







Ann. I. H. Poincaré - AN 24 (2007) 167-169

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Corrigendum

Corrigendum for the comparison theorems in: "A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations"

[Ann. I. H. Poincaré – AN 23 (5) (2006) 695–711]

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In this note, we shall present the correction of the proofs of the comparison results in the paper [1]. In order to show clearly the correct way of the demonstration, we shall simplify the problem to the following. (Problem (I)):

$$F(x, u, \nabla u, \nabla^2 u) - \int_{\mathbf{R}^{\mathbf{N}}} u(x+z) - u(x) - \mathbf{1}_{|z| \leqslant 1} \langle z, \nabla u(x) \rangle q(\mathrm{d}z) = 0 \quad \text{in } \Omega,$$
 (1)

(Problem (II)):

$$F(x, u, \nabla u, \nabla^2 u) - \int_{\{z \in \mathbf{R}^{\mathbf{N}} \mid x + z \in \overline{\Omega}\}} u(x + z) - u(x) - \mathbf{1}_{|z| \leqslant 1} \langle z, \nabla u(x) \rangle q(\mathrm{d}z) = 0 \quad \text{in } \Omega,$$
 (2)

where $\Omega \subset \mathbf{R}^{\mathbf{N}}$ is open, and $q(\mathrm{d}z)$ is a positive Radon measure such that $\int_{|z| \leq 1} |z|^2 q(\mathrm{d}z) + \int_{|z| > 1} 1q(\mathrm{d}z) < \infty$. Although in [1] only (II) was studied, in order to avoid the non-essential technical complexity, here, let us give the explanation mainly for (I). For (I), we consider the Dirichlet B.C.:

$$u(x) = g(x) \quad \forall x \in \Omega^c, \tag{3}$$

where g is a given continuous function in Ω^c . For (II), we assume that Ω is a precompact convex open subset in $\mathbf{R}^{\mathbf{N}}$ with C^1 boundary satisfying the uniform exterior sphere condition, and consider either the Dirichlet B.C.:

$$u(x) = h(x) \quad \forall x \in \partial \Omega,$$
 (4)

where h is a given continuous function on $\partial \Omega$, or the Neumann B.C.:

$$\langle \nabla u(x), \mathbf{n}(x) \rangle = 0 \quad \forall x \in \partial \Omega,$$
 (5)

where $\mathbf{n}(x) \in \mathbf{R}^{\mathbf{N}}$ the outward unit normal vector field defined on $\partial \Omega$. The above problems are studied in the framework of the viscosity solutions introduced in [1]. Under all the assumptions in [1], for (I) the following comparison result holds, and for (II), although the proofs therein are incomplete, the comparison results stated in [1] hold, and we shall show in a future article.

DOI of original article: 10.1016/j.anihpc.2005.09.002. *E-mail address:* arisawa@math.is.tohoku.ac.jp (M. Arisawa). **Theorem 1.1** (Problem (I) with Dirichlet B.C.). Assume that Ω is bounded, and the conditions for F in [1] hold. Let $u \in USC(\mathbf{R}^{\mathbf{N}})$ and $v \in LSC(\mathbf{R}^{\mathbf{N}})$ be respectively a viscosity subsolution and a supersolution of (1) in Ω , which satisfy $u \leq v$ on Ω^c . Then, $u \leq v$ in Ω .

To prove Theorem 1.1, we approximate the solutions u and v by the supconvolution: $u^r(x) = \sup_{y \in \mathbb{R}^N} \{u(y) - u^r(x)\}$ $\frac{1}{2r^2}|x-y|^2$ and the infconvolution: $v_r(x) = \inf_{y \in \mathbb{R}^N} \{v(y) + \frac{1}{2r^2}|x-y|^2\}$ $(x \in \mathbb{R}^N)$, where r > 0.

Lemma 1.2 (Approximation for Problem (I)). Let u and v be respectively a viscosity subsolution and a supersolution of (1). For any v > 0 there exists r > 0 such that u^r and v_r are respectively a subsolution and a supersolution of the following problems.

$$F(x, u, \nabla u, \nabla^2 u) - \int_{\mathbf{R}^N} u(x+z) - u(x) - \mathbf{1}_{|z| \leqslant 1} \langle z, \nabla u(x) \rangle q(\mathrm{d}z) \leqslant \nu, \tag{6}$$

$$F(x, v, \nabla v, \nabla^2 v) - \int_{\mathbf{R}^{\mathbf{N}}} v(x+z) - v(x) - \mathbf{1}_{|z| \leqslant 1} \langle z, \nabla v(x) \rangle q(dz) \geqslant -v, \tag{7}$$

in $\Omega_r = \{x \in \Omega \mid \operatorname{dist}(x, \partial \Omega) > \sqrt{2Mr}\}$, where $M = \max\{\sup_{\overline{\Omega}} |u|, \sup_{\overline{\Omega}} |v|\}$.

Remark that u^r is semiconvex, v_r is semiconcave, and both are Lipschitz continuous in $\mathbb{R}^{\mathbb{N}}$. We then deduce from the Jensen's maximum principle and the Alexandrov's theorem (deep results in the convex analysis, see [2] and [3]), the following lemma, the last claim of which is quite important in the limit procedure in the nonlocal term.

Lemma 1.3. Let U be semiconvex and V be semiconcave in Ω . For $\phi(x, y) = \alpha |x - y|^2$ ($\alpha > 0$) consider $\Phi(x, y) = \alpha |x - y|^2$ $U(x) - V(y) - \phi(x, y)$, and assume that (\bar{x}, \bar{y}) is an interior maximum of Φ in $\overline{\Omega} \times \overline{\Omega}$. Assume also that there is an open precompact subset O of $\Omega \times \Omega$ containing (\bar{x}, \bar{y}) , and that $\mu = \sup_{\Omega} \Phi(x, y) - \sup_{\partial \Omega} \Phi(x, y) > 0$. Then, the following holds.

(i) There exists a sequence of points $(x_m, y_m) \in O$ $(m \in \mathbb{N})$ such that $\lim_{m \to \infty} (x_m, y_m) = (\bar{x}, \bar{y})$, and $(p_m, X_m) \in O$ $J_Q^{2,+}U(x_m), (p'_m, Y_m) \in J_Q^{2,-}V(y_m)$ such that

$$\lim_{m\to\infty} p_m = \lim_{m\to\infty} p'_m = 2\alpha(x_m - y_m) = p,$$

and $X_m \leq Y_m \ \forall m$.

- (ii) For $P_m = (p_m p, -(p'_m p))$, $\Phi_m(x, y) = \Phi(x, y) \langle P_m, (x, y) \rangle$ takes a maximum at (x_m, y_m) in O. (iii) The following holds for any $z \in \mathbf{R}^{\mathbf{N}}$ such that $(x_m + z, y_m + z) \in O$.

$$U(x_m + z) - U(x_m) - \langle p_m, z \rangle \leqslant V(y_m + z) - V(y_m) - \langle p_m', z \rangle. \tag{8}$$

By admitting these lemmas here, let us show how Theorem 1.1 is proved.

Proof of Theorem 1.1. We use the argument by contradiction, and assume that $\max_{\overline{Q}}(u-v) = (u-v)(x_0) = M_0 > 0$ for $x_0 \in \Omega$. Then, we approximate u by u^r (supconvolution) and v by v_r (infconvolution), which are a subsolution and a supersolution of (6) and (7), respectively. Clearly, $\max_{\overline{\Omega}}(u^r - v_r) \ge M_0 > 0$. Let $\overline{x} \in \Omega$ be the maximizer of $u^r - v_r$. In the following, we abbreviate the index and write $u = u^r$, $v = v_r$ without any confusion. As in the PDE theory, consider $\Phi(x, y) = u(x) - v(y) - \alpha |x - y|^2$, and let (\hat{x}, \hat{y}) be the maximizer of Φ . Then, from Lemma 1.3 there exists $(x_m, y_m) \in \Omega$ $(m \in \mathbb{N})$ such that $\lim_{m \to \infty} (x_m, y_m) = (\hat{x}, \hat{y})$, and we can take $(\varepsilon_m, \delta_m)$ a pair of positive

$$u(x_m+z) \leq u(x_m) + \langle p_m, z \rangle + \frac{1}{2} \langle X_m z, z \rangle + \delta_m |z|^2, \qquad v(y_m+z) \geq v(y_m) + \langle p_m', z \rangle + \frac{1}{2} \langle Y_m z, z \rangle - \delta_m |z|^2,$$

for $\forall |z| \leq \varepsilon_m$. From the definition of the viscosity solutions, we have

$$F(x_{m}, u(x_{m}), p_{m}, X_{m}) - \int_{|z| \leq \varepsilon_{m}} \frac{1}{2} \langle (X_{m} + 2\delta_{m}I)z, z \rangle dq(z)$$

$$- \int_{|z| \geq \varepsilon_{m}} u(x_{m} + z) - u(x_{m}) - \mathbf{1}_{|z| \leq 1} \langle z, p_{m} \rangle q(dz) \leq \nu,$$

$$F(y_{m}, v(y_{m}), p'_{m}, Y_{m}) - \int_{|z| \leq \varepsilon_{m}} \frac{1}{2} \langle (Y_{m} - 2\delta_{m}I)z, z \rangle dq(z)$$

$$- \int_{|z| \geq \varepsilon_{m}} v(y_{m} + z) - v(y_{m}) - \mathbf{1}_{|z| \leq 1} \langle z, p'_{m} \rangle q(dz) \geq -\nu.$$

By taking the difference of the above two inequalities, by using (8), and by passing $m \to \infty$ (thanking to (8), it is now available), we can obtain the desired contradiction. The claim $u \le v$ is proved. \Box

Remark 1.1. To prove the comparison results for (II) (in [1]), we do the approximation by the supconvolution: $u^r(x) = \sup_{y \in \overline{\Omega}} \{u(y) - \frac{1}{2r^2}|x - y|^2\}$, and the infconvolution: $v_r(x) = \inf_{y \in \overline{\Omega}} \{v(y) + \frac{1}{2r^2}|x - y|^2\}$ as in Lemma 1.2. Because of the restriction of the domain of the integral of the nonlocal term and the Neumann B.C., a slight technical complexity is added. The approximating problem for (2)–(5) in $\overline{\Omega}$ is as follows.

$$\begin{split} & \min \bigg[F \big(x, u(x), \nabla u(x), \nabla^2 u(x) \big) + \min_{y \in \overline{\Omega}, \, |x-y| \leqslant \sqrt{2M}r} \bigg\{ - \int_{\{z \in \mathbf{R^N} \, | \, y+z \in \overline{\Omega}\}} u(x+z) - u(x) \\ & - \mathbf{1}_{|z| \leqslant 1} \big\langle z, \nabla u(x) \big\rangle q(\mathrm{d}z), \, \min_{y \in \partial \Omega, \, |x-y| \leqslant \sqrt{2M}r} \Big\{ \big\langle \mathbf{n}(y), \nabla u(x) \big\rangle + \rho \Big\} \bigg\} \bigg] \leqslant \nu, \\ & \max \bigg[F \big(x, v(x), \nabla v(x), \nabla^2 v(x) \big) + \max_{y \in \overline{\Omega}, \, |x-y| \leqslant \sqrt{2M}r} \bigg\{ - \int_{\{z \in \mathbf{R^N} \, | \, y+z \in \overline{\Omega}\}} v(x+z) - v(x) \\ & - \mathbf{1}_{|z| \leqslant 1} \big\langle z, \nabla v(x) \big\rangle q(\mathrm{d}z), \, \max_{y \in \partial \Omega, \, |x-y| \leqslant \sqrt{2M}r} \Big\{ \big\langle \mathbf{n}(y), \nabla v(x) \big\rangle - \rho \Big\} \bigg\} \bigg] \geqslant -\nu. \end{split}$$

We deduce the comparison result from this approximation and Lemma 1.3, by using the similar argument as in the proof of Theorem 1.1.

References

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