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Ground states of nonlinear Schrödinger equations with potentials

Solutions d'énergie minimale des équations de Schrödinger non-linéaires avec potentiel

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Abstract

In this paper we study the nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We give general conditions which assure the existence of *ground state solutions*. Under a Nehari type condition, we show that the standard Ambrosetti–Rabinowitz super-linear condition can be replaced by a more natural super-quadratic condition.

Résumé

Dans cet article nous étudions l'équation non-linéaire de Schrödinger :

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Nous donnons les conditions générales qui garantissent l'existence de solutions d'énergie minimale. Sous une condition de type Nehari, nous démontrons que la condition super-linéaire d'Ambrosetti–Rabinowitz peut être remplacée par une condition super-quadratique plus naturelle.

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Keywords: Nonlinear Schrödinger equations; Ground state solutions; The Ambrosetti–Rabinowitz condition

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1. Introduction

We study the nonlinear Schrödinger equation with potentials:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

We are concerned with the existence of *ground state solutions*, i.e., solutions corresponding to the least positive critical value of the variational functional:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x,$$

where $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$.

To establish the existence of ground states, usually besides the growth condition on the nonlinearity and a Nehari type condition, the following superlinear condition due to Ambrosetti–Rabinowitz (e.g., [2,12]) is assumed:

(AR) There is $\mu > 2$ such that for $u \neq 0$ and $x \in \mathbb{R}^N$,

$$0 < \mu F(x,u) \leqslant u f(x,u),$$

where $F(x,u) = \int_0^u f(x,t)\,\mathrm{d}t$.

This condition implies that for some $C > 0$, $F(x,u) \geqslant C|u|^\mu$.

In this paper we show that a weaker and more natural version suffices to assure the existence of a *ground state solution*. Instead of (AR) we assume the following super-quadratic condition

(SQ) $\lim_{|u|\to\infty} \frac{F(x,u)}{u^2} = \infty$, uniformly in x.

We always assume $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) > 0$. We consider two cases of the potentials, one is periodic, i.e., the x-dependence is periodic; the other is when V has a bounded potential well in the sense that $\lim_{|x|\to\infty} V(x)$ exists and is equal to $\sup_{\mathbb{R}^N} V$. The results will be stated and proved in Sections 2 and 3.

We would like to mention earlier results on existence of entire solutions of Schrödinger type equations with or without potentials which was studied in [3,4,9,10] (see references therein). In recent years there have been intensive studies on semiclassical states for nonlinear Schrödinger equations for which in Eq. (1.1) there is a small parameter corresponding to the Plank constant. We refer [1] for references in this direction. Our results do not require smallness of such a parameter. A recent result in [5] is in similar spirit of our Theorem 3.1; but the conditions in [5] and ours are mutually non-inclusive and the methods are different.

For (1.1) in bounded domains or if the potential function $V(x)$ possesses certain compactness condition, one can prove (1.1) have certain solutions. In [8] Liu and Wang first used (SQ) to get the bounds of minimizing sequences on the Nehari manifold, and under coercive condition of $V(x)$ they proved the existence of three solutions: one positive, one negative, and one sign-changing. The results in this current paper are natural generalizations of that in [8] to noncompact cases. In the two cases we do not have compact embedding, which is the main difficulty in this paper. We shall make use of a combination of the techniques in [8,7] with applications of the concentration-compactness principle of Lions [6,11,12].

2. The periodic case

We consider weak solutions of

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We need the following assumptions:

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) \geqslant V_0 > 0$. $V(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$.

(f_1) $f(x,t) \in C^1$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, f_t is a Caratheodory function and there exists $C > 0$, such that

$$\left|f_t(x,t)\right| \leqslant C\left(1+|t|^{2^*-2}\right), \quad \lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{2^*-1}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N.$$

(f_2) $f(x,t) = \mathrm{o}(|t|)$, as $|t| \to 0$, uniformly in x.

(f_3) $\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty$, uniformly in x.

(f_4) $\frac{f(x,t)}{|t|}$ is strictly increasing in t.

Here $2^* = \frac{2N}{N-2}$ for $N \geqslant 3$. For $N = 1, 2$ we assume there is $q > 2$ in the place of 2^* in (f_1). We work in Hilbert space $X = \{u \in H^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty\}$, with norm $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x$. The functional associated with Eq. (1.1) is

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in X.$$

Define

$$\gamma(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)u\,\mathrm{d}x.$$

Theorem 2.1. *Under assumptions* (V_1), (f_1)–(f_4) *Eq.* (1.1) *has a weak solution* $u \in X$, *such that* $\Phi(u) = c > 0$, c *is defined as*

$$c = \inf_{\mathcal{N}} \Phi(u),$$

where $\mathcal{N} = \{u \in X\colon\ u \neq 0,\ \gamma(u) = 0\}$.

First we need a few lemmas.

Lemma 2.2. *Let* (u_n) *be a minimizing sequence for* c. *Then*

(i) *There is* $\beta > 0$ *such that* $\liminf_{n\to\infty} \|u_n\| \geqslant \beta$.
(ii) (u_n) *is bounded in* X.
(iii) *For a subsequence, up to translations,* u_n *converges weakly to* $u \neq 0$.

Proof. (i) The proof is similar to the case with (AR) satisfied. We omit its proofs (see [12]).

(ii) Let (u_n) be a minimizing sequence of c. If (u_n) is not bounded, we define $v_n = u_n/\|u_n\|$, so $\|v_n\| = 1$. Passing to a subsequence, we may assume, $v_n \rightharpoonup v$ in X, $v_n \to v$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \leqslant p < 2^*$, $v_n \to v$ a.e. on $\mathbb{R}^N$.

If $v \neq 0$, we have

$$\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n^2} v_n^2\,\mathrm{d}x = \frac{c + \mathrm{o}(1)}{\|u_n\|^2} > 0.$$

By Fadou's lemma and (f_3) we have a contradiction as follows,

$$\frac{1}{2} \geqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n^2} v_n^2\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2} v_n^2\,\mathrm{d}x = \infty.$$

If $v = 0$, we take $y_n = (y_n^1, y_n^2, \ldots, y_n^N) \in \mathbb{N}^N$ with all y_n^i $(1 \leqslant i \leqslant N)$ being integers. Define translations of v_n by $w_n(x) = v_n(x + y_n)$. Since $V(x)$ and $f(x,u)$ are periodic, we have $\|w_n\| = \|v_n\| = 1$, $|w_n|_p = |v_n|_p$, and $\Phi(w_n) = \Phi(v_n)$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \to w$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \leqslant p < 2^*$, $w_n \to w$ a.e.

on $\mathbb{R}^N$. If there exist y_n, such that $w_n \rightharpoonup w \neq 0$, we will get a contradiction as the case of $v \neq 0$. If for any y_n, $w_n \rightharpoonup 0$, we will get a contradiction by proving $v_n \to 0$ in $L^p(\mathbb{R}^N)$. In this case, we claim for all $p \in (2, 2^*)$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p \,\mathrm{d}x = 0.$$

If this is not true, there exists $p \in (2, 2^*)$, $\delta > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p \,\mathrm{d}x \geqslant \delta > 0,$$

then there exists $z_n \in \mathbb{R}^N$ such that, $\lim_{n\to\infty} \int_{B_2(z_n)} |v_n|^p \,\mathrm{d}x \geqslant \delta/2 > 0$. We can choose $y_n \in \mathbb{N}^N \in B_2(z_n)$ such that $B_1(y_n) \subset B_2(z_n)$ and

$$\lim_{n\to\infty} \int_{B_1(y_n)} |v_n|^p \,\mathrm{d}x \geqslant \frac{\delta}{2} > 0,$$

we have $\lim_{n\to\infty} \int_{B_1(0)} |w_n|^p \,\mathrm{d}x \geqslant \delta/2 > 0$, that is $w_n \rightharpoonup w \neq 0$, a contradiction.

By Lions Lemma (cf. [12, Lemma 1.21]), we get $v_n \to 0$ in $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$. Fix $p \in (2, 2^*)$. By (f_1) and (f_2), for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|f(x,u)| \leqslant \varepsilon(|u| + |u|^{2^*-1}) + C_\varepsilon |u|^{p-1}$. Then $|F(x,u)| \leqslant \varepsilon(|u|^2 + |u|^{2^*}) + C_\varepsilon |u|^p$. Then fixing an $R > \sqrt{2c}$, using Lebesgue Dominated Convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n) \,\mathrm{d}x = \int_{\mathbb{R}^N} \lim_{n\to\infty} F(x, Rv_n) \,\mathrm{d}x = 0.$$

Since by (f_4), $\Phi(tu_n) \leqslant \Phi(u_n)$ for $t \geqslant 0$ we thus have

$$c + \mathrm{o}(1) = \Phi(u_n) \geqslant \Phi(Rv_n) = \frac{1}{2}R^2 - \int_{\mathbb{R}^N} F(x, Rv_n) \,\mathrm{d}x,$$

which is a contradiction. Thus (u_n) is bounded.

(iii) We can assume u_n weakly converges to u. To show $u \neq 0$, again we define translations of u_n as above, assume $y_n = (y_n^1, y_n^2, \ldots, y_n^N) \in \mathbb{N}^N$, with all y_n^i $(1 \leqslant i \leqslant N)$ being integers. $u_n^{y_n} = u_n(x + y_n)$ are all possible translation of u_n. If for some $y_n \subset \mathbb{N}^N$, $u_n^{y_n} \rightharpoonup u \neq 0$ we are done. If for any $y_n \subset \mathbb{N}^N$, $u_n^{y_n} \rightharpoonup 0$, by similar argument as above we can prove $u_n \to 0$ in $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$. Then as $n \to \infty$, $\int_{\mathbb{R}^N} u_n f(x, u_n) \,\mathrm{d}x \to 0$. Thus by (i) we have a contradiction:

$$0 < \beta \leqslant \|u_n\|^2 = \int_{\mathbb{R}^N} u_n f(x, u_n) \,\mathrm{d}x \to 0, \quad \text{as } n \to \infty. \qquad \Box$$

Lemma 2.3. *For each $u \in X \setminus \{0\}$, there exists unique $t = t(u) > 0$, such that $tu \in \mathcal{N}$.*

This is similar to the case of assuming (AR), we omit it.

Lemma 2.4. *Let $(u_n) \subset X$ be a sequence such that $\gamma(u_n) \to 0$ and $\int_{\mathbb{R}^N} f(x, u_n)u_n \to a > 0$ as $n \to \infty$. Then exist $t_n > 0$ such that $t_n u_n \in \mathcal{N}$, $t_n \to 1$, as $n \to \infty$.*

Proof. Since $u_n \neq 0$, by Lemma 2.3, there exists only one $t_n > 0$, such that $t_n u_n \in \mathcal{N}$, i.e.

$$t_n^2 \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)|u_n|^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n \,\mathrm{d}x = 0.$$

By (f_1) and (f_2), $|f(x,u)u| \leqslant \varepsilon(|u|^2 + |u|^{2^*}) + C_\varepsilon |u|^p$, we see t_n cannot go zero, that is $t_n \geqslant t_0 > 0$. By (f_4), $f(x,u)u \geqslant 2F(x,u)$. If $t_n \to \infty$, we get

$$a + \mathrm{o}(1) = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)|u_n|^2\right) \mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(x, t_n u_n) t_n u_n}{t_n^2} \,\mathrm{d}x \geqslant 2 \int_{\mathbb{R}^N} \frac{F(x, t_n u_n)}{t_n^2 u_n^2} u_n^2 \,\mathrm{d}x.$$

By the condition, up to translations, $u_n \to u \neq 0$ a.e. in $\mathbb{R}^N$. We have

$$\int_{\mathbb{R}^N} \frac{F(x, t_n u_n)}{t_n^2 u_n^2} u_n^2 \,\mathrm{d}x \to +\infty, \quad \text{as } n \to \infty$$

a contradiction. Thus $0 < t_0 \leqslant t_n \leqslant C$. Assume $t_n \to T$, now we claim $T = 1$. Since $t_n u_n \in \mathcal{N}$, by $\gamma(u_n) \to 0$ we have

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)|u_n|^2\right) \mathrm{d}x = \int_{\mathbb{R}^N} f(x, u_n) u_n \,\mathrm{d}x + \mathrm{o}(1).$$

Since $t_n \to T$, by (f_1) and (f_3)

$$T^2 \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x)|u_n|^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} f(x, T u_n) T u_n \,\mathrm{d}x = \mathrm{o}(1),$$

that is

$$\mathrm{o}(1) = \int_{\mathbb{R}^N} \frac{f(x, T u_n)}{T u_n} u_n^2 - \frac{f(x, u_n)}{u_n} u_n^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \left(\frac{f(x, T u_n)}{T u_n} - \frac{f(x, u_n)}{u_n}\right) u_n^2 \,\mathrm{d}x.$$

For a subsequence $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ $2 \leqslant p < 2^*$. Up to translations, we may assume $u \neq 0$. Then by (f_4) and Fatou's lemma

$$\int_{\mathbb{R}^N} \left(\frac{f(x, T u)}{T u} - \frac{f(x, u)}{u}\right) u^2 \,\mathrm{d}x = 0,$$

by (f_4) we have $T = 1$. □

Next we construct a special minimizing sequence along which $\int_{\mathbb{R}^N} F(x, u)$ is weakly continuous. Consider Eq. (1.1) on $B_R(0)$,

$$\begin{cases} -\Delta u + V(x) u = f(x, u), & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0). \end{cases} \tag{2.1}$$

We can similarly define $\mathcal{N}_R$, c_R. By the result of [8], c_R is achieved by a positive solution of (2.1) called u_R. It is easy to check that $c_R > c$ and $c_R \to c$ as $R \to \infty$. This implies (u_R) as $R \to \infty$ minimizes c. Let $R_n \to \infty$, $u_n := u_{R_n}$. Fix $p \in (2, 2^*)$.

Lemma 2.5.

(i) $\int_{\mathbb{R}^N} |u_n|^p \to A > 0$.
(ii) *There exist* $x_n \in \mathbb{R}^N$ *such that* $\forall \varepsilon > 0$, $\exists R > 0$, $\liminf \int_{B_R(x_n)} |u_n|^p \geqslant A - \varepsilon$.

Proof. (i) follows from $\gamma(u_n) = 0$ and the fact that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$, $|f(x, u)| \leqslant \varepsilon(|u| + |u|^{2^*-1}) + C_\varepsilon |u|^{p-1}$.

For (ii) we apply the concentration compactness principle to $\int_{\mathbb{R}^N} |u_n|^p$. Then there exist $\alpha \in (0, 1]$, $(x_n) \subset \mathbb{R}^N$, $\forall \varepsilon > 0$, $\exists R > 0$, $\forall r > R$, $r' > R$, have

$$\liminf \int_{B_r(x_n)} |u_n|^p \geqslant \alpha A - \varepsilon, \qquad \liminf \int_{B^c_{r'}(x_n)} |u_n|^p \geqslant (1 - \alpha) A - \varepsilon.$$

Next we claim $\alpha = 1$. Choose $\varepsilon_n \to 0$, $r_n \to \infty$, $r'_n = 4 r_n$. Let ξ be a cut-off function such that $\xi(s) = 0$, for $s \leqslant 1$ or $s \geqslant 4$, $\xi(s) = 1$, for $2 \leqslant s \leqslant 3$, and $|\xi'(s)| \leqslant 2$. Take $\phi(x) = \xi(|x - x_n|/r_n) u_n$. Using equation

$$\int_{B_{R_n}} \left(\nabla u_n \nabla \phi + V(x) u_n \phi - f(x, u_n) \phi\right) \mathrm{d}x = 0,$$

we have,

$$\int_{B_{3r_n}(x_n)\setminus B_{2r_n}(x_n)} \left(|\nabla u_n|^2 + V(x)|u_n|^2\right) \mathrm{d}x + \int_{B_{3r_n}(x_n)\setminus B_{2r_n}(x_n)} f(x,u_n)u_n \,\mathrm{d}x = \mathrm{o}(1).$$

Take another cut-off function η such that $\eta(s) = 1$, for $s \leqslant 2$, $\eta(s) = 0$, for $s \geqslant 3$, and $|\eta'(s)| \leqslant 2$, for $2 \leqslant s \leqslant 3$. Set

$$w_n(x) = \eta\left(\frac{|x-x_n|}{r_n}\right)u_n, \qquad v_n(x) = \left(1 - \eta\left(\frac{|x-x_n|}{r_n}\right)\right)u_n(x).$$

Using equation as above we have

$$\Phi(u_n) = \Phi(w_n) + \Phi(v_n) + \mathrm{o}(1),$$

and

$$\int_{\mathbb{R}^n} |w_n|^p \geqslant \alpha A - \varepsilon_n, \qquad \int_{\mathbb{R}^n} |v_n|^p \geqslant (1-\alpha)A - \varepsilon_n.$$

Finally using w_n to test the equation for (u_n) we get

$$\gamma(w_n) = \left\langle \Phi'(u_n), w_n \right\rangle + \mathrm{o}(1) = \mathrm{o}(1).$$

Similarly $\gamma(v_n) = \mathrm{o}(1)$, by Lemma 2.4, $\exists t_n \to 1$, $s_n \to 1$, such that $t_n w_n \in \mathcal{N}$, $s_n v_n \in \mathcal{N}$. Then

$$c + \mathrm{o}(1) = \Phi(u_n) = \Phi(w_n) + \Phi(v_n) + \mathrm{o}(1) = \Phi(t_n w_n) + \Phi(s_n v_n) + \mathrm{o}(1) \geqslant 2c + \mathrm{o}(1),$$

which is a contradiction. Thus $\alpha = 1$. □

Proof of Theorem 2.1. Let $(u_n) \subset \mathcal{N}$ be the minimizing sequence for c given above. By Lemma 2.2 (u_n) is bounded in X and weak convergent to $u \neq 0$. By Lemma 2.5, $-\int_{R^N} F(x,u_n)$ is weakly continuous. Using the weakly lower semi-continuity we have $\Phi(u) \leqslant c$. If $u \in \mathcal{N}$ we have $\Phi(u) = c$. If $u \notin \mathcal{N}$, by Lemma 2.5, there is $t > 0$ such that $tu_n \in \mathcal{N}$. Then

$$c \leqslant \Phi(tu) \leqslant \liminf_{n\to\infty} \Phi(tu_n) \leqslant \liminf_{n\to\infty} \Phi(u_n) = c.$$

Since $\mathcal{N}$ is smooth, the minimizer is a critical point of Φ. □

3. The potential well case

We consider weak solutions of

$$\begin{cases} -\Delta u + V(x)u = f(u), \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{3.1}$$

for the case where potential function $V(x)$ has a bounded potential well. Since the nonlinearity is autonomous, the conditions on f needs modified slightly. More precisely, we make the following assumptions.

(V_2) $0 < \inf_{\mathbb{R}^N} V(x) \leqslant \lim_{|x|\to\infty} V(x) = \sup_{\mathbb{R}^N} V(x) < \infty$.

(f_1) $f(t) \in C^1$. f_t is a Caratheodory function and there exists $C > 0$, s. t.

$$\left|f_t(t)\right| \leqslant C\left(1 + |t|^{2^*-2}\right), \qquad \lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{2^*-1}} = 0.$$

(f_2) $f(t) = \mathrm{o}(|t|)$, as $|t| \to 0$.

(f_3) $\lim_{|t|\to\infty} \frac{F(t)}{t^2} = \infty$.

(f_4) $\frac{f(t)}{|t|}$ is strictly increasing in t.

Theorem 3.1. *Under assumptions* (V_2), (f_1)–(f_4) *Eq.* (3.1) *has a weak solution* $u \in X$, *such that* $\Phi(u) = c > 0$, *c is defined as*

$$c = \inf_{\mathcal{N}} \Phi(u),$$

where $\mathcal{N} = \{u \in X\colon\ u \neq 0,\ \gamma(u) = 0\}$.

In this section we denote $V_\infty = \lim_{|x|\to\infty} V(x)$. There is an associated problem

$$\begin{cases} -\Delta u + V_\infty u = f(u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We define the energy functional Φ_∞ by replacing V with V_∞, $c_\infty = \inf_{\mathcal{N}_\infty} \Phi_\infty(u)$, here $\mathcal{N}_\infty = \{u \in X/\{0\}\colon \langle \Phi'_\infty(u), u\rangle = 0\}$. Since V_∞ is a constant, by Theorem 2.1, $c_\infty > 0$ is achieved at some $u_\infty \in \mathcal{N}_\infty$.

Lemma 3.2. $0 < c < c_\infty$.

Proof. It is easy to see $c > 0$. Let u_∞ be the minimizer of c_∞. Then $\gamma(u_\infty) < 0$, and there is $t > 0$ such that $tu_\infty \in \mathcal{N}$. We have

$$c \leqslant \Phi(tu_\infty) < \Phi_\infty(tu_\infty) \leqslant \Phi_\infty(u_\infty) = c_\infty. \qquad \square$$

We note that with minor changes Lemma 2.3 and 2.4 still hold.

Lemma 3.3. *Let* (u_n) *be a minimizing sequence for c. Then*

(i) *There is* $\beta > 0$ *such that* $\liminf_{n\to\infty} ||u_n|| \geqslant \beta$.
(ii) (u_n) *is bounded in X.*
(iii) *For a subsequence,* u_n *converges weakly to* $u \neq 0$.

Proof. (i) The same as Lemma 2.2.

(ii) If not, define $v_n = u_n/\|u_n\|$. Passing to a subsequence, we may assume, $v_n \rightharpoonup v$ in X. If $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 \leqslant q < 2^*$, we use the Lebesgue Dominated Convergence theorem to get for any $R > 0$ fixed, $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(Rv_n)\,\mathrm{d}x = 0$. Therefore a contradiction by choosing a large $R > 0$ in $\Phi(u_n) \geqslant \Phi(Rv_n) = \frac{1}{2}R^2 - \int_{\mathbb{R}^N} F(Rv_n)\,\mathrm{d}x$. Thus by the concentration compactness principle there are $y_n \in \mathbb{R}^N$ such that $w_n(x) = v_n(y_n + x) \to w \neq 0$. Then the proof follows from the arguments in Lemma 2.2(ii). Thus (u_n) is bounded.

(iii) We can assume $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. If $u = 0$, we have $\int_{\mathbb{R}^N} (V(x) - V_\infty)|u_n|^2\,\mathrm{d}x \to 0$, as $n \to \infty$. Thus we have $c + \mathrm{o}(1) = \Phi_\infty(u_n) + \mathrm{o}(1)$. Similarly we have $\gamma(u_n) = 0$, $\gamma_\infty(u_n) = \mathrm{o}(1)$. By Lemma 2.4 there exist $t_n \to 1$ such that $t_n u_n \in \mathcal{N}_\infty$. Then we have $c + \mathrm{o}(1) = \Phi_\infty(u_n) + \mathrm{o}(1) = \Phi_\infty(t_n u_n) + \mathrm{o}(1) \geqslant c_\infty + \mathrm{o}(1)$, a contradiction with Lemma 3.2. $\square$

Next consider Eq. (3.1) on $B_R(0)$,

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0) \end{cases} \tag{3.2}$$

we can similarly define $\mathcal{N}_R = \mathcal{N} \cap H^1_0(B_R)$, c_R. By the result of [8], c_R is achieved by a positive solution called u_R. It is easy to check that $c_R > c$ and $c_R \to c$ as $R \to \infty$.

Lemma 3.4. *Let* $u_R \in \mathcal{N}_R$ *be a minimizer of* c_R. *Assume for a subsequence* $R_n \to \infty$, $\int_{B_{R_n}} |u_n|^p \to A \in (0,\infty)$, *where* $u_n = u_{R_n}$. *Then there exists* $(y_n) \subset \mathbb{R}^N$ *s.t. for any* $\varepsilon > 0$, *exists* $r_\varepsilon > 0$, *for all* $r \geqslant r_\varepsilon$,

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant A - \varepsilon.$$

Proof. Note that u_R satisfies (3.3) for all $\varphi \in H_0^1(B_R)$,

$$\int_{B_{R_n}} \big(\nabla u_R \nabla\varphi + V(x) u_R \varphi\big)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u_R)\varphi\,\mathrm{d}x = 0. \tag{3.3}$$

Since (u_n) is bounded in $H^1(\mathbb{R}^N)$, by using the concentration compactness principle, exists $\alpha \in (0,1]$ and $(y_n) \subset \mathbb{R}^N$ s.t. for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$, for all $r' \geqslant r \geqslant r_\varepsilon$,

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \alpha A - \varepsilon, \tag{3.4}$$

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N \setminus B_{r'}(y_n)} |u_n|^p \geqslant (1-\alpha)A - \varepsilon. \tag{3.5}$$

Now suppose $\alpha < 1$, then following exactly the same construction as in Lemma 2.5 we have two sequences w_n and v_n satisfying

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^p \geqslant \alpha A, \qquad \liminf_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p \geqslant (1-\alpha)A, \qquad \Phi(u_n) = \Phi(w_n) + \Phi(v_n) + \mathrm{o}(1).$$

Moreover, if we take $\varphi = w_n$, by (3.3)

$$\gamma(w_n) = \langle \Phi'(u_n), w_n\rangle + \mathrm{o}(1) = \mathrm{o}(1).$$

Similarly, $\gamma(v_n) = \mathrm{o}(1)$. By Lemma 2.4, there exist $t_n \to 1$, $s_n \to 1$, s.t.

$$\widetilde{w}_n = t_n w_n \in \mathcal{N}, \qquad \tilde{v}_n = s_n v_n \in \mathcal{N}.$$

If (y_n) is bounded, then $\liminf_{n\to\infty} \Phi(\widetilde{w}_n) \geqslant c$ and $\liminf_{n\to\infty} \Phi(\tilde{v}_n) \geqslant c_\infty$. If (y_n) is unbounded, then

$$\liminf_{n\to\infty} \Phi(\widetilde{w}_n) \geqslant c_\infty \quad \text{and} \quad \liminf_{n\to\infty} \Phi(\tilde{v}_n) \geqslant c.$$

Altogether, we have

$$\Phi(u_n) = \Phi(w_n) + \Phi(v_n) + \mathrm{o}(1) = \Phi(t_n w_n) + \Phi(s_n v_n) + \mathrm{o}(1),$$

and

$$\liminf \Phi(u_n) \geqslant \liminf \Phi(t_n w_n) + \liminf \Phi(s_n v_n) \geqslant c + c_\infty.$$

A contradiction, so we have $\alpha = 1$. □

Proof of Theorem 3.1. Let $(u_n) \subset \mathcal{N}$ be the minimizing sequence for c given in Lemma 3.4. Let $A = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p\,\mathrm{d}x$. By Lemma 3.3, (u_n) is bounded in X and weakly converges to $u \neq 0$. By Lemma 3.4, there exists $(y_n) \subset \mathbb{R}^N$ s.t. $\forall\varepsilon > 0$, $\exists r > 0$,

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant A - \varepsilon.$$

Then (y_n) must be bounded. Otherwise, $\gamma_\infty(u_n) = \gamma(u_n) + \mathrm{o}(1)$. We find $t_n \to 1$, s.t. $\gamma(t_n u_n) = 0$. Then we have

$$c_\infty \leqslant \liminf \Phi_\infty(t_n u_n) = \liminf \Phi_\infty(u_n) = \liminf \Phi(u_n) = c$$

a contraction with $c < c_\infty$. Now, when (y_n) is bounded, we have $u_n \to u$ in $L^p(\mathbb{R}^N)$. This gives that along this sequence $\Phi(u_n)$ is weakly lower semi-continuous, we have

$$c = \inf_{\mathcal{N}} \Phi(u) \leqslant \Phi(u) \leqslant \liminf \Phi(u_n) = c. \qquad \square$$

Remark 3.5. Though we assume in this section f depends only on t, looking at the proofs we see the arguments can be used with little changes to deal with the following case: $f = b(x)f(t)$ with b satisfying $b \in C^1(\mathbb{R}^N, \mathbb{R})$, $b_1 \leqslant b(x) \leqslant b_2$ for some $b_1, b_2 > 0$, and $b(x) \geqslant \inf_{\mathbb{R}^N} b(x) = \lim_{|x|\to\infty} b(x)$. The precisely statement is the same.

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