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Blow-up solutions of the self-dual Chern–Simons–Higgs vortex equation

Solutions explosives de l'équation auto-duale de vortex de Chern–Simons–Higgs

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Abstract

We apply the variational method and the blow-up analysis to the self-dual Chern–Simons–Higgs vortex equation on a flat torus to obtain two solutions for certain values of the Chern–Simons constant. As the corresponding Chern–Simons constant tends to zero, one of corresponding solutions converges to zero and the other blows up at only one point in the sense of Brezis–Merle provided that the total number of vortex is greater than 2. Further, the below-up solution is of spike type and becomes a critical point of J_ϵ^+ when the total number of vortex is greater than 3. As a consequence, we show the existence of the third solution for some periodic configuration of vortices and some Chern–Simons constant.

Résumé

Nous nous appliquons la méthode variationnelle et l'analyse d'explosion à l'équation auto-duale de vortex de Chern–Simons–Higgs sur un tore plat pour obtenir deux solutions pour certaines valeurs de la constante de Chern–Simons. Lorsque la constante correspondante de Chern–Simons tend vers zéro, une des solutions correspondantes converge vers zéro et l'autre solution explose en seulement un point dans le sens de Brezis–Merle à condition que le nombre de vortex total soit plus grand que 2. De plus, l'explosion est de type "pic" et, quand le nombre de vortex total est plus grand que 3, la solution est un point critique de J_ϵ^+. Nous en déduisons l'existence d'une troisième solution pour une certaine configuration périodique des vortex et une certaine constante de Chern–Simons.

Keywords: Chern–Simons–Higgs vortex equation; Blow-up solutions

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1. Introduction

The Chern–Simons–Higgs model is a $(2+1)$ dimensional gauge model and it was proposed in [19,20] in an attempt to explain the superconductivity of type II. Unlike the Abelian–Higgs (or Ginzburg–Landau) model, the Chern–Simons–Higgs model admits vortices which is charged both electrically and magnetically and is known to have two different type of solutions (see, for instance, [6,14,28] and references therein). Hence it has been studied actively in the mathematical literature(see [5,29,32] and references therein).

The self-dual Chern–Simons–Higgs vortex equation on a flat 2-torus Ω can be written as follows;

$$\Delta u = \frac{1}{\epsilon^2} e^u\big(e^u - 1\big) + \sum_{j=1}^{k} 4\pi m_j \delta_{p_j} \quad \text{in } \Omega. \tag{1.1}$$

Here, $2\epsilon > 0$ is the Chern–Simons constant, $m_j \in \mathbb{N}$, $p_j \in \Omega$, and $j = 1, \dots, k$. The solution u of (1.1) is often called a vortex solution and each p_j $(j = 1, \dots, k)$ is called a vortex point and m_j the multiplicity of p_j. The vortex points are related to the (local) maximum point of the magnetic flux in the Chern–Simons–Higgs model.

Meanwhile, (1.1) can be thought as a formal perturbation of the mean field equation. Indeed, if we let $w = u - 2\ln\epsilon$ then (1.1) can be rewritten as

$$\Delta w = -e^w\big(1 - \epsilon^2 e^w\big) + \sum_{j=1}^{k} 4\pi m_j \delta_{p_j} \quad \text{in } \Omega. \tag{1.2}$$

If $\epsilon = 0$, (1.2) becomes the mean field equation. Indeed, when $k = m_1 = 1$, it was proved by Tarantello [28] that (1.2) admits a family of solutions converging to a solution of the mean field equation as ϵ tends to zero.

Denoting $N = \sum_{j=1}^{k} m_j$ and introducing $v = u - u_0$,

$$\Delta u_0 = \sum_{j=1}^{k} 4\pi m_j \delta_{p_j} - \frac{4\pi N}{|\Omega|} \quad \text{in } \Omega, \qquad \int_\Omega u_0\, dx = 0,$$

we can equivalently write (1.1) in a more favorable form as follows;

$$\Delta v = \frac{1}{\epsilon^2} e^{u_0+v}\big(e^{u_0+v} - 1\big) + \frac{4\pi N}{|\Omega|} \quad \text{in } \Omega. \tag{1.3}$$

A solution v of (1.3) is called of finite energy if v belongs to H^1. Indeed, it is well known that the corresponding physical energy of the solution v is finite if $v \in H^1$ [5,29,32]. Thus, solutions of finite energy are indeed physically meaningful in (1.3) and has been sought in the literature. It was first proved in [5] that there is a critical number $\epsilon_0 = \epsilon_0(m_j, p_j) > 0$ such that if $\epsilon < \epsilon_0$ then (1.3) admits a H^1 solution, and if $\epsilon > \epsilon_0$ then (1.3) admits no H^1 solution. This phenomenon is called a vortex confinement and it also appears in the Abelian–Higgs model [30]. Later, in [28], Tarantello showed that when $\epsilon < \epsilon_0$, there exist at least two H^1 solutions to (1.3). This multiplicity result was physically unexpected since the possible H^1 solutions of (1.3) have the same physical energy as well as the same distribution of vortex provided that the configurations of m_j and p_j $(j = 1, \dots, k)$ are the same. We remind that such multiplicity does not happen in the Abelian–Higgs model by the uniqueness [30]. After that, naturally, the asymptotic behavior of the multiple solutions has been studied on a torus as ϵ tends to 0 [24,25,28].

There are now many existence results for H^1-solutions of (1.3). By using the super-subsolution method, Caffarelli and Yang [5] constructed a maximal solution $\tilde{v}$ in the sense that if v is another solution then $v < \tilde{v}$. Asymptotics for maximal solutions was obtained in [16–18]. It was also pointed out in [5,14,15,24,25,28] that (1.3) admits a variational structure: every solution of (1.3) is a critical point of the associated functional

$$F_\epsilon(v) = \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon^2}\int_\Omega \big(e^{u_0+v} - 1\big)^2\, dx + \frac{4\pi N}{|\Omega|}\int_\Omega v\, dx, \quad v \in H^1(\Omega).$$

Moreover, if we decompose in (1.3)

$$v = w + c, \quad c = \frac{1}{|\Omega|}\int_\Omega v\, dx,$$

we get the following quadratic equation

$$e^{2c}\int\limits_{\Omega} e^{2u_0+2w}\,dx - e^{c}\int\limits_{\Omega} e^{u_0+w}\,dx + 4\pi N\epsilon^2 = 0,$$

which implies that

$$w \in \mathcal{A}_\epsilon \equiv \left\{ w \in H^1(\Omega) \;\Big|\; \int\limits_{\Omega} w\,dx = 0,\ \left(\int\limits_{\Omega} e^{u_0+w}\,dx\right)^2 - 16\pi N\epsilon^2 \int\limits_{\Omega} e^{2u_0+2w}\,dx \geqslant 0 \right\}.$$

Thus we may have two different variational formulations:
For $w \in \mathcal{A}_\epsilon$, define a constant $c_\pm(w)$ by

$$e^{c_\pm(w)} = \frac{\int_\Omega e^{u_0+w}\,dx \pm \sqrt{(\int_\Omega e^{u_0+w}\,dx)^2 - 16\pi N\epsilon^2 \int_\Omega e^{2u_0+2w}\,dx}}{2\int_\Omega e^{2u_0+2w}\,dx}, \tag{1.4}$$

so that

$$F_\epsilon\big(w + c_\pm(w)\big) = J_\epsilon^\pm(w) + \frac{|\Omega|}{2\epsilon^2} - 2\pi N + 4\pi N \ln\big(8\pi N\epsilon^2\big),$$

where

$$J_\epsilon^\pm(w) = \frac{1}{2}\|\nabla w\|_2^2 - 4\pi N \ln \int\limits_{\Omega} e^{u_0+w}\,dx - \frac{4\pi N}{1 \mp \sqrt{1-\epsilon^2 B(w)}} - 4\pi N \ln\big(1 \mp \sqrt{1-\epsilon^2 B(w)}\big),$$

$$B(w) = 16\pi N \int\limits_{\Omega} e^{2u_0+2w}\,dx \Big/ \left(\int\limits_{\Omega} e^{u_0+w}\,dx\right)^2.$$

Once we find a critical point $w_\pm \in \mathcal{A}_\epsilon$ of $J_\epsilon^\pm$ then $w_\pm + c_\pm(w_\pm)$ is a solution of (1.3). In particular, if w^ϵ is an interior infimum of J_ϵ^+ then $w^\epsilon + c_+(w^\epsilon)$ is a local minimum of F_ϵ. If w^ϵ is an interior infimum of J_ϵ^-, then $w^\epsilon + c_-(w^\epsilon)$ is a saddle point of F_ϵ. See [5,15,24,25,28] for details. The merit of this variational formulation is in analyzing the asymptotic behavior of solutions. In fact, in the case of $N = 1$, the Moser–Trudinger inequality enables us to find two interior infimum $w_\pm^\epsilon \in \mathcal{A}_\epsilon$ for $\epsilon > 0$ sufficiently small [5,28]. Moreover, in this case, w_-^ϵ is uniformly bounded in H^1 [28], and consequently, along a subsequence, $u_0 + w_-^\epsilon$ converges to a solution of the mean field equation as $\epsilon \to +0$. It is also proved in [15,24,25] that if $N = 2$, both J_ϵ^+ and J_ϵ^- attain global minimizers in the interior of $\mathcal{A}_\epsilon$. For this case, convergence to the solution of the mean field equation is not known [24,25].

For the case $N \geqslant 3$, it was proved in [14] by the heat flow method that for $\epsilon > 0$ sufficiently small, (1.3) admits at least two solutions $v_{1,\epsilon}$ and $v_{2,\epsilon}$ such that $v_{1,\epsilon} \to -u_0$ and $u_0 + v_{2,\epsilon} \to -\infty$ pointwisely almost everywhere as $\epsilon \to 0$. However, asymptotics for solutions of (1.3) are not completely known for $N \geqslant 2$. We refer to [9,25] for this topic.

In this paper, we consider asymptotics of solutions of (1.1) when $N \geqslant 3$. We construct two kinds of solutions for (1.1) by the variational method for some values of Chern–Simons constant. One kind of solutions converges to 0 as ϵ tends to zero. This solutions become the maximal solutions when ϵ is small enough. The other kind of solutions blows up at a single point in the sense of Brezis–Merle as ϵ tends to zero. In particular, the blow-up solution we find is of spike type, that is, the maximum values of the exponential of the solutions remain bounded and the solutions converge to zero except the maximum point as the Chern–Simons constant tends to zero. Similar kind of spike solutions has been dealt with in the different area (see, for example [3,23,31] and references therein). Furthermore, when $N > 3$, it turns out that the blow-up solution is a critical point of the functional J_ϵ^+. It is well known [28] that, for $\epsilon > 0$ sufficiently small, the maximal solution is a critical point of J_ϵ^+. Therefore, it indicates that when $N > 3$, J_ϵ^+ may have more than one critical point and the structure of the solution space of (1.3) might be complicated. As a corollary of our main theorem, in the case that the distribution of the vortex points are periodic in a torus, we can show that there are solutions blowing up at several points in the sense of Brezis–Merle. Moreover, if the vortex points are distributed periodically with multiplicity 1 or 2, we show that there are at least three solutions for certain values of the Chern–Simons constant. In this respect, under the periodic distribution of single vortex, (1.1) shows all possibilities of Brezis–Merle type alternatives.

This paper is organized as follows. In Section 2, we find solutions for (1.3) for certain values of ϵ by variational method. In Section 3, we present our main result, the asymptotics as $\epsilon \to 0$ of the solutions we find using the results in Section 4. In Section 4, we develop typical blow-up alternatives for (1.2) following [1,2,4,24,25], which is used in Section 3.

2. Existence

Throughout this paper, we fix some notations and definitions. We let $\mathcal{Z} = \{p_1, \ldots, p_k\} \subset \Omega$ the set of vortex points, m_j the multiplicities of the vortex points p_j, $N = \sum_j m_j \geqslant 1$ as before. We also let G the Green function for Ω satisfying

$$-\Delta_x G(x,y) = \delta_y - \frac{1}{|\Omega|}, \quad x, y \in \Omega, \quad \text{and} \quad \int_\Omega G(x,y)\,dx = 0$$

and $\gamma(x,y) = G(x,y) + \frac{1}{2\pi}\ln|x-y|$ be the regular part of the Green function. It is obvious that $u_0(x) = -\sum_{j=1}^k 4\pi m_j G(x, p_j)$. Finally, we denote

$$H^1_\# = \left\{ v \in H^1(\Omega) \,\Big|\, \int_\Omega v\,dx = 0 \right\},$$

$$J(v) = \frac{1}{2}\|\nabla v\|_2^2 - 4\pi N \ln \int_\Omega e^{u_0+v}\,dx \quad \text{for } v \in H^1_\#,$$

$$B(v) = 16\pi N \frac{\int_\Omega e^{2u_0+2v}\,dx}{(\int_\Omega e^{u_0+v}\,dx)^2} \quad \text{for } v \in H^1(\Omega).$$

We also present the Green representation formula for a solution v of (1.3)

$$v = \frac{1}{|\Omega|}\int_\Omega v(y)\,dy + \int_\Omega \epsilon^{-2} G(x,y)\big(e^{u_0+v} - e^{2u_0+2v}\big)(y)\,dy. \tag{2.1}$$

We note that for every $v \in H^1_\#$, $B(v) \geqslant 16\pi N/|\Omega|$ by the Hölder inequality. In fact, it is easy to show that, for any $t > 16\pi N/|\Omega|$, the set

$$S(t) = \big\{ v \in H^1_\# \mid B(v) = t \big\}$$

is nonempty and thus weakly closed in $H^1_\#$ by the Trudinger embedding theorem. Now, we borrow the following lemma from [24,25] to proceed.

Lemma 2.1. *For every $v \in H^1_\#$ and $0 < \tau \leqslant 1$,*

$$\int_\Omega e^{u_0+v}\,dx \leqslant \left(\frac{B(v)}{16\pi N}\right)^{\frac{1-\tau}{\tau}} \left(\int_\Omega e^{\tau(u_0+v)}\,dx\right)^{\frac{1}{\tau}}. \tag{2.2}$$

This lemma could be shown by the Hölder inequality. For the sake of convenience, we denote $J(t) \equiv \inf_{v\in S(t)} J(v)$ from now on.

Lemma 2.2. *For any $t > 16\pi N/|\Omega|$, $J(v)$ attains the infimum on $S(t)$ and $J(t)$ is continuous with respect to $t > 16\pi N/|\Omega|$.*

Proof. Let $v \in S(t)$. Taking $\tau = 1/N$ in (2.2) and using the Moser–Trudinger inequality, we have

$$\int_\Omega e^{u_0+v}\,dx \leqslant Ct^{N-1}\biggl(\int_\Omega e^{\frac{1}{N}(u_0+v)}\,dx\biggr)^N \leqslant Ct^{N-1}\exp\biggl(\frac{1}{16\pi N}\|\nabla v\|_2^2\biggr).$$

This implies that

$$J(v) \geqslant \frac{1}{4}\|\nabla v\|_2^2 - 4\pi N(N-1)\ln t - C. \tag{2.3}$$

Thus, J is coercive on $S(t)$ and attains the infimum on $S(t)$. Now let v_t be a minimizer of J on $S(t)$. By direct calculation,

$$B'(v_t)\varphi = 2B(v_t)\biggl(\frac{\int_\Omega e^{2u_0+2v_t}\varphi\,dx}{\int_\Omega e^{2u_0+2v_t}\,dx} - \frac{\int_\Omega e^{u_0+v_t}\varphi\,dx}{\int_\Omega e^{u_0+v_t}\,dx}\biggr) \quad \text{for } \varphi \in H^1. \tag{2.4}$$

Hence, $B'(v_t) \neq 0$. Choose $\varphi \in H^1$ such that $B'(v_t)\varphi = 1$. Then, applying the implicit function theorem to the function $a \mapsto B(v_t + a\phi)$, we get $\varepsilon_0 > 0$ and

$$a : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}, \qquad \frac{da}{d\varepsilon}\bigg|_{\varepsilon=0} = 1 \tag{2.5}$$

such that $B(v_t + a(\varepsilon)\varphi) = t + \varepsilon$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Thus,

$$J(t+\varepsilon) \leqslant J(v_t + a(\varepsilon)\varphi) \to J(v_t)$$

as $\varepsilon \to 0$ by the continuity of $J(v)$. That is, $\limsup_{\varepsilon\to 0} J(t+\varepsilon) \leqslant J(t)$. Similar argument replacing v_t with $v_{t+\varepsilon}$ gives $J(t) \leqslant \liminf_{\varepsilon\to 0} J(t+\varepsilon)$, which shows the continuity. □

Lemma 2.3. $J(t) = -2\pi N(N-2)\ln t + O(1)$ *for* $N \geqslant 2$ *as* $t \to \infty$.

Proof. Let $v \in S(t)$. As in Lemma 2.2, we plug $\tau = 2/N \leqslant 1$ into (2.2) to have

$$\int_\Omega e^{u_0+v}\,dx \leqslant Ct^{\frac{N-2}{2}}\biggl(\int_\Omega e^{\frac{2}{N}(u_0+v)}\,dx\biggr)^{\frac{N}{2}} \leqslant Ct^{\frac{N-2}{2}}\exp\biggl(\frac{1}{8\pi N}\|\nabla v\|_2^2\biggr). \tag{2.6}$$

Then, (2.6) implies that

$$J(v) \geqslant -2\pi N(N-2)\ln t - C.$$

We show that the growth rate $-2\pi N(N-2)$ is sharp in the above inequality. Without loss of generality, we may assume that u_0 attains a maximum at the origin. Fix a constant $r > 0$ such that the ball $B_{2r}(0) \subset \Omega$. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function such that $\chi \equiv 1$ on $B_r(0)$, and $\chi \equiv 0$ on $[B_{2r}(0)]^c$. Consider the test function

$$\varphi_\varepsilon(x) = -\chi(x)\ln\bigl(|x|^2+\varepsilon^2\bigr)^N, \quad \varepsilon > 0. \tag{2.7}$$

It is easily checked that as $\varepsilon \to 0$,

$$\|\nabla\varphi_\varepsilon\|_2^2 = \int_{|x|\leqslant r} \frac{4N^2|x|^2}{(|x|^2+\varepsilon^2)^2}\,dx + O(1) = 8\pi N^2\ln\frac{1}{\varepsilon} + O(1),$$

$$\int\limits_{\Omega} e^{u_0+\varphi_\varepsilon}\,dx = \int\limits_{|x|\leqslant r} \frac{e^{u_0(x)}}{(|x|^2+\varepsilon^2)^N}\,dx + \mathrm{O}(1)$$
$$= \int\limits_{|y|\leqslant r/\varepsilon} \frac{\varepsilon^{2-2N}e^{u_0(\varepsilon y)}}{(|y|^2+1)^N}\,dy + \mathrm{O}(1) = C\varepsilon^{2-2N} + \mathrm{O}(1),$$
$$\int\limits_{\Omega} e^{2u_0+2\varphi_\varepsilon}\,dx = \int\limits_{|x|\leqslant r} \frac{e^{2u_0(x)}}{(|x|^2+\varepsilon^2)^{2N}}\,dx + \mathrm{O}(1)$$
$$= \int\limits_{|y|\leqslant r/\varepsilon} \frac{\varepsilon^{2-4N}e^{2u_0(\varepsilon y)}}{(|y|^2+1)^{2N}}\,dy + \mathrm{O}(1) = C\varepsilon^{2-4N} + \mathrm{O}(1),$$

and $\int_\Omega \varphi_\varepsilon\,dx = \mathrm{O}(1)$. Let $\bar{\varphi}_\varepsilon = \varphi_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varphi_\varepsilon\,dx \in H^1_\#$. Since 0 is a maximum point of u_0, we have $|e^{u_0(\varepsilon y)} - e^{u_0(0)}| \leqslant C\varepsilon^2|y|^2$ for $|y| \leqslant r/\varepsilon$ and hence

$$B(\bar{\varphi}_\varepsilon) = B(\varphi_\varepsilon) = 16\pi N \frac{C\varepsilon^{2-4N} + \mathrm{O}(1)}{(C\varepsilon^{2-2N} + \mathrm{O}(1))^2} = C_0\varepsilon^{-2} + \mathrm{O}(1).$$

Thus, for t sufficiently large, we can choose $\varepsilon \sim \sqrt{C_0/t} > 0$ such that $t = B(\varphi_\varepsilon)$. Then

$$\inf_{B(v)=t} J(v) \leqslant J(\bar{\varphi}_\varepsilon) \leqslant -4\pi N(N-2)\ln\frac{1}{\varepsilon} + C \leqslant -2\pi N(N-2)\ln t + C$$

as $t \to \infty$. □

For a constant $\mu > 0$, we define a functional $I_\mu : H^1_\# \to \mathbb{R}$ by

$$I_\mu(v) = J(v) + \frac{B(v)}{\mu}. \tag{2.8}$$

Lemma 2.4. *For each $\mu > 0$, I_μ is coercive in $H^1_\#$ and there exists a global minimizer of I_μ.*

Proof. As in Lemma 2.2, we set $\tau = 1/N$ in (2.2) and repeat the calculation. Then, for all $v \in H^1_\#$,

$$I_\mu(v) \geqslant \frac{1}{4}\|\nabla v\|_2^2 - 4\pi N(N-1)\ln B(v) + \frac{B(v)}{\mu} - C$$
$$\geqslant \frac{1}{4}\|\nabla v\|_2^2 + \inf_{t>16\pi N/|\Omega|}\big[(t/\mu) - 4\pi N(N-1)\ln t\big] - C$$
$$\geqslant \frac{1}{4}\|\nabla v\|_2^2 - 4\pi N(N-1)\ln\mu - C,$$

where C depends only on Ω and $\mathcal{Z}$. Thus I_μ is bounded from below and coercive in $H^1_\#$. Since I_μ is lower semicontinuous, there exists a minimizer for each $\mu > 0$. □

For each $\mu > 0$, let $\underline{v}_\mu \in H^1_\#$ be a minimizer of I_μ. By the Lagrange multiplier theorem, the variational equation for I_μ is given by

$$\Delta \underline{v}_\mu = \frac{2}{\mu}B(\underline{v}_\mu)\frac{e^{2u_0+2\underline{v}_\mu}}{\int_\Omega e^{2u_0+2\underline{v}_\mu}\,dx} - \left(4\pi N + \frac{2}{\mu}B(\underline{v}_\mu)\right)\frac{e^{u_0+\underline{v}_\mu}}{\int_\Omega e^{u_0+\underline{v}_\mu}\,dx} + \frac{4\pi N}{|\Omega|} \quad \text{on } \Omega. \tag{2.9}$$

Lemma 2.5. *$B(\underline{v}_\mu)$ is strictly increasing with respect to μ. Furthermore, when $N \geqslant 3$, there exist two constants $C_1, C_2 > 0$ depending only on Ω and $\mathcal{Z}$ such that $C_1\mu \leqslant B(\underline{v}_\mu) \leqslant C_2\mu$ for μ sufficiently large.*

Proof. First, given $\mu_1 > \mu_2 > 0$,

$$I_{\mu_1}(\underline{v}_{\mu_1}) \leqslant I_{\mu_1}(\underline{v}_{\mu_2}) = I_{\mu_2}(\underline{v}_{\mu_2}) + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) B(\underline{v}_{\mu_2})$$
$$\leqslant I_{\mu_2}(\underline{v}_{\mu_1}) + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) B(\underline{v}_{\mu_2})$$

by the minimizing property of $\underline{v}_\mu$. However,

$$I_{\mu_1}(\underline{v}_{\mu_1}) - I_{\mu_2}(\underline{v}_{\mu_1}) = \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) B(\underline{v}_{\mu_1}).$$

From this, we deduce that $B(\underline{v}_\mu)$ is monotonically increasing with respect to μ. Since the equality holds only when $I_{\mu_1}(\underline{v}_{\mu_1}) = I_{\mu_1}(\underline{v}_{\mu_2})$, the equality implies that $\underline{v}_{\mu_2}$ is a minimizer of both I_{μ_1} and I_{μ_2}. But then, for any $\phi \in H^1_\#$,

$$B'(\underline{v}_{\mu_2})\phi = \frac{\mu_1\mu_2}{\mu_2 - \mu_1}(I'_{\mu_1} - I'_{\mu_2})(\underline{v}_{\mu_2})\phi = 0,$$

which is a contradiction by (2.4). Therefore, $B(\underline{v}_\mu)$ is strictly increasing and $\underline{v}_{\mu_1} \neq \underline{v}_{\mu_2}$ if $\mu_1 \neq \mu_2$.

Next, when $N \geqslant 3$,

$$I_\mu(\underline{v}_\mu) \leqslant \inf_{B(v)=\mu} I_\mu(v) = \inf_{B(v)=\mu} J(v) + 1 \leqslant -2\pi N(N-2)\ln\mu + C$$

for μ sufficiently large by Lemma 2.3. On the other hand, (2.6) implies that

$$I_\mu(\underline{v}_\mu) = J(\underline{v}_\mu) + \frac{B(\underline{v}_\mu)}{\mu} \geqslant -2\pi N(N-2)\ln\left(\frac{B(\underline{v}_\mu)}{\mu}\right) + \frac{B(\underline{v}_\mu)}{\mu} - 2\pi N(N-2)\ln\mu - C.$$

Consequently, it follows that

$$\frac{B(\underline{v}_\mu)}{\mu} - 2\pi N(N-2)\ln\left(\frac{B(\underline{v}_\mu)}{\mu}\right) \leqslant C$$

for μ sufficiently large. Then, we have $C_1 < B(\underline{v}_\mu)/\mu < C_2$ for some $C_1, C_2 > 0$ from the asymptotics of the function $t \mapsto t - 2\pi N(N-2)\ln t$. $\square$

Theorem 2.1. *Let $N \geqslant 3$, $\underline{v}_\mu$ be a minimizer of I_μ as before, and*

$$\epsilon = \epsilon_\mu \equiv \sqrt{\frac{8\pi N}{\mu}}\left(2\pi N + \frac{B(\underline{v}_\mu)}{\mu}\right)^{-1} \tag{2.10}$$

for some $\mu > 0$. Then, there exist a solution $\underline{u}_\mu \in H^1$ for (1.3) *with $\epsilon = \epsilon_\mu$. Furthermore, $B(\underline{u}_\mu) \to \infty$ as $\mu \to \infty$ and*

$$\lim_{\mu\to 0} \epsilon_\mu = \lim_{\mu\to\infty} \epsilon_\mu = 0.$$

Proof. By Lemma 2.4 and 2.5, for any $\mu > 0$, there exists $\underline{v}_\mu$ satisfying (2.9). Let us define $c_\mu \in \mathbb{R}$ and $\underline{u}_\mu \in H^1(\Omega)$ by

$$c_\mu = \ln\frac{\int_\Omega e^{u_0 + \underline{v}_\mu}\,dx}{\int_\Omega e^{2u_0 + 2\underline{v}_\mu}\,dx} + \ln\left(\frac{B(\underline{v}_\mu)}{2\pi N\mu + B(\underline{v}_\mu)}\right), \tag{2.11}$$

$$\underline{u}_\mu = \underline{v}_\mu + c_\mu. \tag{2.12}$$

Then, by direct calculation, $\underline{u}_\mu$ is a solution of (1.3) with $\epsilon = \epsilon_\mu$. Since $C_1\mu < B(\underline{v}_\mu) < C_2\mu$ for large enough μ and $B(\underline{u}_\mu) = B(\underline{v}_\mu)$, it follows from (2.10) that

$$\lim_{\mu\to 0} \epsilon_\mu = \lim_{\mu\to\infty} \epsilon_\mu = 0, \qquad \lim_{\mu\to\infty} B(\underline{u}_\mu) = \infty. \qquad \square$$

It is easily checked that $\underline{u}_\mu \in H^1$ is a critical point of F_ϵ with $\epsilon = \epsilon_\mu$, and $c_\mu = (1/|\Omega|)\int_\Omega \underline{u}_\mu\,dx$. Then $c_\mu = c_+(\underline{v}_\mu)$ or $c_\mu = c_-(\underline{v}_\mu)$, where $c_\pm$ is defined in (1.4). We will prove in Section 3 that $c_\mu = c_+(\underline{v}_\mu)$ for μ sufficiently large and $N > 3$, and consequently, $\underline{v}_\mu$ is a critical point of J_ϵ^+ with $\epsilon = \epsilon_\mu$.

3. Asymptotics of the solutions

In this section, we study the asymptotic behavior of $\underline{u}_\mu = \underline{v}_\mu + c_\mu$ as $\mu \to 0$ and $\mu \to \infty$. We first present some preliminary facts.

Lemma 3.1. *Let $u \in H^1$ be a solution of* (1.1). *Then, $u = v + u_0 \leqslant 0$ and*

$$\int_\Omega \frac{1}{\epsilon^2} e^u\left(1 - e^u\right) = 4\pi N.$$

The lemma is well known (see, for example, [5,29]) and can be shown simply by the maximum principle. Now, we consider (1.1) on the whole of $\mathbb{R}^2$ when the distribution of vortex points, $\mathcal{Z} = \{0\}$.

Lemma 3.2. *Let m be a nonnegative integer, and u be a* (*smooth*) *solution of the following equation*

$$\Delta u = e^u\left(e^u - 1\right) + 4\pi m \delta_{p=0} \quad \text{in } \mathbb{R}^2. \tag{3.1}$$

If $e^u(e^u - 1) \in L^1(\mathbb{R}^2)$, either

(i) $u(x) \to 0$ *as* $|x| \to \infty$, *or*
(ii) $u(x) = -\beta \ln|x| + \mathrm{O}(1)$ *near* ∞, *where*

$$\beta = -2m + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^u\left(1 - e^u\right) dx.$$

Assume that u satisfies the boundary condition (ii). *Then we have*

$$\int_{\mathbb{R}^2} e^{2u}\,dx = \pi\left(\beta^2 - 4\beta - 4m^2 - 8m\right) \quad \text{and} \quad \int_{\mathbb{R}^2} e^u\,dx = \pi\left(\beta^2 - 2\beta - 4m^2 - 4m\right). \tag{3.2}$$

In particular, $\int_{\mathbb{R}^2} e^u(1 - e^u)\,dx > 8\pi(1 + m)$.

Proof. This lemma might be well-known. But since we cannot find its' proof in the literature, we present the sketch of the proof here following the argument in [12]. Since $e^u(e^u - 1) \in L^1(\mathbb{R}^2)$, the argument of [4] implies that u is bounded from above and $u \in C_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. Moreover, by [12], $u(x) = -\beta \ln|x| + \mathrm{O}(1)$ near ∞ for some constant $\beta \in \mathbb{R}$ and $u = 2m \ln|x| + \mathrm{O}(1)$ near the origin. Then it follows from the L^1-condition and elliptic estimates that either $\beta = 0$ or $\beta > 2$. In the case that $\beta = 0$, we arrive at (i) by the L^1-condition. In the case that $\beta > 2$, we further have $\nabla u(x) = -\beta \frac{x}{|x|^2} + \mathrm{o}(|x|^{-1})$ near ∞ by [12]. Multiplying (3.1) by $x \cdot \nabla u$ and integrating on the domain $\Sigma = \{x \mid r < |x| < R\}$, we obtain

$$\int_{\partial\Sigma} \left[\frac{1}{2}(x \cdot \nu)|\nabla u|^2 - (x \cdot \nabla u)(\nu \cdot \nabla u) + (x \cdot \nu)\left(\frac{1}{2}e^{2u} - e^u\right)\right] d\sigma = \int_\Sigma \left(e^{2u} - 2e^u\right) dx.$$

Letting $r \to 0$ and $R \to \infty$, we obtain $\int_{\mathbb{R}^2}(2e^u - e^{2u})\,dx = \pi(\beta^2 - 4m^2)$. Meanwhile, integrating (3.1) on Σ and letting $r \to 0$ and $R \to \infty$, we have $\int_{\mathbb{R}^2}(e^u - e^{2u})\,dx = \pi(4m + 2\beta)$. Thus, (3.2) immediately follows. Then the first identity in (3.2) implies that $\beta > 2m + 4$, which in turn implies that $\|e^u(e^u - 1)\|_{L^1(\mathbb{R}^2)} > 8\pi(1 + m)$. $\square$

If u is a solution of (3.1) with $m = 0$ and $e^u(e^u - 1) \in L^1(\mathbb{R}^2)$, we further have the following lemma due to [7,10,27].

Lemma 3.3. *Let u be a solution of* (3.1) *with $m = 0$ and $e^u(e^u - 1) \in L^1(\mathbb{R}^2)$. Then, u is radially symmetric and smooth. Let $u(r; s)$ be the radial solution of* (3.1) *such that $\lim_{r\to 0} u(r; s) = s$ and $\lim_{r\to 0} u_r(r; s) = 0$. Then we further obtain*

(a) $u(\cdot\,; 0) = 0$,
(b) *If $s < 0$, $u(r; s) \to -\infty$ as $r \to \infty$,*
(c) *If $s > 0$, $u(r; s)$ blows up at some $r = r(s) > 0$.*

Moreover, if we define a function $\xi_0 : (-\infty, 0) \to \mathbb{R}_+ = (0, \infty)$ by

$$\xi_0(s) = \int_0^\infty e^{u(r;s)}\bigl(1 - e^{u(r;s)}\bigr) r\,dr \tag{3.3}$$

then $\lim_{s\to 0^-} \xi_0(s) = \infty$, $\lim_{s\to -\infty} \xi_0(s) = 4$, and ξ_0 is continuously differentiable and strictly increasing on the interval $(-\infty, 0)$.

The following is an analogy of the Brezis–Merle type alternatives [1,2,4,24,25] for (1.3). It is not only interesting in itself but also will be used frequently in this section.

Theorem 3.1. *Let v_ϵ, $\epsilon \to 0$ be a sequence of solutions of* (1.3). *Then, up to subsequences, one of the following holds true*:

(i) *$v_\epsilon \to -u_0$ in $C_{\mathrm{loc}}(\Omega\setminus\mathcal{Z})$, or*
(ii) *$v_\epsilon - 2\ln\epsilon$ is bounded uniformly in $C^0(\Omega)$, or*
(iii) *$\limsup_\epsilon \sup_\Omega (u_0 + v_\epsilon) < 0$ and there exist a nonempty finite set $\mathcal{S} = \{q_1, \dots, q_l\} \subset \Omega$ and l number of sequences of points $x_{j,\epsilon} \to q_j$, $j = 1, \dots, l$ such that*

$$(v_\epsilon - 2\ln\epsilon)(x_{j,\epsilon}) \to \infty$$

for any $j = 1, \dots, l$ and $v_\epsilon - 2\ln\epsilon \to -\infty$ uniformly on any compact subset of $\Omega\setminus\mathcal{S}$. Moreover,

$$\frac{1}{\epsilon^2} e^{u_0+v_\epsilon}\bigl(1 - e^{u_0+v_\epsilon}\bigr) \to \sum \alpha_j \delta_{q_j}, \quad \alpha_j \geqslant 8\pi$$

in the sense of measure.

The proof of the above theorem is a bit technical, so we postpone it to Section 4.
In view of the above theorem, we define the blow-up solutions for (1.2) as follows.
For a sequence of solutions $\{w_\epsilon\}$ of (1.2), if there exist $q \in \Omega$ and $x_\epsilon \in \Omega$ satisfying

$$w_\epsilon(x_\epsilon) \to \infty, \quad x_\epsilon \to q$$

as $\epsilon \to 0$, we call $\{w_\epsilon\}$ blow-up solutions of (1.2) following Brezis–Merle [4]. Also, we call q a blow-up point and call the collection of all blow-up points of $\{w_\epsilon\}$ the blow-up set for $\{w_\epsilon\}$.
Now, we consider the asymptotics when $\mu \to 0$.

Lemma 3.4. *Let $\underline{v}_\mu$ be as in Section* 2. *$B(\underline{v}_\mu)$ converges to $16\pi N/|\Omega|$ as $\mu \to 0$.*

Proof. Given any $\delta > 0$, let φ be a smooth function such that $B(\varphi) < 16\pi N/|\Omega| + \delta$. Then (2.3) implies that

$$\frac{B(\underline{v}_\mu)}{\mu} - 4\pi N(N-1)\ln B(\underline{v}_\mu) - C \leqslant I_\mu(\underline{v}_\mu) \leqslant I_\mu(\varphi) = J(\varphi) + \frac{B(\varphi)}{\mu},$$

which in turn implies that

$$B(\underline{v}_\mu) - 4\pi N(N-1)\mu \ln B(\underline{v}_\mu) \leqslant B(\varphi) + C\mu.$$

Since $B(\underline{v}_\mu)$ is monotone by Lemma 2.5, letting $\mu \to 0$ in the above inequality, we get

$$\limsup_{\mu\to 0} B(\underline{v}_\mu) \leqslant B(\varphi) < 16\pi N/|\Omega| + \delta.$$

However, $B(\underline{v}_\mu) \geqslant 16\pi N/|\Omega|$ and $\delta > 0$ is arbitrary, Lemma 3.4 immediately follows. □

The following theorem tells that $\{u_0 + \underline{u}_\mu\}$ satisfies the first alternative in Theorem 3.1 as $\mu \to 0$.

Theorem 3.2. $\|u_0 + \underline{u}_\mu\|_{L^\infty(K)} \to 0$ *as* $\mu \to 0$ *for any compact subset* K *of* $\Omega\backslash\mathcal{Z}$.

Proof. We argue by contradiction, and suppose that there exists a sequence of μ's (still denoted by μ) such that $\mu \to 0$ and $\{u_0 + \underline{u}_\mu\}$ does not satisfy the alternative (i) in Theorem 3.1. Let ϵ_μ as in (2.10). Then $\underline{u}_\mu - 2\ln\epsilon_\mu$ is a solution of the following equation.

$$\Delta v = -e^{u_0+v}\bigl(1 - \epsilon^2 e^{u_0+v}\bigr) + \frac{4\pi N}{|\Omega|}.$$

If (ii) of Theorem 3.1 is the case, RHS of the above equation is uniformly bounded. Thus, by the elliptic theory, we arrive that $\underline{u}_\mu - 2\ln\epsilon_\mu$ converges uniformly to a smooth function ϕ up to subsequences. Then, $B(\underline{v}_\mu) = B(\underline{u}_\mu - 2\ln\epsilon_\mu) \to B(\phi)$. However, $B(\phi) > 16\pi N/|\Omega|$ for any smooth ϕ, which contradicts Lemma 3.4.

If (iii) of Theorem 3.1 is the case, there exists a blow-up set $S = \{q_1, \dots, q_l\}$ and $e^{\underline{u}_\mu - 2\ln\epsilon_\mu} \to 0$ in $C^0_{\mathrm{loc}}(\Omega\backslash S)$ up to subsequences. Thus, denoting $w_\mu = u_0 + \underline{u}_\mu - 2\ln\epsilon_\mu$, for any $r > 0$ small enough,

$$\int_\Omega e^{w_\mu} = \int_{\bigcup B_r(q_i)} e^{w_\mu} + \mathrm{o}(1) \leqslant Cr\biggl(\int_{\bigcup B_r(q_i)} e^{2w_\mu}\biggr)^{1/2} + \mathrm{o}(1) \leqslant Cr\biggl(\int_\Omega e^{2w_\mu}\biggr)^{1/2} + \mathrm{o}(1).$$

But then, since $\|e^{w_\mu}\|_{L^1(\Omega)} \geqslant 4\pi N$ by Lemma 3.1,

$$B(\underline{v}_\mu) = 16\pi N \frac{\int_\Omega e^{2w_\mu}}{(\int_\Omega e^{w_\mu})^2} \leqslant C/r^2$$

as $\mu \to 0$. Taking r small enough, we are led to a contradiction. Theorem 3.2 is proved. □

The following theorem follows from the uniqueness of the solution of (1.1) near the maximal solutions in [13].

Theorem 3.3. *For* $\mu > 0$ *sufficiently small, the function* $\mu \to B(\underline{v}_\mu)$ *is continuous and* $\{\underline{u}_\mu\}$ *becomes the continuous family of maximal solutions. Thus, there exists a constant* $\mu_0 > 0$ *such that if* $\epsilon = \epsilon_\mu$ *for some* $\mu > \mu_0$*, we have two solutions for* (1.1).

Proof. By Lemma 3.2 and [13], when $\mu > 0$ is small enough, $\underline{u}_\mu$ must be the maximal solution of (1.3) for $\epsilon = \epsilon_\mu$. Therefore, there exists a constant μ_1 such that the mappings $\mu \mapsto B(\underline{v}_\mu)$ and $\mu \mapsto \epsilon_\mu$ are (single-valued) continuous for $\mu < \mu_1$. Then, by Theorem 2.1, there always exists $\mu < \mu_1$ for any $\epsilon < \epsilon_{\mu_1}$ such that $\epsilon_\mu = \epsilon$. Meanwhile, there exists $\mu_0 \gg \mu_1$ such that $\epsilon_{\mu_0} < \epsilon_{\mu_1}$ by Theorem 2.1. Consequently, if $\epsilon = \epsilon_\mu$ with $\mu > \mu_0$, we have two solutions for (1.1), one with $\mu < \mu_1$ and the other with $\mu > \mu_0$. □

We now concentrate on the other situation, $\mu \to +\infty$. In this case, Lemma 2.5 imply that either (ii) or (iii) of Theorem 3.1 is the case and thus there is a constant $\nu = \nu(\Omega, \mathcal{Z}) > 0$ such that

$$\sup_{\mu>1} \sup_\Omega (u_0 + \underline{u}_\mu) \leqslant -\nu. \tag{3.4}$$

It will turn out that (iii) of Theorem 3.1 holds in this case. Moreover, the blow-up set consists of a single point q, which should be a maximum point of u_0. To prove it, we need the following lemma dealing with a special case of (iii) of Theorem 3.1, blow-up away from the vortex points.

Lemma 3.5. *Let $w_\epsilon = v_\epsilon - 2\ln\epsilon$ be the blow-up sequence in* (iii) *of Theorem* 3.1 *and q_j, α_j as in* (iii) *of Theorem* 3.1. *Assume that $q_j \notin \mathcal{Z}$. Then, given $r > 0$ small enough, there exist a constant $C > 0$ and a sequence of points $\{x_\epsilon\} \subset B_r(q_j)$ with the property that*

$$w_\epsilon(x_\epsilon) = \max_{|x-q_j|\leqslant r} w_\epsilon(x) \to \infty \quad \text{as } \epsilon \to 0 \tag{3.5}$$

and

$$\max_{|x-q_j|\leqslant r} \big(w_\epsilon(x) + 2\ln|x - x_\epsilon|\big) \leqslant C. \tag{3.6}$$

Moreover, for any sequence $\{R_\epsilon\}$ such that $R_\epsilon \to \infty$,

$$\lim_{\epsilon\to 0} \int\limits_{|y-x_\epsilon|\leqslant R_\epsilon s_\epsilon} e^{w_\epsilon}\big(1 - \epsilon^2 e^{w_\epsilon}\big)(y)\,dy = \alpha_j \tag{3.7}$$

where $s_\epsilon = \exp[-\frac{1}{2} w_\epsilon(x_\epsilon)]$.

Proof. See Section 4. □

Now, we are ready to show our main result.

Theorem 3.4. *Assume that $N \geqslant 3$ and $\underline{u}_\mu$, $\underline{v}_\mu$ as before.*

(i) *As $\mu \to \infty$, along a subsequence, $\underline{u}_\mu - 2\ln\epsilon_\mu \to -\infty$ uniformly on any compact set $K \subset \Omega\backslash\{q\}$ for some $q \in \Omega$, and*

$$\frac{1}{\epsilon_\mu^2} e^{u_0 + \underline{u}_\mu}\big(1 - e^{u_0+\underline{u}_\mu}\big) \to 4\pi N\delta_q \quad \text{in the sense of measure}.$$

Furthermore, $u_0(q) = \max_\Omega u_0$.

(ii) $\lim_{\mu\to\infty} \frac{B(\underline{v}_\mu)}{\mu} = 2\pi N(N-2)$.

(iii) *$\underline{v}_\mu$ is a critical point of the functional J_ϵ^+ with $\epsilon = \epsilon_\mu$ provided that $N > 3$ and μ is sufficiently large.*

Proof. We first show (i). We break it into several steps.

Step 1. $\max_\Omega(\underline{u}_\mu - 2\ln\epsilon_\mu) \to \infty$, and hence $\|\nabla\underline{u}_\mu\|_2 \to \infty$.

If not, there would be a sequence(still denoted by μ) such that $\mu \to \infty$ and $\max_\Omega(\underline{u}_\mu - 2\ln\epsilon_\mu) \leqslant C$ for some constant $C > 0$. Then case (ii) of Theorem 3.1 must hold true. That is, along a subsequence, $\{\underline{u}_\mu - 2\ln\epsilon_\mu\}$ is bounded in $C^0(\Omega)$. It follows that $B(\underline{v}_\mu) = B(\underline{u}_\mu - 2\ln\epsilon_\mu) \leqslant C$, which contradicts Lemma 2.5 and shows Step 1. Step 1 implies that case (iii) of Theorem 3.1 holds true for $\underline{u}_\mu$. In particular, we obtain that $\|\nabla\underline{u}_\mu\|_2 \to \infty$.

Step 2. $|\mathcal{S}| = 1$.

We argue by contradiction, and suppose that, there is a sequence still denoted by $\underline{u}_\mu$ which blows up at more than two points. Let $\mathcal{S} = \{q_1, \ldots, q_l\}$ be the blow-up set for $\underline{u}_\mu$ with $l \geqslant 2$. We take a small constant $r > 0$ such that $B_{2r}(q_i)$'s are mutually disjoint. It follows from Theorem 3.1 and Green's representation formula (2.1) that

$$\underline{v}_\mu = \int\limits_\Omega \frac{1}{\epsilon_\mu^2} G(x,y)\big(e^{u_0+\underline{u}_\mu} - e^{2u_0+2\underline{u}_\mu}\big)(y)\,dy$$

and

$$\underline{v}_\mu \to \sum_{i=1}^{l} \alpha_i G(x, q_i), \quad \alpha_i \geqslant 8\pi \tag{3.8}$$

in $C^1_{\mathrm{loc}}(\Omega\backslash\mathcal{S})$. In particular, $\underline{v}_\mu$ is bounded in $C^1(\Omega\backslash\bigcup_{i=1}^l B_r(q_i))$. Moreover, Theorem 3.1 imply that there is a positive constant c_0 independent of μ such that

$$c_0 \leqslant \int\limits_{B_r(q_i)} \frac{1}{\epsilon_\mu^2} e^{u_0+\underline{u}_\mu}\,dx < \frac{1}{c_0}.$$

Since $\underline{u}_\mu = \underline{v}_\mu + c_\mu$,

$$c_0^2 \int\limits_{B_r(q_i)} e^{u_0+\underline{v}_\mu}\,dx \leqslant \int\limits_{B_r(q_j)} e^{u_0+\underline{v}_\mu}\,dx \leqslant \frac{1}{c_0^2} \int\limits_{B_r(q_i)} e^{u_0+\underline{v}_\mu}\,dx \tag{3.9}$$

for all $1 \leqslant i < j \leqslant l$. Note also that $\|e^{u_0+\underline{v}_\mu}\|_{L^1(\Omega)} \to \infty$ by (3.8). Thus, together with (3.8) and (3.9), we have

$$\ln \int\limits_{B_r(q_i)} e^{u_0+\underline{v}_\mu}\,dx = \ln \int\limits_{\Omega} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1) \tag{3.10}$$

for all $1 \leqslant i \leqslant l$.

For $i = 1, \ldots, l$, we let χ_i be a smooth function such that $\chi_i = 1$ on $B_r(q_i)$, $0 \leqslant \chi_i \leqslant 1$, and $\chi_i = 0$ outside $B_{2r}(q_i)$. Set $\varphi_{j,\mu} = \chi_j \underline{v}_\mu$ for $j = 1, \ldots, l$. It follows from (3.9) that

$$\frac{B(\varphi_{j,\epsilon})}{\mu} \leqslant \frac{C}{\mu} \frac{\int_\Omega e^{2u_0+2\varphi_{j,\mu}}\,dx}{(\int_{\cup B_r(q_i)} e^{u_0+\varphi_{j,\mu}}\,dx)^2} \leqslant \frac{C}{\mu} \frac{\int_\Omega e^{2u_0+2\underline{v}_\mu}\,dx + 1}{(\int_\Omega e^{u_0+\underline{v}_\mu}\,dx)^2} \leqslant \frac{C}{\mu}\big(B(\underline{v}_\mu) + 1\big) \leqslant C.$$

Then (3.10) implies that

$$\begin{aligned}
I_\mu(\underline{v}_\mu) &= \sum_{i=1}^{l} \frac{1}{2} \|\nabla \underline{v}_\mu\|^2_{L^2(B_r(q_i))} - 4\pi N \ln \int\limits_{\Omega} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1) \\
&= \sum_{i=1}^{l} \frac{1}{2} \|\nabla \underline{v}_\mu\|^2_{L^2(B_r(q_i))} - 4\pi N \ln \int\limits_{B_r(q_1)} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1) \\
&= \sum_{i=1}^{l} I_\mu(\varphi_{i,\mu}) + 4\pi N \sum_{i=2}^{l} \ln \int\limits_{B_r(q_i)} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1) \\
&\geqslant l I_\mu(\underline{v}_\mu) + 4\pi N \sum_{i=2}^{l} \ln \int\limits_{B_r(q_i)} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1).
\end{aligned} \tag{3.11}$$

Subsequently,

$$I_\mu(\underline{v}_\mu) \leqslant -\frac{4\pi N}{l-1} \sum_{i=2}^{l} \ln \int\limits_{B_r(q_i)} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1) \leqslant -4\pi N \ln \int\limits_{\Omega} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1),$$

which means that $\|\nabla \underline{v}_\mu\|_2^2$ is uniformly bounded for $\mu > 1$. This contradicts Step 1.

Step 3. The blow-up set is disjoint with $\mathcal{Z}$.

We argue by contradiction, and suppose that a subsequence of $\underline{u}_\mu$(still denoted by $\underline{u}_\mu$) which blows up at $p \in \mathcal{Z}$. Given any $\delta > 0$, fix a constant $r > 0$ small enough such that $e^{u_0}(x) \leqslant \delta$ for $x \in B_r(p)$. For $0 < \tau < 1$ and μ sufficiently large, (iii) of Theorem 3.1 and the Moser–Trudinger inequality imply that

$$\begin{aligned}
\int\limits_{\Omega} e^{\tau(u_0+\underline{v}_\mu)}\,dx &= \big(1 + \mathrm{o}(1)\big) \int\limits_{B_r(p)} e^{\tau(u_0+\underline{v}_\mu)}\,dx \leqslant \delta^\tau \big(1 + \mathrm{o}(1)\big) \int\limits_{B_r(p)} e^{\tau \underline{v}_\mu}\,dx \\
&\leqslant \delta^\tau \big(1 + \mathrm{o}(1)\big) \int\limits_{\Omega} e^{\tau \underline{v}_\mu}\,dx \leqslant C\delta^\tau \exp\left[\frac{\tau^2}{16\pi} \|\nabla \underline{v}_\mu\|_2^2\right].
\end{aligned}$$

Set $\tau = 2/N$. Then (2.2) and Lemma 2.5 imply that

$$I_\mu(\underline{v}_\mu) \geqslant -2\pi N(N-2) \ln B(\underline{v}_\mu) - 4\pi N \ln \delta - C,$$

which contradicts Lemma 2.3 if $\delta > 0$ is sufficiently small.

Step 4. The blow-up point is a maximal point of u_0.

We argue by contradiction again. Suppose that q is the blow-up point for a sequence of solutions $\underline{u}_\mu$ and $u_0(q) < \max_\Omega u_0$. Let q^* be a maximum point of u_0, $x_\mu \to q$ be a maximum point of $\underline{v}_\mu$, and $v_\mu^*(x) = \underline{v}_\mu(x + x_\mu - q^*)$. Let $\delta > 0$ be a small constant. Since $q \notin \mathcal{Z}$ and $x_\mu \to q$,

$$\begin{aligned}\int_\Omega e^{u_0+v_\mu^*}\,dx &= \int_\Omega e^{u_0(x+q^*-x_\mu)+\underline{v}_\mu(x)}\,dx = \int_{B_\delta(q)} e^{u_0(x+q^*-x_\mu)-u_0(x)}e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1)\\ &= \big(e^{u_0(q^*)-u_0(q)} + \mathrm{O}(\delta)\big)\int_{B_\delta(q)} e^{u_0+\underline{v}_\mu}\,dx + \mathrm{O}(1)\\ &= \big(e^{u_0(q^*)-u_0(q)} + \mathrm{O}(\delta)\big)\int_\Omega e^{u_0+\underline{v}_{\mu_n}}\,dx + \mathrm{O}(1)\end{aligned}$$

as $\mu \to +\infty$. Similarly, we obtain

$$\int_\Omega e^{2u_0+2v_\mu^*}\,dx = \big(e^{2u_0(q^*)-2u_0(q)} + \mathrm{O}(\delta)\big)\int_\Omega e^{2u_0+2\underline{v}_\mu}\,dx + \mathrm{o}(1)$$

as $\mu \to +\infty$. Then it follows that

$$\frac{B(v_\mu^*)}{\mu} = \frac{16\pi N}{\mu}\frac{\int_\Omega e^{2u_0+2v_\mu^*}\,dx}{(\int_\Omega e^{u_0+v_\mu^*}\,dx)^2} = \frac{B(\underline{v}_\mu)}{\mu}\big(1+\mathrm{O}(\delta)\big) + \mathrm{o}(1),$$

and consequently, as $\mu \to +\infty$,

$$\begin{aligned}I_\mu(v_\mu^*) &= \frac{1}{2}\|\nabla v_\mu^*\|_2^2 - 4\pi N \ln\int_\Omega e^{u_0+v_\mu^*}\,dx + \frac{B(v_\mu^*)}{\mu}\\ &= \frac{1}{2}\|\nabla \underline{v}_\mu\|_2^2 - 4\pi N\ln\big[\big(e^{u_0(q^*)-u_0(q)} + \mathrm{O}(\delta)\big)C_\mu + \mathrm{o}(1)\big] + \big(1+\mathrm{O}(\delta)\big)\frac{B(\underline{v}_\mu)}{\mu} + \mathrm{o}(1)\\ &= I_\mu(\underline{v}_\mu) - 4\pi N\big(u_0(q^*) - u_0(q) + \mathrm{O}(\delta)\big) + \mathrm{o}(1) < I_\mu(\underline{v}_\mu) = \inf I_\mu\end{aligned}$$

if we choose δ small enough. This yields a contradiction and (ii) is proved.

We now prove (ii). Let x_μ be a maximum point of $\underline{v}_\mu$, q be the only blow-up point of $\underline{u}_\mu - 2\ln\epsilon_\mu$, and

$$s_\mu = \exp\left[-\frac{1}{2}\left(\underline{v}_\mu(x_\mu) - \ln\int_\Omega e^{u_0+\underline{v}_\mu}\,dx\right)\right].$$

It is obvious that $x_\mu \to q$. Recall that $\underline{u}_\mu = \underline{v}_\mu + c_\mu$ where c_μ is defined in (2.11).

For simplicity, we let $t_\mu = B(\underline{v}_\mu)/\mu$. Then it follows from (2.10) that

$$\frac{\epsilon_\mu^2}{s_\mu^2} = \frac{8\pi N}{\mu s_\mu^2(2\pi N + t_\mu)^2} = \frac{e^{\underline{u}_\mu(x_\mu)}}{4\pi N + 2t_\mu}. \tag{3.12}$$

In particular,

$$-2\ln s_\mu + \ln(4\pi N + 2t_\mu) = \underline{u}_\mu(x_\mu) - 2\ln\epsilon_\mu \to \infty.$$

Lemma 2.5 implies that $\ln(4\pi N + 2t_\mu)$ is bounded. Consequently, $s_\mu \to 0$ as $\mu \to \infty$.

We let

$$\varphi_\mu(x) = \underline{v}_\mu(s_\mu x + x_\mu) - \underline{v}_\mu(x_\mu)$$

for $x \in \Omega_\mu = \{x \mid s_\mu x + x_\mu \in \Omega\}$. Then φ_μ satisfies

$$-\Delta\varphi_\mu = (4\pi N + 2t_\mu)e^{u_0(s_\mu x + x_\mu)+\varphi_\mu} - \frac{32\pi N}{\mu s_\mu^2}e^{2u_0(s_\mu x+x_\mu)+2\varphi_\mu} - \frac{4\pi N s_\mu^2}{|\Omega|} \quad \text{in } \Omega_\mu.$$

Since $\underline{u}_\mu(x_\mu) < -\nu$ for some constant $\nu = \nu(\mathcal{Z}, \Omega) > 0$, (3.12) implies that

$$\frac{32\pi N}{\mu s_\mu^2} e^{u_0(s_\mu x + x_\mu) + \varphi_\mu(x)} \leqslant \frac{32\pi N}{\mu s_\mu^2} e^{-\nu - \underline{u}_\mu(x_\mu)} = e^{-\nu}(4\pi N + 2t_\mu).$$

In particular, $32\pi N/(\mu s_\mu^2) \leqslant e^{-u_0(x_\mu)-\nu}(4\pi N + 2t_\mu)$.

Since $\varphi_\mu \leqslant \varphi_\mu(0) = 0$, it follows from Harnack's inequality that φ_μ is bounded in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. Passing to subsequences, we may assume that $t_\mu \to t$ for some constant $t > 0$, $32\pi N/(\mu s_\mu^2) \to c_0^2$ for some constant $c_0 \geqslant 0$, and φ_μ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function φ_* satisfying

$$\begin{aligned}
&-\Delta\varphi_* = e^{u_0(q)+\varphi_*}\big((4\pi N + 2t) - c_0^2 e^{u_0(q)+\varphi_*}\big) \quad \text{in } \mathbb{R}^2,\\
&c_0^2 e^{u_0(q)+\varphi_*} \leqslant e^{-\nu}(4\pi N + 2t).
\end{aligned}$$

By making use of the diagonal process, we can choose a sequence $R_\mu \to \infty$ such that $\|\varphi_\mu - \varphi\|_{C^2(B_{R_\mu})} \to 0$. Then it follows from (3.7) in Lemma 3.5 that

$$\int_{\mathbb{R}^2} e^{u_0(q)+\varphi_*}\big((4\pi N + 2t) - c_0^2 e^{u_0(q)+\varphi_*}\big)\,dx = 4\pi N. \tag{3.13}$$

If $c_0 = 0$, φ_* satisfies the Liouville equation. But then

$$\int_{\mathbb{R}^2} e^{u_0(q)+\varphi_*}(4\pi N + 2t) = 8\pi$$

by [11], which is a contradiction to (3.13) since $N \geqslant 3$. Therefore, $c_0 > 0$. Then (3.12) implies that $\underline{u}_\mu$ is bounded from below. This together with (i) implies that $\underline{u}_\mu$ is of spike type up to subsequences. Next, we let

$$\xi_\mu(x) = \varphi_\mu\left(\frac{\epsilon_\mu x}{s_\mu}\right) + 2\ln\left(\frac{\epsilon_\mu}{s_\mu}\right).$$

Then ξ_μ is bounded in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$, and we may assume that $\xi_\mu \to \xi$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, where

$$\xi(x) = \varphi_*\left(\frac{c_0 x}{4\pi N + 2t}\right) + 2\ln\left(\frac{c_0}{4\pi N + 2t}\right).$$

It is easy to check that $\xi + u_0(q) + \ln(4\pi N + 2t)$ satisfies (3.1) with $m = 0$ and then Lemma 3.2 and (3.13) imply that $\xi(x) = -2N\ln|x| + \mathrm{O}(1)$ near ∞, and consequently,

$$(4\pi N + 2t)^2 \int_{\mathbb{R}^2} e^{2u_0(q)+2\xi}\,dx = 4\pi N(N-2).$$

Let $\tilde{\Omega}_\mu = \{x \mid \epsilon_\mu x + x_\mu \in \Omega\}$. Then it follows that

$$\begin{aligned}
B(\underline{v}_\mu) &= 16\pi N \int_\Omega e^{2u_0 + 2\underline{v}_\mu - 2\ln\int_\Omega e^{u_0+\underline{v}_\mu}\,dx}\,dx = 16\pi N \epsilon_\mu^{-2} \int_{\tilde{\Omega}_\mu} e^{2u_0(\epsilon_\mu x + x_\mu) + 2\xi_\mu}\,dx\\
&= \frac{\mu}{2}(4\pi N + 2t)^2 \int_{\tilde{\Omega}_\mu} e^{2u_0(\epsilon_\mu x + x_\mu) + 2\xi_\mu}\,dx + \mathrm{o}(\mu).
\end{aligned}$$

However, by (3.6) and the fact $\xi_\mu \leqslant \varphi(0) + 2\ln\frac{\epsilon_\mu}{s_\mu} < C$, we have $e^{2\xi_\mu} \leqslant \min\{C, C|x|^{-4}\}$ uniformly on $B_{r/\epsilon_\mu}(0)$ for any small enough $r > 0$. Then, applying the Lebesgue dominated convergence theorem and (ii) above, we have

$$\int_{\tilde{\Omega}_\mu} e^{2u_0(\epsilon_\mu x + x_\mu) + 2\xi_\mu}\,dx = \int_{B_{r/\epsilon_\mu}(0)} e^{2u_0(q) + 2\xi_\mu}\,dx + \mathrm{o}(1) = \int_{\mathbb{R}^2} e^{2u_0(q)+2\xi}\,dx + \mathrm{o}(1).$$

Hence, we have

$$B(\underline{v}_\mu) = \frac{\mu}{2}(4\pi N + 2t)^2\left(\int_{\mathbb{R}^2} e^{2u_0(p)+2\xi}\,dx + \mathrm{o}(1)\right) = \mu\big(2\pi N(N-2) + \mathrm{o}(1)\big).$$

Consequently, $B(\underline{v}_\mu)/\mu \to 2\pi N(N-2)$ up to subsequences. Since it holds for any subsequences, it holds for the original sequence.

Finally, we prove (iii). To see this, we show that $c_\mu = c_+(\underline{v}_\mu)$ where c_μ and $c_+(\underline{v}_\mu)$ are defined in (2.11) and (1.4), respectively. We argue by contradiction, and suppose that there is a sequence $\mu \to \infty$ such that $c_\mu = c_-(\underline{v}_\mu)$. Then,

$$2e^{c_\mu}\int_\Omega e^{2(u_0+\underline{v}_\mu)} = \int_\Omega e^{u_0+\underline{v}_\mu} - \left[\left(\int_\Omega e^{u_0+\underline{v}_\mu}\right)^2 - 16\pi N\epsilon_\mu^2\int_\Omega e^{2u_0+2\underline{v}_\mu}\right]^{1/2}. \tag{3.14}$$

Meanwhile, since $B(\underline{v}_\mu)/\mu \to 2\pi N(N-2)$ by (i), (2.10) implies that

$$\epsilon_\mu^2\mu \to \frac{2}{\pi N(N-1)^2}.$$

Thus, from (3.14),

$$e^{c_\mu} = \left(\frac{1}{N-1} + \mathrm{o}(1)\right)\frac{\int_\Omega e^{u_0+\underline{v}_\mu}\,dx}{\int_\Omega e^{2u_0+2\underline{v}_\mu}\,dx}$$

However, (2.11) implies that

$$e^{c_\mu} = \left(\frac{N-2}{N-1} + \mathrm{o}(1)\right)\frac{\int_\Omega e^{u_0+\underline{v}_\mu}\,dx}{\int_\Omega e^{2u_0+2\underline{v}_\mu}\,dx},$$

which yields a contradiction if $N > 3$. Our claim is proved. □

As a corollary of the above theorem, we now consider the case that the distribution of vortex, $\mathcal{Z}$ is further periodic in Ω. Let $\Omega = [0,a]\times[0,b]$ and let $a, b > 1$ be positive integers. We denote a torus of unit side lengths by $\Omega_0 = [0,1]\times[0,1]$, e_1, e_2 be the basis of the torus Ω_0, and $\mathcal{Z}_0 = \{p_1,\dots,p_k \in \Omega_0\}$. We call $\mathcal{Z}$ is periodic when $\mathcal{Z} = \bigcup_{i,j}(\mathcal{Z}_0 + ie_1 + je_2)$, $i = 0,\dots,a-1$, $j = 0,\dots,b-1$ with the multiplicities satisfying $m(p_l) = m(p_l + ie_1 + je_2)$.

Corollary 3.1. *Let $\mathcal{Z}$ be periodic in $\Omega = [0,a]\times[0,b]$ and the total vortex number of the corresponding $\mathcal{Z}_0$ is greater than* 2. *Then, as $\epsilon \to 0$, there exist at least Q number of different blow-up sequences for* (1.3). *Here, Q is the number of divisors of ab.*

Proof. Let a' and b' be divisors of a and b respectively. Consider the torus $\Omega_{a',b'} = [0,a']\times[0,b']$ with the vortex distribution

$$\mathcal{Z}_{a',b'} = \bigcup_{i=0,\,j=0}^{i=a/a'-1,\,j=b/b'-1} \mathcal{Z}_0 + ie_1 + je_2, \qquad m(p_l) = m(p_l + ie_1 + je_2).$$

Theorem 3.4 tells us that there exist blow-up solutions as $\epsilon \to 0$ for (1.3) in $\Omega_{a',b'}$ with the vortex distribution $\mathcal{Z}_{a',b'}$. Further, this solution blows up at only one point in $\Omega_{a',b'}$. We can extend this solution periodically on the whole of Ω. However, on Ω, this solution blows up exactly at $ab/a'b'$ number of points. Thus, there exist at least one distinct family of blow-up solutions for each different $a'b'$, which finishes the proof. □

Corollary 3.2. *Let $N > 3$, $\mathcal{Z}$ be periodic in Ω and the total vortex number of the corresponding $\mathcal{Z}_0$ is* 1 *or* 2. *Then, for some small enough $\epsilon > 0$, there exist at least three solutions for* (1.3)*, two corresponding to J_ϵ^+ and one corresponding to J_ϵ^-.*

Proof. By [15,28], there exists a solution corresponding to J_ϵ^- for $\mathcal{Z}_0$ on Ω_0 for any small enough $\epsilon > 0$. Extending this solution periodically to the whole of Ω, we have a solution corresponding to J_ϵ^- for $\mathcal{Z}$ on Ω. Meanwhile, there exists a maximal solution corresponding to J_ϵ^+ for any small enough $\epsilon > 0$ for $\mathcal{Z}$ on Ω. And by Theorem 3.4, there exists a blow-up solution which corresponds to J_ϵ^+ for $\mathcal{Z}$ on Ω if $\epsilon = \epsilon_\mu$ for some μ large enough. Thus, there are at least three different solutions if $\epsilon = \epsilon_\mu$ for some large enough μ for $\mathcal{Z}$ on Ω. □

4. Blow-up analysis

In this section, we develop the blow-up analysis for (1.2) following [1,2,4,8,21,22,24] to prove Theorem 3.1 and Lemma 3.5.

Lemma 4.1. *Suppose that there is a sequence of solutions $\{u_\epsilon\}$, $\epsilon \to 0$ of* (1.1) *such that* $\sup_\Omega u_\epsilon \to 0$ *as* $\epsilon \to 0$. *Then we have*

$$\|u_\epsilon\|_{L^\infty(K)} \to 0 \quad \text{as } \epsilon \to 0 \tag{4.1}$$

for any compact set $K \subset \Omega \backslash \mathcal{Z}$.

Proof. Since $u_\epsilon < 0$, $e^{u_\epsilon}(e^{u_\epsilon} - 1)$ is bounded in $L^1(\mathbb{R}^2)$. Choose a sequence of points $\{x_\epsilon\} \subset \Omega$ such that $u_\epsilon(x_\epsilon) = \sup_\Omega u_\epsilon \to 0$. Passing to a subsequence (still denoted by u_ϵ), we may assume that $x_\epsilon \to x_0 \in \Omega$. We consider two cases separately: either $x_0 \notin \mathcal{Z}$ or $x_0 \in \mathcal{Z}$.

Case 1: $x_0 \notin \mathcal{Z}$.

Fix a positive constant $d \leqslant (1/3)\,\mathrm{dist}(x_0, \mathcal{Z})$. Since we can cover K by finite open balls, we have only to prove that

$$\inf_{B_d(x_0)} u_\epsilon \to 0 \quad \text{as } \epsilon \to \infty.$$

We argue by contradiction. Suppose that there exist a positive constant c_0 and a sequence $\{z_\epsilon\} \subset \Omega$ such that $|z_\epsilon - x_0| \leqslant d$ and $u_\epsilon(z_\epsilon) = \inf_{B_d(x_0)} u_\epsilon < -c_0$.

Consider the function ξ_0 defined in (3.3). Fix two constants $s_0, s_1 < 0$ such that $\xi_0(s_0) > 4\pi N$ and $\max\{-c_0, s_0\} < s_1 < 0$. For ϵ sufficiently small, we can choose $y_\epsilon \in B_d(x_0)$ such that $u_\epsilon(y_\epsilon) = s_1$ by the intermediate value theorem.

Let $\hat{u}_\epsilon(x) = u_\epsilon(\epsilon x + y_\epsilon)$ for $x \in \Omega_\epsilon := \{x \in \Omega \mid \epsilon x + y_\epsilon \in B_{2d}(x_0)\}$. We note that $\cup_\epsilon \Omega_\epsilon = \mathbb{R}^2$. For ϵ sufficiently small, by Lemma 3.1, $\hat{u}_\epsilon$ satisfies

$$\Delta \hat{u}_\epsilon = e^{\hat{u}_\epsilon}\left(e^{\hat{u}_\epsilon} - 1\right) \quad \text{in } \Omega_\epsilon,$$
$$\int_{\Omega_\epsilon} e^{\hat{u}_\epsilon}\left(1 - e^{\hat{u}_\epsilon}\right) dx \leqslant 4\pi N.$$

Since $\hat{u}_\epsilon(0) = s_1$ and $\hat{u}_\epsilon < 0$ in Ω_ϵ, it follows from Harnack's inequality (see e.g. [4]) that $\hat{u}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(\Omega_\epsilon)$. Passing to a subsequence, we may assume that $\hat{u}_\epsilon$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function $\hat{u}_*$ which is a solution of

$$\Delta u = e^u\left(e^u - 1\right) \quad \text{in } \mathbb{R}^2,$$
$$\int_{\mathbb{R}^2} e^u\left(1 - e^u\right) dx \leqslant 4\pi N \quad \text{and} \quad u(0) = s_1. \tag{4.2}$$

Then it follows that $\hat{u}_*$ is negative and radially symmetric with respect to some point in $\mathbb{R}^2$ by [27]. Since $\hat{u}_*(0) = s_1$, Lemma 3.3 implies that

$$2\pi \int_0^\infty e^{\hat{u}_*}\left(1 - e^{\hat{u}_*}\right) r\,dr \geqslant \xi_0(s_1) > \xi_0(s_0) > 4\pi N,$$

which leads to a contradiction. Thus, for any sequence satisfying Case 1, there exists a subsequence for which (4.1) holds true.

Case 2: $x_0 \in \mathcal{Z}$.

For the sake of simplicity, we assume that $x_0 = 0 \in \mathcal{Z}$. Fix a small positive constant c such that $\{x \in \mathbb{R}^2 \mid |x| \leqslant c\} \cap \mathcal{Z} = \{0\}$. In view of case 1, it suffices to prove that

$$\sup_{|x|=c} u_\epsilon(x) \to 0 \quad \text{as } \epsilon \to 0. \tag{4.3}$$

We argue by contradiction again. Suppose that, passing to a subsequence,

$$\sup_{\epsilon>0} \sup_{|x|=c} u_\epsilon(x) < -\gamma_1$$

for some constant $\gamma_1 > 0$. We first show that

$$|x_\epsilon|/\epsilon \to +\infty \quad \text{as } \epsilon \to 0. \tag{4.4}$$

If not, we have $\liminf_{\epsilon\to 0} |x_\epsilon|/\epsilon < +\infty$. Passing to a subsequence, we may assume that $|x_\epsilon|/\epsilon \leqslant c_1$ for some constant $c_1 > 0$. Note that $u_\epsilon(x) = 2m_j \ln|x| + v_\epsilon(x)$ near $x = 0$ for some smooth function v_ϵ and $1 \leqslant j \leqslant k$. Let $\hat{v}_\epsilon(x) = v_\epsilon(|x_\epsilon|x) + 2m_j \ln|x_\epsilon|$ for $|x| < c/|x_\epsilon|$. Then $\hat{v}_\epsilon$ satisfies

$$\Delta \hat{v}_\epsilon = \frac{|x_\epsilon|^2}{\epsilon^2}|x|^{2m_j} e^{\hat{v}_\epsilon}\big(|x|^{2m_j} e^{\hat{v}_\epsilon} - 1\big) \quad \text{on } B_{c/|x_\epsilon|}(0),$$
$$\int\limits_{|x|<c/|x_\epsilon|} \frac{|x_\epsilon|^2}{\epsilon^2}|x|^{2m_j} e^{\hat{v}_\epsilon}\big(1 - |x|^{2m_j} e^{\hat{v}_\epsilon}\big)\, dx \leqslant 4\pi N.$$

We note that $\hat{v}_\epsilon(x_\epsilon/|x_\epsilon|) = u_\epsilon(x_\epsilon) \to 0$ as $\epsilon \to 0$. Since $|x|^{2m_j} e^{\hat{v}_\epsilon} \leqslant 1$ by Lemma 3.1 and $|x_\epsilon|/\epsilon \leqslant c_1$, it follows from Harnack's inequality that $\hat{v}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(B_{c/|x_\epsilon|}(0))$. Passing to a subsequence, we may assume that $x_\epsilon/|x_\epsilon| \to \bar{y}_0 \in S^1$, $|x_\epsilon|/\epsilon \to c_0 \geqslant 0$ and $\hat{v}_\epsilon$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function $\hat{v}_*$. Then the function $\hat{u}_* = 2m_j \ln|x| + \hat{v}_*$ satisfies

$$\Delta \hat{u}_* = c_0^2 e^{\hat{u}_*}\big(e^{\hat{u}_*} - 1\big) + 4\pi m_j \delta_{p=0} \quad \text{in } \mathbb{R}^2.$$

Since $\hat{u}_* \leqslant 0$, we have $c_0 > 0$ and since $\hat{u}_*(\bar{y}_0) = \lim_{\epsilon\to 0} u_\epsilon(x_\epsilon) = 0$, we have $\hat{u}_* = 0$ by the strong maximum principle. Thus we arrive at a contradiction and (4.4) is proved.

We continue to prove Case 2. Consider the function ξ_0 defined in (3.3). Fix a constant $s_2 < 0$ such that $\xi_0(s_2) > 4\pi N$ and $-\gamma_1 < s_2 < 0$. For ϵ sufficiently small, we can choose y_ϵ on a line segment joining x_ϵ to $cx_\epsilon/|x_\epsilon|$ such that $u_\epsilon(y_\epsilon) = s_2$ and $|y_\epsilon| \geqslant |x_\epsilon|$ by the intermediate value theorem.

Let $\hat{u}_\epsilon(x) = u_\epsilon(\epsilon x + y_\epsilon)$ for $x \in \widehat{\Omega}_\epsilon := \{x \in \mathbb{R}^2 \mid \epsilon x + y_\epsilon \in B_{|x_\epsilon|/2}(y_\epsilon)\}$. We note that $0 \notin B_{|x_\epsilon|/2}(y_\epsilon)$ and $\bigcup_\epsilon \widehat{\Omega}_\epsilon = \mathbb{R}^2$ by (4.4). Then $\hat{u}_\epsilon$ satisfies

$$\Delta \hat{u}_\epsilon = e^{\hat{u}_\epsilon}\big(e^{\hat{u}_\epsilon} - 1\big) \quad \text{in } \widehat{\Omega}_\epsilon,$$
$$\int\limits_{\widehat{\Omega}_\epsilon} e^{\hat{u}_\epsilon}\big(1 - e^{\hat{u}_\epsilon}\big)\, dx \leqslant 4\pi N.$$

Since $\hat{u}_\epsilon < 0$ and $\hat{u}_\epsilon(0) = s_2$, it follows that $\hat{u}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(\widehat{\Omega}_\epsilon)$. Then the argument in case 1 leads to a contraction again. Therefore, for any sequence satisfying Case 2, there exists a subsequence satisfying (4.1). Thus, (4.1) holds true for the original sequence. □

Lemma 4.1 is an investigation of the case (i) of Theorem 3.1 and, as a corollary, Lemma 4.1 gives the following proposition.

Proposition 4.1. *Let u_ϵ be a sequence of solution of* (1.1) *with $\epsilon \to 0$. Then, up to subsequences, one of the following alternatives holds*:

(i) $\sup_{\epsilon>0} \sup_{x\in\Omega} u_\epsilon(x) < -\nu$ *for some constant $\nu = \nu(\Omega, \mathcal{Z}) > 0$, or*
(ii) $\|u_\epsilon\|_{L^\infty(K)} \to 0$ *for any compact set $K \subset \Omega \setminus \mathcal{Z}$.*

Remark. Recently, it is shown that $\{u_\epsilon\}$ satisfying (ii) are maximal solutions constructed by Caffarelli–Yang [5] if ϵ is sufficiently small, and that the second solution constructed by Tarantello [28] satisfies (i) [13].

In what follows, we study the asymptotic behavior of u_ϵ satisfying (i) of Proposition 4.1. So, let us denote

$$w_\epsilon(x) = u_\epsilon(x) - 2\ln\epsilon \quad \text{for } x \in \Omega. \tag{4.5}$$

w_ϵ satisfies (1.2) and by Lemma 3.1 and Proposition 4.1

$$\left\| e^{w_\epsilon}\left(1-\epsilon^2 e^{w_\epsilon}\right)\right\|_{L^1(\Omega)} = 4\pi N, \qquad w_\epsilon + 2\ln\epsilon < -\nu < 0.$$

Then it is easily checked that

$$4\pi N \leqslant \left\| e^{w_\epsilon}\right\|_{L^1(\Omega)} \leqslant 4\pi N/\left(1-e^{-\nu}\right). \tag{4.6}$$

Thus, if $w_\epsilon \leqslant C$ then it follows that $w_\epsilon - u_0$ is bounded in $L^\infty(\Omega)$ by the Harnack inequality and (4.6). Therefore, from now on, we concentrate on the case

$$\lim_{\epsilon\to 0}\sup_{\Omega} w_\epsilon \to \infty.$$

In this case since Ω is compact, at least certain subsequence of w_ϵ must have one blow-up point. Further, defining $V_\epsilon \equiv (1-\epsilon^2 e^{w_\epsilon})e^{u_0} < C$, $w_\epsilon - u_0$ satisfies the following Liouville equation

$$\Delta(w_\epsilon - u_0) = -V_\epsilon e^{w_\epsilon - u_0} + \frac{4\pi N}{|\Omega|}.$$

Applying a smallness condition theorem like Corollary 3 of [4] to the above equation, we can conclude that $w_\epsilon - u_0$ (hence w_ϵ) is bounded locally uniformly except for some finite set. The following lemma further tells the local mass of such blow-up points.

Lemma 4.2. *Let $q \in \Omega$ be a blow-up point for $\{w_\epsilon\}$. Then we have*

$$\liminf_{\epsilon\to 0} \int_{B_d(q)} e^{w_\epsilon}\left(1-\epsilon^2 e^{w_\epsilon}\right) dx \geqslant 8\pi$$

for any $d > 0$.

Proof. Fix $d > 0$ and choose a sequence of points $\{x_\epsilon\} \subset B_d(q)$ such that $w_\epsilon(x_\epsilon) = \max_{|x-q|\leqslant d} w_\epsilon(x)$, $|x_\epsilon - q| < d/2$ for ϵ small enough. Such x_ϵ exists due to the local uniform boundedness of w_ϵ except for some finite set. We let

$$s_\epsilon = \exp\left[-\frac{1}{2} w_\epsilon(x_\epsilon)\right]$$

and

$$\alpha_q = \liminf_{\epsilon\to 0} \int_{B_d(q)} \left(1-\epsilon^2 e^{w_\epsilon}\right) e^{w_\epsilon}\, dy.$$

Note that $\epsilon^2/s_\epsilon^2 = \exp[w_\epsilon(x_\epsilon) + 2\ln\epsilon] \leqslant e^{-\nu}$ for some constant $\nu > 0$. Passing to a subsequence, we may consider the following three cases separately.

Case 1: $q \notin \mathcal{Z}$.

We may assume that $B_d(q) \cap \mathcal{Z} = \emptyset$. Let

$$\overline{w}_\epsilon(x) = w_\epsilon(s_\epsilon x + x_\epsilon) + 2\ln s_\epsilon \quad \text{for } |x| < d/(2s_\epsilon).$$

For ϵ sufficiently small, $\overline{w}_\epsilon$ satisfies

$$\begin{aligned} &-\Delta \overline{w}_\epsilon = e^{\overline{w}_\epsilon}\left(1 - \frac{\epsilon^2}{s_\epsilon^2} e^{\overline{w}_\epsilon}\right) \quad \text{for } |x| < d/(2s_\epsilon),\\ &\|\Delta \overline{w}_\epsilon\|_{L^1(|x|<d/(2s_\epsilon))} \leqslant \alpha_q. \end{aligned} \tag{4.7}$$

Since $\overline{w}_\epsilon(x) \leqslant \overline{w}_\epsilon(0) = 0$ for $|x| < d/(2s_\epsilon)$, Harnack's inequality implies that $\overline{w}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. Passing to a subsequence, we may assume that $\epsilon^2/s_\epsilon^2 \to c_0^2$ for some constant $c_0 \in [0,1)$, and $\overline{w}_\epsilon \to \overline{w}_*$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$\begin{aligned} &-\Delta \overline{w}_* = e^{\overline{w}_*}\big(1 - c_0^2 e^{\overline{w}_*}\big) \quad \text{in } \mathbb{R}^2, \\ &\int_{\mathbb{R}^2} e^{\overline{w}_*}\big(1 - c_0^2 e^{\overline{w}_*}\big)\, dx \leqslant \alpha_q \leqslant 4\pi N. \end{aligned} \tag{4.8}$$

If $c_0 = 0$, then by [11], $\int_{\mathbb{R}^2} e^{\overline{w}_*} dx = 8\pi$. If $c_0 > 0$ then we can apply Lemma 3.2 to the function $\phi(x) = \overline{w}_*(c_0 x) + 2\ln c_0$, and conclude that $\alpha_q > 8\pi$.

Case 2: $q = p_j \in \mathcal{Z}$ for some $1 \leqslant j \leqslant k$ and $\lim_{\epsilon\to 0} \frac{|x_\epsilon - q|}{s_\epsilon} = \infty$.

For the sake of simplicity, we assume that $q = p_j = 0$. Note that $w_\epsilon(x) = 2m_j \ln|x| + v_\epsilon(x)$ near $x = 0$ for a smooth function v_ϵ. Let

$$\bar{v}_\epsilon(x) = v_\epsilon(s_\epsilon x + x_\epsilon) + 2\ln s_\epsilon + 2m_j \ln|x_\epsilon| \quad \text{for } |x| < |x_\epsilon|/(2s_\epsilon).$$

Then $\bar{v}_\epsilon$ satisfies

$$\begin{aligned} &-\Delta \bar{v}_\epsilon = \left|\frac{s_\epsilon}{|x_\epsilon|}x + \frac{x_\epsilon}{|x_\epsilon|}\right|^{2m_j} e^{\bar{v}_\epsilon}\left(1 - \frac{\epsilon^2}{s_\epsilon^2}\left|\frac{s_\epsilon}{|x_\epsilon|}x + \frac{x_\epsilon}{|x_\epsilon|}\right|^{2m_j} e^{\bar{v}_\epsilon}\right), \quad |x| < |x_\epsilon|/(2s_\epsilon), \\ &\int_{|x|<|x_\epsilon|/(2s_\epsilon)} (-\Delta \bar{v}_\epsilon)\, dx \leqslant \alpha_q \quad \text{and} \quad \frac{\epsilon^2}{s_\epsilon^2}\left|\frac{s_\epsilon}{|x_\epsilon|}x + \frac{x_\epsilon}{|x_\epsilon|}\right|^{2m_j} e^{\bar{v}_\epsilon} \leqslant e^{-\nu} < 1. \end{aligned}$$

Since $\bar{v}_\epsilon(0) = w_\epsilon(x_\epsilon) + 2\log s_\epsilon = 0$ and

$$\begin{aligned} \bar{v}_\epsilon(x) &= w_\epsilon(s_\epsilon x + x_\epsilon) + 2\ln s_\epsilon - 2m_j \ln\left|\frac{s_\epsilon}{|x_\epsilon|}x + \frac{x_\epsilon}{|x_\epsilon|}\right| \\ &\leqslant 2m_j \ln 2 \quad \text{for } |x| < |x_\epsilon|/(2s_\epsilon), \end{aligned}$$

it follows from Harnack's inequality that $\bar{v}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(|x| < |x_\epsilon|/(2s_\epsilon))$. Passing to subsequences, we may assume that $\epsilon^2/s_\epsilon^2 = \epsilon^2 \exp[w_\epsilon(x_\epsilon)] \to c_1^2$ for some constant $c_1 \in [0,1)$, $x_\epsilon/|x_\epsilon| \to \bar{y}_1$ for some $\bar{y}_1 \in S^1$ and $\bar{v}_\epsilon \to \bar{v}_*$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, which satisfies

$$\begin{aligned} &-\Delta \bar{v}_* = e^{\bar{v}_*}\big(1 - c_1^2 e^{\bar{v}_*}\big) \quad \text{in } \mathbb{R}^2, \\ &\int_{\mathbb{R}^2} e^{\bar{v}_*}\big(1 - c_1^2 e^{\bar{v}_*}\big)\, dx \leqslant \alpha_q \quad \text{and} \quad \sup_{\mathbb{R}^2} c_1^2 e^{\bar{v}_*} < 1. \end{aligned} \tag{4.9}$$

Then we can repeat the argument in Case 1 to conclude that $\alpha_q \geqslant 8\pi$ in Case 2 as well.

Case 3: $q = p_j \in \mathcal{Z}$ and $\frac{|x_\epsilon - q|}{s_\epsilon} \leqslant C$ for some constant $C > 0$.

As in case 2, we assume that $q = 0$ and $w_\epsilon(x) = 2m_j \ln|x| + v_\epsilon(x)$ near $x = 0$. Fix a constant $d > 0$ such that $B_d(0) \cap \mathcal{Z} = \{0\}$. Let

$$\hat{v}_\epsilon(x) = v_\epsilon(s_\epsilon x + x_\epsilon) + 2(1 + m_j)\ln s_\epsilon \quad \text{for } |x| \leqslant d/2s_\epsilon.$$

Then it is easily checked that

$$\begin{aligned} &-\Delta \hat{v}_\epsilon = \left|x + \frac{x_\epsilon}{s_\epsilon}\right|^{2m_j} e^{\hat{v}_\epsilon}\left(1 - \frac{\epsilon^2}{s_\epsilon^2}\left|x + \frac{x_\epsilon}{s_\epsilon}\right|^{2m_j} e^{\hat{v}_\epsilon}\right) \quad \text{for } |x| \leqslant d/2s_\epsilon, \\ &\int_{|x|\leqslant d/2s_\epsilon} (-\Delta \hat{v}_\epsilon)\, dx \leqslant \alpha_q. \end{aligned}$$

We note that

$$\left|x + \frac{x_\epsilon}{s_\epsilon}\right|^{2m_j} e^{\hat{v}_\epsilon(x)} = s_\epsilon^2 e^{w_\epsilon(s_\epsilon x + x_\epsilon)} \leqslant 1 \quad \text{for } |x| < \frac{d}{2s_\epsilon}$$

and that

$$\sup_{|x|<d/2s_\epsilon} \frac{\epsilon^2}{s_\epsilon^2}\left|x+\frac{x_\epsilon}{s_\epsilon}\right|^{2m_j} e^{\hat{v}_\epsilon(x)} < 1. \tag{4.10}$$

Note that $\hat{v}_\epsilon(0) = -2m_j \ln\frac{|x_\epsilon|}{s_\epsilon}$ is bounded from below by the assumption and $\hat{v}_\epsilon(-x_\epsilon/s_\epsilon - x_\epsilon/|x_\epsilon|) = w_\epsilon(-s_\epsilon x_\epsilon/|x_\epsilon|) + 2\ln s_\epsilon \leqslant 0$ since $w_\epsilon(x_\epsilon)$ is the maximum of w_ϵ in $B_d(0)$. Hence, it follows from Harnack's inequality that $\hat{v}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(|x| \leqslant d/2s_\epsilon)$. Passing to a subsequence, we may assume that $x_\epsilon/s_\epsilon \to \bar{y}_2$ for some $\bar{y}_2 \in \mathbb{R}^2$, $\epsilon/s_\epsilon \to c_2$ for some $c_2 \in [0,1)$, and $\hat{v}_\epsilon$ converges in $C^2_{\mathrm{loc}}(|x| < d/2s_\epsilon)$ to a function $\hat{v}_* \in C^2_{\mathrm{loc}}(\mathbb{R}^2)$ satisfying

$$\begin{aligned} &-\Delta\hat{v}_* = |x+\bar{y}_2|^{2m_j} e^{\hat{v}_*}\big(1 - c_2^2|x+\bar{y}_2|^{2m_j}e^{\hat{v}_*}\big) \quad \text{in } \mathbb{R}^2,\\ &\int_{\mathbb{R}^2} |x+\bar{y}_2|^{2m_j} e^{\hat{v}_*}\big(1 - c_2^2|x+\bar{y}_2|^{2m_j}e^{\hat{v}_*}\big)\,dx \leqslant \alpha_q, \end{aligned} \tag{4.11}$$

and $\sup_{\mathbb{R}^2} c_2^2|x+\bar{y}_2|^{2m_j}e^{\hat{v}_*} < 1$. Letting $\hat{u}_*(x) = \hat{v}_*(x) + 2m_j\ln|x+\bar{y}_2|$, we have

$$\Delta\hat{u}_* = e^{\hat{u}_*}\big(c_2^2 e^{\hat{u}_*} - 1\big) + 4\pi m_j \delta_{-\bar{y}_2}.$$

If $c_2 = 0$ then all the solutions of (4.11) are completely known, and $\alpha_q \geqslant 8\pi(1+m_j)$. (See [26] for the details.) If $c_2 > 0$ then we can apply Lemma 3.2 to the function $\phi(x) = \hat{u}_*(c_2 x) + 2\ln c_2$, and conclude that $\alpha_q > 8\pi(1+m_j)$.

Thus, Lemma 4.2 is proved. □

Since w_ϵ is bounded locally uniformly except for some finite set, taking subsequences repeatedly if necessary, we can assume $\{w_\epsilon\}$ is bounded locally uniformly except for some blow-up set $\mathcal{S}$. Then, we can prove the following lemma following the argument in [2] (Theorem 4) and [4].

Lemma 4.3. *Let $\{w_\epsilon\}$ be a blow-up sequence of solutions of* (1.2) *with $\epsilon \to 0$ and $\mathcal{S} = \{q_1,\dots,q_l\} \subset \Omega$ be the blow-up set for $\{w_\epsilon\}$. Then $\sup_{x\in K}(w_\epsilon(x) - u_0(x)) \to -\infty$ for any compact subset K of $\Omega\backslash\mathcal{S}$.*

Moreover, $e^{w_\epsilon}(1-\epsilon^2 e^{w_\epsilon}) \to \sum_{j=1}^{l}\alpha_j\delta_{q_j}$ in the sense of measure with $\alpha_j \geqslant 8\pi$.

Proof. Let $d > 0$ be a small constant and $\{x_{j,\epsilon}\}$ be l number of sequences of points such that $x_{j,\epsilon} \to q_j$, $B_{2d}(q_j) \cap B_{2d}(q_i) = \emptyset$ for $j \neq i$, and $w_\epsilon(x_{j,\epsilon}) = \max_{|x-x_{j,\epsilon}|\leqslant d} w_\epsilon(x) \to \infty$ for $j = 1,\dots,l$. We shall prove that

$$\max_{r\leqslant|x-q_j|\leqslant d}(w_\epsilon - u_0)(x) \to -\infty$$

for any $r \in (0,d]$ and $q_j \in \mathcal{S}$. We argue by contradiction. The detailed proof can be found in [2], and we sketch the proof here. For simplicity, we assume that $q_j = 0$. Suppose that $\sup_{r\leqslant|x|\leqslant d}(w_\epsilon - u_0)(x)$ is bounded from below for some $r \in (0,d]$. Then it follows from Harnack's inequality that there is an $r_0 \in (0,d)$ such that $\inf_{|x|=r_0}(w_\epsilon - u_0)(x) \geqslant C$ for some constant $C > 0$. Elliptic estimates imply that, along a subsequence, $w_\epsilon - u_0 \to \xi$ in $C^2_{\mathrm{loc}}(B_d\backslash\{0\})$. Then $e^{w_\epsilon}(1-\epsilon^2 e^{w_\epsilon}) \to e^{u_0+\xi} + \alpha_j\delta_{p=0}$ in the sense of measure for some constant $\alpha_j \geqslant 8\pi$ by Lemma 4.2. Moreover, Green's representation formula implies that $\xi(x) = -\frac{\alpha_j}{2\pi}\ln|x| + \phi + \eta$ with $\eta \in C^1(|x| < r_0)$ and

$$\phi(x) = \frac{1}{2\pi}\int_{|y|\leqslant r_0} \ln\frac{1}{|y-x|} e^{(u_0+\xi)(y)}\,dy.$$

Let $m = m_j$ if $0 = p_j \in \mathcal{Z}$ and $m = 0$ if $0 \notin \mathcal{Z}$. Then, $|u_0(x) - 2m\ln|x|| \leqslant C$ for $|x| \leqslant r_0$.

Since $e^{u_0+\xi} \in L^1(|x| \leqslant r_0)$, it follows that $\phi \in L^p(|x| \leqslant r_0)$ for any $p \in (1,\infty)$ and

$$\phi(x) \geqslant -\frac{1}{2\pi}\|e^{u_0+\xi}\|_{L^1(|y|\leqslant r_0)}\ln(2r_0) \quad \text{for } |x| < r_0.$$

Using $e^{u_0+\xi} \in L^1(|x| \leqslant r_0)$ again, we have $2m - \frac{\alpha_j}{2\pi} > -2$.

We let $\varphi_\epsilon(x) = w_\epsilon(x) - 2m\ln|x|$. Then φ_ϵ satisfies

$$-\Delta\varphi_\epsilon = |x|^{2m}e^{\varphi_\epsilon} - \epsilon^2|x|^{4m}e^{2\varphi_\epsilon} \quad \text{for } |x| \leqslant r_0. \tag{4.12}$$

Multiplying (4.12) by $(x \cdot \nabla\varphi_\epsilon)$ and integrating over $\{|x| \leqslant r\}$ with $0 < r < r_0$, we obtain

$$0 \leqslant \int\limits_{|x|\leqslant r} \epsilon^2|x|^{4m}e^{2\varphi_\epsilon(x)}\,dx = \int\limits_{|x|=r}\left[\frac{1}{r}(x\cdot\nabla\varphi_\epsilon)^2 - \frac{r}{2}|\nabla\varphi_\epsilon|^2 + r^{1+2m}e^{\varphi_\epsilon} - \frac{\epsilon^2}{2}r^{1+4m}e^{2\varphi_\epsilon}\right]d\sigma$$
$$- (2+2m)\int\limits_{|x|\leqslant r}|x|^{2m}e^{\varphi_\epsilon}\big(1-\epsilon^2|x|^{2m}e^{\varphi_\epsilon}\big)(x)\,dx.$$

Letting $\epsilon \to 0$, we have

$$(2+2m)\alpha_j + (2+2m)\int\limits_{|x|\leqslant r} e^{u_0+\xi}\,dx \leqslant \int\limits_{|x|=r}\left[\frac{1}{r}(x\cdot\nabla\varphi)^2 - \frac{r}{2}|\nabla\varphi|^2 + r^{1+2m}e^{\varphi}\right]d\sigma,$$

where $\varphi(x) = \xi(x) + u_0(x) - 2m\ln|x|$. Since $\phi \in L^p(|x| \leqslant r_0)$ for any $p \in (1,\infty)$, Hölder inequality implies that $\phi \in L^\infty(|x| \leqslant r_0)$. Then it follows that $e^{(u_0+\xi)(x)} = \mathrm{O}(|x|^{-\alpha_j/2\pi+2m})$ as $|x| \to 0$, and $|x|^{1+2m}e^{\varphi(x)} \leqslant C|x|^{\tau-1}$ for some constant $\tau > 0$. Moreover, it follows from the argument in [2] that $|\nabla\phi(x)| \leqslant C(|x|^{\tau-1}+1)$ for some $\tau > 0$. Then we conclude that

$$\nabla\varphi = -\frac{\alpha_j x}{2\pi|x|^2} + \nabla h,$$

with $|\nabla h(x)| \leqslant C(|x|^{\tau-1}+1)$ for some $\tau > 0$. Letting $r \to 0$ in the above inequality, we then obtain that $(2+2m)\alpha_j \leqslant \alpha_j^2/4\pi$, which contradicts the inequality $2m - \frac{\alpha_j}{2\pi} > -2$.

Therefore, it follows from Harnack's inequality that $w_\epsilon - u_0 \to -\infty$ uniformly on any compact subset of $\Omega\backslash\mathcal{S}$. Since $e^{w_\epsilon}(1-\epsilon^2 e^{w_\epsilon})$ is nonnegative and bounded in $L^1(\Omega)$, along a subsequence, $e^{w_\epsilon}(1-\epsilon^2e^{w_\epsilon})$ converges to a nonnegative measure. However, this measure must be supported on $\mathcal{S}$ since $w_\epsilon \to -\infty$ uniformly in $C^0_{\mathrm{loc}}(\Omega\backslash\mathcal{S})$. Then the measure should be a sum of Dirac measures and Lemma 4.2 implies that each Dirac mass should be greater than or equal to 8π. □

Together with Proposition 4.1 and the above lemma, we now prove Theorem 3.1.

Proof of Theorem 3.1. If either case (i) or (ii) of Theorem 3.1 is not the case, by Proposition 4.1, we have $\sup_{\epsilon\to 0}\sup_\Omega(v_\epsilon + u_0) < -\nu$ for some constant $\nu > 0$ and $\limsup_{\epsilon\to+0}|v_\epsilon - 2\ln\epsilon| = +\infty$. Now, we show that $\limsup_{\epsilon\to 0}(v_\epsilon - 2\ln\epsilon) = +\infty$. If not, the RHS of (1.3) is uniformly bounded. Then, Harnack's inequality imply that, along a subsequence, $\sup_\Omega(v_\epsilon - 2\ln\epsilon) \to -\infty$. But then $\|\epsilon^{-2}e^{v_\epsilon+u_0}(1-e^{v_\epsilon+u_0})\|_{L^1(\Omega)} \to 0$, which leads to a contradiction. Thus, $\limsup_{\epsilon\to 0}(v_\epsilon - 2\ln\epsilon) = \infty$.

Now, let $w_\epsilon = u_0 + v_\epsilon - 2\ln\epsilon$. Since Ω is compact, a sequence of maximum points x_ϵ of w_ϵ converges up to subsequences. Thus, for this subsequence, the limit of x_ϵ is a blow-up point and this subsequence becomes a blow-up sequence. Consequently, by Lemma 4.3 we arrive at case (iii). □

Remark. When $N = 1$, by the above theorem, case (iii) above cannot be realized. When $N = 2$, if the case (iii) above is realized, the blow-up happens at only one point and, in view of Lemma 4.2, the suitable scaled subsequence of solutions ($\overline{w}_\epsilon$ in Lemma 4.2) converges to the solution of the Liouville equation in $\mathbb{R}^2$.

Next, by making use of the Pohozaev identity as well as the argument in [22], we deliver the proof of Lemma 3.5.

Proof of Lemma 3.5. We take x_ϵ to be a maximum point of w_ϵ in $B_r(q_j)$, namely, $w_\epsilon(x_\epsilon) = \max_{|x-q_j|\leqslant r} w_\epsilon(x)$. By (iii) of Theorem 3.1, we have $x_\epsilon \to q_j$. Hence we can assume $|x_\epsilon - q_j| < r/2$ without loss of generality. Under this situation, we need to show (3.6) and (3.7). We break it into two parts.

Part 1. Proof of (3.6).

We argue by contradiction. Suppose that there is a sequence $\{y_\epsilon\}$ such that $|y_\epsilon - q_j| \leqslant r$ and

$$w_\epsilon(y_\epsilon) + 2\ln|y_\epsilon - x_\epsilon| = \max_{|x-q_j|\leqslant r}\big(w_\epsilon(x) + 2\ln|x - x_\epsilon|\big) \to \infty$$

as $\epsilon \to 0$. It is easy to check that $y_\epsilon \neq x_\epsilon$ and $w_\epsilon(y_\epsilon) \to \infty$. Thus $y_\epsilon \to q_j$ by Lemma 4.3. Let $d_\epsilon \equiv |x_\epsilon - y_\epsilon| \to 0$ and

$$\overline{w}_\epsilon(x) \equiv w_\epsilon(d_\epsilon x + x_\epsilon) + 2\ln d_\epsilon, \quad |x| < r/(2d_\epsilon).$$

Then $\overline{w}_\epsilon$ satisfies

$$\begin{aligned} &-\Delta \overline{w}_\epsilon = e^{\overline{w}_\epsilon}\left(1 - \frac{\epsilon^2}{d_\epsilon^2} e^{\overline{w}_\epsilon}\right) \quad \text{for } |x| < \frac{r}{2d_\epsilon}, \\ &\max_{|x|<r/(2d_\epsilon)} \overline{w}_\epsilon(x) + 2\ln(\epsilon/d_\epsilon) < 0, \end{aligned} \tag{4.13}$$

and $\|\Delta \overline{w}_\epsilon\|_{L^1(|x|<r/(2d_\epsilon))} \leqslant C$ by Proposition 4.1. We note that

$$\frac{\epsilon^2}{d_\epsilon^2} \leqslant \frac{1}{d_\epsilon^2} e^{-w_\epsilon(x_\epsilon)} \leqslant e^{-w_\epsilon(y_\epsilon) - 2\ln|y_\epsilon - x_\epsilon|} \to 0$$

and $(\epsilon^2/d_\epsilon^2)e^{\overline{w}_\epsilon} \leqslant e^{-\nu} < 1$ for $|x| < r/2d_\epsilon$. We also note that $\overline{w}_\epsilon((y_\epsilon - x_\epsilon)/d_\epsilon) = w_\epsilon(y_\epsilon) + 2\ln|y_\epsilon - x_\epsilon| \to \infty$. By passing to a subsequence, we may assume that $(y_\epsilon - x_\epsilon)/d_\epsilon \to z_1 \in \mathbb{R}^2$ with $|z_1| = 1$. Then the proof of Lemma 4.3 implies that, along a subsequence, there is a finite blow-up set $\mathcal{S}^* = \{z_1, \ldots, z_t\}$ for $\overline{w}_\epsilon$ such that $\overline{w}_\epsilon \to -\infty$ uniformly on any compact subset of $\mathbb{R}^2 \backslash \mathcal{S}^*$, and moreover

$$e^{\overline{w}_\epsilon}\left(1 - \frac{\epsilon_\epsilon^2}{d_\epsilon^2} e^{\overline{w}_\epsilon}\right) \to \sum_{j=1}^t m_j^* \delta_{z_j}, \quad m_j^* \geqslant 8\pi$$

in the sense of measure on any $K \subset\subset \mathbb{R}^2 \backslash \mathcal{S}^*$. Since $\overline{w}_\epsilon(0) = w_\epsilon(x_\epsilon) + 2\ln d_\epsilon \geqslant w_\epsilon(y_\epsilon) + 2\ln d_\epsilon$, we have $\overline{w}_\epsilon(0) \to \infty$. It follows that $0 \in \mathcal{S}^*$ and $|\mathcal{S}^*| \geqslant 2$.

Fix a point $p_0 \in \mathbb{R}^2 \backslash \mathcal{S}^*$. Then, Green's representation formula (2.1) of the equation (4.13) becomes

$$\begin{aligned} \overline{w}_\epsilon(x) - \overline{w}_\epsilon(p_0) &= u_0(d_\epsilon x + x_\epsilon) - u_0(d_\epsilon p_0 + x_\epsilon) + \frac{1}{2\pi} \int_{B_\epsilon} \ln \frac{|p_0 - y|}{|x - y|} \left(e^{\overline{w}_\epsilon} - \frac{\epsilon^2}{d_\epsilon^2} e^{2\overline{w}_\epsilon}\right) dy \\ &\quad + \int_{B_r(q_j)} \big[\gamma(d_\epsilon x + x_\epsilon, y) - \gamma(d_\epsilon p_0 + x_\epsilon, y)\big]\big(e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon}\big)\, dy \\ &\quad + \int_{[B_r(q_j)]^c} \big[G(d_\epsilon x + x_\epsilon, y) - G(d_\epsilon p_0 + x_\epsilon, y)\big]\big(e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon}\big)\, dy, \end{aligned}$$

where $B_\epsilon = \{y \mid d_\epsilon y + x_\epsilon \in B_r(q_j)\}$.

Now, fix a compact subset K of $\mathbb{R}^2 \backslash \mathcal{S}^*$. Since $x_\epsilon \to q_j \notin \mathcal{Z}$ and $d_\epsilon \to 0$ as $\epsilon \to 0$,

$$\begin{aligned} &\max_{x\in K}\big|u_0(d_\epsilon x + x_\epsilon) - u_0(d_\epsilon p_0 + x_\epsilon)\big| + \max_{x\in K}\big|\gamma(d_\epsilon x + x_\epsilon, y) - \gamma(d_\epsilon p_0 + x_\epsilon, y)\big| \\ &\quad + \max_{x\in K, y\notin B_r(q_j)}\big|G(d_\epsilon x + x_\epsilon, y) - G(d_\epsilon p_0 + x_\epsilon, y)\big| \to 0. \end{aligned}$$

We also note that $\max_{x\in K} \big|\ln|p_0 - y| - \ln|x - y|\big| \to 0$ uniformly as $|y| \to \infty$. Therefore, it follows that

$$\overline{w}_\epsilon - \overline{w}_\epsilon(p_0) \to \sum_{j=1}^t \frac{m_j^*}{2\pi} \ln \frac{|p_0 - z_j|}{|x - z_j|}$$

uniformly in $C^0_{\mathrm{loc}}(K)$. Similarly, we obtain that $\nabla \overline{w}_\epsilon$ converges to $\sum_{j=1}^t \frac{m_j^*}{2\pi} \frac{z_j - x}{|z_j - x|^2}$ uniformly on K.

Now, we determine the location of $\{z_1, \ldots, z_t\}$ as follows. Fix a unit vector $\xi \in \mathbb{R}^2$ and choose a small number $\delta > 0$ such that $B_{2\delta}(z_j) \cap \mathcal{S}^* = \{z_j\}$ for $1 \leqslant j \leqslant t$. Multiplying by $\xi \cdot \nabla \overline{w}_\epsilon$ both sides of (4.13) and integrating on $\{x \mid |x - z_j| \leqslant \delta\}$, we obtain

$$\int_{|x-z_j|=\delta} \left[\frac{1}{2}(\xi \cdot \nu)|\nabla \overline{w}_\epsilon|^2 - (\xi \cdot \nabla \overline{w}_\epsilon)(\nu \cdot \nabla \overline{w}_\epsilon)\right] d\sigma = \int_{|x-z_j|=\delta} (\xi \cdot \nu) e^{\overline{w}_\epsilon}\left(1 - \frac{\epsilon^2}{2d_\epsilon^2} e^{\overline{w}_\epsilon}\right) d\sigma$$

for any $1 \leqslant j \leqslant t$. Letting $\epsilon \to 0$ in the above equation, we obtain

$$LHS = \int_{|x-z_j|=\delta} \left[\frac{1}{2}(\xi\cdot\nu)|\nabla H_j^*|^2 + \frac{m_j^*}{2\pi\delta}\xi\cdot\nabla H_j^* - (\xi\cdot\nabla H_j^*)(\nu\cdot\nabla H_j^*)\right] d\sigma = RHS = 0,$$

where H_j^* is defined by

$$H_j^*(x) = \sum_{i\neq j} m_i^* \ln\frac{|p_0 - z_i|}{|x - z_i|}, \quad 1 \leqslant j \leqslant t.$$

Letting $\delta \to 0$, we obtain $\xi\cdot\nabla H_j^*(z_j) = 0$ for $1 \leqslant j \leqslant t$. Since $\xi \in \mathbb{R}^2$ is arbitrary, $\nabla H_j^*(z_j) = 0$ for all $1 \leqslant j \leqslant t$.

On the other hand, by direct calculation

$$\nabla H_j^*(z_j) = \sum_{i\neq j} \frac{m_i^*}{|z_i - z_j|^2}(z_i - z_j).$$

Hence, considering the element of $\mathcal{S}^*$ whose first component is the largest one in $\mathcal{S}^*$ (denoted by z_1), $\nabla_1 H_1^*(z_1) < 0$ since $|\mathcal{S}^*| \geqslant 2$, which yields a contradiction.

Part 2. Proof of (3.7).

Let $R_\epsilon \to \infty$ be given. Fix a constant $\delta > 0$ such that $B_\delta(q_j)$'s are mutually disjoint and

$$\overline{w}_\epsilon(x) = w_\epsilon(s_\epsilon x + x_\epsilon) + 2\ln s_\epsilon \quad \text{for } |x| < \delta/s_\epsilon.$$

Then $\overline{w}_\epsilon$ satisfies

$$-\Delta\overline{w}_\epsilon = e^{\overline{w}_\epsilon}\left(1 - \frac{\epsilon^2}{s_\epsilon^2}e^{\overline{w}_\epsilon}\right) \quad \text{for } |x| \leqslant \delta/s_\epsilon,$$
$$\int_{|x|\leqslant\delta/s_\epsilon} e^{\overline{w}_\epsilon}\left(1 - \frac{\epsilon^2}{s_\epsilon^2}e^{\overline{w}_\epsilon}\right) dx \leqslant 4\pi N,$$

and $\epsilon^2/s_\epsilon^2 \leqslant e^{-\nu} < 1$. Since $\overline{w}_\epsilon \leqslant \overline{w}_\epsilon(0) = 0$, it follows from Harnack's inequality that $\overline{w}_\epsilon$ is bounded in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$. Passing to subsequences, we may assume that $\epsilon^2/s_\epsilon^2 \to c_0^2$ for some constant $c_0 \in [0, 1)$, and that $\overline{w}_\epsilon$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to $\overline{w}$ which is a solution of

$$-\Delta v = e^v\left(1 - c_0^2 e^v\right) \quad \text{in } \mathbb{R}^2,$$
$$\int_{\mathbb{R}^2} e^v\left(1 - c_0^2 e^v\right) dx \leqslant 4\pi N \quad \text{and} \quad v \leqslant v(0) = 0. \tag{4.14}$$

by Lemma 3.1. Now, since $\overline{w}_\epsilon \to \overline{w}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, we can choose $\{r_\epsilon\}$ such that $r_\epsilon \leqslant R_\epsilon$, $r_\epsilon \to \infty$, and

$$\|\overline{w}_\epsilon - \overline{w}\|_{C^2(B_{r_\epsilon}(0))} \to 0. \tag{4.15}$$

Without loss of generality, we may assume that $r_\epsilon s_\epsilon \to 0$. If $0 < c_0 < 1$, it follows from Lemma 3.2 that $\overline{w} \to -\infty$ near ∞. Then Lemma 3.3 imply that $\overline{w}$ is radially symmetric, and $\overline{w}$ is the unique solution of (4.14). if $c_0 = 1$ then the argument in [11] implies that $\overline{w}$ is radially symmetric and it is the unique solution of (4.14).

Let

$$\hat{\alpha}_j = \liminf_{\epsilon\to 0} \int_{|y-x_\epsilon|\leqslant r_\epsilon s_\epsilon} e^{w_\epsilon}\left(1 - \epsilon^2 e^{w_\epsilon}\right)(y)\, dy.$$

It suffices to prove that $\hat{\alpha}_j = \alpha_j$. By (4.14) and (4.15),

$$\int_{\mathbb{R}^2} e^{\overline{w}}\left(1 - c_0^2 e^{\overline{w}}\right) dx = \lim_{\epsilon\to 0} \int_{|x|\leqslant r_\epsilon} e^{\overline{w}}\left(1 - c_0^2 e^{\overline{w}}\right) dx$$
$$= \lim_{\epsilon\to 0} \int_{|x-x_\epsilon|\leqslant r_\epsilon s_\epsilon} e^{w_\epsilon}\left(1 - \epsilon^2 e^{w_\epsilon}\right)(y)\, dy = \hat{\alpha}_j.$$

Recall that w_ϵ satisfies that

$$-\Delta w_\epsilon = e^{w_\epsilon}\left(1-\epsilon^2 e^{w_\epsilon}\right) \quad \text{for } |x-x_\epsilon| \leqslant \delta. \tag{4.16}$$

Let $A_\epsilon = \{x \mid r_\epsilon s_\epsilon \leqslant |x-x_\epsilon| \leqslant \delta\}$. Multiplying (4.16) by $(x-x_\epsilon)\cdot\nabla w_\epsilon$ and integrating on A_ϵ, we obtain

$$\begin{aligned}
&\int_{|x-x_\epsilon|=r_\epsilon s_\epsilon} \left[\frac{1}{2} r_\epsilon s_\epsilon |\nabla w_\epsilon|^2 - \frac{1}{r_\epsilon s_\epsilon}\big((x-x_\epsilon)\cdot\nabla w_\epsilon\big)^2 - r_\epsilon s_\epsilon\left(e^{w_\epsilon} - \frac{\epsilon^2}{2}e^{2w_\epsilon}\right)\right] d\sigma \\
&\quad = \int_{|x-x_\epsilon|=\delta} \left[\frac{\delta}{2}|\nabla w_\epsilon|^2 - \frac{1}{\delta}\big((x-x_\epsilon)\cdot\nabla w_\epsilon\big)^2 - \delta\left(e^{w_\epsilon} - \frac{\epsilon^2}{2}e^{2w_\epsilon}\right)\right] d\sigma \\
&\qquad + \int_{A_\epsilon} \left(2e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon}\right) dx.
\end{aligned} \tag{4.17}$$

We first estimate the second integral in (4.17). Lemma 4.3 implies that $w_\epsilon \to -\infty$ uniformly on any compact subset of $B_{2\delta}(q_j)\backslash\{q_j\}$. Moreover, there is a harmonic function $H_j \in C^\infty(B_{2\delta}(q_j))$ such that $\nabla w_\epsilon(x) \to -\frac{\alpha_j}{2\pi}\frac{x-q_j}{|x-q_j|^2} + \nabla H_j(x)$ in $C^0_{\mathrm{loc}}(B_{2\delta}(q_j)\backslash\{q_j\})$. Indeed, H_j is given by

$$H_j(x) = u_0(x) - \sum_{i\neq j}\frac{\alpha_i}{2\pi}\ln|x-q_i| + \sum_{i=1}^{l}\alpha_i\gamma(x,q_i), \quad x \in B_{2\delta}(q_j).$$

Therefore, it follows that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{|x-x_\epsilon|=\delta} \left[\frac{\delta}{2}|\nabla w_\epsilon|^2 - \frac{1}{\delta}\big((x-x_\epsilon)\cdot\nabla w_\epsilon\big)^2 - \delta\left(e^{w_\epsilon} - \frac{\epsilon^2}{2}e^{2w_\epsilon}\right)\right] d\sigma \\
&\quad = -\frac{\alpha_j^2}{4\pi} + \int_{|x-q_j|=\delta}\left[\frac{\delta}{2}|\nabla H_j|^2 - \frac{1}{\delta}\big((x-q_j)\cdot\nabla H_j\big)^2 + \frac{\alpha_j}{2\pi}(\nu\cdot\nabla H_j)\right] d\sigma = -\frac{\alpha_j^2}{4\pi}.
\end{aligned}$$

Next, we estimate the first integral in (4.17). Let

$$\hat{w}_\epsilon(x) = w_\epsilon(r_\epsilon s_\epsilon x + x_\epsilon) + 2\ln(r_\epsilon s_\epsilon) \quad \text{for } |x| \leqslant \frac{\delta}{r_\epsilon s_\epsilon}.$$

Note that $\hat{w}_\epsilon(0) = w_\epsilon(x_\epsilon) + 2\ln(r_\epsilon s_\epsilon) = 2\ln r_\epsilon \to \infty$, and that $\hat{w}_\epsilon$ satisfies

$$-\Delta\hat{w}_\epsilon = e^{\hat{w}_\epsilon} - \frac{\epsilon^2}{r_\epsilon^2 s_\epsilon^2}e^{2\hat{w}_\epsilon} \quad \text{in } B_{\delta/(r_\epsilon s_\epsilon)}(0).$$

Recall that $\overline{w}_\epsilon(x) = w_\epsilon(s_\epsilon x + x_\epsilon) + 2\ln s_\epsilon$, and that $\overline{w}$ is the unique solution of (4.14). Thus, $\overline{w}(x) \leqslant -4\ln|x| + C$ near ∞. It follows from (4.15) that

$$\hat{w}_\epsilon(x) = \overline{w}(r_\epsilon x) + 2\ln r_\epsilon + \mathrm{o}(1) \leqslant -2\ln r_\epsilon - 4\ln|x| + C_d \to -\infty$$

uniformly on $\{x \mid d \leqslant |x| \leqslant 1\}$ for any constant $0 < d \leqslant 1$. Moreover, (3.6) implies that

$$\hat{w}_\epsilon(x) = w_\epsilon(r_\epsilon s_\epsilon x + x_\epsilon) + 2\ln|r_\epsilon s_\epsilon x| - 2\ln|x| \leqslant -2\ln|x| + C$$

for $0 < |x| < \delta/(2r_\epsilon s_\epsilon)$. Note that $\epsilon^2/(r_\epsilon^2 s_\epsilon^2) \to 0$. Then the proof of Lemma 4.3 implies that

$$e^{\hat{w}_\epsilon} - \frac{\epsilon^2}{r_\epsilon^2 s_\epsilon^2}e^{2\hat{w}_\epsilon} \to \hat{\alpha}_j\delta_{p=0}, \quad \hat{\alpha}_j \geqslant 8\pi$$

in the sense of measure on any compact subset of $\mathbb{R}^2$.

Fix any point $p_0 \in \mathbb{R}^2$ such that $|p_0| = 1$. Then (2.1) implies that

$$\begin{aligned}
\hat{w}_\epsilon(x) - \hat{w}_\epsilon(p_0) &= \frac{1}{2\pi} \int_{\widehat{B}_\epsilon} \ln \frac{|y-p_0|}{|y-x|} \left(e^{\hat{w}_\epsilon} - \frac{\epsilon^2}{r_\epsilon^2 s_\epsilon^2} e^{2\hat{w}_\epsilon} \right) dy \\
&\quad + \int_{B_\delta(q_j)} \left[\gamma(r_\epsilon s_\epsilon x + x_\epsilon, y) - \gamma(r_\epsilon s_\epsilon p_0 + x_\epsilon, y) \right] \left(e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon} \right) dy \\
&\quad + \int_{[B_\delta(q_j)]^c} \left[G(r_\epsilon s_\epsilon x + x_\epsilon, y) - G(r_\epsilon s_\epsilon p_0 + x_\epsilon, y) \right] \left(e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon} \right) dy \\
&\quad + u_0(r_\epsilon s_\epsilon x + x_\epsilon) - u_0(r_\epsilon s_\epsilon p_0 + x_\epsilon),
\end{aligned}$$

where $\widehat{B}_\epsilon = \{x \mid r_\epsilon s_\epsilon x + x_\epsilon \in B_\delta(q_j)\}$.

Then it follows that $\hat{w}_\epsilon - \hat{w}_\epsilon(p_0) \to -\frac{\hat{\alpha}_j}{2\pi} \ln|x|$ uniformly in $C^1_{\mathrm{loc}}(\widehat{B}_\epsilon \backslash \{0\})$. Therefore we conclude that

$$\begin{aligned}
&\int_{|x-x_\epsilon|=r_\epsilon s_\epsilon} \left[\frac{r_\epsilon s_\epsilon}{2} |\nabla w_\epsilon|^2 - \frac{1}{r_\epsilon s_\epsilon} \big((x-x_\epsilon)\cdot \nabla w_\epsilon \big)^2 - r_\epsilon s_\epsilon \left(e^{w_\epsilon} - \frac{\epsilon^2}{2} e^{2w_\epsilon} \right) \right] d\sigma \\
&\quad = \int_{|x|=1} \left[\frac{1}{2} \left| \nabla \hat{w}_\epsilon(x) \right|^2 - \left[x \cdot \nabla \hat{w}_\epsilon(x) \right]^2 - \left(e^{\hat{w}_\epsilon} - \frac{\epsilon^2}{2 r_\epsilon^2 s_\epsilon^2} e^{2\hat{w}_\epsilon} \right) \right] d\sigma \to -\frac{\hat{\alpha}_j^2}{4\pi},
\end{aligned}$$

where we used the fact that $\epsilon/(r_\epsilon s_\epsilon) \to 0$.

Finally, we estimate the last integral in (4.17). Since $w_\epsilon(x) \leqslant -2\ln|x-x_\epsilon| + C$ on A_ϵ, it follows that

$$\int_{A_\epsilon} \left(2e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon} \right) dx = 2 \int_{A_\epsilon} \left(e^{w_\epsilon} - \epsilon^2 e^{2w_\epsilon} \right) dx + \epsilon^2 \int_{A_\epsilon} e^{2w_\epsilon}\, dx = 2(\alpha_j - \hat{\alpha}_j) + \mathrm{o}(1).$$

Letting $\epsilon \to 0$, we obtain from (4.17) that

$$\left(\alpha_j^2 - \hat{\alpha}_j^2 \right) - 8\pi(\alpha_j - \hat{\alpha}_j) = 0.$$

Since $\alpha_j \geqslant \hat{\alpha}_j \geqslant 8\pi$, it follows that $\alpha_j = \hat{\alpha}_j$. $\square$

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