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On a Liouville phenomenon for entire weak supersolutions of elliptic partial differential equations

Autour d'un phénomène de Liouville pour les sursolutions entières faibles d'équations aux derivées partielles elliptiques

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In fond memory of Professor Heinz Bauer

Abstract

We study a new Liouville-type phenomenon for entire weak supersolutions of elliptic partial differential equations of the form $A(u) = 0$ on $\mathbb{R}^n$, $n \geqslant 2$. Typical examples of the operator $A(u)$ are the p-Laplacian for $p > 1$, the mean curvature operator, and their well-known modifications.

Résumé

Ce travail est consacré à l'étude d'un nouveau phénomène de type de Liouville pour les sursolutions entières faibles d'équations aux derivées partielles elliptiques de la forme $A(u) = 0$ sur $\mathbb{R}^n$, $n \geqslant 2$. Des exemples typiques de l'opérateur $A(u)$ sont le p-laplacien pour $p > 1$, l'opérateur de courbure moyenne, et leurs modifications bien connues.

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1. Introduction

Liouville's well-known theorem says that any superharmonic function on $\mathbb{R}^2$ bounded below by a constant is itself a constant. On the other hand it is also well known that for $n \geqslant 3$ there exist non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant. The purpose of this work is to determine for $n \geqslant 3$ the 'sharp distance at infinity' between the non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant and this constant itself in

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the form of a theorem of Liouville type and to characterize basic properties of quasilinear elliptic partial differential operators which make it possible to obtain such a theorem for supersolutions of quasilinear elliptic partial differential equations of the form

$$A(u)=0 \tag{1}$$

on $\mathbb{R}^n$, $n\geqslant 2$. Typical examples of the operator $A(u)$ are the p-Laplacian

$$\Delta_p(u):=\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr),\quad p>1, \tag{2}$$

its well-known modification (see, e.g., [8, p. 155])

$$\tilde{\Delta}_p(u):=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right),\quad p>1, \tag{3}$$

the mean curvature operator

$$\Xi(u):=\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{4}$$

and its well-known modifications.

Note that a Liouville theorem for solutions of linear uniformly elliptic second-order partial differential equations on $\mathbb{R}^n$, $n>2$, was first obtained, as a direct consequence of a Harnack inequality, in [1] under some continuity assumptions on the coefficients of the equations and in [12] without continuity assumptions on the coefficients of the equations. In the case of quasilinear uniformly elliptic second-order partial differential equations on $\mathbb{R}^n$, $n\geqslant 2$, a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [14]. Note also that a Liouville theorem for mappings of $\mathbb{R}^n$, $n>2$, with bounded distortion was first obtained in [13] by using the same Harnack inequality from [14]. Finally, in the case of linear uniformly elliptic second-order partial differential equations on $\mathbb{R}^2$, a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [7].

2. Definitions

Let $A(u)$ be a differential operator defined formally by

$$A(u)=\sum_{i=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x_i}A_i(x,u,\nabla u). \tag{5}$$

Here and in what follows, $n\geqslant 2$. We assume that the functions $A_i(x,\eta,\xi)$, $i=1,\dots,n$, satisfy the usual Carathéodory conditions on $\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n$; namely, they are continuous in η and ξ for almost all $x\in\mathbb{R}^n$ and measurable in x for any $\eta\in\mathbb{R}^1$ and $\xi\in\mathbb{R}^n$.

Definition 1. Let $\alpha>1$ be a given number. The operator $A(u)$ given by (5) belongs to the class $\mathcal{A}(\alpha)$ if for all $\eta\in\mathbb{R}^1$, all $\xi,\psi\in\mathbb{R}^n$, and almost all $x\in\mathbb{R}^n$ the following two inequalities hold:

$$0\leqslant\sum_{i=1}^{n}\xi_iA_i(x,\eta,\xi), \tag{6}$$

with equality only if $\xi=0$, and

$$\left|\sum_{i=1}^{n}\psi_iA_i(x,\eta,\xi)\right|^{\alpha}\leqslant\mathcal{K}|\psi|^{\alpha}\left(\sum_{i=1}^{n}\xi_iA_i(x,\eta,\xi)\right)^{\alpha-1}, \tag{7}$$

with $\mathcal{K}$ a certain positive constant.

It is easy to see that condition (7) is fulfilled whenever the inequality

$$\left(\sum_{i=1}^{n}A_i^2(x,\eta,\xi)\right)^{\alpha/2}\leqslant\mathcal{K}\left(\sum_{i=1}^{n}\xi_iA_i(x,\eta,\xi)\right)^{\alpha-1} \tag{8}$$

holds for all $\eta \in \mathbb{R}^1$, all $\xi, \psi \in \mathbb{R}^n$, and almost all $x \in \mathbb{R}^n$. Hence, the operator $A(u)$ given by (5) and satisfying conditions (6) and (8) belongs to the class $\mathcal{A}(\alpha)$.

Remark 1. Conditions (7) and (8) on the behavior of the coefficients of partial differential operators were introduced in [10].

It is not difficult to verify that for any given $p > 1$ the differential operators (2) and (3) as well as the differential operator $A(u)$ given by (5) and satisfying the well-known growth conditions

$$\left(\sum_{i=1}^{n} A_i^2(x, \eta, \xi)\right)^{1/2} \leqslant \mathcal{K}_1 |\xi|^{p-1} \tag{9}$$

and

$$|\xi|^p \leqslant \mathcal{K}_2 \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi), \tag{10}$$

with $\mathcal{K}_1, \mathcal{K}_2$ positive constants, belong to the class $\mathcal{A}(\alpha)$ with $\alpha = p$.

It is also easy to see that linear divergent elliptic partial differential operators of the form

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) \tag{11}$$

with $a_{ij}(x)$ measurable bounded coefficients and with the (possibly non-uniformly) positive-definite quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \tag{12}$$

belong to the class $\mathcal{A}(\alpha)$ with $\alpha = 2$ but do not satisfy condition (10) for any fixed $p > 1$.

In connection with this we give another example of an operator that belongs to the class $\mathcal{A}(\alpha)$ with a certain $\alpha > 1$ but does not satisfy condition (10). Let $a(x, \eta, \xi)$ be a positive bounded function that satisfies the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. It is easy to see that for a given $p > 1$ the weighted p-Laplacian

$$\bar{\Delta}_p(u) := \operatorname{div}\big(a(x, u, \nabla u)|\nabla u|^{p-2}\nabla u\big) \tag{13}$$

belongs to the class $\mathcal{A}(\alpha)$ with $\alpha = p$ but does not satisfy condition (10) for any fixed $p > 1$ if the function $a(x, \eta, \xi)$ is only assumed to be positive.

It can happen that an operator $A(u)$ given by (5) belongs simultaneously to several different classes $\mathcal{A}(\alpha)$. For example, the mean curvature operator $\varXi(u)$ given by (4) belongs to the classes $\mathcal{A}(\alpha)$ for all $1 < \alpha \leqslant 2$; similarly its modification for $p \geqslant 2$,

$$\varXi_p(u) := \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{\sqrt{1 + |\nabla u|^2}}\right), \tag{14}$$

belongs to the classes $\mathcal{A}(\alpha)$ for all $\alpha \in (p-1, p]$ and $p \geqslant 2$. Obviously, operators given by (4) and (14) do not satisfy conditions (9)–(10) for any fixed $p \geqslant 1$.

Definition 2. Let $\alpha > 1$ be a given number, and let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$. A measurable function $u : \mathbb{R}^n \to \mathbb{R}^1$ is called an entire weak supersolution of Eq. (1) on $\mathbb{R}^n$ if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $|\nabla u| \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, and the integral inequality

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i} A_i(x, u, \nabla u)\, \mathrm{d}x \geqslant 0 \tag{15}$$

holds for every non-negative function $\varphi \in W^{1,\alpha}(\mathbb{R}^n)$ with compact support.

3. Results

Theorem 1. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $\alpha \geqslant n$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant. Then $u(x)$ is a constant on $\mathbb{R}^n$.*

Theorem 2. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant c and such that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Then either $u(x) = c$ on $\mathbb{R}^n$ or the relation*

$$\liminf_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-c\big)\Big]r^{\frac{n-\alpha}{\alpha-1-\nu}} = +\infty \tag{16}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Theorem 3. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant c. Then either $u(x) = c$ on $\mathbb{R}^n$ or the relation*

$$\liminf_{r\to+\infty} r^{-\alpha}\int\limits_{r\leqslant|x|\leqslant 2r}\big(u(x)-c\big)^{\alpha-1-\nu}\,\mathrm{d}x = +\infty \tag{17}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Due to the arbitrariness of the constant c in Theorems 2 and 3, the statements of these theorems can be reformulated in a slightly different form.

Theorem 2′. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant and such that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Then either $u(x)$ is a constant on $\mathbb{R}^n$ or relation* (16) *holds with any fixed real number c such that $u(x) \geqslant c$ on $\mathbb{R}^n$ and any fixed $\nu \in (0, \alpha - 1)$.*

Theorem 3′. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant. Then either $u(x)$ is a constant on $\mathbb{R}^n$ or relation* (17) *holds with any fixed real number c such that $u(x) \geqslant c$ on $\mathbb{R}^n$ and any fixed $\nu \in (0, \alpha - 1)$.*

Remark 2. It is important to note that for any given $n \geqslant 2$ and $\alpha > 1$ such that $n > \alpha$ the function

$$u(x) = \big(1 + |x|^{\frac{\alpha}{\alpha-1}}\big)^{\frac{\alpha-n}{\alpha}} \tag{18}$$

is an entire weak supersolution of the equation

$$\Delta_p(u) = 0 \tag{19}$$

with $p = \alpha$ that is bounded below and is such that relations (16) and (17) hold with any fixed $\nu \in (0, \alpha - 1)$ and, at the same time, the relations

$$\lim_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-0\big)\Big]r^{\frac{n-\alpha}{\alpha-1}} = C_1 \tag{20}$$

and

$$\lim_{r\to+\infty} r^{-\alpha}\int\limits_{r\leqslant|x|\leqslant 2r}\big(u(x)-0\big)^{\alpha-1}\,\mathrm{d}x = C_2, \tag{21}$$

with C_1, C_2 certain positive constants, also hold.

Remark 3. The results of this work were announced in [5]. To prove these results we further develop an approach that was proposed for solving similar problems in [6].

Remark 4. The results of Theorem 1 are new only for $\alpha = n$. Similar results to those of Theorem 1 for entire weak continuous supersolutions of (1) on $\mathbb{R}^n$ for $\alpha = n$ were first obtained in [11]. For $\alpha > n$, the results of Theorem 1 for entire weak supersolutions of (1) on $\mathbb{R}^n$, which in this case are continuous on $\mathbb{R}^n$ by the well-known Sobolev imbedding theory, were also first obtained in [11]. Here, we give a new proof of these results from [11] by developing an approach from [6] which does not explicitly use the continuity of entire weak supersolutions of (1) on $\mathbb{R}^n$.

Remark 5. In the case when $\alpha = p$ and $A(u) = \Delta_p(u)$, Theorem 1 coincides with well-known Liouville-type theorems for entire superharmonic and p-superharmonic functions locally bounded on $\mathbb{R}^n$ (see, e.g., [2, p. 68] and [3, p. 179]). Also, in this case, the results of Theorems 2 and 3 correlate well with certain results in the theory of entire superharmonic and p-superharmonic functions (see, e.g., [2, pp. 131, 139] and [3, pp. 133, 135]).

4. Proofs

Proof of Theorem 2. The statement of Theorem 2 follows immediately from Theorem 3. In fact, let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant c, i.e., $u(x) \geqslant c$ on $\mathbb{R}^n$, and such that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, by Theorem 3, either $u(x) = c$ on $\mathbb{R}^n$ or relation (17) holds with any fixed $\nu \in (0, \alpha - 1)$. Further, via the trivial inequality

$$r^{-\alpha} \int\limits_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)^{\alpha-1-\nu} \,\mathrm{d}x \leqslant r^{-\alpha}\Bigl[\sup_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)^{\alpha-1-\nu}\Bigr] \int\limits_{r\leqslant|x|\leqslant 2r} \mathrm{d}x, \tag{22}$$

which obviously holds for any $r > 0$, it follows from (17) that

$$\liminf_{r\to+\infty} \Bigl[\sup_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)^{\alpha-1-\nu}\Bigr] r^{n-\alpha} = +\infty. \tag{23}$$

Then, since

$$\sup_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)^{\alpha-1-\nu} \leqslant \Bigl[\sup_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)\Bigr]^{\alpha-1-\nu} \tag{24}$$

and

$$\Bigl[\sup_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)\Bigr]^{\alpha-1-\nu} r^{n-\alpha} = \Bigl(\Bigl[\sup_{r\leqslant|x|\leqslant 2r} \bigl(u(x) - c\bigr)\Bigr] r^{\frac{n-\alpha}{\alpha-1-\nu}}\Bigr)^{\alpha-1-\nu}, \tag{25}$$

the validity of (16) follows immediately from that of (23) and (25). □

In what follows, a 'smooth' function is a C^∞-function on $\mathbb{R}^n$, $B(r)$ is an open ball on $\mathbb{R}^n$ of radius $r > 0$ centered at the origin, and $\overline{B(r)}$ is the closure of $B(r)$.

Proof of Theorem 3. Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant c, i.e., $u(x) \geqslant c$ on $\mathbb{R}^n$. Let r and ε be positive numbers, and let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth function which equals 1 on $\overline{B(r)}$ and 0 outside $B(2r)$. Substituting, without loss of generality, $\varphi(x) = (u(x) - c + \varepsilon)^{-\nu}\zeta^\alpha(x)$ as a test function in inequality (15), where $\nu \in (0, \alpha - 1)$ is arbitrary, and integrating by parts, we find

$$\begin{aligned} &\alpha \int\limits_{B(2r)\setminus B(r)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu}\zeta^{\alpha-1} \,\mathrm{d}x \\ &\quad\geqslant \nu \int\limits_{B(2r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu-1}\zeta^{\alpha} \,\mathrm{d}x. \end{aligned} \tag{26}$$

Estimating the left-hand side of (26) by using condition (7) on the coefficients of the operator $A(u)$, we have

$$\alpha\mathcal{K}^{1/\alpha}\int\limits_{B(2r)\setminus B(r)}\left(\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha}|\nabla\zeta|(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x$$
$$\geqslant\left|\alpha\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x\right|. \tag{27}$$

Further, estimating the left-hand side of (27) by Hölder's inequality, we arrive at

$$\alpha\mathcal{K}^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\right)^{1/\alpha}$$
$$\times\left(\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x\right)^{(\alpha-1)/\alpha}$$
$$\geqslant\left|\alpha\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x\right|. \tag{28}$$

In turn, (26) and (28) imply the inequality

$$\alpha\mathcal{K}^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\right)^{1/\alpha}$$
$$\times\left(\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x\right)^{(\alpha-1)/\alpha}$$
$$\geqslant\nu\int\limits_{B(2r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x \tag{29}$$

and, therefore, the inequality

$$\alpha^{\alpha}\mathcal{K}\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\geqslant\nu^{\alpha}\int\limits_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x. \tag{30}$$

It is easy to see that the right-hand side of (30) increases monotonically if $\varepsilon>0$ decreases strongly monotonically to zero. Therefore, it follows from (30) that the inequality

$$\alpha^{\alpha}\mathcal{K}\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{-\nu+\alpha-1}\,\mathrm{d}x\geqslant\nu^{\alpha}\int\limits_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\delta)^{-\nu-1}\,\mathrm{d}x \tag{31}$$

holds with any $\delta>0$ and any $\varepsilon\in(0,\delta]$. Since for any sequence $\varepsilon_k>0$ monotonically decreasing to zero as $k\to+\infty$ the sequence of functions

$$\Phi_k(x):=|\nabla\zeta|^{\alpha}(u-c+\varepsilon_k)^{\alpha-1-\nu} \tag{32}$$

measurable on $\mathbb{R}^n$ converges a.e. on $\mathbb{R}^n$ to the function

$$\Phi(x):=|\nabla\zeta|^{\alpha}(u-c)^{\alpha-1-\nu} \tag{33}$$

measurable on $\mathbb{R}^n$, since for sufficiently large k

$$\left|\Phi_k(x)\right|\leqslant|\nabla\zeta|^{\alpha}(u-c+1)^{\alpha-1-\nu} \tag{34}$$

on $\mathbb{R}^n$, and since the function

$$|\nabla\zeta|^\alpha (u-c+1)^{\alpha-1-\nu} \tag{35}$$

is locally integrable on $\mathbb{R}^n$, then, by Lebesgue's theorem (see, e.g., [4, p. 303]), for $\varepsilon=\varepsilon_k>0$ monotonically decreasing to zero we can pass to the limit as $k\to+\infty$ on the left-hand side of (31). As a result, we obtain the inequality

$$\alpha^\alpha \mathcal{K} \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x \geqslant \nu^\alpha \int\limits_{B(r)} \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\delta)^{-\nu-1}\,\mathrm{d}x, \tag{36}$$

which holds with any $\delta>0$. Then, for any $r>0$ and any sequence $\varepsilon_k>0$ monotonically decreasing to zero as $k\to+\infty$, it follows from (36), by letting $\delta=\varepsilon_k$ and

$$\Psi_k(x) := \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon_k)^{-\nu-1}, \tag{37}$$

that the sequence of integrals

$$\int\limits_{B(r)} \Psi_k(x)\,\mathrm{d}x \tag{38}$$

is bounded above by the positive constant

$$c_1 = \mathcal{K}\left(\frac{\alpha}{\nu}\right)^\alpha \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x, \tag{39}$$

which does not depend on ε_k. Hence, since

$$\Psi_1(x)\leqslant\Psi_2(x)\leqslant\cdots\leqslant\Psi_k(x)\leqslant\cdots \tag{40}$$

on $\mathbb{R}^n$, then by Beppo Levi's theorem (see, e.g., [4, p. 305]), for any $r>0$ there exists a function $\Theta_r : B(r)\to\mathbb{R}^1$ integrable on $B(r)$ and such that the sequence of functions $\Psi_k(x)$ converges a.e. to $\Theta_r(x)$ on $B(r)$ and

$$\lim_{k\to+\infty} \int\limits_{B(r)} \Psi_k(x)\,\mathrm{d}x = \int\limits_{B(r)} \Theta_r(x)\,\mathrm{d}x. \tag{41}$$

Further, it is easy to see that the family of functions $\{\Theta_r\}_{r>0}$ uniquely determines a function $\Psi:\mathbb{R}^n\to\mathbb{R}^1$ which is non-negative, measurable, locally integrable on $\mathbb{R}^n$ and is such that $\Psi(x)=\Theta_r(x)$ on $B(r)$ for all $r>0$. Therefore, the sequence of functions $\Psi_k(x)$ given by (37) converges a.e. to $\Psi(x)$ on $\mathbb{R}^n$ for any sequence $\varepsilon_k>0$ monotonically decreasing to zero as $k\to+\infty$. Then, by choosing $\delta=\varepsilon_k$ in (36), where the sequence $\varepsilon_k>0$ converges monotonically to zero as $k\to+\infty$, and passing to the limit on the right-hand side of (36), we find, due to (41), the inequality

$$\alpha^\alpha \mathcal{K} \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x \geqslant \nu^\alpha \int\limits_{B(r)} \Psi(x)\,\mathrm{d}x. \tag{42}$$

We divide the rest of the proof into three cases according to the behavior of the right-hand side of (42), which can monotonically approach zero, $+\infty$, or some positive number I as r strongly monotonically approaches $+\infty$.

If the right-hand side of (42) approaches zero as $r\to+\infty$, then, due to the non-negativity of the function $\Psi(x)$, we have that $\Psi(x)=0$ on $\mathbb{R}^n$. Further, since by (37) and (40) the inequality

$$\sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon_k)^{-\nu-1} \leqslant \Psi(x) \tag{43}$$

holds on $\mathbb{R}^n$ for any sequence $\varepsilon_k>0$ monotonically decreasing to zero as $k\to+\infty$, then, again, due to the non-negativity of the left-hand side of (43), we obtain that

$$\sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon_k)^{-\nu-1} = 0 \tag{44}$$

on $\mathbb{R}^n$. Hence, by condition (6) on the coefficients of the operator $A(u)$, the supersolution $u(x) = \text{const.}$ on $\mathbb{R}^n$, and, therefore, either $u(x) = c$ on $\mathbb{R}^n$ or relation (17) holds with any fixed $\nu \in (0, \alpha - 1)$.

If the right-hand side of (42) approaches $+\infty$ as $r \to +\infty$, then, due to monotonicity, (42) yields that

$$\liminf_{r \to +\infty} \int\limits_{B(2r)\setminus B(r)} |\nabla \zeta|^\alpha (u - c)^{\alpha - 1 - \nu} \,\mathrm{d}x = +\infty. \tag{45}$$

Finally, if the right-hand side of (42) monotonically approaches a certain positive number I as r approaches $+\infty$, i.e.,

$$\lim_{r \to +\infty} \nu^\alpha \int\limits_{B(r)} \Psi(x) \,\mathrm{d}x = I > 0, \tag{46}$$

we again consider inequality (29), just noting here that, due to monotonicity,

$$\int\limits_{B(2r_k)\setminus B(r_k)} \Psi(x) \,\mathrm{d}x \to 0 \tag{47}$$

for any sequence $r_k > 0$ such that $r_k \to +\infty$. First, we have from (29) the inequality

$$\begin{aligned} &\alpha \mathcal{K}^{1/\alpha} \Biggl(\int\limits_{B(2r)\setminus B(r)} |\nabla \zeta|^\alpha (u - c + \varepsilon)^{\alpha - 1 - \nu} \,\mathrm{d}x \Biggr)^{1/\alpha} \\ &\quad \times \Biggl(\int\limits_{B(2r)\setminus B(r)} \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu - 1} \,\mathrm{d}x \Biggr)^{(\alpha - 1)/\alpha} \\ &\quad \geqslant \nu \int\limits_{B(r)} \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu - 1} \,\mathrm{d}x. \end{aligned} \tag{48}$$

In (48), let $\varepsilon = \varepsilon_k > 0$ converge monotonically to zero as $k \to +\infty$. Then, by Lebesgue's theorem (see, e.g., [4, p. 303]), we can pass to the limit on both sides of (48). Namely, we know from the above that for any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$ the sequences of functions $\Phi_k(x)$ and $\Psi_k(x)$ measurable and locally integrable on $\mathbb{R}^n$ and given, respectively, by (32) and (37), converge a.e. on $\mathbb{R}^n$, respectively, to the functions $\Phi(x)$ and $\Psi(x)$ measurable and locally integrable on $\mathbb{R}^n$. Further, arguing as above and letting $\varepsilon = \varepsilon_k > 0$ monotonically decrease to zero as $k \to +\infty$, by Lebesgue's theorem (see, e.g., [4, p. 303]) we can pass to the limit on both sides of (48). As a result, we arrive at the inequality

$$\alpha \mathcal{K}^{1/\alpha} \Biggl(\int\limits_{B(2r)\setminus B(r)} |\nabla \zeta|^\alpha (u - c)^{\alpha - 1 - \nu} \,\mathrm{d}x \Biggr)^{1/\alpha} \Biggl(\int\limits_{B(2r)\setminus B(r)} \Psi(x) \,\mathrm{d}x \Biggr)^{(\alpha - 1)/\alpha} \geqslant \nu \int\limits_{B(r)} \Psi(x) \,\mathrm{d}x. \tag{49}$$

In (49), for $r = r_k > 0$ monotonically increasing to $+\infty$, by passing to the limit as $r_k \to +\infty$, we obtain from (46), (47), and (49) that

$$\lim_{r_k \to +\infty} \int\limits_{B(2r_k)\setminus B(r_k)} |\nabla \zeta|^\alpha (u - c)^{\alpha - 1 - \nu} \,\mathrm{d}x = +\infty. \tag{50}$$

Thus, due to the arbitrariness in the choice of the sequence r_k in (50), we again arrive at relation (45).

Now, without loss of generality, we choose in (45) the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/(2r))$, where $\psi : [0, +\infty) \to [0, 1]$ is a smooth function that equals 1 on $[0, 1/2]$ and 0 on $[1, +\infty)$ and is such that the inequality

$$|\nabla \zeta| \leqslant c_2 r^{-1} \tag{51}$$

holds on $\mathbb{R}^n$ with a certain positive constant c_2 for an arbitrary $r > 0$. Relation (17) then follows immediately from (45) and (51). □

Proof of Theorem 1. Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $\alpha \geqslant n$. Let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant c, i.e., $u(x) \geqslant c$ on $\mathbb{R}^n$. Let r, R, and ε be positive numbers such that $R > r$, and let $\zeta : \mathbb{R}^n \to [0,1]$ be a smooth function which equals 1 on $\overline{B(r)}$ and 0 outside $B(R)$. Substituting, without loss of generality, $\varphi(x) = (u(x) - c + \varepsilon)^{-\nu}\zeta^{\alpha}(x)$ as a test function in inequality (15), where $\nu > \alpha - 1$ is an arbitrary positive number, and integrating by parts, we have the inequality

$$\alpha \int_{B(R)\setminus B(r)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x \geqslant \nu \int_{B(R)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x. \tag{52}$$

Further, we repeat the proof of Theorem 3 word for word from (26) to (30). As a result, we arrive at the inequality

$$\alpha^{\alpha}\mathcal{K} \int_{B(R)\setminus B(r)} |\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x \geqslant \nu^{\alpha} \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x. \tag{53}$$

It follows immediately from (53) that the inequality

$$\alpha^{\alpha}\varepsilon^{\alpha-1-\nu}\mathcal{K} \int_{B(R)\setminus B(r)} |\nabla\zeta|^{\alpha}\,\mathrm{d}x \geqslant \nu^{\alpha} \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x \tag{54}$$

holds with any fixed $\varepsilon > 0$ and $\nu > \alpha - 1$.

Now, first let $\alpha > n$. In (54), choosing $R = 2r$ and the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0,+\infty) \to [0,1]$ is a smooth function that equals 1 on $[0,1/2]$ and 0 on $[1,+\infty)$ and is such that the inequality (51) holds on $\mathbb{R}^n$ with a certain positive constant c_2 for an arbitrary $R > 0$, we obtain from (51) and (54) the inequality

$$c_3 r^{n-\alpha} \geqslant \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x, \tag{55}$$

which holds with a certain positive constant c_3 that does not depend on r. Passing to the limit as $r \to +\infty$ in (55), we find, due to the non-negativity of the integrand, that the equality

$$\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1} = 0 \tag{56}$$

holds on $\mathbb{R}^n$, and, therefore, by condition (6) on the coefficients of the operator $A(u)$, that $u(x) = \text{const.}$ on $\mathbb{R}^n$.

If $\alpha = n$, we choose in (54) the function $\zeta(x)$ in the form $\zeta(x) = \psi(\frac{\ln(|x|/r)}{\ln(R/r)})$ with arbitrary $R > r > 1$, where $\psi : [-\infty,+\infty) \to [0,1]$ is a smooth function which equals 1 on $[-\infty,0]$ and 0 on $[1,+\infty)$. It is not difficult to understand (see, e.g., [9, p. 12]) that the inequality

$$\left|\nabla\zeta(x)\right| \leqslant \frac{c_4}{|x|\ln(R/r)} \tag{57}$$

holds on $\mathbb{R}^n$ with a certain positive constant c_4 for arbitrary $R > r > 1$. It then follows from (54) and (57) that the inequality

$$c_5 \int_{B(R)\setminus B(r)} \left(|x|\ln(R/r)\right)^{-n}\,\mathrm{d}x \geqslant \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x \tag{58}$$

holds, and, therefore, so does the inequality

$$c_6\left(\ln(R/r)\right)^{-n+1} \geqslant \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x \tag{59}$$

with arbitrary $R > r > 1$ and certain positive constants c_5 and c_6 that do not depend on R. Passing to the limit as $R \to +\infty$ in (59), we find that the equality

$$\int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu-1}\, \mathrm{d}x = 0 \tag{60}$$

holds with an arbitrary $r > 1$. Passing to the limit as $r \to +\infty$ in (60), we again obtain, due to the non-negativity of the integrand in (60) and by condition (6), that $u(x) = \text{const.}$ on $\mathbb{R}^n$. $\square$

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