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Necessary conditions for a minimum at a radial cavitating singularity in nonlinear elasticity

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For Jerry Ericksen on the occasion of his eightieth birthday.

Abstract

It is still not known if the radial cavitating minimizers obtained by Ball [J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, Phil. Trans. R. Soc. Lond. A 306 (1982) 557–611] (and subsequently by many others) are global minimizers of any physically reasonable nonlinearly elastic energy. We therefore consider in this paper the related problem of obtaining necessary conditions for these radial solutions to be minimizers with respect to nonradial perturbations. A standard blowup argument applied to either an inner or an outer variation yields an apparently new inequality that, for most constitutive relations, has yet to be verified. However, in the special case of a compressible neo-Hookean material, $W(\mathbf{F}) = \frac{\mu}{2}|\mathbf{F}|^2 + h(\det \mathbf{F})$, we show that the inequality produced by an outer variation clearly holds whilst that produced by an inner variation is a well-known inequality (first proven by Brezis, Coron, and Lieb [H. Brezis, J.-M. Coron, E.H. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (1986) 649–705]) which arises in the theory of nematic liquid crystals:

$$\int_B \left|\nabla \mathbf{n}(\mathbf{x})\right|^2 d\mathbf{x} \geqslant \int_B \left|\nabla\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\right|^2 d\mathbf{x} = 8\pi$$

for all $\mathbf{n} \in W^{1,2}(B, \partial B)$ (so that $|\mathbf{n}| = 1$ a.e.) that satisfy $\mathbf{n}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ on ∂B, where B is the unit ball in $\mathbb{R}^3$.

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1. Introduction

For more than twenty years a major open problem in nonlinear elasticity has been whether or not the radial hole creating minimizers discovered[1] by John Ball [4] are indeed global minimizers of the energy. It is therefore of interest to obtain conditions that are *necessary for radial minimizers to be global minimizers* of the elastic energy:[2]

$$E(\mathbf{u}) = \int_B W\big(\nabla\mathbf{u}(\mathbf{x})\big)\,d\mathbf{x},$$

where $B \subset \mathbb{R}^n$ is the unit ball and W is a continuous, nonnegative, extended real-valued mapping on $n \times n$ matrices $\mathbf{F} = \nabla\mathbf{u}(\mathbf{x})$ that satisfies $W(\mathbf{F}) = +\infty$ whenever the determinant of $\mathbf{F}$ satisfies $\det\mathbf{F} \leqslant 0$.

A result of [19,21] shows that a necessary condition for a mapping $\mathbf{u}: \Omega \to \mathbb{R}^n$ to be a local (in $W^{1,p} \cap L^\infty$, $1 \leqslant p \leqslant \infty$) minimizer of the energy E is that, at each point $\mathbf{x}_0$ where $\mathbf{u}$ is continuously differentiable, W must be $W^{1,p}$-quasiconvex[3] at $\mathbf{F}_0 := \nabla\mathbf{u}(\mathbf{x}_0)$: that is,

$$\int_\Omega W\big(\mathbf{F}_0 + \nabla\mathbf{v}(\mathbf{x})\big)\,d\mathbf{x} \geqslant \int_\Omega W(\mathbf{F}_0)\,d\mathbf{x} \tag{1.1}$$

for every bounded open set $\Omega \subset \mathbb{R}^n$ whose boundary has measure zero and every $\mathbf{v}$ in the Sobolev space $W_0^{1,p}(\Omega;\mathbb{R}^n)$.

The standard derivation[4] of (1.1) for Lipschitz $\mathbf{v}$ (the case $p = \infty$) is to use test functions of the form $\mathbf{u}_\epsilon(\mathbf{x}) := \mathbf{u}(\mathbf{x}) + \epsilon\mathbf{v}(\frac{\mathbf{x}-\mathbf{x}_0}{\epsilon})$ and the change of variables $\mathbf{y} = \frac{\mathbf{x}-\mathbf{x}_0}{\epsilon}$ to obtain

$$\lim_{\epsilon\to 0}\left(\frac{E(\mathbf{u}_\epsilon) - E(\mathbf{u})}{\epsilon^n}\right) \geqslant 0. \tag{1.2}$$

The bounded convergence theorem allows one to interchange the limit and integral in the above inequality and deduce (1.1). In this argument, essential use is made of the assumption that $\mathbf{u}$ is continuously differentiable at $\mathbf{x}_0$ to prove that the limit exists.[5]

In the study of radial cavitation, $\Omega = B$ and one considers deformations of the form

$$\mathbf{u} = \mathbf{u}_r(\mathbf{x}) := \frac{r(R)}{R}\mathbf{x}, \quad R := |\mathbf{x}|,$$

where $r: [0,1] \to [0,\infty)$. If $r(0) > 0$, then the above deformation produces a spherical hole of radius $r(0)$ at the center of the deformed ball. It follows (from (3.2)) that, when cavitation occurs,

$$\big|\nabla\mathbf{u}_r(\mathbf{x})\big| \to \infty \quad \text{as } |\mathbf{x}| \to 0.$$

Moreover, since $\mathbf{u}_r$ is not even continuous at the origin, let alone continuously differentiable, the standard derivation of the necessary condition (1.1) fails at the point $\mathbf{x}_0 = \mathbf{0}$.

Our approach to circumvent this difficulty is to localize around $\mathbf{x}_0 = \mathbf{0}$ (in the reference configuration) and points $\mathbf{y}_0$ on the cavity surface (in the deformed configuration) through the use of inner and outer variations, respectively. In the case of inner variations we consider test functions of the form

$$\mathbf{u}_\epsilon(\mathbf{x}) := \mathbf{u}_r\left(\mathbf{x} + \epsilon\mathbf{w}\left(\frac{\mathbf{x}-\mathbf{x}_0}{\epsilon}\right)\right),$$

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[1] Ball's work is based upon that of Gent and Lindley [11] who find a critical load at which an infinitesimal hole in an infinite body grows to finite size. See [7,14] for prior related approaches in the context of elastoplasticity. See Horgan and Polignone [16] for a survey of the literature on cavitation in nonlinear elasticity.

[2] A typical model elastic energy density is the generalized compressible neo-Hookean material: $W(\mathbf{F}) = \frac{\mu}{p}|\mathbf{F}|^p + h(\det\mathbf{F})$, where p is between 1 and n (the dimension of the space), $\mu > 0$, and h is continuous, convex, and blows up as its argument goes to zero or infinity.

[3] See Ball and Murat [6] and Morrey [23,24].

[4] This proof is due to Ball [3, Theorem 3.1]. See also, e.g., [5, Theorem 2.2] or [26, Theorem 17.1.4].

[5] When $\mathbf{v}$ is not Lipschitz but merely lies in the Sobolev space $W^{1,p}$, $1 \leqslant p < \infty$, the alternative derivation of James and Spector [19] is necessitated by the condition that $W(\mathbf{F}) \to +\infty$ as $\det\mathbf{F} \to 0^+$ (see [19, Remark 4.7]. Results of Ball and Murat [6, Theorems 4.1(iii) and 4.5(ii)] for the model energy given in footnote 2 (with $\mu = 0$ and $\mu > 0$, respectively) show that h must be constant in order for E to be sequentially weakly lower semicontinuous (SWLSC) in materials that allow for hole formation ($p < n$). For such materials this is clearly incompatible with a continuous stored energy W that satisfies $W(\mathbf{F}) \to +\infty$ as $\det\mathbf{F} \to 0^+$. See [28] and Müller and Spector [25] for recovery of SWLSC and Conti and De Lellis [9] for interesting related results.

and, in the case of outer variations, we consider

$$\mathbf{u}_\epsilon(\mathbf{x}) := \mathbf{u}_r(\mathbf{x}) + \epsilon\mathbf{w}\left(\frac{\mathbf{u}_r(\mathbf{x}) - \mathbf{y}_0}{\epsilon}\right),$$

where $|\mathbf{y}_0| = r(0)$ so that $\mathbf{y}_0$ lies on the surface of the cavity produced by the map $\mathbf{u}_r$.

In this paper we take $n = 3$ and use the above approach to obtain necessary conditions for $\mathbf{u}_r$ to be a minimizer with respect to nonradial perturbations. We prove that the limit

$$\lim_{\epsilon\to 0}\left(\frac{E(\mathbf{u}_\epsilon) - E(\mathbf{u}_r)}{g(\epsilon)}\right) \geqslant 0$$

indeed exists and has a nice form for suitable functions g, where the choice of g depends[6] on W.

In particular, if

$$W(\mathbf{F}) = |\mathbf{F}|^p + |\operatorname{adj}\mathbf{F}|^q + h(\det\mathbf{F})$$

we use the above approach to show that a necessary condition for a radial cavitating minimizer to be a local minimizer is

$$\int_B \left|\nabla\left(\frac{\mathbf{w}(\mathbf{y})}{|\mathbf{w}(\mathbf{y})|}\right)\right|^p d\mathbf{y} \geqslant \int_B \left|\nabla\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\right|^p d\mathbf{y} = \frac{(4\pi)2^{p/2}}{3-p} \tag{1.3}$$

in the case $3 > p > 2q \geqslant 2$, or

$$\int_B \frac{1}{|\mathbf{w}(\mathbf{y})|^{2q}}\left|\big(\operatorname{adj}\nabla\mathbf{w}(\mathbf{y})\big)\left(\frac{\mathbf{w}(\mathbf{y})}{|\mathbf{w}(\mathbf{y})|}\right)\right|^q d\mathbf{y} \geqslant \int_B \frac{1}{|\mathbf{y}|^{2q}}\,d\mathbf{y} = \frac{4\pi}{3-2q} \tag{1.4}$$

in the case $1 \leqslant p < 2q < 3$, for a suitable class of mappings $\mathbf{w} \in W^{1,p}(B;\mathbb{R}^3)$ that satisfy $\mathbf{w} = \mathbf{i}$ on ∂B.

The inequality (1.4) appears to be new and, to our knowledge, has yet to be proved or disproved. However, there is an extensive literature[7] on (1.3) that shows it is satisfied for all mappings $\mathbf{w} \in W^{1,p}(B;\mathbb{R}^3)$, $1 \leqslant p < 3$, that satisfy $\mathbf{w} = \mathbf{i}$ on ∂B. It is intriguing that (1.3) with $p = 2$ appears to first occur in a paper of Brezis, Coron, and Lieb [8] where $\mathbf{n} = \mathbf{w}/|\mathbf{w}|$ is the director in a nematic liquid crystal.

2. Preliminaries

In the following, Ω will denote a nonempty, bounded, open subset of $\mathbb{R}^3$ whose boundary, $\partial\Omega$, has 3-dimensional Lebesgue measure zero. By $L^p(\Omega)$ and $W^{1,p}(\Omega)$ we denote the usual Lebesgue and Sobolev spaces, respectively. We use the notation $L^p(\Omega;\mathbb{R}^3)$, etc; for vector-valued maps. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Given $\mathbf{f} \in W^{1,p}(\Omega;\mathbb{R}^3)$ we write $\mathbf{v} \in \mathbf{f} + W_0^{1,p}(\Omega;\mathbb{R}^3)$ provided $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^3)$ and $\mathbf{v} - \mathbf{f} \in W_0^{1,p}(\Omega;\mathbb{R}^3)$. If $U \subset \Omega$ is an open set we write $U \subset\subset \Omega$ provided there is a compact set K_U such that $U \subset K_U \subset \Omega$.

We write

$$B(\mathbf{a}, r) := \big\{\mathbf{x} \in \mathbb{R}^3\colon\ |\mathbf{x} - \mathbf{a}| < r\big\}$$

for the (open) ball of radius r centered at $\mathbf{a} \in \mathbb{R}^3$. In particular we write $B := B(\mathbf{0}, 1)$ for the unit ball centered at the origin. Given a unit vector $\mathbf{e}_0 \in \mathbb{R}^3$, we write

$$\mathcal{HB} := \{\mathbf{z} \in B\colon\ \mathbf{z}\cdot\mathbf{e}_0 > 0\}$$

for the (open) half-ball with plane face perpendicular to $\mathbf{e}_0$. We let Lin denote the set of all linear maps from $\mathbb{R}^3$ into $\mathbb{R}^3$ with inner product and norm

$$\mathbf{L} : \mathbf{M} = \operatorname{trace}\big(\mathbf{L}\mathbf{M}^{\mathrm{T}}\big), \qquad |\mathbf{L}|^2 = \mathbf{L} : \mathbf{L},$$

[6] For an inner variation of a generalized compressible neo-Hookean material in three-dimensions g is simply $g(\epsilon) = \epsilon^{3-p}$. For an outer variation, g depends on the inverse of the radial minimizer (see Remark 3.2).

[7] See, e.g., [1,2,8,10,12,13,15,17].

respectively. We write Lin^+ for those $\mathbf{L} \in \mathrm{Lin}$ with positive determinant. The mapping $\mathrm{adj}: \mathrm{Lin} \to \mathrm{Lin}$ will be the unique continuous function that satisfies

$$\mathbf{L}(\mathrm{adj}\,\mathbf{L}) = (\det \mathbf{L})\mathbf{I}$$

for all $\mathbf{L} \in \mathrm{Lin}$, where $\det \mathbf{L}$ is the determinant of $\mathbf{L}$ and $\mathbf{I} \in \mathrm{Lin}$ is the identity mapping. Thus, with respect to any orthonormal basis, the matrix corresponding to $\mathrm{adj}\,\mathbf{L}$ is the transpose of the cofactor matrix corresponding to $\mathbf{L}$.

In nonlinear elasticity one is interested in globally invertible maps that preserve orientation since, in general, matter can neither interpenetrate itself nor reverse its orientation. A mapping $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^3)$ is called *one-to-one almost everywhere* if there is a Lebesgue null set $N \subset \Omega$ such that $\mathbf{u}|_{\Omega \setminus N}$ is injective. We write

$$\mathrm{Def}(\Omega) := \{\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3): \det \nabla\mathbf{u} > 0 \text{ a.e. and } \mathbf{u} \text{ is one-to-one a.e.}\} \tag{2.1}$$

for those Sobolev mappings that are allowable *deformations* of the body.

3. Elasticity; Radial minimizers

We consider a homogeneous body that for convenience will be identified with the ball $B_o := B(\mathbf{0}, R_o) \subset \mathbb{R}^3$ of radius $R_o > 0$ centered at the origin that it occupies in a fixed homogeneous reference configuration. We assume that the body is hyperelastic with continuous *stored energy* density $W: \mathrm{Lin} \to \mathbb{R} \cup \{+\infty\}$. The quantity $W(\nabla\mathbf{u}(\mathbf{x}))$ gives the energy stored per unit volume in B_o at any point $\mathbf{x} \in B_o$ when the body is deformed by a smooth deformation $\mathbf{u}$. Further, we assume that $W(\mathbf{F}) = +\infty$ whenever $\det \mathbf{F} \leqslant 0$.

The problem of interest is the determination of minimizers for the total elastic energy

$$E(\mathbf{u}, B_o) = \int_{B_o} W(\nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x} \tag{3.1}$$

among orientation preserving, injective $\mathbf{u}$ that satisfy the boundary condition $\mathbf{u}(\mathbf{x}) = \lambda\mathbf{x}$ for $\mathbf{x} \in \partial B_o$, for each $\lambda > 0$. In particular we will be concerned with mappings that are *radial*, i.e., $\mathbf{u}$ of the form

$$\begin{aligned} \mathbf{u}_r(\mathbf{x}) &= \frac{r(R)}{R}\mathbf{x}, \quad R := |\mathbf{x}|, \\ \nabla\mathbf{u}_r(\mathbf{x}) &= \frac{r(R)}{R}\mathbf{I} + \left(r'(R) - \frac{r(R)}{R}\right)\frac{\mathbf{x}}{|\mathbf{x}|} \otimes \frac{\mathbf{x}}{|\mathbf{x}|}. \end{aligned} \tag{3.2}$$

The stored-energy functions we will consider are those that satisfy[8]

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|, |\mathrm{adj}\,\mathbf{F}|, \det \mathbf{F}) + h(\det \mathbf{F}) \tag{3.3}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$, where $h \in C^2(\mathbb{R}^+, [0, \infty))$ is a strictly convex function that satisfies

$$\lim_{t \to 0^+} h(t) = \lim_{t \to +\infty} \frac{h(t)}{t} = +\infty \tag{3.4}$$

and, for a unique fixed $H > 1$,

$$h'(H) = 0. \tag{3.5}$$

In addition we will require that $\Psi \in C^2(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, [0, \infty))$ satisfies

$$\Psi(|\mathbf{F}|, |\mathrm{adj}\,\mathbf{F}|, \det \mathbf{F}) \leqslant c|\mathbf{F}|^p \quad \text{for all } \mathbf{F} \in \mathrm{Lin}^+ \tag{3.6}$$

for some $p \in [1, 3)$ and $c > 0$.

[8] Eq. (3.3), with $h = 0$, is a consequence of the response of the material being independent of observer and the material in its reference configuration being isotropic.

Remark 3.1. Under various additional hypotheses on Ψ (see [4,22,27,29]) for each $\lambda \in (0, \infty)$ there exists

$$r_\lambda \in \mathcal{A}_\lambda := \{ r \in W^{1,1}((0, R_o)) \colon r(R_o) = \lambda R_o,\ r(0) \geqslant 0,\ r' > 0 \text{ a.e.} \} \tag{3.7}$$

at which the total elastic energy (3.1), satisfying (3.3)–(3.6), attains its infimum. Moreover, $r_\lambda \in C^1([0, R_o]) \cap C^2((0, R_o])$ and is a solution of the corresponding Euler equation. Furthermore, there exists $\lambda_{\text{crit}} \geqslant \sqrt[3]{H}$ such that

$$r_\lambda(R) = \lambda R \quad \text{for } \lambda \leqslant \lambda_{\text{crit}}$$

and

$$r_\lambda(0) > 0 \quad \text{for } \lambda > \lambda_{\text{crit}}. \tag{3.8}$$

In addition, for $\lambda > \lambda_{\text{crit}}$ the function r_λ satisfies

$$r'_\lambda(0) = 0, \qquad \lim_{R \to 0^+} r'_\lambda(R) \left[\frac{r_\lambda(R)}{R} \right]^2 = H, \tag{3.9}$$

$$H < r'_\lambda(R) \left[\frac{r_\lambda(R)}{R} \right]^2 < \lambda^3 \quad \text{for } R \in (0, R_o]. \tag{3.10}$$

In other words, for $\lambda > \lambda_{\text{crit}}$, the deformation that minimizes the total elastic energy is not the homogeneous deformation $\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x}$, which is the minimizer in $W^{1,s}$ for $s > 3$. Moreover, the minimizer among radial deformations exists and creates a new hole at the center of the ball.

Remark 3.2. It follows from (3.10) that $r'_\lambda > 0$ on $(0, R_o)$. Now fix $\lambda > \lambda_{\text{crit}}$ and define $c_\lambda = r_\lambda(0)$ to be the newly created cavity radius in the deformed configuration. Then, the map $R \mapsto r_\lambda(R)$ has a unique inverse $r \mapsto R_\lambda(r)$. This map is smooth and satisfies $R_\lambda(c_\lambda) = 0$. Let

$$\phi(r) = \frac{1}{3} R_\lambda(r)^3, \qquad \phi'(r) = R'_\lambda(r) R_\lambda(r)^2 = \frac{R_\lambda(r)^2}{r'_\lambda(R_\lambda(r))}. \tag{3.11}$$

Then, by $(3.9)_2$ and $(3.11)_2$,

$$\lim_{r \to c_\lambda^+} \phi'(r) = \lim_{R \to 0^+} \frac{R^2}{r'_\lambda(R)} = \frac{c_\lambda^2}{H} > 0. \tag{3.12}$$

Thus, although all of the derivatives of R_λ are infinite at the surface of the cavity, the cube of R_λ has a finite derivative. We will make use of this fact in Section 5.

Remark 3.3. If $\Psi = 0$, then the unique minimizer of the energy among deformations that are both radial and injective is given by

$$r_\lambda(R)^3 := \begin{cases} \lambda^3 R^3 & \text{if } \lambda^3 \leqslant H, \\ H R^3 + (\lambda^3 - H) R_o^3 & \text{if } \lambda^3 > H. \end{cases}$$

4. Inner variations

In the remainder of the paper we will assume that Ψ is independent of $\det \mathbf{F}$, that is,

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|, |\operatorname{adj} \mathbf{F}|) + h(\det \mathbf{F}).$$

We suppose further that there exist $m > 0$ and $C > 1$ such that for all $t > 0$

$$C^{-1}(t^m + t^{-m}) \leqslant h(t) \leqslant C(t^m + t^{-m}). \tag{4.1}$$

We also assume that there exists a positive integer M and constants $K_i > 0$, $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, $i = 1, 2, \ldots, M$, such that

$$\lim_{\varepsilon \to 0^+} \varepsilon^p \Psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon^2} \right) = \Psi_\infty(x, y) := \sum_{i=1}^{M} K_i x^{\alpha_i} y^{\beta_i} \tag{4.2}$$

uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}^+$, where $p \in [1, 3)$ is the growth exponent in (3.6). It follows from (4.2) that $\alpha_i + 2\beta_i = p$ for each i.

Remark 4.1. A standard hypothesis used to obtain existence of minimizers in nonlinear elasticity is that the stored energy W be polyconvex. A well-known result on the composition of convex functions yields sufficient conditions for W to be polyconvex: Ψ is convex with $z \mapsto \Psi(z,s)$ and $z \mapsto \Psi(s,z)$ nondecreasing for every $s > 0$.

Theorem 4.2. *Assume that the radial minimizer*

$$\mathbf{u}_r(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \quad R := |\mathbf{x}| \tag{4.3}$$

given by Remark 3.1 *is a local minimizer of the total elastic energy*

$$E\big(\mathbf{u}, B(\mathbf{0}, R_o)\big) = \int_{B(\mathbf{0},R_o)} \big[\Psi\big(|\nabla\mathbf{u}(\mathbf{x})|, |\operatorname{adj}\nabla\mathbf{u}(\mathbf{x})|\big) + h\big(\det\nabla\mathbf{u}(\mathbf{x})\big)\big]\,d\mathbf{x} \tag{4.4}$$

in the $W^{1,p}(B(\mathbf{0},R_o);\mathbb{R}^3) \cap \mathcal{L}^\infty(B(\mathbf{0},R_o);\mathbb{R}^3)$ *topology, where* $\lambda > \lambda_{\text{crit}}$ *and* h *satisfying* (4.1) *is as in Section* 3.

Let $B := B(\mathbf{0},1)$ *be the unit ball in* $\mathbb{R}^3$ *and suppose that* $\mathbf{w} \in \operatorname{Def}(B)$ *is a deformation of* B *that satisfies* $\mathbf{w} = \mathbf{i}$ *on* ∂B *and has finite energy* $E(\mathbf{w}, B) < +\infty$. *Suppose, in addition, that there is a Lebesgue null set* $N = N_\mathbf{w} \subset B$ *and an open neighborhood* $U = U_\mathbf{w} \subset\subset B$, *both of which may depend on* $\mathbf{w}$, *such that either*

$$\mathbf{0} \in U, \qquad \mathbf{w}(B \backslash N) \cap U = \varnothing, \tag{4.5}$$

or

$$\mathbf{0} \in \mathbf{w}(U), \qquad \mathbf{w}|_U \text{ is a diffeomorphism}, \qquad \mathbf{w}(B \setminus (U \cup N)) \cap \mathbf{w}(U) = \varnothing. \tag{4.6}$$

Then

$$\begin{aligned}&\sum_{i=1}^{M} K_i \int_B \left|\nabla\left(\frac{\mathbf{w}(\mathbf{y})}{|\mathbf{w}(\mathbf{y})|}\right)\right|^{\alpha_i} \left(\left|\big(\operatorname{adj}\nabla\mathbf{w}(\mathbf{y})\big)\left(\frac{\mathbf{w}(\mathbf{y})}{|\mathbf{w}(\mathbf{y})|^3}\right)\right|\right)^{\beta_i} d\mathbf{y} \\ &\quad \geqslant \sum_{i=1}^{M} K_i \int_B \left|\nabla\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\right|^{\alpha_i} \frac{1}{|\mathbf{y}|^{2\beta_i}}\,d\mathbf{y} = \frac{4\pi}{3-p}\sum_{i=1}^{M} K_i 2^{(\alpha_i/2)}.\end{aligned} \tag{4.7}$$

Remark 4.3. The purpose of conditions (4.5) and (4.6) is to ensure that the composition of the radial energy minimizer $\mathbf{u}_r$ with $\mathbf{w}$ is a deformation with finite energy. Since $\mathbf{u}_r$ is known to be C^2 away from the origin the only difficulty occurs at points $\mathbf{x}$ where $\mathbf{w}(\mathbf{x}) = \mathbf{0}$. Condition (4.5) ensures that this does not occur since, except for a null set, $\mathbf{w}$ maps no points into a neighborhood of the origin, i.e, $\mathbf{w}$ is a cavitating map that creates one or more new holes, one of which includes the origin. Condition (4.6) instead requires that $\mathbf{w}$ be a diffeomorphism onto some neighborhood of the origin and that, except for a null set, no other points map into this neighborhood. In particular, (4.6) is satisfied if $\mathbf{w}$ itself is a diffeomorphism of the unit ball.

Corollary 4.4. *Let*[9] $\Psi(x,y) = x^p$, $1 \leqslant p < 3$. *Under the hypotheses of Theorem* 4.2, *a necessary condition for the radial minimizer* $\mathbf{u}_r$ *given by* (4.3) *to be a local minimizer of the total elastic energy* (4.4) *is that*

$$\int_{B(\mathbf{0},1)} |\nabla\mathbf{v}(\mathbf{y})|^p\,d\mathbf{y} \geqslant \int_{B(\mathbf{0},1)} \left|\nabla\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\right|^p d\mathbf{y} = \frac{(4\pi)2^{(p/2)}}{3-p} \tag{4.8}$$

for all $\mathbf{v} \in W^{1,p}(B, \partial B)$ *that are in the closure of the set*

$$\left\{\frac{\mathbf{w}}{|\mathbf{w}|}:\ \mathbf{w} \in \operatorname{Def}(B),\ \mathbf{w} = \mathbf{i} \text{ on } \partial B \text{ and } \mathbf{w} \text{ satisfies (4.5) or (4.6)}\right\}.$$

[9] Or, more generally, $\Psi_\infty(x,y) = x^p$.

Remark 4.5. It is known that (4.8) is satisfied for all $\mathbf{v} \in W^{1,p}(B, \partial B)$, $1 \leqslant p < 3$, that satisfy $\mathbf{v}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ on ∂B. For $p = 2$ this inequality was obtained by Brezis, Coron, and Lieb [8] who noted its significance within the theory of nematic liquid crystals. For $p \in \{1, 2\}$ see Coron and Gulliver [10]. For $p \in [2, 3)$ see Hardt, Lin, and Wang [13]. The final interval $p \in (1, 2]$ was obtained by Hong [15] in 2001. See also Lin [20], Almgren, Browder, and Lieb [1], Avellaneda and Lin [2] and Jäger and Kaul [17].

Proof of Theorem 4.2. Let $\mathbf{w}$ be as above. Then, for every sufficiently small $\varepsilon > 0$, the mapping $\mathbf{v}_\varepsilon : B(\mathbf{0}, R_\mathrm{o}) \to \mathbb{R}^3$ defined by

$$\mathbf{v}_\varepsilon(\mathbf{z}) := \begin{cases} \varepsilon \mathbf{w}(\frac{\mathbf{z}}{\varepsilon}) & \text{if } \mathbf{z} \in B(\mathbf{0}, \varepsilon), \\ \mathbf{z} & \text{otherwise,} \end{cases} \tag{4.9}$$

satisfies $\mathbf{v}_\varepsilon \in \mathrm{Def}(B(\mathbf{0}, R_\mathrm{o}))$ and $\mathbf{v}_\varepsilon = \mathbf{i}$ on $\partial B(\mathbf{0}, R_\mathrm{o})$.

We fix $\lambda > \lambda_{\text{crit}}$ and let (4.3) be a radial minimizer of E in $\mathcal{A}_\lambda$ given by Remark 3.1. By hypothesis $\mathbf{u}_r$ is also a local minimizer of E: thus if $\mathbf{u}_r \circ \mathbf{v}_\varepsilon$ is sufficiently close to $\mathbf{u}_r$ in $W^{1,p} \cap \mathcal{L}^\infty$ then, in view of (4.5) or (4.6), $\mathbf{u}_r \circ \mathbf{v}_\varepsilon \in \mathrm{Def}(B(\mathbf{0}, R_\mathrm{o}))$ and

$$\int_{B(\mathbf{0},R_\mathrm{o})} W\big(\nabla(\mathbf{u}_r \circ \mathbf{v}_\varepsilon)(\mathbf{z})\big)\, d\mathbf{z} \geqslant \int_{B(\mathbf{0},R_\mathrm{o})} W\big(\nabla(\mathbf{u}_r \circ \mathbf{i})(\mathbf{z})\big)\, d\mathbf{z}.$$

Thus by $(4.9)_2$

$$\int_{B(\mathbf{0},\varepsilon)} \big[W\big(\nabla_\mathbf{z}(\mathbf{u}_r \circ \mathbf{v}_\varepsilon)(\mathbf{z})\big) - W\big(\nabla_\mathbf{z}\mathbf{u}_r(\mathbf{z})\big)\big]\, d\mathbf{z} \geqslant 0.$$

However, $\nabla_\mathbf{z}(\mathbf{u}_r \circ \mathbf{v}_\varepsilon)(\mathbf{z}) = \nabla_\mathbf{x}\mathbf{u}_r(\varepsilon\mathbf{w}(\frac{\mathbf{z}}{\varepsilon}))\nabla_\mathbf{y}\mathbf{w}(\frac{\mathbf{z}}{\varepsilon})$ and so the change of variables $\mathbf{z} = \varepsilon\mathbf{y}$, $d\mathbf{z} = \varepsilon^3\, d\mathbf{y}$, and $\nabla_\mathbf{y} = \varepsilon\nabla_\mathbf{z}$ in the last inequality yields

$$\int_{B(\mathbf{0},1)} \big[W\big(\nabla_\mathbf{x}\mathbf{u}_r\big(\varepsilon\mathbf{w}(\mathbf{y})\big)\nabla_\mathbf{y}\mathbf{w}(\mathbf{y})\big) - W\big(\nabla_\mathbf{z}\mathbf{u}_r(\varepsilon\mathbf{y})\big)\big]\, d\mathbf{y} \geqslant 0. \tag{4.10}$$

We will next make use of (4.1)–(4.3) and the results in Appendix A in order to simplify our computation of the integrands in (4.10). By (3.2)

$$\nabla_\mathbf{x}\mathbf{u}_r\big(\varepsilon\mathbf{w}(\mathbf{y})\big) = \frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\mathbf{I} + \left(r'\big(\varepsilon|\mathbf{w}|\big) - \frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\right)\mathbf{e}_\mathbf{w} \otimes \mathbf{e}_\mathbf{w}, \tag{4.11}$$

where $\mathbf{w} := \mathbf{w}(\mathbf{y})$ and $\mathbf{e}_\mathbf{w} := \mathbf{w}(\mathbf{y})/|\mathbf{w}(\mathbf{y})|$ is a unit vector for a.e. $\mathbf{y}$. Then (4.11) and Lemma A.1 yield

$$\begin{aligned} \big|\nabla_\mathbf{x}\mathbf{u}_r(\varepsilon\mathbf{w})\mathbf{G}_\mathbf{w}\big| &= \big(\big|\nabla_\mathbf{x}\mathbf{u}_r(\varepsilon\mathbf{w})\mathbf{G}_\mathbf{w}\big|^2\big)^{1/2} \\ &= \left[\left[\frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\right]^2 |\mathbf{G}_\mathbf{w}|^2 + \left(r'\big(\varepsilon|\mathbf{w}|\big)^2 - \left[\frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\right]^2\right)\big|(\mathbf{G}_\mathbf{w})^\mathrm{T}\mathbf{e}_\mathbf{w}\big|^2\right]^{1/2} \end{aligned} \tag{4.12}$$

where $\mathbf{G}_\mathbf{w} := \nabla_\mathbf{y}\mathbf{w}(\mathbf{y})$, while (4.11) and Lemma A.3 imply

$$\big|\mathrm{adj}\big(\nabla_\mathbf{x}\mathbf{u}_r(\varepsilon\mathbf{w})\mathbf{G}_\mathbf{w}\big)\big| = \left[\frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\right]\left[r'\big(\varepsilon|\mathbf{w}|\big)^2|\mathbf{A}_\mathbf{w}|^2 + \left(\left[\frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\right]^2 - r'\big(\varepsilon|\mathbf{w}|\big)^2\right)\big|\mathbf{A}_\mathbf{w}\mathbf{e}_\mathbf{w}\big|^2\right]^{1/2}, \tag{4.13}$$

where $\mathbf{A}_\mathbf{w} := \mathrm{adj}(\nabla_\mathbf{y}\mathbf{w}(\mathbf{y}))$. Also,

$$\det\big(\nabla_\mathbf{x}\mathbf{u}_r(\varepsilon\mathbf{w})\mathbf{G}_\mathbf{w}\big) = r'\big(\varepsilon|\mathbf{w}|\big)\left[\frac{r(\varepsilon|\mathbf{w}|)}{\varepsilon|\mathbf{w}|}\right]^2 \det\mathbf{G}_\mathbf{w}. \tag{4.14}$$

Our next task is to multiply (4.10) by an appropriate power of ε and take the limit as ε approaches zero. We note that it suffices to make this computation for the first integral in (4.10) since the second follows in a similar manner. We start by proceeding pointwise on the individual terms in the integrand. By (3.8) and $(3.9)_1$, for a.e. $\mathbf{y} \in B(\mathbf{0}, 1)$,

$$\lim_{\varepsilon\to 0^+} r\big(\varepsilon\big|\mathbf{w}(\mathbf{y})\big|\big) = r(0) > 0, \qquad \lim_{\varepsilon\to 0^+} r'\big(\varepsilon\big|\mathbf{w}(\mathbf{y})\big|\big) = 0. \tag{4.15}$$

Therefore, if we make use of (4.12) and (4.15),

$$\lim_{\varepsilon\to 0^+} \varepsilon\left|\nabla_{\mathbf{x}}\mathbf{u}_r(\varepsilon\mathbf{w})\mathbf{G}_{\mathbf{w}}\right| = r(0)\left(\frac{|\nabla\mathbf{w}|^2}{|\mathbf{w}|^2} - \frac{|(\nabla\mathbf{w})^{\mathrm{T}}\mathbf{w}|^2}{|\mathbf{w}|^4}\right)^{1/2} = r(0)\left|\nabla\left(\frac{\mathbf{w}(\mathbf{y})}{|\mathbf{w}(\mathbf{y})|}\right)\right| \tag{4.16}$$

and similarly, by (4.13) and (4.15),

$$\lim_{\varepsilon\to 0^+} \varepsilon^2\left|\operatorname{adj}\left(\nabla_{\mathbf{x}}\mathbf{u}_r(\varepsilon\mathbf{w})\mathbf{G}_{\mathbf{w}}\right)\right| = \left[\frac{r(0)}{|\mathbf{w}(\mathbf{y})|}\right]^2\left|\left(\operatorname{adj}\nabla\mathbf{w}(\mathbf{y})\right)\left[\frac{\mathbf{w}(\mathbf{y})}{|\mathbf{w}(\mathbf{y})|}\right]\right|. \tag{4.17}$$

Next, in view of (3.10) and (4.14),

$$H\det\nabla\mathbf{w}(\mathbf{y}) \leqslant \left[\det\nabla_{\mathbf{x}}\mathbf{u}_r\left(\varepsilon\mathbf{w}(\mathbf{y})\right)\right]\det\nabla\mathbf{w}(\mathbf{y}) \leqslant \lambda^3\det\nabla\mathbf{w}(\mathbf{y})$$

and consequently, since h is convex and nonnegative,

$$0 \leqslant h\left(\det\left(\nabla_{\mathbf{x}}\mathbf{u}_r\left(\varepsilon\mathbf{w}(\mathbf{y})\right)\nabla\mathbf{w}(\mathbf{y})\right)\right) \leqslant h\left(H\det\nabla\mathbf{w}(\mathbf{y})\right) + h\left(\lambda^3\det\nabla\mathbf{w}(\mathbf{y})\right). \tag{4.18}$$

Thus, if we integrate (4.18) over B and make use of (4.1) we find that

$$0 \leqslant \int_B h\left(\det\left(\nabla_{\mathbf{x}}\mathbf{u}_r\left(\varepsilon\mathbf{w}(\mathbf{y})\right)\nabla\mathbf{w}(\mathbf{y})\right)\right)d\mathbf{y} \leqslant 2C^2\lambda^{3m}\int_B h\left(\det\nabla\mathbf{w}(\mathbf{y})\right)d\mathbf{y} < +\infty \tag{4.19}$$

since $\lambda > \lambda_{\mathrm{crit}} \geqslant H > 1$ and $\mathbf{w}$ has finite energy. Therefore, it follows from (4.19) that if we multiply (4.10) by ϵ^s for any $s > 0$ and let ϵ approach zero, both of the terms involving h in (4.10) (see (4.4)) will go to zero.

To complete the proof we now multiply (4.10) by ϵ^p and let $\epsilon = \epsilon_n$ be any sequence that converges to zero to conclude with the aid of the previous paragraph, (3.8), (4.16), (4.17), the dominated convergence theorem, and (4.2) that (4.7) is satisfied.

Finally, we must show that $\mathbf{u}_r \circ \mathbf{v}_\varepsilon \to \mathbf{u}_r$ strongly in $W^{1,p} \cap \mathcal{L}^\infty$ as $\varepsilon \to 0$ in order to complete the proof. The convergence in $\mathcal{L}^\infty(B(\mathbf{0}, R_{\mathrm{o}}); \mathbb{R}^3)$, and hence $\mathcal{L}^p(B(\mathbf{0}, R_{\mathrm{o}}); \mathbb{R}^3)$, is straightforward. The computation that produced (4.10) shows that $\nabla(\mathbf{u}_r \circ \mathbf{v}_\varepsilon) \to \nabla\mathbf{u}_r$ in $\mathcal{L}^p(B(\mathbf{0}, R_{\mathrm{o}}); \mathbb{R}^3)$ is equivalent to

$$\lim_{\varepsilon\to 0^+} \varepsilon^3 \int_{B(\mathbf{0},1)} \overline{W}\left(\nabla_{\mathbf{x}}\mathbf{u}_r\left(\varepsilon\mathbf{w}(\mathbf{y})\right)\nabla_{\mathbf{y}}\mathbf{w}(\mathbf{y}) - \nabla_{\mathbf{z}}\mathbf{u}_r(\varepsilon\mathbf{y})\right)d\mathbf{y} = 0, \tag{4.20}$$

where $\overline{W}(\mathbf{F}) := |\mathbf{F}|^p$. However, (4.20) follows from (4.12), (4.16), and the dominated convergence theorem, since $p < 3$. □

5. Outer variations

Recall that $B := B(\mathbf{0}, 1)$ is the (open) unit ball in $\mathbb{R}^3$ and

$$\mathrm{Def}(B) := \left\{\mathbf{u} \in W^{1,p}\left(B; \mathbb{R}^3\right):\ \det\nabla\mathbf{u} > 0 \text{ a.e. and } \mathbf{u} \text{ is one-to-one a.e.}\right\}$$

is the set of deformations of B. Let $\mathbf{e}_0 \in \mathbb{R}^3$ with $|\mathbf{e}_0| = 1$ and, for any $\delta > 0$,

$$\mathcal{HB}_\delta := \left\{\mathbf{z} \in B:\ \mathbf{z}\cdot\mathbf{e}_0 > \delta\right\}, \qquad \mathcal{HB} := \left\{\mathbf{z} \in B:\ \mathbf{z}\cdot\mathbf{e}_0 > 0\right\}. \tag{5.1}$$

Define

$$\mathrm{Def}_{\mathbf{i}}(\mathcal{HB}) := \left\{\mathbf{w} \in \mathrm{Def}(B):\ \text{there exists } \delta > 0 \text{ such that } \mathbf{w} = \mathbf{i} \text{ on } \overline{B}\setminus\mathcal{HB}_\delta\right\} \tag{5.2}$$

and note that $\mathbf{w} = \mathbf{i}$ on ∂B and that $\delta = \delta_{\mathbf{w}}$ will in general depend on $\mathbf{w}$.

Theorem 5.1. *Assume that the radial minimizer*

$$\mathbf{u}_r(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \quad R := |\mathbf{x}| \tag{5.3}$$

given by Remark 3.1 *is a local minimizer of the total elastic energy*

$$E\big(\mathbf{u}, B(\mathbf{0}, R_o)\big) = \int_{B(\mathbf{0},R_o)} \big[\Psi\big(|\nabla\mathbf{u}(\mathbf{x})|, |\operatorname{adj}\nabla\mathbf{u}(\mathbf{x})|\big) + h\big(\det\nabla\mathbf{u}(\mathbf{x})\big)\big]\,d\mathbf{x} \tag{5.4}$$

in the $W^{1,p}(B(\mathbf{0}, R_o); \mathbb{R}^3) \cap \mathcal{L}^\infty(B(\mathbf{0}, R_o); \mathbb{R}^3)$ *topology, where* $\lambda > \lambda_{\text{crit}}$ *and* h *satisfying* (4.1) *is as in Section* 3. *Suppose that* $\mathbf{w} \in \operatorname{Def}_{\mathbf{i}}(\mathcal{HB})$ *is a deformation of* B *that has finite energy* $E(\mathbf{w}, B) < +\infty$. *Then*

$$\begin{aligned}\sum_{i=1}^{M} K_i \int_{\mathcal{HB}} (\mathbf{z}\cdot\mathbf{e}_0)^{-p/3}\big(|\nabla\mathbf{w}(\mathbf{z})|^2 - |(\nabla\mathbf{w}(\mathbf{z}))\mathbf{e}_0|^2\big)^{\alpha_i/2}\big|(\operatorname{adj}\nabla\mathbf{w}(\mathbf{z}))^{\mathrm{T}}\mathbf{e}_0\big|^{\beta_i}\,d\mathbf{z}\\ \geqslant \sum_{i=1}^{M} K_i \int_{\mathcal{HB}} (\mathbf{z}\cdot\mathbf{e}_0)^{-p/3}(2)^{\alpha_i/2}\,d\mathbf{z} = \frac{18\pi}{(3-p)(9-p)}\sum_{i=1}^{M} K_i 2^{(\alpha_i/2)}.\end{aligned} \tag{5.5}$$

Theorem 5.2. *Let* $\Psi(x, y) = x^p$ *or* $\Psi(x, y) = y^p$, $p \geqslant 1$. *Then, in either case, inequality* (5.5) *is satisfied for all* $\mathbf{w} \in \mathbf{i} + W_0^{1,p}(\mathcal{HB}; \mathbb{R}^3)$.

Remark 5.3. It is clear from the proof that Theorem 5.2 remains valid if one instead assumes that Ψ_∞ is equal to x^p or y^p, $p \in [1, 3)$, or $K_x x^p + K_y y^{p/2}$, $p \in [2, 3)$, where K_x and K_y are nonnegative. This will be the case when Ψ is convex, as one might require in a general theorem on existence of minimizers (see Remark 4.1), since Ψ_∞ will consequently be convex and the conditions $p \in [1, 3)$ and $\alpha_i + 2\beta_i = p$ reduces Ψ_∞ to one of these forms.

Remark 5.4. It is unfortunate that our blowup analysis using outer variations does not allow for perturbations $\mathbf{w}$ that vary on the newly formed free surface of the cavity. One might instead anticipate a condition similar to Ball and Marsden's [5] quasiconvexity at the boundary. Our analysis is also unable to recover the results of James and Spector [18] who construct stored energies for which the radial minimizer is not a local minimizer (in $W^{1,p}$, $1 \leqslant p < 2$) due to the creation of further long thin holes near the cavity and perpendicular to its surface. The analysis in [18] relies crucially on using finite values of the adjugate term, while our analysis uses, at best, the growth of the adjugate term at infinity.

Proof of Theorem 5.2. The proof is a slight variant of the standard proof that a polyconvex function is quasiconvex. In the first case we note that $\mathbf{F} \mapsto (|\mathbf{F}|^2 - |\mathbf{F}\mathbf{e}_0|^2)^{p/2}$ is convex for $p \geqslant 1$. Thus for $\mathbf{w} \in \mathbf{i} + C_0^\infty(\mathcal{HB}; \mathbb{R}^3)$ we find that the single term on the left-hand side of (5.5) is greater than or equal to the single term on the right-hand side plus

$$\int_{\mathcal{HB}} p2^{(p-2)/2}(\mathbf{z}\cdot\mathbf{e}_0)^{-p/3}(\mathbf{I} - \mathbf{e}_0 \otimes \mathbf{e}_0) : \big(\nabla\mathbf{w}(\mathbf{z}) - \mathbf{I}\big)\,d\mathbf{z}. \tag{5.6}$$

We now observe that the integrand in (5.6) only involves derivatives of $\mathbf{w}$ in directions perpendicular to the unit vector $\mathbf{e}_0$ and hence an integration by parts in planes perpendicular to $\mathbf{e}_0$, together with the boundary condition $\mathbf{w} = \mathbf{i}$ on $\partial(\mathcal{HB})$, shows that the integral in (5.6) is equal to zero. The desired result then follows by the density of C_0^∞ in $W_0^{1,p}$.

Similarly in the second case, $\mathbf{M} \mapsto |\mathbf{M}^{\mathrm{T}}\mathbf{e}_0|^p$ is convex for $p \geqslant 1$. Thus for $\mathbf{w} \in \mathbf{i} + C_0^\infty(\mathcal{HB}; \mathbb{R}^3)$ we find that the single term on the left-hand side of (5.5) is greater than or equal to the single term on the right-hand side plus

$$\int_{\mathcal{HB}} p(\mathbf{z}\cdot\mathbf{e}_0)^{-p/3}(\mathbf{e}_0 \otimes \mathbf{e}_0) : \big(\operatorname{adj}\nabla\mathbf{w}(\mathbf{z}) - \mathbf{I}\big)\,d\mathbf{z}.$$

The result will now follow as above upon observing that $(\operatorname{adj}\nabla\mathbf{w})^{\mathrm{T}}\mathbf{e}_0$ only depends on derivatives of $\mathbf{w}$ in directions perpendicular to $\mathbf{e}_0$, while $(\operatorname{adj}\nabla\mathbf{w})\mathbf{e}_0$ is a divergence: without loss of generality assume $\mathbf{e}_0 = (1, 0, 0)$, then

$$\begin{aligned}(\mathbf{e}_0 \otimes \mathbf{e}_0) : \operatorname{adj}\nabla\mathbf{w} &= w^2_{,2}w^3{}_{,3} - w^2{}_{,3}\,w^3{}_{,2}\\ &= (w^2 w^3{}_{,3})_{,2} - (w^2 w^3{}_{,2})_{,3}. \qquad \Box\end{aligned}$$

Proof of Theorem 5.1. We first fix $\lambda > \lambda_{\text{crit}}$ and let (5.3) be a radial minimizer of E given by Remark 3.1. Define $\mathbf{y}_0 := r(0)\mathbf{e}_0$ so that $\mathbf{y}_0$ is a point on the newly formed cavity surface. Let $\mathbf{w} \in \text{Def}_{\mathbf{i}}(\mathcal{HB})$ have finite energy and let $\delta = \delta_{\mathbf{w}}$ be as (5.2). Then, for every sufficiently small $\varepsilon > 0$, the mapping $\mathbf{v}_\varepsilon : B(\mathbf{0}, \lambda R_{\text{o}}) \to \mathbb{R}^3$ defined by

$$\mathbf{v}_\varepsilon(\mathbf{y}) := \begin{cases} \mathbf{y}_0 + \varepsilon \mathbf{w}(\frac{\mathbf{y}-\mathbf{y}_0}{\varepsilon}) & \text{if } \mathbf{y} \in B(\mathbf{y}_0, \varepsilon), \\ \mathbf{y} & \text{otherwise,} \end{cases} \tag{5.7}$$

satisfies $\mathbf{v}_\varepsilon \in \text{Def}(B(\mathbf{0}, \lambda R_{\text{o}}))$ and $\mathbf{v}_\varepsilon = \mathbf{i}$ on $\partial B(\mathbf{0}, \lambda R_{\text{o}})$.

By hypothesis, $\mathbf{u}_r$ is a local minimizer of E: thus if $\mathbf{v}_\varepsilon \circ \mathbf{u}_r$ is sufficiently close to $\mathbf{u}_r$ in $W^{1,p} \cap \mathcal{L}^\infty$ then

$$\int\limits_{B_{\text{o}}} W\big(\nabla_{\mathbf{x}}(\mathbf{v}_\varepsilon \circ \mathbf{u}_r)(\mathbf{x})\big)\, d\mathbf{x} \geqslant \int\limits_{B_{\text{o}}} W\big(\nabla_{\mathbf{x}} \mathbf{u}_r(\mathbf{x})\big)\, d\mathbf{x}. \tag{5.8}$$

However, by Remark 3.1, $\mathbf{u}_r : B_{\text{o}} \backslash \{\mathbf{0}\} \to B(\mathbf{0}, \lambda R_{\text{o}}) \backslash \overline{B(\mathbf{0}, r_\lambda(0))}$ is a diffeomorphism. The inverse map $(\mathbf{u}_r)^{-1}$: $B(\mathbf{0}, \lambda R_{\text{o}}) \backslash \overline{B(\mathbf{0}, r_\lambda(0))} \to B_{\text{o}} \backslash \{\mathbf{0}\}$ satisfies

$$\begin{aligned} (\mathbf{u}_r)^{-1}(\mathbf{y}) &= \frac{R(r)}{r}\mathbf{y}, \quad r := |\mathbf{y}|, \\ \nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y}) &= \frac{R(r)}{r}\mathbf{I} + \left(R'(r) - \frac{R(r)}{r}\right)\frac{\mathbf{y}}{|\mathbf{y}|} \otimes \frac{\mathbf{y}}{|\mathbf{y}|}, \end{aligned}$$

where $r \mapsto R(r)$ is the inverse map to $R \mapsto r(R)$.

The change of variables $\mathbf{x} = (\mathbf{u}_r)^{-1}(\mathbf{y})$ in (5.8) yields

$$\int\limits_{\mathbf{u}_r(B_{\text{o}})} \frac{W(\nabla_{\mathbf{y}}\mathbf{v}_\varepsilon(\mathbf{y})[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})]^{-1})}{(\det[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})])^{-1}}\, d\mathbf{y} \geqslant \int\limits_{\mathbf{u}_r(B_{\text{o}})} \frac{W([\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})]^{-1})}{(\det[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})])^{-1}}\, d\mathbf{y}$$

and hence, in view of $(5.7)_2$,

$$\int\limits_{\mathbf{u}_r(B_{\text{o}}) \cap B(\mathbf{y}_0, \varepsilon)} \frac{W(\nabla_{\mathbf{y}}\mathbf{v}_\varepsilon(\mathbf{y})[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})]^{-1}) - W([\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})]^{-1})}{(\det[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})])^{-1}}\, d\mathbf{y} \geqslant 0, \tag{5.9}$$

where

$$\big[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y})\big]^{-1} = \frac{r}{R(r)}\mathbf{I} + \left(\frac{1}{R'(r)} - \frac{r}{R(r)}\right)\frac{\mathbf{y}}{|\mathbf{y}|} \otimes \frac{\mathbf{y}}{|\mathbf{y}|}. \tag{5.10}$$

The change of variables $\mathbf{y} = \mathbf{y}_0 + \varepsilon \mathbf{z}$, $d\mathbf{y} = \varepsilon^3\, d\mathbf{z}$, and $\nabla_{\mathbf{z}} = \varepsilon \nabla_{\mathbf{y}}$ in (5.9) and (5.10) yields, with the aid of $(5.7)_1$,

$$\int\limits_{\mathcal{HB}_\delta \cap ((\mathbf{u}_r(B_{\text{o}}) - \mathbf{y}_0)/\varepsilon)} \frac{W(\nabla_{\mathbf{z}}\mathbf{w}(\mathbf{z})\mathbf{K}_\varepsilon(\mathbf{z})) - W(\mathbf{K}_\varepsilon(\mathbf{z}))}{\det[\mathbf{K}_\varepsilon(\mathbf{z})]}\, d\mathbf{z} \geqslant 0, \tag{5.11}$$

where

$$\begin{aligned} \mathbf{K}_\varepsilon(\mathbf{z}) &:= \big[\nabla_{\mathbf{y}}(\mathbf{u}_r)^{-1}(\mathbf{y}_0 + \varepsilon\mathbf{z})\big]^{-1} \\ &= \frac{r_\varepsilon(\mathbf{z})}{R(r_\varepsilon(\mathbf{z}))}\mathbf{I} + \left(\frac{1}{R'(r_\varepsilon(\mathbf{z}))} - \frac{r_\varepsilon(\mathbf{z})}{R(r_\varepsilon(\mathbf{z}))}\right)\mathbf{e}_\varepsilon(\mathbf{z}) \otimes \mathbf{e}_\varepsilon(\mathbf{z}), \end{aligned} \tag{5.12}$$

$$r_\varepsilon(\mathbf{z}) := |\mathbf{y}_0 + \varepsilon\mathbf{z}|, \qquad \mathbf{e}_\varepsilon(\mathbf{z}) = \frac{\mathbf{y}_0 + \varepsilon\mathbf{z}}{|\mathbf{y}_0 + \varepsilon\mathbf{z}|}, \tag{5.13}$$

and we have made use of the fact that $\mathbf{w} = \mathbf{i}$ on $B \backslash \mathcal{HB}_\delta$.

We will next make use of (4.1), (4.2), as well the results in Appendix A and (5.12) in order to simplify our computation of the integrands in (5.11). By (5.12) and Lemma A.1

$$\big|\mathbf{G}_{\mathbf{w}}\mathbf{K}_\varepsilon(\mathbf{z})\big| = \left[\left[\frac{r_\varepsilon(\mathbf{z})}{R(r_\varepsilon(\mathbf{z}))}\right]^2 |\mathbf{G}_{\mathbf{w}}|^2 + \left(\frac{1}{[R'(r_\varepsilon(\mathbf{z}))]^2} - \left[\frac{r_\varepsilon(\mathbf{z})}{R(r_\varepsilon(\mathbf{z}))}\right]^2\right)\big|\mathbf{G}_{\mathbf{w}}\mathbf{e}_\varepsilon(\mathbf{z})\big|^2\right]^{1/2} \tag{5.14}$$

where $\mathbf{G}_{\mathbf{w}} := \nabla_{\mathbf{z}}\mathbf{w}(\mathbf{z})$, while (5.12) and Lemma A.3 imply

$$\begin{aligned}&\left|\operatorname{adj}\left(\mathbf{G}_{\mathbf{w}}\mathbf{K}_\varepsilon(\mathbf{z})\right)\right|\\&\quad=\left[\frac{r_\varepsilon(\mathbf{z})}{R(r_\varepsilon(\mathbf{z}))}\right]\left[\frac{1}{[R'(r_\varepsilon(\mathbf{z}))]^2}|\mathbf{A}_{\mathbf{w}}|^2+\left(\left[\frac{r_\varepsilon(\mathbf{z})}{R(r_\varepsilon(\mathbf{z}))}\right]^2-\frac{1}{[R'(r_\varepsilon(\mathbf{z}))]^2}\right)\left|(\mathbf{A}_{\mathbf{w}})^{\mathrm{T}}\mathbf{e}_\varepsilon(\mathbf{z})\right|^2\right]^{1/2}\end{aligned}\tag{5.15}$$

where $\mathbf{A}_{\mathbf{w}} := \operatorname{adj}(\nabla_{\mathbf{z}}\mathbf{w}(\mathbf{z}))$. Also,

$$\det\left(\mathbf{G}_{\mathbf{w}}\mathbf{K}_\varepsilon(\mathbf{z})\right)=(\det\mathbf{G}_{\mathbf{w}})\frac{r_\varepsilon(\mathbf{z})^2}{R'(r_\varepsilon(\mathbf{z}))R(r_\varepsilon(\mathbf{z}))^2}.\tag{5.16}$$

Our next task is to multiply (5.11) by an appropriate function of ε, which has limit zero at $\varepsilon = 0$, and take the limit as ε approaches zero. By (3.10) and (5.16),

$$\begin{aligned}&H\leqslant\det\mathbf{K}_\varepsilon(\mathbf{z})\leqslant\lambda^3,\\&H\det\nabla\mathbf{w}(\mathbf{z})\leqslant\det\mathbf{K}_\varepsilon(\mathbf{z})\det\nabla\mathbf{w}(\mathbf{z})\leqslant\lambda^3\det\nabla\mathbf{w}(\mathbf{z}).\end{aligned}\tag{5.17}$$

Therefore, if we multiply (5.11) by any function of ε, which has limit zero at $\varepsilon = 0$, and let ε approach zero, we conclude from the similar argument in Section 4 (cf. (4.18) and (4.19)) that both of the terms involving h in (5.11) (see (5.4)) will go to zero.

The function of ε that we will multiply (5.11) by is $R(|\mathbf{y}_0| + \varepsilon)^p$. We first proceed pointwise and note that, by (3.11), L'Hôpital's rule, the chain rule, and (3.12)

$$\lim_{\varepsilon\to0^+}\left[\frac{R(|\mathbf{y}_0+\varepsilon\mathbf{z}|)}{R(|\mathbf{y}_0|+\varepsilon)}\right]^3=\lim_{\varepsilon\to0^+}\left[\frac{\phi'(|\mathbf{y}_0+\varepsilon\mathbf{z}|)}{\phi'(|\mathbf{y}_0|+\varepsilon)}\frac{(\mathbf{y}_0+\varepsilon\mathbf{z})\cdot\mathbf{z}}{|\mathbf{y}_0+\varepsilon\mathbf{z}|}\right]=\mathbf{e}_0\cdot\mathbf{z},\tag{5.18}$$

($\mathbf{e}_0 = \mathbf{y}_0/|\mathbf{y}_0|$) while by $(3.9)_1$ and the fact that $R(|\mathbf{y}_0|) = 0$,

$$\lim_{\varepsilon\to0^+}\frac{1}{R'(r_\varepsilon(\mathbf{z}))}=0.\tag{5.19}$$

Thus, in view of (5.13), (5.14), (5.18), and (5.19),

$$\lim_{\varepsilon\to0^+}R\left(|\mathbf{y}_0|+\varepsilon\right)^{\alpha_i}|\mathbf{G}_{\mathbf{w}}\mathbf{K}_\varepsilon|^{\alpha_i}=|\mathbf{y}_0|^{\alpha_i}(\mathbf{z}\cdot\mathbf{e}_0)^{-\alpha_i/3}\left(\left|\nabla\mathbf{w}(\mathbf{z})\right|^2-\left|\left(\nabla\mathbf{w}(\mathbf{z})\right)\mathbf{e}_0\right|^2\right)^{\alpha_i/2}\tag{5.20}$$

and similarly, by (5.13), (5.15), (5.18), and (5.19),

$$\lim_{\varepsilon\to0^+}R\left(|\mathbf{y}_0|+\varepsilon\right)^{2\beta_i}|\operatorname{adj}(\mathbf{G}_{\mathbf{w}}\mathbf{K}_\varepsilon)|^{\beta_i}=|\mathbf{y}_0|^{2\beta_i}(\mathbf{z}\cdot\mathbf{e}_0)^{-2\beta_i/3}\left|\left(\operatorname{adj}\nabla\mathbf{w}(\mathbf{z})\right)^{\mathrm{T}}\mathbf{e}_0\right|^{\beta_i}.\tag{5.21}$$

In order to apply the dominated convergence theorem to take the limit as ϵ goes to zero in (5.11) we require an upper bound on the ratio

$$\frac{R(|\mathbf{y}_0|+\varepsilon)}{R(|\mathbf{y}_0+\varepsilon\mathbf{z}|)}\tag{5.22}$$

by an integrable function of $\mathbf{z}$. Define $\Phi_{\mathrm{n}}(\varepsilon) := R(|\mathbf{y}_0| + \varepsilon)^3 = \phi(|\mathbf{y}_0| + \varepsilon)$ and $\Phi_{\mathrm{d}}(\varepsilon, \mathbf{z}) := R(|\mathbf{y}_0 + \varepsilon\mathbf{z}|)^3 = \phi(|\mathbf{y}_0 + \varepsilon\mathbf{z}|)$ (see (3.11)) the cube of the function in the numerator and denominator, respectively. Then, by the mean value theorem applied to each of the functions $\varepsilon \mapsto \Phi_{\mathrm{n}}(\varepsilon)$ and $\varepsilon \mapsto \Phi_{\mathrm{d}}(\varepsilon, \mathbf{z})$, the chain rule, and the fact that $\Phi_{\mathrm{n}}(0) = \Phi_{\mathrm{d}}(0, \mathbf{z}) = R(|\mathbf{y}_0|)^3 = 0$,

$$0<\frac{R(|\mathbf{y}_0|+\varepsilon)^3}{R(|\mathbf{y}_0+\varepsilon\mathbf{z}|)^3}=\frac{\Phi_{\mathrm{n}}(\varepsilon)}{\Phi_{\mathrm{d}}(\varepsilon,\mathbf{z})}=\frac{\Phi_{\mathrm{n}}'(c^{\mathrm{n}})\varepsilon}{\Phi_{\mathrm{d}}'(c^{\mathrm{d}},\mathbf{z})\varepsilon}=\frac{\phi'(c^{\mathrm{n}})}{\phi'(c^{\mathrm{d}})}\frac{|\mathbf{y}_0+\varepsilon\mathbf{z}|}{(\mathbf{y}_0+\varepsilon\mathbf{z})\cdot\mathbf{z}},$$

where $c^{\mathrm{n}} \in (0, \varepsilon)$ and $c^{\mathrm{d}} = c^{\mathrm{d}}(\mathbf{z}) \in (0, \varepsilon)$. In view of (3.12), ϕ' is bounded away from zero for sufficiently small ε. Thus for such ε we find that the function given in (5.22) is bounded, uniformly in $\mathbf{z}$ on the set $\mathcal{HB}_\delta$.

To complete the argument we multiply (5.11) by $R(|\mathbf{y}_0| + \varepsilon)^p$ and let $\epsilon = \epsilon_n$ be any sequence that converges to zero to conclude with the aid of the sentence following (5.17), (3.8), (5.20), (5.21), the dominated convergence theorem, (4.2), and the fact that $\alpha_i + 2\beta_i = p$ that (5.5) is satisfied.

Finally, the proof that $\mathbf{v}_\varepsilon \circ \mathbf{u}_r \to \mathbf{u}_r$ strongly in $W^{1,p} \cap \mathcal{L}^\infty$ as $\varepsilon \to 0$ is similar to that in the previous section. □

Note added in proof

Theorems 5.1 and 5.2 are valid for a more general class of deformations of the unit ball B: those $\mathbf{w} \in \mathrm{Def}(B)$ that are equal to the identity on ∂B and whose restriction to a neighborhood of $\partial(\mathcal{HB})\backslash\partial B$ is contained in $W^{1,q}$ for some $q > 3p/(3-p)$. To see this, note that the only point in the proof where we originally believed that we needed a stronger hypothesis is in the bound of Eq. (5.22) above, uniformly in ϵ, by an integrable function. However, the existence of such a function for the above $\mathbf{w}$ can be deduced, from (5.22) and the displayed equation that follows it, upon converting the integrals on the left-hand side of (5.5) into spherical coordinates with the flat surface of $\partial(\mathcal{HB})$ taken as the x–y plane.

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Appendix A

Lemma A.1. *Let*

$$\mathbf{F} = \alpha\mathbf{I} + (\beta-\alpha)\mathbf{e}\otimes\mathbf{e},$$

where $\mathbf{e} \in \mathbb{R}^3$ *is a unit vector. Then for any* $\mathbf{M} \in \mathrm{Lin}$

$$|\mathbf{MF}|^2 = \alpha^2|\mathbf{M}|^2 + \left(\beta^2-\alpha^2\right)|\mathbf{Me}|^2, \tag{A.1}$$

$$|\mathbf{FM}|^2 = \alpha^2|\mathbf{M}|^2 + \left(\beta^2-\alpha^2\right)|\mathbf{M}^{\mathrm{T}}\mathbf{e}|^2. \tag{A.2}$$

Proof. We note that

$$\mathbf{MF} = \alpha\mathbf{M} + (\beta-\alpha)\mathbf{Me}\otimes\mathbf{e},$$
$$(\mathbf{MF})^{\mathrm{T}} = \alpha\mathbf{M}^{\mathrm{T}} + (\beta-\alpha)\mathbf{e}\otimes\mathbf{Me}$$

and hence

$$\mathbf{MF}(\mathbf{MF})^{\mathrm{T}} = \alpha^2\mathbf{MM}^{\mathrm{T}} + \left[2\alpha(\beta-\alpha) + (\beta-\alpha)^2\right]\mathbf{Me}\otimes\mathbf{Me}.$$

The desired result (A.1) now follows if one takes the trace. The proof of (A.2) is similar. □

Lemma A.2. *Let*

$$\mathbf{A} = \lambda\mathbf{e}\otimes\mathbf{e} + \mu(\mathbf{I}-\mathbf{e}\otimes\mathbf{e}),$$

where $\mathbf{e} \in \mathbb{R}^3$ *is a unit vector. Then*

$$\mathrm{adj}\,\mathbf{A} = \mu^2\mathbf{e}\otimes\mathbf{e} + \lambda\mu(\mathbf{I}-\mathbf{e}\otimes\mathbf{e}). \tag{A.3}$$

Proof. Since $n = 3$ it follows that the determinant of $\mathbf{A}$ satisfies $\det\mathbf{A} = \lambda\mu^2$. However, if we multiply (A.3) by $\mathbf{A}$ we conclude that the result is equal to

$$\lambda\mu^2\mathbf{e}\otimes\mathbf{e} + \lambda\mu^2(\mathbf{I}-\mathbf{e}\otimes\mathbf{e}) = \lambda\mu^2\mathbf{I} = (\det\mathbf{A})\mathbf{I},$$

which completes the proof since $\mathrm{adj}\,\mathbf{A}$ is the unique linear transformation that satisfies $(\mathrm{adj}\,\mathbf{A})\mathbf{A} = (\det\mathbf{A})\mathbf{I}$ for all invertible $\mathbf{A}$. □

Lemma A.3. *Let*

$$\mathbf{F} = \alpha\mathbf{I} + (\beta - \alpha)\mathbf{e} \otimes \mathbf{e},$$

where $\mathbf{e} \in \mathbb{R}^3$ *is a unit vector. Then for any* $\mathbf{M} \in \mathrm{Lin}$

$$\left|\operatorname{adj}(\mathbf{MF})\right|^2 = \alpha^2\left(\beta^2|\operatorname{adj}\mathbf{M}|^2 + \left(\alpha^2 - \beta^2\right)\left|(\operatorname{adj}\mathbf{M})^{\mathrm{T}}\mathbf{e}\right|^2\right), \tag{A.4}$$

$$\left|\operatorname{adj}(\mathbf{FM})\right|^2 = \alpha^2\left(\beta^2|\operatorname{adj}\mathbf{M}|^2 + \left(\alpha^2 - \beta^2\right)\left|(\operatorname{adj}\mathbf{M})\mathbf{e}\right|^2\right). \tag{A.5}$$

Proof. We first note that $\operatorname{adj}(\mathbf{MF}) = (\operatorname{adj}\mathbf{F})(\operatorname{adj}\mathbf{M})$, while by the previous lemma

$$\operatorname{adj}\mathbf{F} = (\alpha\beta)\mathbf{I} + (\alpha^2 - \alpha\beta)\mathbf{e} \otimes \mathbf{e}.$$

Thus if we combine these equations and apply (A.2) of Lemma A.1 we find that

$$\begin{aligned}\left|\operatorname{adj}(\mathbf{MF})\right|^2 &= \left|(\operatorname{adj}\mathbf{F})(\operatorname{adj}\mathbf{M})\right|^2 \\ &= \left((\alpha\beta)^2|\operatorname{adj}\mathbf{M}|^2 + \left(\alpha^4 - (\alpha\beta)^2\right)\left|(\operatorname{adj}\mathbf{M})^{\mathrm{T}}\mathbf{e}\right|^2\right) \\ &= \alpha^2\left(\beta^2|\operatorname{adj}\mathbf{M}|^2 + \left(\alpha^2 - \beta^2\right)\left|(\operatorname{adj}\mathbf{M})^{\mathrm{T}}\mathbf{e}\right|^2\right).\end{aligned}$$

The proof of (A.5) is similar. □

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