u \subseteq \Omega$ satisfying $|\Omega \setminus \Omega_u| = 0$. A proof of such a regularity result was claimed by Campanato [4,6,7], but this work contained an irreparable flaw, which has unfortunately propagated into the literature. Therefore the problem of relaxing the assumption $\omega(\varrho) \leqslant \varrho^{\beta/2}$ to the assumption (1.3); that is *establishing a low-order partial regularity theory for vectorial elliptic problems*; has remained a mostly unresolved issue. Campanato [5,7] provided a positive answer in the low dimensional case, where $n \leqslant p+2$ (see also [23] for the variational case). A partial positive resolution has also been provided by Foss [17], who established full Hölder continuity for solutions under additional structural assumptions. The closest successful attempt to producing a general low-order partial regularity theory is due to Duzaar and Gastel [9], who considered Dini continuous coefficients; that is they assumed the convergence condition $\int_0 \sqrt{\omega(\varrho)}/\varrho \, d\rho < \infty$ (see also [10,12] for variational integrals, and [32]). This assumption is, in some sense, the weakest continuity assumption that still allows one to successfully carry out a certain dyadic type of iteration, which is critical for the usual partial regularity proofs. Our objective is to provide a low-order partial regularity theory in full generality. Indeed we have

Theorem 1.1. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the system* (1.1) *under the assumptions* (1.2) *and* (1.3)*. Then there exists an open subset $\Omega_u \subseteq \Omega$ such that $|\Omega \setminus \Omega_u| = 0$ and $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, and*

$$\Omega \setminus \Omega_u \subseteq \Big\{ x_0 \in \Omega \colon \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \big|Du - (Du)_{x_0,\varrho}\big|^p \, dx > 0 \text{ or } \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2 \, dx > 0 \Big\},$$

for every $\beta \in (0,2)$. Moreover, for every $\alpha \in (0,1)$ there exists an open subset $\Omega_u^\alpha \subset \Omega$ such that $|\Omega \setminus \Omega_u^\alpha| = 0$, $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$ and such that for every $\beta \in (0,2)$ the following inclusion holds:

$$\Omega \setminus \Omega_u^\alpha \subseteq \left\{x_0 \in \Omega\colon \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \left|Du - (Du)_{x_0,\varrho}\right|^p dx \geqslant s \text{ or } \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2\, dx \geqslant s\right\},$$

where $s > 0$ depends only upon $n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot)$. In particular s is otherwise independent of the solution u and of the vector field a.

The inclusion for the singular set Ω_u^α described in the previous theorem tells us that a sort of "quantization of singularities" property holds for problems such as (1.1): the "energy" represented by the integrals $\fint_{B_\varrho(x_0)} |Du - (Du)_{x_0,\varrho}|^p\, dx$ and $\varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2\, dx$ must exceed at every scale a certain quantity s to allow for a singularity. The number s is "universal" in the sense that it depends neither on the system considered nor on the solution u but just essentially on the ellipticity data ν, L, and on the rate of Hölder continuity chosen $\alpha \in (0,1)$. Similar quantization phenomena typically occur when considering harmonic maps, or harmonic and p-harmonic flows. As for different type of inclusions for the singular set $\Omega \setminus \Omega_u$ see also Remarks 3.2–3.3 below.

Having established partial Hölder regularity results for systems, the remaining task is to deal with variational integrals

$$\mathcal{F}[u] := \int_\Omega F(x, u, Du)\, dx, \tag{1.4}$$

where we shall always consider a continuous integrand $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$. In the vectorial case $N > 1$ a central assumption for the integrand F is quasiconvexity; that is

$$\int_{(0,1)^n} \left[F\big(x, u, z + D\varphi(y)\big) - F(x, u, z)\right] dy \geqslant 0, \quad \text{for every } \varphi \in C_c^\infty\big((0,1)^n, \mathbb{R}^N\big), \tag{1.5}$$

whenever $x \in \Omega$, $u \in \mathbb{R}^N$, and $z \in \mathbb{R}^{N\times n}$. This is a far-reaching extension of the notion of convexity that turns out to be essentially equivalent to the lower semicontinuity of variational integrals as originally noted by Morrey [28], and later extended to the optimal assumptions by Acerbi and Fusco [1], and plays an important role in modern mathematical materials science [3,29]. The assumptions that we are going to impose on F are that the partial map $z \mapsto F(\cdot,\cdot,z)$ is of class C^2 and that F satisfies

$$\begin{cases} \nu|z|^p \leqslant F(x, v, z) \leqslant L(1+|z|)^p, \\ \nu \int_{(0,1)^n} [(1+|z|+|D\varphi(y)|)^{p-2}|D\varphi(y)|^2]\, dy \leqslant \int_{(0,1)^n} [F(x, u, z + D\varphi(y)) - F(x, u, z)]\, dy, \\ |F(x, u, z) - F(x_0, v, z)| \leqslant L\omega(|x - x_0|^p + |u - v|^p)(1+|z|)^p, \\ |F_{zz}(x, u, z_2) - F_{zz}(x, u, z_1)| \leqslant L\mu(\frac{|z_2 - z_1|}{1+|z_1|+|z_2|})(1+|z_1|+|z_2|)^{p-2}, \end{cases} \tag{1.6}$$

for all $x, x_0 \in \Omega$, $u, v \in \mathbb{R}^N$ and $z, z_1, z_2, \lambda \in \mathbb{R}^{N\times n}$. The functions $\omega(\cdot)$ and $\mu(\cdot)$ are as those in (1.2). The second inequality $(1.6)_2$ is required to hold whenever $\varphi \in C_c^\infty((0,1)^n, \mathbb{R}^N)$, and is the so called *uniform, strict quasiconvexity*, a suitable reinforcement of (1.5) which facilitates proving the partial Hölder continuity of the gradient [15,2,10]. This assumption, in many respects, emulates the role of a non-degenerate convexity. Indeed when $z \mapsto F(\cdot,\cdot,z)$ is convex it can be verified (see for example [16]) that $(1.6)_2$ implies

$$\nu_2\big(1+|z|\big)^{p-2}|\lambda|^2 \leqslant \big\langle F_{zz}(x, v, z)\lambda, \lambda\big\rangle \leqslant L_2\big(1+|z|\big)^{p-2}|\lambda|^2$$

for suitable constants $\nu_2 \equiv (n, p, \nu)$ and $L_2 \equiv L_2(n, p, L)$, which is the analog of $(1.2)_2$.

Let us momentarily turn our attention once again to the scalar case $N = 1$. In this setting, the quasiconvexity hypothesis reduces to the ordinary notion of convexity, and under the additional assumptions in (1.6) the following is known: if the dependence upon the "coefficients" (x, u) of F is Hölder continuous; i.e. $\omega(\varrho) \leqslant \varrho^{\beta/p}$; then for any local minimizer of the functional $\mathcal{F}[\cdot]$ in (1.4), we find that $Du \in C^{0,\beta/2}_{\mathrm{loc}}(\Omega)$ (see [20], Chapter 8). Assuming only continuity with respect to (x, u), that is (1.3), it has been established that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for every $\alpha \in (0,1)$, exactly as in the case of equations. See [25–27] and related references, and especially [8] for the last result.

Returning to the vectorial setting, it is known that the gradient of a local minimizer is partially regular; that is Du is locally Hölder continuous outside a negligible closed, singular set; provided that $\omega(\varrho) \leqslant \varrho^{\beta/p}$ for some $\beta \in (0,1)$. One may again check [10,12] for the case of Dini-continuous coefficients, and [17] for a case with a particular structure. On the other hand nothing is known when assuming just continuity of coefficients, that is (1.3). We establish the expected result:

Theorem 1.2. *Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional $\mathcal{F}[\cdot]$ in* (1.4) *under the assumptions* (1.6) *and* (1.3). *Then there exists an open subset $\Omega_u \subseteq \Omega$ such that $|\Omega \setminus \Omega_u| = 0$ and $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, and*

$$\Omega \setminus \Omega_u \subseteq \left\{ x_0 \in \Omega \colon \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \left| Du - (Du)_{x_0,\varrho} \right|^p dx > 0 \text{ or } \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^p \, dx > 0 \right\},$$

for every $\beta \in (0,p)$. Moreover, for every $\alpha \in (0,1)$ there exists an open subset $\Omega_u^\alpha \subset \Omega$ such that $|\Omega \setminus \Omega_u^\alpha| = 0$, $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$ and such that for every $\beta \in (0,p)$ the following inclusion holds:

$$\Omega \setminus \Omega_u^\alpha \subseteq \left\{ x_0 \in \Omega \colon \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \left| Du - (Du)_{x_0,\varrho} \right|^p dx \geqslant s \text{ or } \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^p \, dx \geqslant s \right\},$$

where $s > 0$ depends only upon $n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot)$. In particular s is otherwise independent of the minimizer u considered and of the integrand F.

Note that the previous result is new already in the case of vectorial functionals which are convex in the gradient variable.

Finally, we make a few remarks regarding the techniques used. As is often the case for proofs of regularity, our arguments ultimately rely upon comparisons to solutions of linearized systems to establish the decay of some excess functional. An important feature in our paper is the use a suitable "hybrid" excess functional $E(x_0,\varrho)$ (see (3.3)). This is an integral functional that describes, in a ball centered at x_0, and with radius ϱ, both the oscillations of the gradient Du via a quantity $C(x_0,\varrho)$, and the oscillations of the function map u, via a quantity $M(x_0,\varrho)$. The quantity $C(x_0,\varrho)$ differs from the typical excess functional used to capture the oscillatory behavior of the gradient in that it has been "re-normalized" by the quantity $(1 + |(Du)_{x_0,\varrho}|)$. Our reason for this is that we are ultimately unable to establish any Lipschitz estimates for the gradient of a solution; indeed, such estimates are not even expected when assuming only (1.3). Thus it is too much to expect a decay estimate for the usual type of excess functional, since such decay estimates do imply Lipschitz estimates. Yet, some decay in the oscillations of the gradient is necessary in order to make comparisons to solutions of the linearized problem. Thus, we are led to consider the "re-normalized" excess functional $C(x_0,\varrho)$. The part of the excess functional controlling the oscillations of u, that is $M(x_0,\varrho)$, is actually incorporated into $\omega(\cdot)$ to reflect the amount of regularity that we assume for the "coefficients" in our problem. The partial Hölder continuity of solutions is then achieved by considering those points where the excess functional is small, and it is at these points where we can employ the typical linearization/comparison argument to show the decay of the excess functional. The smallness assumption leads to the inclusions of Theorems 1.1 and 1.2. Due to the particular form of the excess adopted we found it appropriate to utilize the method of A-harmonic approximation, a brilliant technique introduced by Duzaar and Steffen in the setting of Geometric Measure Theory [14], and later applied in the non-parametric setting [11,10,9,21]. This method facilitates a rapid and elegant implementation of the linearization techniques required for partial regularity, and in our case allows us to easily by-pass certain technical problems arising when dealing with the particular form of the excess $E(x_0,\varrho)$ used. The philosophy underlying our arguments is essentially the same for both systems and functionals: we shall first give the proof for the case of systems which is easier, and then we pass to the case of quasiconvex functionals, which necessitates considerably more care. In particular, the form of the excess adopted for the case of functionals is more delicate, and influenced by the fact that we need to use certain higher integrability results for minimizers (see (4.6)).

2. Notations, technical preliminaries

In this paper, c denotes a positive constant, possibly varying from expression to expression. On occasion, we will denote a specific occurrence of a constant by c_1, c_* so that it may be later referenced. For convenience, unless otherwise stated, all such constants will be assumed to be larger than one. We shall define $B_\varrho(x_0) := \{x \in \mathbb{R}^n\colon |x - x_0| < \varrho\}$; when no ambiguity will arise, or when the center is unimportant in the context, we shall also denote $B_\varrho(x_0) \equiv B_\varrho$. Adopting a similar convention regarding centers, if $g \in L^1(B_\varrho(x_0))$ we shall put:

$$(g)_\varrho \equiv (g)_{x_0,\varrho} := \fint_{B_\varrho(x_0)} g(x)\,dx.$$

We recall that a weak solution u to the system (1.1), under the assumptions considered in (1.2), is a $W^{1,p}(\Omega, \mathbb{R}^N)$-map such that

$$\int_\Omega a(x, u, Du) D\varphi\,dx = 0, \quad \text{for every } \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^N), \tag{2.1}$$

while a $W^{1,p}(\Omega, \mathbb{R}^N)$-map u is a local minimizer of the functional $\mathcal{F}[\cdot]$ in (1.4) provided,

$$\mathcal{F}[u] \leqslant \mathcal{F}[v], \quad \text{for every } v \in u + W_0^{1,p}(\Omega, \mathbb{R}^N).$$

Remark 2.1. In many papers the assumption $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is weakened to $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. This may also be done for each of the results we prove. To avoid distracting complications, however, we will simply assume that $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and leave it to the reader to make the necessary adjustments.

Concerning the assumptions (1.2) and (1.6) let us first observe that while $(1.2)_{1,2}$ are quite standard, the one in $(1.2)_3$ is a bit less. A typical instance of $(1.6)_3$, usually adopted in the literature for super-quadratic growth problems, see for instance [13], is

$$\left|a_z(x, u, z_2) - a_z(x, u, z_1)\right| \leqslant L|z_2 - z_1|^\alpha \left(1 + |z_1| + |z_2|\right)^{p-2-\alpha},$$

for $\alpha \in (0, p-2) \cap (0, 1)$, when $p > 2$. This means that $\mu(t) := t^\alpha$, while on the other hand we are assuming that $\mu(\cdot)$ is bounded in $(1.2)_3$. Of course there is no loss of generality in such an assumption, since in $(1.2)_3$ the argument of the function $\mu(\cdot)$ is always less than or equal than one. The same kind of observations apply when considering (1.6). Therefore in the following, by enlarging the constant L if required, we shall always assume that

$$\omega(t) \leqslant 1, \quad \mu(t) \leqslant 1, \quad \text{for every } t \geqslant 0. \tag{2.2}$$

Moreover, since both $\omega(\cdot)$ and $\mu(\cdot)$ are concave and hence sub-linear, we shall very often use

$$\omega(t+s) \leqslant \omega(t) + \omega(s), \quad \omega(ct) \leqslant c\omega(t), \quad \mu(ct) \leqslant c\mu(t), \quad s, t \geqslant 0,\ c \geqslant 1. \tag{2.3}$$

Let $\mathcal{A}$ be a bilinear form on $\mathbb{R}^{N\times n}$ with constant coefficients, which is strictly elliptic in the sense of Legendre–Hadamard, with ellipticity constant $\nu > 0$ and upper bound L; that is

$$\nu|\xi|^2|\eta|^2 \leqslant \mathcal{A}(\xi \otimes \eta, \xi \otimes \eta) \leqslant L|\xi|^2|\eta|^2, \quad \text{for every } \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^N. \tag{2.4}$$

We recall that a map h on B_ϱ is termed $\mathcal{A}$-harmonic [14] if and only if:

$$\fint_{B_\varrho} \mathcal{A}(Dh, D\varphi)\,dx = 0, \quad \text{for every } \varphi \in C_0^1(B_\varrho, \mathbb{R}^N).$$

We now state a version of the A-harmonic approximation lemma from Duzaar and Steffen [14] (see also [11,13]).

Lemma 2.1 *(A-harmonic approximation). For each $\nu, L, \varepsilon > 0$, there exists a positive number $\delta(n, N, \nu, L, \varepsilon) \leqslant 1$ with the following property: If $\mathcal{A}$ is a bilinear form on $\mathbb{R}^{N\times n}$ satisfying* (2.4)*, $\varrho > 0$, and $w \in W^{1,2}(B_\varrho, \mathbb{R}^N)$, with $\fint_{B_\varrho} |Dw|^2\,dx \leqslant 1$, is approximatively $\mathcal{A}$-harmonic in the sense that*

$$\left|\fint_{B_\varrho} \mathcal{A}(Dw, D\varphi)\,dx\right| \leqslant \delta(n, N, \nu, L, \varepsilon)\|D\varphi\|_{L^\infty(B_\varrho)} \quad \textit{for every } \varphi \in C_0^1(B_\varrho, \mathbb{R}^N), \tag{2.5}$$

then there exists an $\mathcal{A}$-harmonic function $h \in W^{1,2}(B_\varrho, \mathbb{R}^N)$ such that

$$\fint_{B_\varrho} |Dh|^2\,dx \leqslant 1 \quad \text{and} \quad \varrho^{-2}\fint_{B_\varrho} |w-h|^2\,dx \leqslant \varepsilon. \tag{2.6}$$

Given $u \in L^2(B_\varrho(x_0, \mathbb{R}^N)$, we denote by $P_{x_0,\varrho}$ the unique affine function minimizing the functional $P \mapsto \int_{B_\varrho(x_0)} |u-P|^2\,dx$ amongst all affine functions $P : \mathbb{R}^n \to \mathbb{R}^N$. Note that

$$P_{x_0,\varrho}(x) = (u)_{x_0,\varrho} + Q_{x_0,\varrho}(x-x_0), \tag{2.7}$$

where

$$Q_{x_0,\varrho} = (n+2)\varrho^{-2} \fint_{B_\varrho(x_0)} u(x) \otimes (x-x_0)\,dx \tag{2.8}$$

is the *momentum* of u. From [13,24], we recall the following facts:

Lemma 2.2. *Let $p \geqslant 2$. There exists a constant $c \equiv c(n,p)$ such that the following assertions hold: for every $u \in L^p(B_\varrho(x_0), \mathbb{R}^N)$ we have*

$$|Q_{x_0,\varrho} - Q_{x_0,\theta\varrho}|^p \leqslant \frac{c}{(\theta\varrho)^p} \fint_{B_{\theta\varrho}(x_0)} |u - P_{x_0,\varrho}|^p\,dx. \tag{2.9}$$

For every $u \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ we have

$$\left|Q_{x_0,\varrho} - (Du)_{x_0,\varrho}\right|^p \leqslant c \fint_{B_\varrho(x_0)} \left|Du - (Du)_{x_0,\varrho}\right|^p dx. \tag{2.10}$$

The next result is known as Ekeland's variational principle (see for instance [20], Chapter 5).

Theorem 2.1. *Let (X,d) be a complete metric space, and $J : X \to [0,\infty]$ be a lower semicontinuous functional not identically ∞. Suppose that $u \in X$ satisfies*

$$J(u) < \inf_{w \in X} J(w) + \sigma.$$

Then there exists $v \in X$ such that

$$d(u,v) \leqslant 1, \quad \text{and} \quad J(v) \leqslant J(w) + \sigma d(v,w) \quad \text{for every } w \in X.$$

The last lemma concerns a well-known iteration result; see [20], Lemma 7.3 for a proof.

Lemma 2.3. *Let $\varphi : [0,\varrho] \to \mathbb{R}$ be a positive, non-decreasing function satisfying*

$$\varphi\big(\theta^{k+1}\varrho\big) \leqslant \theta^\gamma \varphi\big(\theta^k\varrho\big) + B\big(\theta^k\varrho\big)^n, \quad \text{for every } k \in \mathbb{N},$$

where $\theta \in (0,1)$, and $\gamma \in (0,n)$. Then there exists $c \equiv c(n,\theta,\gamma)$ such that for every $t \in (0,\varrho)$ the following holds

$$\varphi(t) \leqslant c\left\{\left(\frac{t}{\varrho}\right)^\gamma \varphi(\varrho) + Bt^\gamma\right\}.$$

Finally, we define the Morrey space $L^{q,\gamma}(\Omega, \mathbb{R}^{N\times n})$ for $q \geqslant 1$, $\gamma \in [0,n]$, as the space of those maps $w : \Omega \to \mathbb{R}^N$ such that

$$\sup_{B_\varrho \Subset \Omega} \varrho^{-\gamma} \int_{B_\varrho} |w|^q\,dx < \infty.$$

The local variant $L^{q,\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ is as usual defined by saying that $w \in L^{q,\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ if and only if $w \in L^{q,\gamma}(\Omega', \mathbb{R}^{N\times n})$, for every open subset $\Omega' \Subset \Omega$. See [20], Chapter 2, for more information on such spaces.

3. Systems

3.1. The Caccioppoli inequality

A standard preliminary tool used to obtain partial regularity is the so called Caccioppoli inequality. Here we present a version which differs slightly from those usually presented in the literature in that it exhibits the exact dependence on certain gradient averages. We need this exact dependence for the sequel.

Proposition 3.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to* (1.1) *under the assumptions* (1.2)*, let $B_\varrho \equiv B_\varrho(x_0) \Subset \Omega$ be a ball, and let $P(x) := A(x - x_0) + \tilde{u}$ be a polynomial, with $A \in \mathbb{R}^{N \times n}$ and $\tilde{u} \in \mathbb{R}^N$. Then there exists a constant $c \equiv c(n, N, p, \nu, L)$ such that*

$$\begin{aligned}
&\fint_{B_{\varrho/2}} \left[\left(1+|A|\right)^{p-2}|Du - A|^2 + |Du - A|^p\right] dx \\
&\quad \leqslant c \fint_{B_\varrho} \left[\left(1+|A|\right)^{p-2}\left|\frac{u-P}{\varrho}\right|^2 + \left|\frac{u-P}{\varrho}\right|^p\right] dx \\
&\qquad + c\left(1+|A|\right)^p \fint_{B_\varrho} \left[\omega(\varrho^2) + \omega\left(|u-\tilde{u}|^2\right) + \omega\left(\varrho^2|A|^2\right)\right] dx. \qquad (3.1)
\end{aligned}$$

Proof. For the sake of completeness we provide some details of the proof rather than just sketching the modifications to the usual arguments, since this would lead no significant gain in shortness. All the balls are centered at x_0. Let us define $v(u) := u(x) - P(x)$, and take a smooth cut-off function $\eta \in C_c^\infty(B_\varrho)$ such that $\eta \equiv 1$ on $B_{\varrho/2}$ and $0 \leqslant \eta \leqslant 1$ with $|D\eta| \leqslant 4/\varrho$. Testing (2.1) with $\eta^p v$ yields

$$\fint_{B_\varrho} \eta^p a(x, u, Du)(Du - A)\, dx = -p \fint_{B_\varrho} \eta^{p-1} a(x, u, Du)(v \otimes D\eta)\, dx.$$

Before going on let us observe that since $\eta \leqslant 1$ and $p \geqslant 2$ we have

$$\eta^{2p-2} \leqslant \eta^p. \qquad (3.2)$$

Therefore, using that $\fint_{B_\varrho} a(x_0, \tilde{u}, A) D\varphi\, dx = 0$ we obtain

$$\begin{aligned}
(I) &:= \fint_{B_\varrho} \eta^p\left[a(x, u, Du) - a(x, u, A)\right](Du - A)\, dx \\
&= -p \fint_{B_\varrho} \eta^{p-1}\left[a(x, u, Du) - a(x, u, A)\right](v \otimes D\eta)\, dx \\
&\quad - \fint_{B_\varrho} \left[a(x, u, A) - a\left(x, \tilde{u} + A(x - x_0), A\right)\right] D\varphi\, dx \\
&\quad - \fint_{B_\varrho} \left[a(x, \tilde{u} + A(x - x_0), A) - a(x_0, \tilde{u}, A)\right] D\varphi\, dx \\
&=: (II) + (III) + (IV).
\end{aligned}$$

Now, a standard monotonicity property implied by the left-hand side of $(1.2)_1$ together with the fact that $p \geqslant 2$ implies that

$$c^{-1} \fint_{B_\varrho} \eta^p\left[\left(1+|A|\right)^{p-2}|Du - A|^2 + |Du - A|^p\right] dx \leqslant (I),$$

with $c \equiv c(n, N, p, \nu) > 0$. On the other hand using the inequality on the right-hand side of $(1.2)_1$, and then using Young's inequality with $\sigma \in (0,1)$ and taking into account (3.2) gives us

$$\begin{aligned}
|(II)| &\leqslant c(L) \fint_{B_\varrho} \eta^{p-1}\big(1+|A|+|Du-A|\big)^{p-2}|Du-A||v||D\eta|\,dx \\
&\leqslant \sigma \fint_{B_\varrho} \eta^p\big[\big(1+|A|\big)^{p-2}|Du-A|^2+|Du-A|^p\big]\,dx \\
&\quad + c(p,L,\sigma)\bigg[\fint_{B_\varrho}(1+|A|)^{p-2}|D\eta|^2|v|^2\,dx + \fint_{B_\varrho}|D\eta|^p|v|^p\,dx\bigg].
\end{aligned}$$

Applying Young's inequality with conjugate exponents $(p, p/(p-1))$, and using that $\omega(\cdot)^{p/(p-1)} \leqslant \omega(\cdot)$ as $\omega(\cdot) \leqslant 1$, we have

$$\begin{aligned}
|(III)|+|(IV)| &\leqslant c\big(1+|A|\big)^{p-1} \fint_{B_\varrho}\big[\omega(\varrho^2)+\omega\big(|u-\tilde{u}|^2\big)+\omega\big(\varrho^2|A|^2\big)\big]|D\varphi|\,dx \\
&\leqslant \sigma \fint_{B_\varrho}\eta^p|Du-A|^p\,dx + \fint_{B_\varrho}|D\eta|^p|v|^p\,dx \\
&\quad + c(n,N,p,L,\sigma)\big(1+|A|\big)^p \fint_{B_\varrho}\big[\omega(\varrho^2)+\omega\big(|u-\tilde{u}|^2\big)+\omega\big(\varrho^2|A|^2\big)\big]\,dx.
\end{aligned}$$

Of course we have taken into account $(1.2)_2$ and then used (2.3) repeatedly. Collecting the estimates for the terms $(I), \ldots, (IV)$ and choosing $\sigma \equiv \sigma(n, N, p, \nu, L)$ small enough in order to re-absorb the integrals multiplying σ into the left-hand side, the assertion follows in a standard way upon recalling that $\eta \equiv 1$ on $B_{\varrho/2}$. $\square$

3.2. Excess functionals

For a map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, and a ball $B_\varrho(x_0) \Subset \mathbb{R}^n$, let us now introduce the excess functional $E(u, x_0, \varrho)$. First we define the "re-normalized" Campanato-type excess

$$C(x_0,\varrho) \equiv C(u,x_0,\varrho) := \fint_{B_\varrho}\bigg[\frac{|Du-(Du)_\varrho|^2}{(1+|(Du)_\varrho|)^2}+\frac{|Du-(Du)_\varrho|^p}{(1+|(Du)_\varrho|)^p}\bigg]\,dx,$$

where of course $(Du)_\varrho \equiv (Du)_{x_0,\varrho}$, and then we define the Morrey-type excess

$$M(x_0,\varrho) \equiv M(u,x_0,\varrho) := \varrho^\beta \fint_{B_\varrho}|Du|^2\,dx, \qquad \beta \in (0,2).$$

Finally, the "hybrid excess functional" $E(x_0, \varrho)$ is defined as

$$E(x_0,\varrho) \equiv E(u,x_0,\varrho) := C(x_0,\varrho) + \sqrt{\omega\big(M(x_0,\varrho)\big)} + \sqrt{\omega(\varrho)}. \tag{3.3}$$

3.3. Excess decay

The main goal in this section is to establish a suitable mixed decay estimate for $C(x_0, \varrho)$.

Proposition 3.2. *For each $\beta \in (0,2)$ and $\theta \in (0,1/4)$, there exist two positive numbers*

$$\varepsilon_0 = \varepsilon_0\big(n,N,p,\nu,L,\beta,\theta,\mu(\cdot)\big) > 0, \quad \textit{and} \quad \varepsilon_1 \equiv \varepsilon_1(n,p,\beta,\theta) > 0, \tag{3.4}$$

such that the following is true: *If* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *is weak solution to* (1.1)*, under the assumptions* (1.2) *and* (1.3)*, and* $B_\varrho(x_0) \Subset \Omega$ *is a ball where the smallness conditions*

$$E(x_0, \varrho) < \varepsilon_0 \quad \textit{and} \quad \varrho < \varepsilon_1 \tag{3.5}$$

are satisfied, then

$$C(x_0, \theta\varrho) \leqslant c_* \theta^2 E(x_0, \varrho). \tag{3.6}$$

The constant c_* *depends only upon* n, N, p *and* ν, L.

Proof. *Step 1: Approximate A-harmonicity*. Let us take $x_0 \in \Omega$ and $\varrho > 0$ as in the statement; from now on all the averages will be referred to balls centered at x_0: therefore $(u)_{x_0,\varrho} \equiv (u)_\varrho$, $(Du)_{x_0,\varrho} \equiv (Du)_\varrho$. In the same way all the balls in the following will be centered at x_0. We will argue under the initial smallness assumption

$$\varrho \leqslant 1. \tag{3.7}$$

Take a map $\varphi \in C^1_0(B_\varrho, \mathbb{R}^N)$; without loss of generality, up to considering $\varphi/\|D\varphi\|_{L^\infty(B_\varrho)}$ and then scaling back, we shall assume that $\|D\varphi\|_{L^\infty(B_\varrho)} \leqslant 1$. Using that u weakly solves (1.1) we have

$$(I) := \fint_{B_\varrho} \big[a\big(x_0, (u)_\varrho, Du\big) - a\big(x, u(x), Du\big)\big] D\varphi \, dx = \fint_{B_\varrho} a\big(x_0, (u)_\varrho, Du\big) D\varphi \, dx. \tag{3.8}$$

Noting that $\int_{B_\varrho} a(x_0, (u)_\varrho, (Du)_\varrho) D\varphi \, dx = 0$, we continue with

$$\begin{aligned}
\left| \fint_{B_\varrho} a\big(x_0, (u)_\varrho, Du\big) D\varphi \, dx \right| &= \left| \fint_{B_\varrho} \big[a\big(x_0, (u)_\varrho, Du\big) - a\big(x_0, (u)_\varrho, (Du)_\varrho\big)\big] D\varphi \, dx \right| \\
&= \left| \fint_{B_\varrho} \int_0^1 a_z\big(x_0, (u)_\varrho, (Du)_\varrho + t\big(Du - (Du)_\varrho\big)\big)(Du - (Du)_\varrho, D\varphi) \, dt \, dx \right| =: |(II)|,
\end{aligned}$$

and therefore the previous equality implies

$$\begin{aligned}
&\left| \fint_{B_\varrho} a_z\big(x_0, (u)_\varrho, (Du)_\varrho\big)\big(Du - (Du)_\varrho, D\varphi\big) \, dx \right| \\
&\quad \leqslant \left| \fint_{B_\varrho} \int_0^1 \big[a_z\big(x_0, (u)_\varrho, (Du)_\varrho\big) - a_z\big(x_0, (u)_\varrho, (Du)_\varrho + t\big(Du - (Du)_\varrho\big)\big)\big] \right. \\
&\qquad \left. \times \big(Du - (Du)_\varrho, D\varphi\big) \, dt \, dx \right| + |(II)| =: |(III)| + |(I)|.
\end{aligned} \tag{3.9}$$

Note that the last equality actually defines the quantity $|(III)|$. With $c \equiv c(n, p, L)$ we may now estimate

$$\begin{aligned}
|(I)| &\overset{(1.2)_2}{\leqslant} c \fint_{B_\varrho} \omega\big(\varrho^2 + \big|u - (u)_\varrho\big|^2\big)\big(1 + |Du|\big)^{p-1} \, dx \\
&\leqslant c \fint_{B_\varrho} \omega\big(\varrho^2 + \big|u - (u)_\varrho\big|^2\big)\big|Du - (Du)_\varrho\big|^{p-1} \, dx + c\big(1 + \big|(Du)_\varrho\big|\big)^{p-1} \fint_{B_\varrho} \omega\big(\varrho^2 + \big|u - (u)_\varrho\big|^2\big) \, dx \\
&=: (IV) + (V).
\end{aligned}$$

Using in a standard way the concavity of $\omega(\cdot)$ and Jensen's and Poincaré's inequalities together with (2.2) and (2.3), we have

$$
\begin{aligned}
(V) &\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\omega\big(\varrho^2\big)+c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_\varrho}\omega\big(\big|u-(u)_\varrho\big|^2\big)\,dx\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\omega\big(\varrho^2\big)+c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\omega\bigg(\fint_{B_\varrho}\big|u-(u)_\varrho\big|^2\,dx\bigg)\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\bigg[\omega\big(\varrho^2\big)+\omega\bigg(c(n)\varrho^2\fint_{B_\varrho}|Du|^2\,dx\bigg)\bigg]\\
&\overset{(3.7)}{\leqslant} c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}E(x_0,\varrho),
\end{aligned}
$$

where $c\equiv c(n,p,L)$. Upon using Young's inequality, we may estimate (IV) in the same way we estimated (V):

$$
\begin{aligned}
(IV) &= c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_\varrho}\omega\big(\varrho^2+\big|u-(u)_\varrho\big|^2\big)\frac{|Du-(Du)_\varrho|^{p-1}}{(1+|(Du)_\varrho|)^{p-1}}\,dx\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_\varrho}\omega\big(\varrho^2+\big|u-(u)_\varrho\big|^2\big)\,dx+c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|^{p}}{(1+|(Du)_\varrho|)^{p}}\,dx\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}E(x_0,\varrho),
\end{aligned}
$$

and again $c\equiv c(n,p,L)$. To estimate (III) we use assumption $(1.2)_3$ and Hölder's inequality to write

$$
\begin{aligned}
|(III)| &\overset{(1.2)_3}{\leqslant} c\fint_{B_\varrho}\big|Du-(Du)_\varrho\big|\big(1+\big|(Du)_\varrho\big|+\big|Du-(Du)_\varrho\big|\big)^{p-2}\mu\bigg(\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|^{p-1}}{(1+|(Du)_\varrho|)^{p-1}}\cdot\mu\bigg(\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\\
&\quad+c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|}{(1+|(Du)_\varrho|)}\cdot\mu\bigg(\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\bigg(\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|^{p}}{(1+|(Du)_\varrho|)^{p}}\,dx\bigg)^{\frac{p-1}{p}}\bigg(\fint_{B_\varrho}\mu^p\bigg(\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\bigg)^{\frac{1}{p}}\\
&\quad+c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\bigg(\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|^{2}}{(1+|(Du)_\varrho|)^{2}}\,dx\bigg)^{\frac{1}{2}}\bigg(\fint_{B_\varrho}\mu^2\bigg(\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\bigg)^{\frac{1}{2}}\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\mu\bigg(\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\,dx\bigg)^{\frac{1}{p}}\big[E(x_0,R)\big]^{\frac{p-1}{p}}\\
&\quad+c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\mu\bigg(\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|}{1+|(Du)_\varrho|}\,dx\bigg)^{\frac{1}{2}}\big[E(x_0,R)\big]^{\frac{1}{2}}\\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\big[\mu\big(\sqrt{E(x_0,\varrho)}\big)^{1/2}+\mu\big(\sqrt{E(x_0,\varrho)}\big)^{1/p}\big]\big[E(x_0,\varrho)^{1/2}+E(x_0,\varrho)^{1-1/p}\big].
\end{aligned}
\tag{3.10}
$$

Observe that we have used Jensen's inequality, the concavity of $\mu(\cdot)$, and that $\mu(\cdot) \leqslant 1$ to estimate $\mu(\cdot)^2 \leqslant \mu(\cdot)$ and $\mu(\cdot)^p \leqslant \mu(\cdot)$. For each of our estimates so far the constant c depends on n, p, L. Now, taking into account (3.9) and the estimates for the terms $(II), \dots, (V)$ we have finally proved

$$\left| \fint_{B_\varrho} \mathcal{A}(Dw, D\varphi)\,dx \right| \leqslant c_1 H\big(E(x_0,\varrho)\big)\|D\varphi\|_{L^\infty(B_\varrho)}, \quad \text{for every } \varphi \in C_0^1\big(B_\varrho, \mathbb{R}^N\big),$$

where c_1 depends on n, p, and L, while

$$H\big(E(x_0,\varrho)\big) := \big[\mu\big(\sqrt{E(x_0,\varrho)}\big)^{1/2} + \mu\big(\sqrt{E(x_0,\varrho)}\big)^{1/p} + \sqrt{E(x_0,\varrho)}\big]\big[1 + E(x_0,\varrho)^{1/2-1/p}\big], \tag{3.11}$$

and

$$w := \frac{u - (Du)_\varrho(x-x_0)}{\sqrt{E(x_0,\varrho)}(1+|(Du)_\varrho|)}, \qquad \mathcal{A} := \frac{a_z(x_0,(u)_\varrho,(Du)_\varrho)}{(1+|(Du)_\varrho|)^{p-2}}. \tag{3.12}$$

We note that by $(1.6)_1$ the tensor $\mathcal{A}$ satisfies (2.4), which is an assumption required by Lemma 2.1. Taking $\varepsilon > 0$ to be fixed later, we determine $\delta \equiv \delta(n,N,\nu,L,\varepsilon) > 0$ according to Lemma 2.1. Now assume the smallness condition

$$H\big(E(x_0,\varrho)\big) \leqslant \delta/c_1. \tag{3.13}$$

By its very definition the map w satisfies $\fint_{B_\varrho} |Dw|^2\,dx \leqslant 1$. With (3.13) in force inequality (2.5) is satisfied, so we may apply Lemma 2.1 to get the existence of an $\mathcal{A}$-harmonic map $h \in C^\infty(B_\varrho, \mathbb{R}^N)$ such that

$$\fint_{B_\varrho} |Dh|^2\,dx \leqslant 1 \quad \text{and} \quad \varrho^{-2} \fint_{B_\varrho} |w-h|^2\,dx \leqslant \varepsilon. \tag{3.14}$$

Step 2: Intermediate decay estimate. Being an $\mathcal{A}$-harmonic map, h also satisfies

$$\varrho^{-2} \sup_{B_{\varrho/2}} |Dh|^2 + \sup_{B_{\varrho/2}} \big|D^2 h\big|^2 \leqslant \frac{c}{\varrho^2} \fint_{B_\varrho} |Dh|^2\,dx \overset{(3.14)}{\leqslant} \frac{c}{\varrho^2}, \tag{3.15}$$

with $c = c(n,N,\nu,L)$ (see for instance [20], Chapter 10). With $\theta \in (0,1/4)$ to be specified later, we can apply Taylor's theorem to h at x_0 to deduce

$$\sup_{x \in B_{2\theta\varrho}} \big|h(x) - h(x_0) - Dh(x_0)(x-x_0)\big|^2 \leqslant c\varrho^{-2}(2\theta\varrho)^4 = c\theta^4\varrho^2. \tag{3.16}$$

We now choose

$$\varepsilon = \theta^{n+4}. \tag{3.17}$$

Thus we have, by the triangle inequality together with (3.14) and (3.16), that

$$\begin{aligned}(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} \big|w(x) - h(x_0) - Dh(x_0)(x-x_0)\big|^2\,dx &\leqslant 2(2\theta\varrho)^{-2}\big[(2\theta)^{-n}\varrho^2\varepsilon + c\theta^4\varrho^2\big] \\ &= 2^{-n-1}\theta^{-n-2}\varepsilon + c\theta^2 \overset{(3.17)}{\leqslant} c\theta^2.\end{aligned} \tag{3.18}$$

Recalling the definition of w in (3.12) we obtain

$$\begin{aligned}&(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} \big|u(x) - (Du)_\varrho(x-x_0) - \sqrt{E(x_0,\varrho)}\big(1+\big|(Du)_\varrho\big|\big)\big[h(x_0) + Dh(x_0)(x-x_0)\big]\big|^2\,dx \\ &\quad \leqslant c\theta^2\big(1+\big|(Du)_\varrho\big|\big)^2 E(x_0,\varrho)\end{aligned} \tag{3.19}$$

where the constant c depends only upon n, N, p and ν, L. Denoting by $P_{2\theta\varrho}$ the affine function minimizing $Q \mapsto \int_{B_{2\theta\varrho}} |u-Q|^2\,dx$ amongst all the affine functions Q (see Lemma 2.2 in Section 2), we easily deduce from (3.19) that

$$(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2\,dx \leqslant c\theta^2\big(1+\big|(Du)_\varrho\big|\big)^2 E(x_0,\varrho) \tag{3.20}$$

with $c = c(n, N, p, \nu, L)$. We next derive an estimate for the term $(2\theta\varrho)^{-p} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p \, dx$ which is needed for an application of Caccioppoli's inequality; of course we are going to do this only when $p > 2$, otherwise (3.20) will suffice. To this end, we let p^* be the usual Sobolev conjugate (that is $p^* := \frac{np}{n-p}$ if $p < n$ and $p^* :=$ "any exponent $> p$" if $p \geqslant n$). Select $t \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p^*}. \tag{3.21}$$

With this choice of t, we use in turn the L^p-interpolation inequality, the definition of $P_{2\theta\varrho}$, the estimate found in (3.20) and Sobolev's–Poincaré inequality, to argue

$$\begin{aligned} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p \, dx &\leqslant \left(\fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2 \, dx \right)^{(1-t)\frac{p}{2}} \left(\fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^{p^*} \, dx \right)^{t\frac{p}{p^*}} \\ &\leqslant c\varrho^p \theta^{(2-t)p} \left(1 + \left|(Du)_\varrho\right|\right)^{(1-t)p} E(x_0, \varrho)^{\frac{(1-t)p}{2}} \left(\fint_{B_{2\theta\varrho}} \left|D(u - P_{2\theta\varrho})\right|^p \, dx \right)^t, \end{aligned} \tag{3.22}$$

where the constant c depends on n, N, p, and ν, L. In order to estimate the last integral appearing in the previous estimate, let us denote by P_ϱ the unique affine function which minimizes $P \mapsto \int_{B_\varrho} |u - P|^2 \, dx$. Using in turn Minkowski's inequality, (2.9), Poincaré's inequality and (2.10) yields

$$\begin{aligned} \left(\int_{B_{2\theta\varrho}} \left|D(u - P_{2\theta\varrho})\right|^p \, dx \right)^{\frac{1}{p}} &\leqslant \left(\int_{B_{2\theta\varrho}} \left|D(u - P_\varrho)\right|^p \, dx \right)^{\frac{1}{p}} + |B_{2\theta\varrho}|^{\frac{1}{p}} |DP_{2\theta\varrho} - DP_\varrho| \\ &\leqslant c|B_\varrho|^{\frac{1}{p}} \left(\fint_{B_\varrho} |D(u - P_\varrho)|^p \, dx \right)^{\frac{1}{p}} \\ &\quad + c\theta^{-1-\frac{n}{p}} |B_{2\theta\varrho}|^{\frac{1}{p}} \left(\varrho^{-p} \fint_{B_\varrho} |u - P_\varrho|^p \, dx \right)^{\frac{1}{p}} \\ &\leqslant c\theta^{-1} |B_\varrho|^{\frac{1}{p}} \left(\fint_{B_\varrho} \left|D(u - P_\varrho)\right|^p \, dx \right)^{\frac{1}{p}} \\ &\leqslant c\theta^{-1} |B_\varrho|^{\frac{1}{p}} \left(\fint_{B_\varrho} \left|Du - (Du)_\varrho\right|^p \, dx \right)^{\frac{1}{p}} \\ &\leqslant c\theta^{-1} |B_\varrho|^{\frac{1}{p}} \left(1 + \left|(Du)_\varrho\right|\right) E(x_0, \varrho)^{\frac{1}{p}}, \end{aligned}$$

with $c = c(n, p)$. Inserting this in (3.22), we find that

$$(2\theta\varrho)^{-p} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p \, dx \leqslant c\theta^{(1-2t)p - tn} E(x_0, \varrho)^{\frac{p-2}{2}(1-t)} \left(1 + \left|(Du)_\varrho\right|\right)^p E(x_0, \varrho). \tag{3.23}$$

Observe that when $p = 2$ we get $t = 0$ in (3.21) and therefore the latter inequality reduces to (3.20). Now when $p > 2$ we assume the smallness condition

$$E(x_0, \varrho) \leqslant \theta^{\frac{2[(2t-1)p + tn + 2]}{(1-t)(p-2)}}, \quad p > 2. \tag{3.24}$$

Therefore (3.23) becomes

$$(2\theta\varrho)^{-p} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p \, dx \leqslant c\theta^2 \left(1 + \left|(Du)_\varrho\right|\right)^p E(x_0, \varrho), \tag{3.25}$$

where again $c \equiv c(n, N, p, \nu, L)$. Now in both (3.20) and (3.25) we want to replace $(Du)_\varrho$ by $(Du)_{\theta\varrho}$. We argue as follows:

$$\begin{aligned}
1 + \left|(Du)_\varrho\right| &\leqslant 1 + \left|(Du)_{\theta\varrho} - (Du)_\varrho\right| + \left|(Du)_{\theta\varrho}\right| \\
&\leqslant \fint_{B_{\theta\varrho}} \left|Du - (Du)_\varrho\right| dx + 1 + \left|(Du)_{\theta\varrho}\right| \\
&\leqslant \frac{(1 + |(Du)_\varrho|)}{\theta^n} \fint_{B_\varrho} \frac{|Du - (Du)_\varrho|}{(1 + |(Du)_\varrho|)} dx + \left(1 + \left|(Du)_{\theta\varrho}\right|\right) \\
&\leqslant \theta^{-n}\sqrt{E(x_0, \varrho)}\left(1 + \left|(Du)_\varrho\right|\right) + \left(1 + \left|(Du)_{\theta\varrho}\right|\right),
\end{aligned}$$

so that imposing the smallness condition

$$\sqrt{E(x_0, \varrho)} \leqslant \theta^n/8 \tag{3.26}$$

we get

$$1 + \left|(Du)_\varrho\right| \leqslant 2\left(1 + \left|(Du)_{\theta\varrho}\right|\right). \tag{3.27}$$

In a completely similar manner we also deduce that

$$1 + \left|(Du)_{2\theta\varrho}\right| \leqslant 2\left(1 + \left|(Du)_{\theta\varrho}\right|\right). \tag{3.28}$$

Indeed first note that

$$1 + \left|(Du)_\varrho\right| \leqslant 2\left(1 + \left|(Du)_{2\theta\varrho}\right|\right),$$

exactly as (3.27). Then, as immediately before (3.26), but also using the last inequality, we get

$$\begin{aligned}
1 + \left|(Du)_{2\theta\varrho}\right| &\leqslant 1 + \left|(Du)_{\theta\varrho}\right| + \left|(Du)_{2\theta\varrho} - (Du)_\varrho\right| + \left|(Du)_{\theta\varrho} - (Du)_\varrho\right| \\
&\leqslant 1 + \left|(Du)_{\theta\varrho}\right| + 2\theta^{-n}\left(1 + \left|(Du)_\varrho\right|\right)\sqrt{E(x_0, \varrho)} \\
&\leqslant 1 + \left|(Du)_{\theta\varrho}\right| + 4\theta^{-n}\left(1 + \left|(Du)_{2\theta\varrho}\right|\right)\sqrt{E(x_0, \varrho)},
\end{aligned}$$

and (3.28) again follows by (3.26). Now, using (3.27) in (3.20) and (3.25) gives us

$$(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2 dx \leqslant c\theta^2\left(1 + \left|(Du)_{\theta\varrho}\right|\right)^2 E(x_0, \varrho), \tag{3.29}$$

and

$$(2\theta\varrho)^{-p} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p dx \leqslant c\theta^2\left(1 + \left|(Du)_{\theta\varrho}\right|\right)^p E(x_0, \varrho), \tag{3.30}$$

respectively, where $c \equiv c(n, N, p, \nu, L)$.

Step 3: Full decay estimate for the Campanato-type excess. We are now going to apply the Caccioppoli's inequality (3.1), taking as a polynomial $P \equiv P_{2\theta\varrho}$. Keeping in mind (2.7), this yields

$$\begin{aligned}
&\fint_{B_{\theta\varrho}} \left[\left(1 + |Q_{2\theta\varrho}|\right)^{p-2}|Du - Q_{2\theta\varrho}|^2 + |Du - Q_{2\theta\varrho}|^p\right] dx \\
&\quad\leqslant c \fint_{B_{2\theta\varrho}} \left[\left(1 + |Q_{2\theta\varrho}|\right)^{p-2}\left|\frac{u - P_{2\theta\varrho}}{2\theta\varrho}\right|^2 + \left|\frac{u - P_{2\theta\varrho}}{2\theta\varrho}\right|^p\right] dx \\
&\qquad + c\left(1 + |Q_{2\theta\varrho}|\right)^p \fint_{B_{2\theta\varrho}} \left[\omega(\varrho^2) + \omega\left(|u - (u)_{2\theta\varrho}|^2\right) + \omega\left(\varrho^2|Q_{2\theta\varrho}|^2\right)\right] dx,
\end{aligned} \tag{3.31}$$

with $c \equiv c(n, N, p, \nu, L)$; for the meaning of $Q_{x_0,2\theta\varrho} \equiv Q_{2\theta\varrho}$ see (2.7)–(2.8) in Section 2. Now, first we are going to estimate the last integral appearing in (3.31). Using Lemma 2.2

$$\begin{aligned}|Q_{2\theta\varrho}|^2 &\leqslant 2\big|Q_{2\theta\varrho} - (Du)_{2\theta\varrho}\big|^2 + 2\big|(Du)_{2\theta\varrho}\big|^2\\ &\leqslant c \fint_{B_{2\theta\varrho}} \big|Du - (Du)_{2\theta\varrho}\big|^2\,dx + c \fint_{B_{2\theta\varrho}} |Du|^2\,dx\\ &\leqslant \frac{c_2}{\theta^n}\fint_{B_\varrho}\big|Du - (Du)_\varrho\big|^2\,dx + \frac{c_3}{\theta^n}\fint_{B_\varrho}|Du|^2\,dx,\end{aligned}$$

with c_2, c_3 depending only on n, N, p, so that we have

$$\begin{aligned}\varrho^2|Q_{2\theta\varrho}|^2 &\leqslant \frac{c_2\varrho^2(1+|(Du)_\varrho|)^2}{\theta^n}\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|^2}{(1+|(Du)_\varrho|)^2}\,dx + \frac{c_3\varrho^2}{\theta^n}\fint_{B_\varrho}|Du|^2\,dx\\ &\leqslant \varrho^2 + \frac{\varrho^2}{2}\fint_{B_\varrho}|Du|^2\,dx + \frac{\varrho^\beta}{2}\fint_{B_\varrho}|Du|^2\,dx\\ &\overset{(3.7)}{\leqslant} \varrho + M(x_0,\varrho),\end{aligned}\tag{3.32}$$

where the second to last inequality holds provided we assume the smallness conditions

$$E(x_0,\varrho) \leqslant \theta^n/(4c_2) \quad\text{and}\quad \varrho \leqslant (\theta^n/2c_3)^{1/(2-\beta)}.\tag{3.33}$$

We recall again that both $c_2, c_3 \equiv c_2, c_3(n, p)$. Using Poincaré's inequality and (3.33), we have, by possibly increasing the value of c_3 but otherwise keeping the same dependence on the constants, that

$$\fint_{B_{2\theta\varrho}} \big|u-(u)_{2\theta\varrho}\big|^2\,dx \leqslant \frac{c}{\theta^n}\fint_{B_\varrho}\big|u-(u)_\varrho\big|^2\,dx \leqslant \frac{c_3\varrho^2}{\theta^n}\fint_{B_\varrho}|Du|^2\,dx \leqslant M(x_0,\varrho),\tag{3.34}$$

so that the concavity of $\omega(\cdot)$ together with Jensen's inequality implies

$$\fint_{B_{2\theta\varrho}} \omega\big(\big|u-(u)_{2\theta\varrho}\big|^2\big)\,dx \leqslant \omega\bigg(\fint_{B_{2\theta\varrho}}\big|u-(u)_{2\theta\varrho}\big|^2\,dx\bigg) \leqslant \omega\big(M(x_0,\varrho)\big).\tag{3.35}$$

Collecting (3.32), (3.34), (3.35), and using (2.3), we arrive at

$$\fint_{B_{2\theta\varrho}} \big[\omega(\varrho^2) + \omega\big(\big|u-(u)_{2\theta\varrho}\big|^2\big) + \omega\big(\varrho^2|Q_{2\theta\varrho}|^2\big)\big]\,dx \leqslant 2\omega(\varrho) + 2\omega\big(M(x_0,\varrho)\big).\tag{3.36}$$

Our next task is to replace all the momenta Q appearing in (3.31) with suitable averages of Du. Arguing in a way similar to the one we followed for (3.32), we have, by again using Lemma 2.2 and possibly increasing the value of c_3 in (3.33), that

$$\begin{aligned}|Q_{2\theta\varrho}|^p &\leqslant c\big|Q_{2\theta\varrho}-(Du)_{2\theta\varrho}\big|^p + c\big|(Du)_{2\theta\varrho}\big|^p\\ &\leqslant \frac{c(1+|(Du)_\varrho|)^p}{\theta^n}\fint_{B_\varrho}\frac{|Du-(Du)_\varrho|^p}{(1+|(Du)_\varrho|)^p}\,dx + c\big|(Du)_{2\theta\varrho}\big|^p\\ &\overset{(3.28),(3.33)}{\leqslant} c\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p.\end{aligned}\tag{3.37}$$

Now we impose the smallness condition

$$\sqrt{\omega\big(M(x_0,\varrho)\big)} + \sqrt{\omega(\varrho)} \leqslant \theta^2.\tag{3.38}$$

Taking into account (3.36), (3.37), and the latter estimate we have immediately

$$\begin{aligned}&\big(1+|Q_{2\theta\varrho}|\big)^p \fint_{B_{2\theta\varrho}} \big[\omega\big(\varrho^2\big)+\omega\big(\big|u-(u)_{2\theta\varrho}\big|^2\big)+\omega\big(\varrho^2|Q_{2\theta\varrho}|^2\big)\big]\,dx\\ &\quad\leqslant c\theta^2\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p E(x_0,\varrho),\end{aligned}\tag{3.39}$$

with $c\equiv c(n,p)$. This completes the estimation of the last integral appearing in (3.31). Our next aim is to replace all the momenta $Q_{2\theta\varrho}$ appearing in (3.31) by suitable averages of Du. First we observe that, trivially

$$\fint_{B_{\theta\varrho}} \big(1+\big|Q_{2\theta\varrho}\big|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2\,dx \leqslant \fint_{B_{\theta\varrho}} \big(1+\big|Q_{2\theta\varrho}\big|\big)^{p-2}\big|Du-Q_{2\theta\varrho}\big|^2\,dx \tag{3.40}$$

and

$$\fint_{B_{\theta\varrho}} \big|Du-(Du)_{\theta\varrho}\big|^p\,dx \leqslant 2^p \fint_{B_{\theta\varrho}} \big|Du-Q_{2\theta\varrho}\big|^p\,dx. \tag{3.41}$$

Then, when $p>2$, we use Young's inequality with conjugate exponents $(p/2,\ p/(p-2))$ to write

$$\begin{aligned}&\fint_{B_{\theta\varrho}} \big(1+\big|(Du)_{\theta\varrho}\big|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2\,dx\\
&\quad\leqslant c\fint_{B_{\theta\varrho}} \big(1+\big|Q_{2\theta\varrho}\big|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2\,dx + c\big|Q_{\theta\varrho}-(Du)_{\theta\varrho}\big|^p + c|Q_{\theta\varrho}-Q_{2\theta\varrho}|^p\\
&\qquad + c\fint_{B_{\theta\varrho}} \big|Du-(Du)_{\theta\varrho}\big|^p\,dx\\
&\quad\leqslant c\fint_{B_{\theta\varrho}} \big(1+|Q_{2\theta\varrho}|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2\,dx + c\fint_{B_{\theta\varrho}} \big|Du-(Du)_{\theta\varrho}\big|^p\,dx\\
&\qquad + c(2\theta\varrho)^{-p}\fint_{B_{2\theta\varrho}} |u-P_{2\theta\varrho}|^p\,dx.\end{aligned}\tag{3.42}$$

where in the last inequality we repeatedly used Lemma 2.2. Combining (3.40)–(3.42) with (3.31) and (3.39), and finally estimating $(1+|Q_{2\theta\varrho}|)^{p-2}\leqslant c(1+|(Du)_{\theta\varrho}|)^{p-2}$ via (3.37), we have

$$\begin{aligned}&\fint_{B_{\theta\varrho}} \big[\big(1+\big|(Du)_{\theta\varrho}\big|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2+\big|Du-(Du)_{\theta\varrho}\big|^p\big]\,dx\\
&\quad\leqslant c\fint_{B_{2\theta\varrho}} \bigg[\big(1+\big|(Du)_{\theta\varrho}\big|\big)^{p-2}\bigg|\frac{u-P_{2\theta\varrho}}{\theta\varrho}\bigg|^2+\bigg|\frac{u-P_{2\theta\varrho}}{\theta\varrho}\bigg|^p\bigg]\,dx + c\theta^2\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p E(x_0,\varrho),\end{aligned}\tag{3.43}$$

with $c\equiv c(n,N,p,\nu,L)$. Now we estimate the second integral in the latter formula via (3.29) and (3.30); an elementary manipulation yields

$$\fint_{B_{\theta\varrho}} \big[\big(1+\big|(Du)_{\theta\varrho}\big|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2+\big|Du-(Du)_{\theta\varrho}\big|^p\big]\,dx \leqslant c_*\theta^2\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p E(x_0,\varrho), \tag{3.44}$$

with finally $c_*\equiv c_*(n,N,p,\nu,L)$, and (3.6) follows upon dividing everything by $(1+|(Du)_{\theta\varrho}|)^p$. All the previous computations hold provided the smallness conditions (3.7), (3.13), (3.24), (3.26), (3.33) and (3.38) hold true. In order to have them satisfied we make use of assumptions (1.3) and (3.5), but we have to verify the precise dependence, described in the statement, for the constants $\varepsilon_0,\varepsilon_1$ appearing in (3.4). The choice of ε, in order to apply the A-harmonic approximation lemma, is made in (3.17); therefore $\varepsilon \equiv \varepsilon(n, \theta)$. This in turn influences the choice of $\delta \equiv \delta(n, N, \nu, L, \varepsilon) \equiv \delta(n, N, \nu, L, \theta)$ in Step 1, keeping in mind (3.13). Indeed since c_1 depends only on n, N, p, ν, L, in order to simultaneously meet (3.7) and (3.13), we have to take $E(x_0, \varrho) \leqslant \varepsilon_0$ where ε_0 at this stage depends on $n, N, p, \nu, L, \mu(\cdot)$, and ultimately also on θ. The same dependence occurs when imposing (3.24) and (3.26) as far as $E(x_0, \varrho)$ and ε_0 are concerned. When imposing the second inequality in (3.33), and also when imposing (3.7), we need to use the second assumption in (3.5). The indicated dependence of ε_1 is seen by noting that c_3 in (3.32) only depends on n, N, p, and that θ and β are involved in the second inequality in (3.33). Note that $\varepsilon_0, \varepsilon_1 \searrow 0$ when $\theta \searrow 0$; the same happens when $\beta \nearrow 2$. □

3.4. Iteration

Here we give the proof of Theorem 1.1 via a suitable iteration procedure.

Proof of Theorem 1.1. *Step 1: Choice of the constants.* Fix $\beta \in (0, 2)$ and $\alpha \in (0, 1)$ as in the statement of Theorem 1.1; then take $\gamma \equiv \gamma(\alpha) \in (n-2, n)$ such that

$$\alpha = 1 - \frac{n-\gamma}{2}. \tag{3.45}$$

We then choose $\theta \in (0, 1/4)$ such that

$$\theta := \min\left\{\left(\frac{1}{24c_*}\right)^{\frac{p}{2}}, \left(\frac{1}{24}\right)^{\frac{p}{\beta}}, \left(\frac{1}{8}\right)^{\frac{p}{n-\gamma}}\right\} \tag{3.46}$$

where $c_* \equiv c_*(n, N, p, \nu, L) \geqslant 1$ is the constant appearing in (3.6). Therefore this choice fixes $\theta \equiv \theta(n, N, p, \nu, L, \alpha, \beta)$. Now fix the new constant

$$\varepsilon_2 := \min\left\{\frac{\theta^n}{24^p}, \frac{\varepsilon_0}{24^p}\right\}, \tag{3.47}$$

where ε_0 appears in (3.4), and θ has been fixed in (3.46). Taking into account the dependence of ε_0 in (3.4), this fixes $\varepsilon_2 \equiv \varepsilon_2(n, N, p, \nu, L, \alpha, \beta, \mu(\cdot))$. Once ε_2 has been fixed, by (1.3) we determine $\delta_1 > 0$ such that

$$t \in [0, \delta_1] \Longrightarrow \sqrt{\omega(t)} < \varepsilon_2, \tag{3.48}$$

and keeping in mind the dependence of ε_2 this fixes $\delta_1 \equiv \delta_1(n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot))$. Finally we are going to choose the radii size. We define the maximal radius

$$\varrho_m := \min\left\{\delta_1^{\frac{1}{\beta}}, \delta_1, \varepsilon_1\right\} > 0, \tag{3.49}$$

where ε_1 appears in (3.4), with θ fixed in (3.46); note that $\varrho_m \leqslant 1$. Taking into account the dependence of ε_1 this is turn fixes $\varrho_m \equiv \varrho_m(n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot))$. From now on all radii ϱ considered in the following will be picked in such a way to satisfy $\varrho \leqslant \varrho_m$.

Step 2: An almost BMO estimate. Let us consider $x_0 \in \Omega$ and a positive radius $\varrho \leqslant \varrho_m$, for which it happens that

$$C(x_0, \varrho) < \varepsilon_2 \quad \text{and} \quad M(x_0, \varrho) < \delta_1. \tag{3.50}$$

Let us show that this implies that for every $k = 0, 1, 2, \ldots$

$$(\mathbf{I})_k \quad C(x_0, \theta^k\varrho) < \varepsilon_2 \quad \text{and} \quad M(x_0, \theta^k\varrho) < \delta_1.$$

We shall of course proceed by induction: we assume $(\mathbf{I})_k$ and prove $(\mathbf{I})_{k+1}$. We have, using ω_n to denote the volume of the n-dimensional unit ball

$$\begin{aligned}\int_{B_{\theta^k\varrho}} \left|Du - (Du)_{\theta^k\varrho}\right|^2 dx &= \left(1 + \left|(Du)_{\theta^k\varrho}\right|\right)^2 \int_{B_{\theta^k\varrho}} \frac{|Du - (Du)_{\theta^k\varrho}|^2}{(1 + |(Du)_{\theta^k\varrho}|)^2}\, dx \\ &\leqslant \omega_n\left(1 + \left|(Du)_{\theta^k\varrho}\right|\right)^2 C\left(x_0, \theta^k\varrho\right)\left(\theta^k\varrho\right)^n\end{aligned}$$

$$
\begin{aligned}
&\overset{(\mathbf{I})_k}{<} \omega_n\big(1+\big|(Du)_{\theta^k\varrho}\big|\big)^2\varepsilon_2\big(\theta^k\varrho\big)^n \\
&< 2\varepsilon_2\int_{B_{\theta^k\varrho}}|Du|^2\,dx+2\omega_n\varepsilon_2\big(\theta^k\varrho\big)^n,
\end{aligned}
\tag{3.51}
$$

and therefore

$$
\big(\theta^k\varrho\big)^\beta \fint_{B_{\theta^k\varrho}}\big|Du-(Du)_{\theta^k\varrho}\big|^2\,dx \leqslant 2\varepsilon_2 M\big(x_0,\theta^k\varrho\big)+2\varepsilon_2\big(\theta^k\varrho\big)^\beta. \tag{3.52}
$$

Using the latter estimate we may now prove the second inequality in (3.50) as follows:

$$
\begin{aligned}
M\big(x_0,\theta^{k+1}\big) &\leqslant 2\big(\theta^{k+1}\varrho\big)^\beta \fint_{B_{\theta^{k+1}\varrho}}\big|Du-(Du)_{\theta^k\varrho}\big|^2\,dx+2\big(\theta^{k+1}\varrho\big)^\beta\big|(Du)_{\theta^k\varrho}\big|^2 \\
&\leqslant 2\theta^{\beta-n}\big(\theta^k\varrho\big)^\beta \fint_{B_{\theta^k\varrho}}\big|Du-(Du)_{\theta^k\varrho}\big|^2\,dx+2\theta^\beta M\big(x_0,\theta^k\varrho\big) \\
&\overset{(3.52)}{\leqslant} 4\theta^{\beta-n}\varepsilon_2 M\big(x_0,\theta^k\varrho\big)+4\theta^{\beta-n}\varepsilon_2\big(\theta^k\varrho\big)^\beta+2\theta^\beta M\big(x_0,\theta^k\varrho\big) \\
&\overset{(3.46)\text{–}(3.47)}{\leqslant} \frac{M(x_0,\theta^k\varrho)}{8}+\frac{\varrho^\beta}{8}+\frac{M(x_0,\theta^k\varrho)}{8} \\
&\overset{(\mathbf{I})_k}{<} \delta_1.
\end{aligned}
\tag{3.53}
$$

As for $C(x_0,\theta^{k+1}\varrho)$, we recall that $(\mathbf{I})_k$ and (3.48) imply $\sqrt{\omega(M(x_0,\theta^k\varrho))}<\varepsilon_2$, while $\varrho<\varrho_m$ and (3.49) imply $\sqrt{\omega(\theta^k\varrho)}\leqslant\sqrt{\omega(\varrho)}\leqslant\varepsilon_2$ so that, using the definition in (3.3) we have $E(x_0,\theta^k\varrho)<3\varepsilon_2<\varepsilon_0$. Therefore, taking also into account (3.49), we may apply Proposition 3.2, using (3.46) we have $C(x_0,\theta^{k+1}\varrho)\leqslant c_*\theta^2E(x_0,\theta^k\varrho)\leqslant c_*\theta^2 3\varepsilon_2<\varepsilon_2$, and the proof of $(\mathbf{I})_{k+1}$ is complete. Therefore $(\mathbf{I})_k$ holds for every $k\in\mathbb{N}$.

Step 3: Final iteration and partial regularity. Again we consider a ball $B_\varrho(x_0)\Subset\Omega$ such that (3.50) holds; therefore $(\mathbf{I})_k$ holds too for every $k\in\mathbb{N}$. With ω_n still denoting the volume of the n-dimensional unit ball, we have

$$
\begin{aligned}
\int_{B_{\theta^{k+1}\varrho}}|Du|^2\,dx &\leqslant 2\omega_n\big(\theta^{k+1}\varrho\big)^n\big|(Du)_{\theta^k\varrho}\big|^2+2\int_{B_{\theta^{k+1}\varrho}}\big|Du-(Du)_{\theta^k\varrho}\big|^2\,dx \\
&\leqslant 2\theta^n\int_{B_{\theta^k\varrho}}|Du|^2\,dx+2\int_{B_{\theta^k\varrho}}\big|Du-(Du)_{\theta^k\varrho}\big|^2\,dx \\
&\overset{(3.51)}{\leqslant} 4\big(\theta^n+\varepsilon_2\big)\int_{B_{\theta^k\varrho}}|Du|^2\,dx+4\omega_n\varepsilon_2\big(\theta^k\varrho\big)^n \\
&\overset{(3.47)}{\leqslant} 8\theta^n\int_{B_{\theta^k\varrho}}|Du|^2\,dx+4\omega_n\varepsilon_2\big(\theta^k\varrho\big)^n \\
&\overset{(3.46)}{\leqslant} \theta^\gamma\int_{B_{\theta^k\varrho}}|Du|^2\,dx+4\omega_n\varepsilon_2\big(\theta^k\varrho\big)^n,
\end{aligned}
\tag{3.54}
$$

where γ is as in (3.45). Letting $\varphi(t):=\int_{B_t}|Du|^2\,dx$, and estimating $\varepsilon_2\leqslant 1$, we have proved that

$$
\varphi\big(\theta^{k+1}\varrho\big)\leqslant\theta^\gamma\varphi\big(\theta^k\varrho\big)+4\omega_n\big(\theta^k\varrho\big)^n. \tag{3.55}
$$

Therefore we may apply Lemma 2.3 in order to obtain

$$
\varphi(t)\leqslant c_4\left[\left(\frac{t}{\varrho}\right)^\gamma\varphi(\varrho)+t^\gamma\right],\quad \text{for every } t\leqslant\varrho, \tag{3.56}
$$

where the constant c_4 depends on n, γ, θ, and therefore ultimately on $n, N, p, \nu, L, \alpha, \beta$. Exploiting the latter inequality gives us

$$\int_{B_t(x_0)} |Du|^2\,dx \leqslant \frac{c_4}{\varrho^\gamma}\left[\int_\Omega |Du|^2\,dx + 1\right]t^\gamma \quad \text{for every } t \leqslant \varrho. \tag{3.57}$$

Observe that (3.57) holds uniformly for all those points $x_0 \in \Omega$ satisfying (3.50) with given radius ϱ. Now we conclude with partial regularity in an almost standard way; denote by Ω_u the set of regular points of u in the sense that

$$\Omega_u := \bigl\{x_0 \in \Omega\colon\ u \in C^{0,t}\bigl(A(x_0), \mathbb{R}^N\bigr) \text{ for every } t \in (0,1) \text{ and some } A(x_0)\bigr\},$$

where $A(x_0)$ denotes an open neighborhood of x_0. Accordingly, we fix $\alpha \in (0,1)$ and define

$$\Omega_u^\alpha := \bigl\{x_0 \in \Omega\colon\ u \in C^{0,\alpha}\bigl(A(x_0), \mathbb{R}^N\bigr) \text{ for some } A(x_0)\bigr\}.$$

Having (3.50) in mind, let us fix

$$s := \min\{\varepsilon_2, \delta_1\} \tag{3.58}$$

and, recalling the dependence upon the various constants of the numbers ε_2 and δ_1, note that s depends on $n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot)$. Indeed, the number s determined in (3.58) is going to be the one that appears in the statement of Theorem 1.1 when considering Ω_u^α with a fixed choice of α, β. Take a point $x_0 \in \Omega$ such that

$$\liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \bigl|Du - (Du)_\varrho\bigr|^p\,dx < s \quad \text{and} \quad \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2\,dx < s; \tag{3.59}$$

we are going to show that $x_0 \in \Omega_u^\alpha$. Using (3.59) and the definition (3.58) we can find $0 < \varrho \leqslant \varrho_m$ such that the smallness conditions in (3.50) are satisfied; observe that at this stage the radius $\varrho > 0$ depends on everything, that is $\varrho \equiv \varrho(n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot), x_0, u)$. In turn, if (3.50) is satisfied for a fixed radius $\varrho \leqslant \varrho_m$ at the point x_0, then, by the absolute continuity of the integral it is also satisfied in a whole neighborhood of the point x_0, say a ball $B_T(x_0)$; we can of course assume that $T \leqslant \varrho$, therefore we have that (3.57) holds for every $\varrho < T/4$ and with x_0 replaced by any $y \in B_{T/4}(x_0)$. But this means that Du belongs to the Morrey space $L^{2,\gamma}(B_{T/4}(x_0), \mathbb{R}^{N\times n})$, and therefore, $u \in C^{0,\alpha}(B_{T/4}(x_0), \mathbb{R}^N)$ where $\alpha := 1 - (n-\gamma)/2$ exactly as in (3.45), by the well known Morrey–Campanato embedding theorem. In conclusion, whenever x_0 is a point such that (3.59) is valid, we find that $u \in C^{0,\alpha}$ in the neighborhood $B_{T/4}(x_0)$ of x_0, and this works for every $\alpha < 1$, provided s is chosen accordingly as a function depending on α too, and eventually restricting the size of the ball $B_{T/4}(x_0)$. From this, and upon observing that (3.59) is satisfied at almost every point, it follows that Ω_u has full measure: $|\Omega \setminus \Omega_u| = 0$. Observe that $\Omega_u, \Omega_u^\alpha$ are open subsets by their very definitions. The inclusion for $\Omega \setminus \Omega_u^\alpha$ in the statement of Theorem 1.1 clearly follows from the last argumentation when fixing the values of α and β, and accordingly the value of the number s. As for the inclusion regarding $\Omega \setminus \Omega_u$, trivially, we have that if $x_0 \in \Omega$ is a point such that

$$\liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \bigl|Du - (Du)_\varrho\bigr|^p\,dx = 0 \quad \text{and} \quad \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2\,dx = 0. \tag{3.60}$$

then (3.59) is satisfied for every choice of s, and ultimately for every choice of the Hölder degree of continuity $\alpha \in (0,1)$ and $\beta \in (0,2)$. As a consequence the first inclusion for $\Omega \setminus \Omega_u$ in Theorem 1.1 follows too. □

A straightforward by-product of the previous proof is the following:

Theorem 3.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (1.1) *under the assumptions* (1.2) *and* (1.3). *Then $Du \in L^{2,\gamma}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{N\times n})$ for every $\gamma \in (0,n)$, where $\Omega_u \subseteq \Omega$ is an open subset such that $|\Omega \setminus \Omega_u| = 0$.*

Remark 3.1. While the solution u is found to be in $C^{0,\alpha}(\Omega_u, \mathbb{R}^N)$ for every $\alpha \in (0,1)$, the Hölder semi-norm $[u]_{0,\alpha}$ blows-up as $\alpha \nearrow 1$. This can be checked by carefully tracing the constant dependence, especially when considering a regular point $x_0 \in \Omega_u$, and observing that the radius of the ball we find $B_{T/4}(x_0)$ decreases as α approaches 1.

Remark 3.2. By carefully checking the proof of Theorem 1.1 one can build slightly better inclusions for the singular sets $\Omega \setminus \Omega_u$ and $\Omega \setminus \Omega_u^\alpha$ than the ones presented in Theorem 1.1, which are of standard type. Indeed, in order to fulfill the first inequality in (3.50) we ask for (3.59), while, as $p \geqslant 2$, we could just ask for the weaker inequality

$$\liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \frac{|Du - (Du)_\varrho|^p}{(1 + |(Du)_\varrho|)^p}\, dx < s$$

to be satisfied. In fact, proceeding exactly as in the proof of Theorem 1.1 we would get

$$\Omega \setminus \Omega_u \subseteq \left\{ x_0 \in \Omega\colon \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \frac{|Du - (Du)_\varrho|^p}{(1 + |(Du)_\varrho|)^p}\, dx > 0 \text{ or } \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2\, dx > 0 \right\}, \tag{3.61}$$

which naturally reflects the fact that, since we are proving Hölder continuity estimates and not Lipschitz ones, we do not mind the fact that Du blows-up in a regular point, a circumstance that even helps fulfilling the first limit condition in (3.61). A similar argumentation, and a similar inclusion, hold for $\Omega \setminus \Omega_u^\alpha$ too.

Remark 3.3. Better inclusions for the singular set can be obtained when $p \geqslant n$. Indeed, when $p > n$ Sobolev–Morrey embedding theorem ensures that u is everywhere Hölder continuous with exponent $1 - n/p$. The same conclusion follows in the borderline case $p = n$ using the higher integrability of the gradient (see [20], Chapter 6) that is $Du \in L^{q_1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ for some $q_1 > n$, and then again u is everywhere Hölder continuous with exponent $1 - n/q_1$. In such cases the strength of Theorem 1.1 relies in ensuring that u is Hölder continuous *with every exponent* $\alpha < 1$, but unfortunately, only outside a negligible singular set $\Omega \setminus \Omega_u$. Nevertheless a better inclusion for $\Omega \setminus \Omega_u$ is available:

$$\Omega \setminus \Omega_u \subseteq \left\{ x_0 \in \Omega\colon \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \frac{|Du - (Du)_\varrho|^p}{(1 + |(Du)_\varrho|)^p}\, dx > 0 \right\}. \tag{3.62}$$

Indeed, without going back to Theorem 1.1 using the fact that u is already Hölder continuous directly into the proof, let us first notice that by Hölder's inequality we have that

$$\varrho^\beta \fint_{B_\varrho(x_0)} |Du|^2\, dx \leqslant \left(\varrho^{\frac{p\beta}{2}} \fint_{B_\varrho(x_0)} |Du|^p\, dx \right)^{\frac{2}{p}}. \tag{3.63}$$

Then let us recall the standard Caccioppoli's inequality valid for solutions to systems satisfying conditions (1.2) (see for instance [20], Chapter 6), that is

$$\fint_{B_\varrho(x_0)} |Du|^p\, dx \leqslant c \fint_{B_{2\varrho}(x_0)} \frac{|u - (u_\varrho)|^p}{\varrho^p}\, dx + c,$$

for $c \equiv c(n, N, p, \nu, L)$, and $B_{2\varrho}(x_0) \Subset \Omega$. Therefore, by $p \geqslant n$, using that u is locally Hölder continuous with exponent $1 - n/q_1$ by Sobolev–Morrey embedding theorem, where $q_1 > p$ is such that $Du \in L^{q_1}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$, we have

$$\varrho^{\frac{p\beta}{2}} \fint_{B_\varrho(x_0)} |Du|^p\, dx \leqslant c\big[\varrho^{\frac{p}{2}(\beta - \frac{2n}{q_1})} + \varrho^{\frac{p\beta}{2}}\big],$$

where $c \equiv c(n, N, p, \nu, L, \|Du\|_{L^{q_1}})$. Now choosing $\beta > 2n/q_1$, that is possible as $q_1 > p \geqslant n$ and $\beta \in (0, 2)$, we have that the left-hand side converges to zero for $\varrho \searrow 0$ whenever $x_0 \in \Omega$; now (3.62) follows from (3.61), taking into account (3.63).

We explicitly remark that in the other low-dimensional case $n \leqslant p + 2$ the partial Hölder continuity u, but not with every exponent, has been correctly proved by Campanato [5,7] in the case of elliptic systems, with a corresponding estimate for the Hausdorff dimension of the singular set $\Omega \setminus \Omega_u$. Notice that Campanato's arguments are based on a suitable combination of Freezing and difference quotient techniques, and for this reason do not apply to the case of quasiconvex functionals, to which, on the contrary, the arguments outlined for the case $n \leqslant p$ apply; see also Remark 4.1 below. Campanato's arguments can be nevertheless extended to the convex variational case; see [23], Section 8.

Remark 3.4. For the sake of simplicity we have confined ourselves to homogeneous systems of the type (1.1); in fact, the analysis of this case already provides all the main ideas of the new approach proposed here. By combining existing methods with ours we could also treat non-homogeneous systems of the type

$$-\operatorname{div} a(x,u,Du) = b(x,u,Du), \tag{3.64}$$

under suitable growth assumptions on the vector field $b : \Omega \times \mathbb{R}^N, \mathbb{R}^{N\times n} \to \mathbb{R}^N$. For instance, we could allow the so called *controllable growth conditions*; that is assume that $b(x,u,z) \leqslant L(1+|z|)^q$, where $q < p$ is suitable growth exponent as in [4,7]. We could also allow critical growth conditions; that is assume $|b(x,u,z)| \leqslant \tilde{L}(1+|z|)^p$. For this case, the traditional smallness assumption $2\tilde{L}\|u\|_{L^\infty} < \nu$ must be assumed, see for instance [22] and the approach and references in [11]. Without such an assumption even partial regularity fails, as is clear by considering the harmonic maps system in [30].

4. Quasiconvex functionals

4.1. The Caccioppoli inequality, higher integrability

The following Caccioppoli type inequality is taken from [19], Proposition 4.1, re-stated using our notation.

Proposition 4.1. *Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\cdot]$, under the assumptions* (1.6). *Let $B_\varrho \equiv B_\varrho(x_0) \Subset \Omega$ be a ball, and let $P(x) := A(x-x_0)+\tilde{u}$ be a polynomial, with $A \in \mathbb{R}^{N\times n}$ and $\tilde{u} \in \mathbb{R}^N$. Then there exists a constant $c \equiv c(n,N,p,\nu,L)$ such that*

$$\begin{aligned}
&\fint_{B_{\varrho/2}} \left[\left(1+|A|\right)^{p-2}|Du-A|^2 + |Du-A|^p\right] dx \\
&\quad \leqslant c \fint_{B_\varrho} \left[\left(1+|A|\right)^{p-2}\left|\frac{u-P}{\varrho}\right|^2 + \left|\frac{u-P}{\varrho}\right|^p\right] dx \\
&\qquad + c \fint_{B_\varrho} \omega\left(\varrho^p + |u-\tilde{u}|^p + |u-P|^p\right)\left(1+|A|+|Du|\right)^p dx.
\end{aligned} \tag{4.1}$$

We shall need the following higher integrability results for minimizers of integral functionals, see for instance [18,20]. They are basically a consequence of Caccioppoli's inequality, and the celebrated Gehring's lemma. In the rest of the paper a special role will be played by the exponents $p < q \leqslant q_1$ introduced in the following twin lemmata.

Proposition 4.2. *Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\cdot]$ under the assumption* $(1.6)_1$, *then there exists a higher integrability exponent $q_1 \equiv q_1(n,N,p,\nu,L) > p$ and a constant $c \equiv c(n,N,p,\nu,L)$ such that $u \in W^{1,q_1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, and moreover, for any ball $B_\varrho(x_0) \Subset \Omega$,*

$$\left(\fint_{B_{\varrho/2}(x_0)} |Du|^{q_1}\,dx\right)^{\frac{1}{q_1}} \leqslant c\left(\fint_{B_\varrho(x_0)} \left(1+|Du|\right)^p dx\right)^{\frac{1}{p}}. \tag{4.2}$$

The following is the higher integrability up-to-the-boundary [18]:

Proposition 4.3. *Let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\cdot]$ under the assumption* $(1.6)_1$, *and let $v_0 \in u + W^{1,p}_0(B_{\varrho/2}(x_0),\mathbb{R}^N)$ be a solution of the following Dirichlet problem:*

$$v_0 \mapsto \min_w \fint_{B_{\varrho/2}(x_0)} G(Dw)\,dx, \quad w \in u + W^{1,p}_0\left(B_{\varrho/2}(x_0),\mathbb{R}^N\right). \tag{4.3}$$

where $G:\mathbb{R}^{N\times n}\to\mathbb{R}$ is continuous and satisfies $\nu|z|^p\leqslant G(z)\leqslant L(1+|z|)^p$, and $B_\varrho(x_0)\Subset\Omega$ is a ball. Then there exists another higher integrability exponent $q\equiv q(n,N,p,\nu,L)\in(p,q_1]$, and a constant $c\equiv c(n,N,p,\nu,L)$ such that

$$\left(\fint_{B_{\varrho/2}(x_0)}|Dv_0|^q\,dx\right)^{\frac{1}{q}}\leqslant c\left(\fint_{B_{\varrho/2}(x_0)}|Dv_0|^p\,dx\right)^{\frac{1}{p}}+c\left(\fint_{B_{\varrho/2}(x_0)}\big(1+|Du|\big)^{q_1}\,dx\right)^{\frac{1}{q_1}}. \tag{4.4}$$

4.2. Excess functionals

In this section we shall give the symbols $E(x_0,\varrho)$ and $M(x_0,\varrho)$ a different meaning from the one given in Section 3.1. More precisely, with $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ and $B_\varrho(x_0)\Subset\Omega$, we define as before

$$C(x_0,\varrho)\equiv C(u,x_0,\varrho):=\fint_{B_\varrho}\left[\frac{|Du-(Du)_\varrho|^2}{(1+|(Du)_\varrho|)^2}+\frac{|Du-(Du)_\varrho|^p}{(1+|(Du)_\varrho|)^p}\right]dx,$$

where $(Du)_\varrho\equiv(Du)_{x_0,\varrho}$, while we redefine

$$M(x_0,\varrho)\equiv M(u,x_0,\varrho):=\varrho^\beta\fint_{B_\varrho}|Du|^p\,dx,\quad \beta\in(0,p). \tag{4.5}$$

Finally we put

$$E(x_0,\varrho)\equiv E(u,x_0,\varrho):=C(x_0,\varrho)+\big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{q-p}{qp}}+\big[\omega(\varrho)\big]^{\frac{q-p}{qp}}, \tag{4.6}$$

where $q\equiv q(n,N,p,\nu,L)>p$ is the higher integrability exponent appearing in Proposition 4.3.

4.3. Preliminary comparison

The freezing technique cannot be used directly as in the case of systems, therefore we need a comparison result.

Proposition 4.4. *Let $u\in W^{1,p}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}[\cdot]$ under the assumptions* (1.6)*, and let $B_\varrho(x_0)\Subset\Omega$ be a ball such that*

$$E(x_0,\varrho)+\varrho\leqslant 1. \tag{4.7}$$

Then there exists a map $v\in u+W_0^{1,p}(B_{\varrho/2}(x_0),\mathbb{R}^N)$ such that

$$\fint_{B_{\varrho/2}(x_0)}|Dv-Du|^p\,dx\leqslant K(x_0,\varrho), \tag{4.8}$$

and

$$\fint_{B_{\varrho/2}(x_0)}G(Dv)\,dx\leqslant\fint_{B_{\varrho/2}(x_0)}G(Dv+D\varphi)\,dx+c_e\big[K(x_0,\varrho)\big]^{1-\frac{1}{p}}\left(\fint_{B_{\varrho/2}(x_0)}|D\varphi|^p\,dx\right)^{\frac{1}{p}}, \tag{4.9}$$

for every $\varphi\in W_0^{1,p}(B_{\varrho/2}(x_0),\mathbb{R}^N)$. Here c_e is a constant depending only on n,N,p,ν,L, the exponent $q>p$ is the one defined in Proposition 4.3*, the integrand G is defined by*

$$G(z):=F\big(x_0,(u)_{x_0,\varrho/2},z\big), \tag{4.10}$$

and

$$K(x_0,\varrho):=\big(1+\big|(Du)_{x_0,\varrho}\big|\big)^p\big\{\big[\omega(\varrho^p)\big]^{\frac{q-p}{q}}+\big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{q-p}{q}}\big\}. \tag{4.11}$$

Proof. All the balls in the following will be centered at x_0, and as already clear from the statement, the averages of the maps considered are related to such balls. Let $v_0 \in u + W_0^{1,p}(B_{\varrho/2}, \mathbb{R}^n)$ be a solution of the Dirichlet problem in (4.3); using the minimality of v_0 and the growth conditions $(1.6)_1$ yields

$$\fint_{B_{\varrho/2}} |Dv_0|^p \, dx \leqslant L/\nu \fint_{B_{\varrho/2}} \big(1 + |Du|\big)^p \, dx. \tag{4.12}$$

Using the minimality of both u and v_0 we have:

$$\begin{aligned}
\fint_{B_{\varrho/2}} G(Du)\, dx &= \fint_{B_{\varrho/2}} \big[F\big(x_0, (u)_{\varrho/2}, Du\big) - F(x, u, Du)\big]\, dx \quad (=: (I)) \\
&\quad + \fint_{B_{\varrho/2}} F(x, u, Du)\, dx \\
&\leqslant (I) + \fint_{B_{\varrho/2}} F(x, v_0, Dv_0)\, dx \\
&\leqslant (I) + (II) + (III) + \min_w \fint_{B_{\varrho/2}} G(Dw)\, dx,
\end{aligned} \tag{4.13}$$

where

$$(II) := \fint_{B_{\varrho/2}} \big[F(x, v_0, Dv_0) - F(x_0, (v_0)_{\varrho/2}\, Dv_0)\big]\, dx,$$

and

$$(III) := \fint_{B_{\varrho/2}} \big[F\big(x_0, (v_0)_{\varrho/2}, Dv_0\big) - F\big(x_0, (u)_{\varrho/2}, Dv_0\big)\big]\, dx.$$

In the following we shall repeatedly use the elementary estimation

$$\begin{aligned}
\fint_{B_\varrho} \big(1 + |Du|^p\big)\, dx &\leqslant c\big(1 + \big|(Du)_\varrho\big|\big)^p \fint_{B_\varrho} \frac{|Du - (Du)_\varrho|^p}{(1 + |(Du)_\varrho|)^p}\, dx + c\big(1 + \big|(Du)_\varrho\big|\big)^p \\
&\leqslant c\big[E(x_0, \varrho) + 1\big]\big(1 + \big|(Du)_\varrho\big|\big)^p \\
&\overset{(4.7)}{\leqslant} c\big(1 + |(Du)_\varrho|\big)^p.
\end{aligned} \tag{4.14}$$

Now, using $(1.6)_3$, (2.3) and the previous estimate we find that

$$\begin{aligned}
|(I)| &\leqslant c \fint_{B_{\varrho/2}} \big[\omega\big(\varrho^p\big) + \omega\big(\big|u - (u)_{\varrho/2}\big|^p\big)\big]\big(1 + |Du|^p\big)\, dx \\
&\leqslant c\omega\big(\varrho^p\big)\big(1 + \big|(Du)_\varrho\big|\big)^p + c \fint_{B_{\varrho/2}} \omega\big(\big|u - (u)_{\varrho/2}\big|^p\big)\big(1 + |Du|^p\big)\, dx.
\end{aligned} \tag{4.15}$$

Using in turn the Hölder and Poincaré inequalities as well as (2.2)–(2.3), we may write

$$\begin{aligned}
&\fint_{B_{\varrho/2}} \omega\big(\big|u - (u)_{\varrho/2}\big|^p\big)\big(1 + |Du|^p\big)\, dx \\
&\quad \leqslant c\bigg(\fint_{B_{\varrho/2}} \big[\omega\big(\big|u - (u)_{\varrho/2}\big|^p\big)\big]^{\frac{q_1}{q_1 - p}}\, dx\bigg)^{\frac{q_1 - p}{q_1}} \bigg(\fint_{B_{\varrho/2}} \big(1 + |Du|^{q_1}\big)\, dx\bigg)^{\frac{p}{q_1}}
\end{aligned}$$

$$\overset{(2.2),(4.2)}{\leqslant} c\biggl(\ \fint_{B_{\varrho/2}} \omega\bigl(\bigl|u-(u)_{\varrho/2}\bigr|^p\bigr)\,dx\biggr)^{\frac{q_1-p}{q_1}} \fint_{B_\varrho} \bigl(1+|Du|\bigr)^p\,dx$$

$$\leqslant c\biggl[\omega\biggl(c(n,p)\varrho^p \fint_{B_\varrho} |Du|^p\,dx\biggr)\biggr]^{\frac{q_1-p}{q_1}} \fint_{B_\varrho} \bigl(1+|Du|\bigr)^p\,dx$$

$$\overset{(4.14)}{\leqslant} c\bigl[\omega\bigl(M(x_0,\varrho)\bigr)\bigr]^{\frac{q-p}{q}}\bigl(1+\bigl|(Du)_\varrho\bigr|\bigr)^p. \tag{4.16}$$

Note that in the last line we used the fact that $q \leqslant q_1$ and $\omega(\cdot) \leqslant 1$; finally we used the second inequality in (2.3). In conclusion, merging (4.15)–(4.16) yields

$$\bigl|(I)\bigr| \leqslant c(n,N,p,\nu,L)K(x_0,\varrho).$$

In a similar way, we may use $(1.6)_3$ to get

$$\bigl|(II)\bigr| \leqslant c \fint_{B_{\varrho/2}} \bigl[\omega\bigl(\varrho^p\bigr)+\omega\bigl(\bigl|v_0-(v_0)_{\varrho/2}\bigr|^p\bigr)\bigr]\bigl(1+|Dv_0|^p\bigr)\,dx$$

$$\leqslant c\omega\bigl(\varrho^p\bigr) \fint_{B_{\varrho/2}} \bigl(1+|Dv_0|^p\bigr)\,dx + c \fint_{B_{\varrho/2}} \omega\bigl(\bigl|v_0-(v_0)_{\varrho/2}\bigr|^p\bigr)\bigl(1+|Dv_0|^p\bigr)\,dx.$$

Using (4.12) and then (4.14), we have

$$\omega\bigl(\varrho^p\bigr) \fint_{B_{\varrho/2}} \bigl(1+|Dv_0|^p\bigr)\,dx \leqslant c\omega\bigl(\varrho^p\bigr)\bigl(1+\bigl|(Du)_\varrho\bigr|\bigr)^p. \tag{4.17}$$

Then, as with (4.16)

$$\fint_{B_{\varrho/2}} \omega\bigl(\bigl|v_0-(v_0)_{\varrho/2}\bigr|^p\bigr)\bigl(1+|Dv_0|^p\bigr)\,dx$$

$$\leqslant c\biggl[\omega\biggl(\ \fint_{B_{\varrho/2}} \bigl|v_0-(v_0)_{\varrho/2}\bigr|^p\,dx\biggr)\biggr]^{\frac{q-p}{q}} \biggl(\ \fint_{B_{\varrho/2}} \bigl(1+|Dv_0|^q\bigr)\,dx\biggr)^{\frac{p}{q}}$$

$$\overset{(4.4)}{\leqslant} c\biggl[\omega\biggl(c(n,p)\varrho^p \fint_{B_{\varrho/2}} |Dv_0|^p\,dx\biggr)\biggr]^{\frac{q-p}{q}} \cdot \biggl[\ \fint_{B_{\varrho/2}} \bigl(1+|Dv_0|^p\bigr)\,dx + \biggl(\ \fint_{B_{\varrho/2}} \bigl(1+|Du|^{q_1}\bigr)\,dx\biggr)^{\frac{p}{q_1}}\biggr]$$

$$\overset{(4.12)}{\leqslant} c\biggl[\omega\biggl(c\varrho^p \fint_{B_\varrho} \bigl(1+|Du|^p\bigr)\,dx\biggr)\biggr]^{\frac{q-p}{q}} \biggl(\ \fint_{B_{\varrho/2}} \bigl(1+|Du|^{q_1}\bigr)\,dx\biggr)^{\frac{p}{q_1}}$$

$$\overset{(4.7),(4.2)}{\leqslant} c\bigl\{\bigl[\omega\bigl(\varrho^p\bigr)\bigr]^{\frac{q-p}{q}}+\bigl[\omega\bigl(M(x_0,\varrho)\bigr)\bigr]^{\frac{q-p}{q}}\bigr\} \fint_{B_\varrho} (1+|Du|^p)\,dx$$

$$\overset{(4.14)}{\leqslant} c\bigl\{\bigl[\omega\bigl(\varrho^p\bigr)\bigr]^{\frac{q-p}{q}}+\bigl[\omega\bigl(M(x_0,\varrho)\bigr)\bigr]^{\frac{q-p}{q}}\bigr\}\bigl(1+\bigl|(Du)_\varrho\bigr|\bigr)^p. \tag{4.18}$$

Merging (4.17) and (4.18) yields

$$\bigl|(II)\bigr| \leqslant c(n,N,p,\nu,L)K(x_0,\varrho).$$

Arguing as we did for (4.18), and recalling that $u \equiv v_0$ on $\partial B_{\varrho/2}$ in order to apply Poincaré inequality, we find that

$$
\begin{aligned}
|(III)| &\leqslant c \fint_{B_{\varrho/2}} \omega\big(\big|(u)_{\varrho/2}-(v_0)_{\varrho/2}\big|^p\big)\big(1+|Dv_0|^p\big)\,dx \\
&\leqslant c\bigg[\omega\bigg(\fint_{B_{\varrho/2}} |u-v_0|^p\,dx\bigg)\bigg]^{\frac{q-p}{q}}\bigg(\fint_{B_{\varrho/2}}\big(1+|Dv_0|^q\big)\,dx\bigg)^{\frac{p}{q}} \\
&\leqslant c\bigg[\omega\bigg(c\varrho^p\fint_{B_{\varrho/2}}\big(|Du|^p+|Dv_0|^p\big)\,dx\bigg)\bigg]^{\frac{q-p}{q}}\bigg(\fint_{B_{\varrho/2}}\big(1+|Dv_0|^q\big)\,dx\bigg)^{\frac{p}{q}} \\
&\leqslant c\bigg[\omega\bigg(c\varrho^p\fint_{B_{\varrho}}\big(1+|Du|^p\big)\,dx\bigg)\bigg]^{\frac{q-p}{q}}\bigg(\fint_{B_{\varrho/2}}\big(1+|Du|^{q_1}\big)\,dx\bigg)^{\frac{p}{q_1}} \\
&\leqslant c\big\{\big[\omega\big(\varrho^p\big)\big]^{\frac{q-p}{q}}+\big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{q-p}{q}}\big\}\fint_{B_\varrho}\big(1+|Du|^p\big)\,dx. \qquad (4.19)
\end{aligned}
$$

Upon taking into account (4.14), we get

$$|(III)| \leqslant c(n,N,p,\nu,L)K(x_0,\varrho).$$

Gathering the estimates found for (I), (II) and (III) and inserting them into (4.13) yields

$$\fint_{B_{\varrho/2}} G(Du)\,dx \leqslant \min_w \fint_{B_{\varrho/2}} G(Dw)\,dx + c_e K(x_0,\varrho), \tag{4.20}$$

where $w \in u + W_0^{1,p}(B_{\varrho/2},\mathbb{R}^N) =: X$, and $c_e \equiv c_e(n,N,p,\nu,L)$. We may now consider the complete metric space (X,d) where

$$d(u_1,u_2) := \big[K(x_0,\varrho)\big]^{-\frac{1}{p}}\bigg(\fint_{B_{\varrho/2}}|Du_1-Du_2|^p\,dx\bigg)^{\frac{1}{p}}, \quad u_1,u_2 \in X.$$

With such a choice and in view of (4.20), we apply Ekeland's variational principle, that is Theorem 2.1, with $\sigma := c_e K(x_0,\varrho)$. This yields the map $v \in X$ satisfying (4.8)–(4.9). □

4.4. Excess decay

We shall next prove a decay estimate involving the various excess functionals introduced. The proof will again be based on the A-harmonic approximation lemma as in Proposition 3.2, but this will ultimately require the use of Proposition 4.4. We suggest to the reader to read the proof of Proposition 4.5 after the one for Proposition 3.2, since some of the arguments introduced there will be used again in a suitably modified form.

Proposition 4.5. *For $\beta \in (0,p)$ and $\theta \in (0,1/8)$, there exist two positive numbers*

$$\varepsilon_0 = \varepsilon_0\big(n,N,p,\nu,L,\beta,\theta,\mu(\cdot)\big) > 0, \quad \text{and} \quad \varepsilon_1 \equiv \varepsilon_1(n,p,\beta,\theta) > 0, \tag{4.21}$$

such that the following is true: If $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}[\cdot]$ in (1.4)*, under the assumptions* (1.6) *and* (1.3)*, and $B_\varrho(x_0) \Subset \Omega$ is a ball where the smallness conditions*

$$E(x_0,\varrho) < \varepsilon_0 \quad \text{and} \quad \varrho < \varepsilon_1 \tag{4.22}$$

are satisfied, then

$$C(x_0,\theta\varrho) \leqslant c_*\theta^2 E(x_0,\varrho). \tag{4.23}$$

The constant c_* depends only upon n, N, p and ν, L.

Proof. *Step 1: Approximate A-harmonicity.* Again all the balls will be centered at x_0. Moreover, in the following we argue under the smallness condition

$$E(x_0,\varrho)+\varrho\leqslant 1. \tag{4.24}$$

The map $v\in u+W_0^{1,p}(B_{\varrho/2},\mathbb{R}^N)$ found in Proposition 4.4 is the minimizer of the functional

$$\xi\mapsto \fint_{B_{\varrho/2}} G(D\xi)\,dx+c_e\big[K(x_0,\varrho)\big]^{1-1/p}\bigg(\fint_{B_{\varrho/2}}|D\xi-Dv|^p\,dx\bigg)^{\frac{1}{p}}$$

with $\xi\in u+W_0^{1,p}(B_{\varrho/2},\mathbb{R}^N)$, $G(z)$ defined in (4.10), $K(x_0,\varrho)$ defined in (4.11), and $c_e\equiv c_e(n,N,p,\nu,L)$. It is then easy to see, applying the procedure usually adopted when deriving the Euler–Lagrange equation for a variational integral, that the map v satisfies the following *Euler–Lagrange variational inequality*:

$$\bigg|\fint_{B_{\varrho/2}} G_z(Dv)D\varphi\,dx\bigg|\leqslant c_e\big[K(x_0,\varrho)\big]^{1-1/p}\bigg(\fint_{B_{\varrho/2}}|D\varphi|^p\,dx\bigg)^{\frac{1}{p}}$$

for every $\varphi\in W_0^{1,p}(B_{\varrho/2},\mathbb{R}^N)$. In what follows, we shall take $\varphi\in C_0^1(B_{\varrho/2},\mathbb{R}^N)$, and without loss of generality, up to considering $\varphi/\|D\varphi\|_{L^\infty(B_{\varrho/2})}$ and then scaling back, we shall assume that $\|D\varphi\|_{L^\infty(B_{\varrho/2})}\leqslant 1$. Therefore the last inequality yields

$$\bigg|\fint_{B_{\varrho/2}} G_z(Dv)D\varphi\,dx\bigg|\leqslant c_e\big[K(x_0,\varrho)\big]^{1-1/p}. \tag{4.25}$$

Now, using that $\fint_{B_{\varrho/2}} G_z((Du)_\varrho)D\varphi\,dx=0$, and taking into account (4.25) we have

$$\begin{aligned}(I)&:=\bigg|\fint_{B_{\varrho/2}}\int_0^1 G_{zz}\big((Du)_\varrho+t\big(Dv-(Du)_\varrho\big)\big)\big(Dv-(Du)_\varrho,D\varphi\big)\,dt\,dx\bigg|\\&=\bigg|\fint_{B_{\varrho/2}}\big[G_z(Dv)-G_z\big((Du)_\varrho\big)\big]D\varphi\,dx\bigg|\leqslant c_e\big[K(x_0,\varrho)\big]^{1-1/p}.\end{aligned}$$

Using the previous relation together with assumption $(1.6)_4$, we have

$$\begin{aligned}&\bigg|\fint_{B_{\varrho/2}} G_{zz}\big((Du)_\varrho\big)\big(Dv-(Du)_\varrho,D\varphi\big)\,dx\bigg|\\&\quad\leqslant\bigg|\fint_{B_{\varrho/2}}\int_0^1\big[G_{zz}\big((Du)_\varrho\big)-G_{zz}\big((Du)_\varrho+t\big(Dv-(Du)_\varrho\big)\big)\big]\big(Dv-(Du)_\varrho,D\varphi\big)\,dt\,dx\bigg|+|(I)|\\&\quad\leqslant c\fint_{B_{\varrho/2}}\big|Dv-(Du)_\varrho\big|\big(1+\big|(Du)_\varrho\big|+\big|Dv-(Du)_\varrho\big|\big)^{p-2}\mu\bigg(\frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\\&\qquad+c_e\big[K(x_0,\varrho)\big]^{1-1/p}=:(III)+c_e\big[K(x_0,\varrho)\big]^{1-1/p}.\end{aligned} \tag{4.26}$$

Before going on with the estimation of *(III)* we derive a few preliminary estimates. Using (4.8) we find that

$$\begin{aligned}\fint_{B_{\varrho/2}}\big|Dv-(Du)_\varrho\big|^p\,dx&\leqslant c\fint_{B_{\varrho/2}}|Dv-Du|^p\,dx+c\fint_{B_{\varrho/2}}\big|Du-(Du)_\varrho\big|^p\,dx\\&\leqslant cK(x_0,\varrho)+c\big(1+\big|(Du)_\varrho\big|\big)^pC(x_0,\varrho)\\&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^pE(x_0,\varrho).\end{aligned} \tag{4.27}$$

Note that we have estimated $K(x_0,\varrho) \leqslant c(1+|(Du)_\varrho|)^p E(x_0,\varrho)$. In the same way, via Hölder's inequality and again (4.8)

$$\begin{aligned}
\fint_{B_{\varrho/2}} \big|Dv-(Du)_\varrho\big|^2\,dx &\leqslant c\Big(\fint_{B_{\varrho/2}} |Dv-Du|^p\,dx\Big)^{2/p} + c\fint_{B_{\varrho/2}} \big|Du-(Du)_\varrho\big|^2\,dx \\
&\leqslant c\big[K(x_0,\varrho)\big]^{2/p} + c\big(1+\big|(Du)_\varrho\big|\big)^2 C(x_0,\varrho) \\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^2 E(x_0,\varrho).
\end{aligned} \tag{4.28}$$

For the last inequality, we have used the estimate (recall that $\omega(\cdot)\leqslant 1$)

$$\begin{aligned}
\big[K(x_0,\varrho)\big]^{2/p} &\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^2\big\{\big[\omega(\varrho^p)\big]^{\frac{q-p}{qp}} + \big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{q-p}{qp}}\big\} \\
&\leqslant c\big(1+\big|(Du)_\varrho\big|\big)^2 E(x_0,\varrho).
\end{aligned}$$

Connecting (4.27) and (4.28) we deduce that

$$\fint_{B_{\varrho/2}} \bigg[\frac{|Dv-(Du)_\varrho|^2}{(1+|(Du)_\varrho|)^2} + \frac{|Dv-(Du)_\varrho|^p}{(1+|(Du)_\varrho|)^p}\bigg]\,dx \leqslant c_k E(x_0,\varrho) \tag{4.29}$$

where $c_k \equiv c_k(n,N,p,\nu,L)$.

Now we proceed to estimate (III), making use of (4.29). We have, compare also with (3.10),

$$\begin{aligned}
(III) \;\leqslant\;& c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_{\varrho/2}} \frac{|Dv-(Du)_\varrho|^{p-1}}{(1+|(Du)_\varrho|)^{p-1}}\cdot\mu\bigg(\frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx \\
&+ c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\fint_{B_{\varrho/2}} \frac{|Dv-(Du)_\varrho|}{(1+|(Du)_\varrho|)}\cdot\mu\bigg(\frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx \\
\leqslant\;& c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\bigg(\fint_{B_{\varrho/2}} \frac{|Dv-(Du)_\varrho|^p}{(1+|(Du)_\varrho|)^p}\,dx\bigg)^{\frac{p-1}{p}}\cdot\bigg(\fint_{B_{\varrho/2}} \mu^p\bigg(\frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\bigg)^{\frac{1}{p}} \\
&+ c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\bigg(\fint_{B_{\varrho/2}} \frac{|Dv-(Du)_\varrho|^2}{(1+|(Du)_\varrho|)^2}\,dx\bigg)^{\frac{1}{2}}\bigg(\fint_{B_{\varrho/2}} \mu^2\bigg(\frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\bigg)\,dx\bigg)^{\frac{1}{2}} \\
\leqslant\;& c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\mu\bigg(\fint_{B_{\varrho/2}} \frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\,dx\bigg)^{\frac{1}{p}}\big[E(x_0,R)\big]^{\frac{p-1}{p}} \\
&+ c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\mu\bigg(\fint_{B_{\varrho/2}} \frac{|Dv-(Du)_\varrho|}{1+|(Du)_\varrho|}\,dx\bigg)^{\frac{1}{2}}\big[E(x_0,R)\big]^{\frac{1}{2}} \\
\overset{(4.29)}{\leqslant}\;& c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}\big[\mu\big(c\sqrt{E(x_0,\varrho)}\big)^{1/2} + \mu\big(c\sqrt{E(x_0,\varrho)}\big)^{1/p}\big]\big[E(x_0,\varrho)^{1/2} + E(x_0,\varrho)^{1-1/p}\big].
\end{aligned}$$

Collecting the estimates found for (III) with (4.26), and recalling that (4.11) yields

$$\begin{aligned}
\big[K(x_0,\varrho)\big]^{1-1/p} \;&\leqslant\; c(p)\big(1+\big|(Du)_\varrho\big|\big)^{p-1}E(x_0,\varrho)^{p-1} \\
&\overset{(4.24)}{\leqslant} c\big(1+\big|(Du)_\varrho\big|\big)^{p-1}E(x_0,\varrho)
\end{aligned}$$

we may therefore conclude that

$$\bigg|\fint_{B_{\varrho/2}} \mathcal{A}(Dw,D\varphi)\,dx\bigg| \leqslant c_6 H\big(E(x_0,\varrho)\big)\|D\varphi\|_{L^\infty(B_{\varrho/2})}, \quad \text{for every } \varphi\in C_0^1\big(B_{\varrho/2},\mathbb{R}^N\big),$$

where, recalling the definition of $G(\cdot)$ in (4.10), we have set

$$w := \frac{v - (Du)_\varrho (x - x_0)}{\sqrt{c_k E(x_0, \varrho)}(1 + |(Du)_\varrho|)}, \qquad \mathcal{A} := \frac{F_{zz}(x_0, (u)_{\varrho/2}, (Du)_\varrho)}{(1 + |(Du)_\varrho|)^{p-2}}, \tag{4.30}$$

and the functional $H(E(x_0, \varrho))$ has been defined in (3.11), taking of course into account the current definition of the excess functional $E(x_0, \varrho)$ in (4.6). The constant c_k appears in (4.29), and depends on n, N, p, ν, L. Therefore, so does c_6. Observe that $\mathcal{A}$ satisfies assumption (2.4) by $(1.6)_2$–$(1.6)_4$, and this allows us to apply the A-harmonic approximation Lemma 2.1. Indeed, with $\varepsilon > 0$ to be fixed later, we determine the corresponding $\delta(n, N, \nu, L, \varepsilon) > 0$ according to (2.1). Then we use (4.29) to see that $\fint_{B_{\varrho/2}} |Dw|^2\,dx \leqslant 1$. Finally upon assuming the smallness condition

$$H\big(E(x_0, \varrho)\big) \leqslant \delta / c_6, \tag{4.31}$$

we infer the existence of an $\mathcal{A}$-harmonic map $h \in C^\infty(B_{\varrho/2}, \mathbb{R}^N)$ such that

$$\fint_{B_{\varrho/2}} |Dh|^2\,dx \leqslant 1 \quad \text{and} \quad (\varrho/2)^{-2} \fint_{B_{\varrho/2}} |w - h|^2\,dx \leqslant \varepsilon. \tag{4.32}$$

Step 2: Intermediate decay estimate. We may now proceed as in Step 2 from the proof of Proposition 3.2: recalling that $\theta \in (0, 1/8)$, and taking $\varepsilon = \theta^{n+4}$ as in (3.17), we are led to (3.18), with the current meaning of w in (4.30). Letting

$$\tilde{P} := (Du)_\varrho (x - x_0) + \sqrt{c_k E(x_0, \varrho)}\big(1 + \big|(Du)_\varrho\big|\big)\big[h(x_0) + Dh(x_0)(x - x_0)\big]$$

we obtain from (3.19)

$$(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |v - \tilde{P}|^2\,dx \leqslant c\theta^2\big(1 + \big|(Du)_\varrho\big|\big)^2 E(x_0, \varrho),$$

so that we also have

$$(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |u - \tilde{P}|^2\,dx \leqslant c(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |v - u|^2\,dx + c\theta^2\big(1 + \big|(Du)_\varrho\big|\big)^2 E(x_0, \varrho). \tag{4.33}$$

Now let us assume the smallness condition

$$\big[\omega\big(M(x_0, \varrho)\big)\big]^{\frac{q-p}{qp}} + \big[\omega(\varrho)\big]^{\frac{q-p}{qp}} \leqslant \theta^{n+4}. \tag{4.34}$$

Since $u \equiv v$ on $\partial B_{\varrho/2}$, Poincaré's inequality implies that

$$\begin{aligned}
(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |v - u|^2\,dx &\leqslant c\theta^{-n-2}\varrho^{-2} \fint_{B_{\varrho/2}} |v - u|^2\,dx \\
&\leqslant c\theta^{-n-2} \fint_{B_{\varrho/2}} |Dv - Du|^2\,dx \overset{(4.8)}{\leqslant} c\theta^{-n-2}\big[K(x_0, \varrho)\big]^{2/p} \\
&\overset{(4.34)}{\leqslant} c\theta^2\big(1 + \big|(Du)_\varrho\big|\big)^2 \Big\{\big[\omega\big(M(x_0, \varrho)\big)\big]^{\frac{q-p}{qp}} + \big[\omega(\varrho)\big]^{\frac{q-p}{qp}}\Big\} \\
&\overset{(4.24)}{\leqslant} c\theta^2\big(1 + \big|(Du)_\varrho\big|\big)^2 E(x_0, \varrho).
\end{aligned} \tag{4.35}$$

Denoting by $P_{2\theta\varrho}$ the affine function minimizing $Q \mapsto \int_{B_{2\theta\varrho}} |u - Q|^2\,dx$ amongst all the affine functions Q (see Lemma 2.2 in Section 2), and gathering (4.33)–(4.35) we easily deduce

$$(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2\,dx \leqslant c\theta^2\big(1 + \big|(Du)_\varrho\big|\big)^2 E(x_0, \varrho), \tag{4.36}$$

where $c \equiv c(n, N, p, \nu, L)$. This last estimate is analogous to (3.20). Now, in the case $p > 2$ we shall proceed as in Proposition 3.2 after inequality (3.20) in Step 2. Assuming the smallness conditions (3.24) and (3.26), that we hereby reproduce

$$E(x_0, \varrho) \leqslant \theta^{\frac{2[(2t-1)p+tn+2]}{(1-t)(p-2)}}, \qquad E(x_0, \varrho) \leqslant \theta^{2n}/4, \tag{4.37}$$

we can interpolate as in (3.23) arriving at (3.25), with the obvious current meaning of u and $E(x_0, \varrho)$. Moreover we also get, as in (3.27)–(3.28), that

$$\big(1 + \big|(Du)_\varrho\big|\big) + \big(1 + \big|(Du)_{2\theta\varrho}\big|\big) \leqslant 4\big(1 + \big|(Du)_{\theta\varrho}\big|\big). \tag{4.38}$$

Using this inequality, we finally arrive at

$$(2\theta\varrho)^{-2} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^2 \, dx \leqslant c\theta^2 \big(1 + \big|(Du)_{\theta\varrho}\big|\big)^2 E(x_0, \varrho), \tag{4.39}$$

and

$$(2\theta\varrho)^{-p} \fint_{B_{2\theta\varrho}} |u - P_{2\theta\varrho}|^p \, dx \leqslant c\theta^2 \big(1 + \big|(Du)_{\theta\varrho}\big|\big)^p E(x_0, \varrho), \tag{4.40}$$

where $c \equiv c(n, N, p, \nu, L)$.

Step 3: Full decay estimate for the Campanato-type excess. We are now going to apply the Caccioppoli's inequality for minimizers (4.1), taking as a polynomial $P \equiv P_{2\theta\varrho}$; this yields

$$\begin{aligned}
&\fint_{B_{\theta\varrho}} \big[\big(1 + |Q_{2\theta\varrho}|\big)^{p-2} |Du - Q_{2\theta\varrho}|^2 + |Du - Q_{2\theta\varrho}|^p\big] \, dx \\
&\quad \leqslant c \fint_{B_{2\theta\varrho}} \bigg[\big(1 + |Q_{2\theta\varrho}|\big)^{p-2} \bigg|\frac{u - P_{2\theta\varrho}}{2\theta\varrho}\bigg|^2 + \bigg|\frac{u - P_{2\theta\varrho}}{2\theta\varrho}\bigg|^p\bigg] \, dx \\
&\qquad + c \fint_{B_{2\theta\varrho}} \big[\omega\big(\varrho^p + \big|u - (u)_{2\theta\varrho}\big|^p + |u - P_{2\theta\varrho}|^p\big)\big]\big(1 + |Q_{2\theta\varrho}| + |Du|\big)^p \, dx =: (IV).
\end{aligned} \tag{4.41}$$

Here $c \equiv c(n, N, p, \nu, L)$; for the meaning of $Q_{x_0, 2\theta\varrho} \equiv Q_{2\theta\varrho}$ see (2.7)–(2.8) in Section 2. For the forthcoming estimates we need to argue under a smallness condition of the type (3.33), namely

$$E(x_0, \varrho) \leqslant \theta^n/(4^p c_2) \quad \text{and} \quad \varrho \leqslant (\theta^n/2c_3)^{1/(p-\beta)}, \tag{4.42}$$

where now c_2, c_3 are two constants that will be increased as needed a finite number of times through the end of the proof; they will anyway always depend on n, N, p, ν, L. We start by estimating the last integral in (4.41) in the obvious way

$$\begin{aligned}
(IV) &\leqslant c \fint_{B_{2\theta\varrho}} \omega\big(\varrho^p\big)\big(1 + |Q_{2\theta\varrho}| + |Du|\big)^p \, dx \\
&\quad + c \fint_{B_{2\theta\varrho}} \omega\big(\big|u - (u)_{2\theta\varrho}\big|^p\big)\big(1 + |Q_{2\theta\varrho}| + |Du|\big)^p \, dx \\
&\quad + c \fint_{B_{2\theta\varrho}} \omega\big(|u - P_{2\theta\varrho}|^p\big)\big(1 + |Q_{2\theta\varrho}| + |Du|\big)^p \, dx =: (V) + (VI) + (VII).
\end{aligned} \tag{4.43}$$

In order to estimate the last three integrals first note that (4.37) implies that

$$\fint_{B_{2\theta\varrho}} \left(1+|Du|\right)^p dx \leqslant \theta^{-n} \fint_{B_\varrho} \left(1+|Du|\right)^p dx$$
$$\leqslant c(p)\theta^{-n}\left[\theta^n + E(x_0,\varrho)\right]\left(1+\left|(Du)_\varrho\right|\right)^p \leqslant c\left(1+\left|(Du)_\varrho\right|\right)^p. \tag{4.44}$$

In a similar way, as $4\theta \leqslant 1$, we also estimate

$$\fint_{B_{4\theta\varrho}} \left(1+|Du|\right)^p dx \leqslant c(p)\left(1+\left|(Du)_\varrho\right|\right)^p. \tag{4.45}$$

Observe that under the assumption (4.42) and by using (4.38) we may proceed as in (3.32)–(3.37), with the exponent 2 replaced by p; in particular we have the following analogues of the inequalities (3.32) and (3.37):

$$\varrho^p|Q_{2\theta\varrho}|^p \leqslant \varrho + M(x_0,\varrho), \qquad |Q_{2\theta\varrho}|^p \leqslant c\left(1+\left|(Du)_{\theta\varrho}\right|\right)^p, \tag{4.46}$$

respectively; moreover, by the cancavity of $\omega(\cdot)$ and Jensen's and Poincaré's inequalities

$$\fint_{B_{2\theta\varrho}} \omega\left(\left|u-(u)_{2\theta\varrho}\right|^p\right) dx \leqslant \omega\left(M(x_0,\varrho)\right). \tag{4.47}$$

Therefore taking into account that $\omega(\cdot) \leqslant 1$ and using inequalities (4.44) and (4.46) we see that

$$(V) \leqslant c\omega\left(\varrho^p\right)\left(1+|Q_{2\theta\varrho}|+\left|(Du)_\varrho\right|\right)^p \leqslant c\left[\omega(\varrho)\right]^{\frac{q-p}{q}}\left(1+\left|(Du)_{\theta\varrho}\right|\right)^p. \tag{4.48}$$

Now we work to estimate (VI). Using Hölder's inequality and taking $q_1 > p$ as provided by Proposition 4.2 we find that

$$\begin{aligned}
(VI) \quad &\leqslant \quad c\left(\fint_{B_{2\theta\varrho}} \left[\omega\left(\left|u-(u)_{2\theta\varrho}\right|^p\right)\right]^{\frac{q_1}{q_1-p}} dx\right)^{\frac{q_1-p}{q_1}} \left(\fint_{B_{2\theta\varrho}} \left(1+|Q_{2\theta\varrho}|+|Du|\right)^{q_1} dx\right)^{\frac{p}{q_1}} \\
&\overset{(4.2)}{\leqslant} \quad c\left(\fint_{B_{2\theta\varrho}} \omega\left(\left|u-(u)_{2\theta\varrho}\right|^p\right) dx\right)^{\frac{q_1-p}{q_1}} \fint_{B_{4\theta\varrho}} \left(1+|Q_{2\theta\varrho}|+|Du|\right)^p dx \\
&\overset{(4.47),(4.45)}{\leqslant} c\left[\omega\left(M(x_0,\varrho)\right)\right]^{\frac{q-p}{q}}\left(1+|Q_{2\theta\varrho}|+\left|(Du)_\varrho\right|\right)^p \\
&\overset{(4.38),(4.46)}{\leqslant} c\left[\omega\left(M(x_0,\varrho)\right)\right]^{\frac{q-p}{q}}\left(1+\left|(Du)_{\theta\varrho}\right|\right)^p,
\end{aligned} \tag{4.49}$$

with $c \equiv c(n,N,p,\nu,L)$. Finally, using the structure of $P_{2\theta\varrho}$, see (2.7), and (2.3), we have

$$(VII) \leqslant c(VI) + c\fint_{B_{2\theta\varrho}} \omega\left(\varrho^p|Q_{2\theta\varrho}|^p\right)\left(1+|Q_{2\theta\varrho}|+|Du|\right)^p dx. \tag{4.50}$$

First, as already for (VI), we have

$$\begin{aligned}
&\fint_{B_{2\theta\varrho}} \omega\left(\varrho^p|Q_{2\theta\varrho}|^p\right)\left(1+|Q_{2\theta\varrho}|+|Du|\right)^p dx \\
&\overset{(4.46)_1}{\leqslant} c(p)\omega\left(\varrho+M(x_0,\varrho)\right)\left(|Q_{2\theta\varrho}|^p + \fint_{B_{2\theta\varrho}} \left(1+|Du|\right)^p dx\right) \\
&\overset{(4.44),(4.46)_2}{\leqslant} c\omega\left(\varrho+M(x_0,\varrho)\right)\left[\left(1+\left|(Du)_{\theta\varrho}\right|\right)^p + \left(1+\left|(Du)_\varrho\right|\right)^p\right] \\
&\overset{(4.38)}{\leqslant} c\left[\omega(\varrho)+\omega\left(M(x_0,\varrho)\right)\right]\left(1+\left|(Du)_{\theta\varrho}\right|\right)^p \\
&\overset{\omega(\cdot)\leqslant 1}{\leqslant} c\left\{\left[\omega\left(M(x_0,\varrho)\right)\right]^{\frac{q-p}{q}} + \left[\omega(\varrho)\right]^{\frac{q-p}{q}}\right\}\left(1+\left|(Du)_{\theta\varrho}\right|\right)^p.
\end{aligned}$$

Inserting the last inequality into (4.50), using (4.49) we have that

$$(VII) \leqslant c\big\{\big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{q-p}{q}} + \big[\omega(\varrho)\big]^{\frac{q-p}{q}}\big\}\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p. \tag{4.51}$$

Inserting (4.48), (4.49) and (4.51) into (4.43), we conclude that

$$(IV) \leqslant c\big\{\big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{q-p}{q}} + \big[\omega(\varrho)\big]^{\frac{q-p}{q}}\big\}\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p. \tag{4.52}$$

Now let us impose the new smallness condition

$$\big[\omega\big(M(x_0,\varrho)\big)\big]^{\frac{(q-p)(p-1)}{qp}} + \big[\omega(\varrho)\big]^{\frac{(q-p)(p-1)}{qp}} \leqslant \theta^2, \tag{4.53}$$

so that upon taking into account the definition of $E(x_0,\varrho)$ in (4.6), inequality (4.52) yields

$$(IV) \leqslant c\theta^2\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p E(x_0,\varrho).$$

Finally, using the last inequality together with (4.41) we obtain

$$\begin{aligned}
&\fint_{B_{\theta\varrho}} \big[\big(1+|Q_{2\theta\varrho}|\big)^{p-2}|Du-Q_{2\theta\varrho}|^2 + |Du-Q_{2\theta\varrho}|^p\big]\,dx \\
&\quad\leqslant c\fint_{B_{2\theta\varrho}} \bigg[\big(1+|Q_{2\theta\varrho}|\big)^{p-2}\bigg|\frac{u-P_{2\theta\varrho}}{\theta\varrho}\bigg|^2 + \bigg|\frac{u-P_{2\theta\varrho}}{\theta\varrho}\bigg|^p\bigg]\,dx + c\theta^2\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p E(x_0,\varrho),
\end{aligned} \tag{4.54}$$

where $c \equiv c(n,N,p,\nu,L)$. This inequality is a sort of analogue of (3.43), therefore we can argue as we did in (3.40)–(3.43) in order to properly replace $Q_{2\theta\varrho}$ by $(Du)_{\theta\varrho}$ in the left-hand side. We finally arrive at

$$\begin{aligned}
&\fint_{B_{\theta\varrho}} \big[\big(1+\big|(Du)_{\theta\varrho}\big|\big)^{p-2}\big|Du-(Du)_{\theta\varrho}\big|^2 + \big|Du-(Du)_{\theta\varrho}\big|^p\big]\,dx \\
&\quad\leqslant c\fint_{B_{2\theta\varrho}} \bigg[\big(1+\big|(Du)_{\theta\varrho}\big|\big)^{p-2}\bigg|\frac{u-P_{2\theta\varrho}}{\theta\varrho}\bigg|^2 + \bigg|\frac{u-P_{2\theta\varrho}}{\theta\varrho}\bigg|^p\bigg]\,dx + c\theta^2\big(1+\big|(Du)_{\theta\varrho}\big|\big)^p E(x_0,\varrho),
\end{aligned}$$

with $c \equiv c(n,N,p,\nu,L)$. Combining this last inequality with (4.39)–(4.40), and eventually dividing by $(1 + |(Du)_{\theta\varrho}|)^p$ yields (4.23) with c_* depending only on n, N, p, ν, L. This argument is valid provided that the smallness conditions (4.24), (4.31), (4.34), (4.37), (4.42), and (4.53) hold. Such conditions can be satisfied using (4.22). Indeed all the constants involved and the exponent q depend only upon n, N, p, ν, L; the verification of the precise dependence of the constants is at this point completely similar to the one performed at the end of Proposition 3.2. □

4.5. Iteration

The proof of Theorem 1.2 is achieved via an iteration procedure utilizing Proposition 4.5.

Proof of Theorem 1.2. Here we shall give the necessary modification to the arguments of the Section 3.4.

Step 1: Choice of the constants. Fix $\alpha \in (0,1)$ and $\beta \in (0,p)$ as in the statement of Theorem 1.2; then select $\gamma \equiv \gamma(\alpha) \in (\max\{0, n-p\}, n)$ such that

$$\alpha = 1 - \frac{n-\gamma}{p}. \tag{4.55}$$

We then choose $\theta \in (0, 1/8)$ such that (3.46) holds, where this time c_* is the constant appearing in (4.23); again $\theta \equiv \theta(n,N,p,\nu,L,\alpha,\beta)$. Then we fix $\varepsilon_2 \equiv \varepsilon_2(n,N,p,\nu,L,\alpha,\beta,\mu(\cdot))$ as in (3.47), where ε_0 is this time defined in (4.21), with θ fixed in (3.46). With ε_2 selected, we use (1.3) to fix δ_1 such that

$$t \in [0,\delta_1] \Longrightarrow \big[\omega(t)\big]^{\frac{q-p}{qp}} < \varepsilon_2, \tag{4.56}$$

where the exponent $q > p$ appears in Proposition 4.3. Keeping in mind the dependence of ε_2, and the dependence of the exponent q in Proposition 4.3, this fixes $\delta_1 \equiv \delta_1(n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot))$. Finally we choose the maximal radius as in (3.49), where ε_1 appears in (4.21). Taking into account the dependence of ε_1 this in turn fixes the maximal radius $\varrho_m \equiv \varrho_m(n, N, p, \nu, L, \alpha, \beta, \omega(\cdot), \mu(\cdot))$, and in the following $\varrho \leqslant \varrho_m$ will always hold whenever considering a radius ϱ.

Step 2: An almost BMO estimate. Again we consider a ball $B_\varrho(x_0) \Subset \Omega$ with $\varrho \leqslant \varrho_m$ such that

$$C(x_0, \varrho) < \varepsilon_2 \quad \text{and} \quad M(x_0, \varrho) < \delta_1, \tag{4.57}$$

where, with $\beta \in (0, p)$ being fixed, $M(x_0, \varrho)$ is as in (4.5). Therefore we can find a positive radius $\varrho \leqslant \varrho_m$ such that (3.50) holds; once again $\varrho > 0$ depends on everything, including x_0, u. Let us show that (4.57) implies $(\mathbf{I})_k$ for every $k = 0, 1, 2, \ldots$; as usual, by induction. Assuming $(\mathbf{I})_k$, exactly as for (3.51) we can prove:

$$\int_{B_{\theta^k \varrho}} \left|Du - (Du)_{\theta^k \varrho}\right|^p dx \leqslant 2^{p-1}\varepsilon_2 \int_{B_{\theta^k \varrho}} |Du|^p\, dx + 2^{p-1}\omega_n \varepsilon_2 \left(\theta^k \varrho\right)^n, \tag{4.58}$$

and then

$$\left(\theta^k \varrho\right)^\beta \fint_{B_{\theta^k \varrho}} \left|Du - (Du)_{\theta^k \varrho}\right|^2 dx \leqslant 2^{p-1}\varepsilon_2 M\left(x_0, \theta^k \varrho\right) + 2^{p-1}\varepsilon_2 \left(\theta^k \varrho\right)^\beta.$$

Using the last estimate, the choices of θ and of ε_2, and arguing exactly as we did for (3.53), we get $M(x_0, \theta^{k+1}) < \delta_1$. As for $C(x_0, \theta^{k+1}\varrho)$, we recall that $(\mathbf{I})_k$ and (4.56) imply $[\omega(M(x_0, \theta^k \varrho))]^{\frac{q-p}{qp}} < \varepsilon_2$, while $\varrho < \varrho_m$ and (3.49) imply $[\omega(\theta^k \varrho))]^{\frac{q-p}{qp}} \leqslant [\omega(\varrho))]^{\frac{q-p}{qp}} \leqslant \varepsilon_2$ so that, by the definition in (4.6) we conclude that $E(x_0, \theta^k \varrho) < 3\varepsilon_2 < \varepsilon_0$. Taking into account (3.49), we may now apply Proposition 4.5, and then using (3.46) we see that $C(x_0, \theta^{k+1}\varrho) \leqslant c_* \theta^2 E(x_0, \theta^k \varrho) \leqslant c_* \theta^2 3\varepsilon_2 < \varepsilon_2$, and the proof of $(\mathbf{I})_{k+1}$ is complete. Therefore $(\mathbf{I})_k$ holds for every $k \in \mathbb{N}$.

Step 3: Final iteration and partial regularity. Proceeding as in (3.54) and considering a ball $B_\varrho(x_0) \Subset \Omega$ where (4.57) are satisfied, using the bounds imposed on δ_1, ε_2, and using (4.58) we get

$$\int_{B_{\theta^{k+1} \varrho}} |Du|^p\, dx \leqslant \theta^\gamma \int_{B_{\theta^k \varrho}} |Du|^p\, dx + 4^p \omega_n \varepsilon_2 \left(\theta^k \varrho\right)^n.$$

Putting this time $\varphi(t) := \int_{B_t} |Du|^p\, dx$, we have proved that (3.55) holds again with 4 replaced by 4^p. Then with $\gamma < n$ fixed as in (4.55), we may iterate as in Section 3.4 to once again arrive at (3.56), with c_4 ultimately depending on $n, N, p, \nu, L, \alpha, \beta, \gamma$. This gives us

$$\int_{B_t(x_0)} |Du|^p\, dx \leqslant \frac{c_4}{\varrho^\gamma}\left[\int_\Omega |Du|^p\, dx + 1\right] t^\gamma \quad \text{for every } t \leqslant \varrho. \tag{4.59}$$

From this point on the proof goes on exactly as in the case of systems, taking again s as in (3.58), and just replacing in the integrals, and in the Morrey spaces used, the exponent 2 by the new one p. In particular, in order to satisfy (4.57) we shall take a point $x_0 \in \Omega$ such that

$$\liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} \left|Du - (Du)_\varrho\right|^p dx < s \quad \text{and} \quad \liminf_{\varrho \searrow 0} \varrho^\beta \fint_{B_\varrho(x_0)} |Du|^p\, dx < s. \tag{4.60}$$

This point is eventually seen to be a regular point, and for almost every point the previous relations are satisfied, from which the partial regularity follows. See again the end of Section 3.4. □

In analogy with the case for systems, we have as a by-product

Theorem 4.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functionals $\mathcal{F}[\cdot]$ in* (1.4) *under the assumptions* (1.6) *and* (1.3)*. Then $Du \in L^{p,\gamma}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^{N\times n})$ for every $\gamma \in (0, n)$, where $\Omega_u \subseteq \Omega$ is an open subset such that $|\Omega \setminus \Omega_u| = 0$.*

Remark 4.1. The content of Remarks 3.2-3.3 applies to the case of minima of functionals too. Remark 3.3 applies to the case of quasiconvex functionals as far as the case $n \leqslant p$ is concerned.

Remark 4.2. Theorems 1.2 and also 3.1 extend to almost minimizers of integral functionals, sometimes also called ω-minima. This concept, that originally arose in the setting of Geometric Measure Theory, is aimed at generalizing the notion of local minimizer by permitting an additional error term when verifying the minimality condition (see [14,10] for further discussion). Following [10], we shall say that a map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is an almost minimizer of the functional $\mathcal{F}[\cdot]$ in (1.4) provided there exists a bounded, concave and non-decreasing function $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lambda(0) = 0$ and

$$\int_{B_\varrho} F(x, u, Du)\, dx \leqslant \int_{B_\varrho} F(x, u + \varphi, Du + D\varphi)\, dx + \lambda(\varrho) \int_{B_\varrho} \left(1 + |Du|^p + |D\varphi|^p\right) dx, \tag{4.61}$$

holds for every $B_\varrho \Subset \Omega$, and every $\varphi \in W_0^{1,p}(B_\varrho, \mathbb{R}^N)$. For such maps a partial regularity theory for the gradient of solutions holds, see for instance [10,12] and the references therein, while following the techniques in this paper a lower order regularity theory can be obtained. See also [12] for a significant particular case. More specifically, let us first note that, up to passing to the concave envelope of $t \mapsto \inf\{\omega(t^p), \lambda(t)\}$, we may assume that $\lambda(t) \leqslant \omega(t^p)$; then Proposition 4.1 holds for almost minimizers, see for instance [12,20]. The only remaining modification occurs in Proposition 4.4 where in (4.13) the additional term

$$(IV) := \omega\left(\varrho^p\right) \fint_{B_{\varrho/2}} \left(1 + |Du|^p + |Dv_0|^p\right) dx,$$

appears on the right-hand side when using the almost minimality of u. The term $\omega(\varrho^p) \fint_{B_{\varrho/2}} (1 + |Du|^p)\, dx$ can be treated as (I), while $\omega(\varrho^p) \fint B_{\varrho/2}(1 + |Du|^p)\, dx$ can be treated using (4.12) and then just as (I) again. The rest of the proof remains unchanged.

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