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A NONEXISTENCE RESULT FOR YAMABE TYPE PROBLEMS ON THIN ANNULI

UN RÉSULTAT DE NON-EXISTENCE POUR UN PROBLÈME DE TYPE YAMABE SUR DES ANNEAUX MINCES

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ABSTRACT. – Given any constant $C > 0$, we show that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, the problem P_ε: $-\Delta u_\varepsilon = u_\varepsilon^{(n+2)/(n-2)}$, $u_\varepsilon > 0$ in A_ε; $u_\varepsilon = 0$ on ∂A_ε, has no solution u_ε, whose energy, $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2$, is less than C, where A_ε is a ringshaped open set in $\mathbb{R}^n$ and $n \geqslant 4$. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Etant donné une constante positive C arbitraire, nous montrons qu'il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon < \varepsilon_0$, le problème P_ε: $-\Delta u_\varepsilon = u_\varepsilon^{(n+2)/(n-2)}$, $u_\varepsilon > 0$ dans A_ε; $u_\varepsilon = 0$ sur ∂A_ε, ne possède pas de solution u_ε dont l'energie, $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2$, est plus petite que C, où A_ε est un ouvert de $\mathbb{R}^n$ ayant la forme d'un anneau et $n \geqslant 4$. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction and the main results

Let us consider the nonlinear elliptic problem

$$P(\Omega) \quad \begin{cases} -\Delta u = u^p, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where Ω is a bounded regular domain in $\mathbb{R}^n$, $n \geqslant 3$ and $p+1 = 2n/(n-2)$ is the critical Sobolev exponent.

The interest in this type of equation comes from its resemblance to some nonlinear problems in geometry (Yamabe problem, Harmonic maps, ...) and physics (Yang–Mills equations, The n body problem, ...) where some lack of compactness occurs (see Brezis [6]). It is well known that if Ω is starshaped, $P(\Omega)$ has no solution (see Pohozaev [13]) and if Ω has nontrivial topology, in the sense that $H_{2k-1}(\Omega; Q) \neq 0$ or $H_k(\Omega; Z/2Z) \neq 0$ for some $k \in \mathbb{N}$, Bahri and Coron [3] have shown that $P(\Omega)$ has a solution. Nevertheless, Ding [9] (see also Dancer [8]) gave the example of contractible domain on which $P(\Omega)$ has a solution. Then, the question related to existence or nonexistence of solution of $P(\Omega)$ remained open.

In this paper, we study the problem $P(\Omega)$ when $\Omega = A_\varepsilon$ is a ringshaped open set in $\mathbb{R}^n$ and $\varepsilon \to 0$. More precisely, let f be any smooth function:

$$f : \mathbb{R}^{n-1} \to [1, 2], \quad (\theta_1, \dots, \theta_{n-1}) \to f(\theta_1, \dots, \theta_{n-1})$$

which is periodic of period π with respect to $\theta_1, \dots, \theta_{n-2}$ and of period 2π with respect to θ_{n-1}. We set

$$S_1(f) = \{x \in \mathbb{R}^n \mid r = f(\theta_1, \dots, \theta_{n-1})\}$$

where $(r, \theta_1, \dots, \theta_{n-1})$ are the polar coordinates of x. For ε positive small enough, we introduce the following map

$$g_\varepsilon : S_1(f) \to g_\varepsilon(S_1(f)) = S_2(f), \quad x \mapsto g_\varepsilon(x) = x + \varepsilon n_x$$

where n_x is the outward normal to $S_1(f)$ at x. We denote by $(A_\varepsilon)_{\varepsilon>0}$ the family of annulus shaped open sets in $\mathbb{R}^n$ such that $\partial A_\varepsilon = S_1(f) \cup S_2(f)$. Our main result is the following theorem.

THEOREM 1.1. – *Assume that $n \geqslant 4$. Let C be any positive constant. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, the problem P_ε: $-\Delta u_\varepsilon = u_\varepsilon^{(n+2)/(n-2)}$, $u_\varepsilon > 0$ in A_ε, $u_\varepsilon = 0$ on ∂A_ε, has no solution such that $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \leqslant C$.*

Remark 1.2. – We believe the result to be true also for $n = 3$ (see Remark 1.4 below).

The proof of Theorem 1.1 is given in two principal steps:

Step 1. We suppose that P_ε has a solution u_ε which satisfies $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \leqslant C$, C being a given constant. We study the asymptotic behavior of u_ε when ε tends to zero. We prove that u_ε blows up at p points ($p \in \mathbb{N}^*$), then the location of blow up points is studied. In order to formulate the result of this step, we need to introduce some notations.

We denote by G_ε the Green's function of Laplace operator defined by $\forall x \in A_\varepsilon$

$$-\Delta G_\varepsilon(x, .) = c_n \delta_x \quad \text{in } A_\varepsilon, \qquad G_\varepsilon(x, .) = 0 \quad \text{on } \partial A_\varepsilon \tag{1.1}$$

where δ_x is the Dirac mass at x and $c_n = (n-2)\text{meas}(S^{n-1})$. We denote by H_ε the regular part of G_ε, that is,

$$H_\varepsilon(x_1, x_2) = |x_1 - x_2|^{2-n} - G_\varepsilon(x_1, x_2), \quad \text{for } (x_1, x_2) \in A_\varepsilon \times A_\varepsilon. \tag{1.2}$$

For $p \in \mathbb{N}^*$ and $x = (x_1, \ldots, x_p) \in A_\varepsilon^p$, we denote by $M = M_\varepsilon$ the matrix defined by

$$M = (m_{ij})_{1 \leqslant i,j \leqslant p}, \quad \text{where } m_{ii} = H_\varepsilon(x_i, x_i),\ m_{ij} = -G_\varepsilon(x_i, x_j),\ i \neq j, \tag{1.3}$$

and define $\rho_\varepsilon(x)$ as the least eingenvalue of $M(x)$ ($\rho_\varepsilon(x) = -\infty$ if $x_i = x_j$ for some $i \neq j$). For $a \in \mathbb{R}^n$ and $\lambda > 0$, $\delta_{(a,\lambda)}$ denotes the function

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{\frac{n-2}{2}}}{(1 + \lambda^2 |x-a|^2)^{\frac{n-2}{2}}}. \tag{1.4}$$

It is well known that if c_0 is suitably chosen ($c_0 = (n(n-2))^{\frac{n-2}{4}}$) the function $\delta_{(a,\lambda)}$ are the only solutions of equation

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in } \mathbb{R}^n \tag{1.5}$$

and they are also the only minimizers for the Sobolev inequality

$$S = \inf\left\{ |\nabla u|^2_{L^2(\mathbb{R}^n)} |u|^{-2}_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)},\ \text{s.t. } \nabla u \in L^2,\ u \in L^{\frac{2n}{n-2}},\ u \neq 0 \right\}. \tag{1.6}$$

We also denote by $P_\varepsilon \delta_{(a,\lambda)}$ the projection of $\delta_{(a,\lambda)}$ on $H_0^1(A_\varepsilon)$, that is,

$$-\Delta P_\varepsilon \delta_{(a,\lambda)} = -\Delta \delta_{(a,\lambda)} \quad \text{in } A_\varepsilon,\ P_\varepsilon \delta_{(a,\lambda)} = 0 \text{ on } \partial A_\varepsilon.$$

Lastly, we define on $H_0^1(A_\varepsilon) \setminus \{0\}$ the functional

$$J_\varepsilon(u) = \frac{\int_{A_\varepsilon} |\nabla u|^2}{(\int_{A_\varepsilon} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \tag{1.7}$$

whose positive critical points, up a multiplicative constant, are solutions of P_ε.

Now we are able to state the main result of step 1.

THEOREM 1.3. – *Let u_ε be a solution of problem P_ε, assume $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \leqslant C$, where C is a positive constant independent of ε. Then, after passing to a subsequence, there exist $p \in \mathbb{N}^*$, $(a_{1,\varepsilon}, \ldots, a_{p,\varepsilon}) \in A_\varepsilon^p$, $(\lambda_{1,\varepsilon}, \ldots, \lambda_{p,\varepsilon}) \in (\mathbb{R}_+^*)^p$ such that*:

(i)

$$\left| \nabla \left(u_\varepsilon - \sum_{i=1}^p P_\varepsilon \delta_{(a_{i,\varepsilon}, \lambda_{i,\varepsilon})} \right) \right|_{L^2(A_\varepsilon)} \to 0,$$

$$\lambda_{i,\varepsilon} \to +\infty, \quad \lambda_{i,\varepsilon} d_{i,\varepsilon} \to +\infty, \quad \varepsilon_{ij} \to 0$$

when $\varepsilon \to 0$, where $d_{i,\varepsilon} = d(a_{i,\varepsilon}, \partial A_\varepsilon)$ and

$$\varepsilon_{ij} = \left(\frac{\lambda_{i,\varepsilon}}{\lambda_{j,\varepsilon}} + \frac{\lambda_{j,\varepsilon}}{\lambda_{i,\varepsilon}} + \lambda_{i,\varepsilon}.\lambda_{j,\varepsilon} |a_{i,\varepsilon} - a_{j,\varepsilon}|^2 \right)^{-\frac{n-2}{2}}.$$

(ii) *Moreover* $p \geqslant 2$ *and if* $n \geqslant 4$, *then we have*: $\exists k \leqslant p$, $\exists i_1, \ldots, i_k \in \{1, 2, \ldots, p\}$ *such that*

$$d^{n-2}\rho_\varepsilon(a_{i_1,\varepsilon}, \ldots, a_{i_k,\varepsilon}) \to 0, \qquad d^{n-1}\nabla\rho_\varepsilon(a_{i_1,\varepsilon}, \ldots, a_{i_k,\varepsilon}) \to 0$$

$\forall m, l \in \{1, \ldots, k\}$ $|a_{i_m,\varepsilon} - a_{i_l,\varepsilon}| \leqslant C_0 d$, *where* $d = \min\{d(a_{i_l,\varepsilon}, \partial A_\varepsilon) \mid 1 \leqslant l \leqslant k\}$ *and* C_0 *is a positive constant independent of* ε.

Remark 1.4. – We believe the result of part (ii) in the Theorem 1.3 to be true for $n = 3$. For $n = 3$ our method also proves easily $d^{n-2}\rho_\varepsilon(a_{i_1,\varepsilon}, \ldots, a_{i_k,\varepsilon}) \to 0$, but for the proof of $d^{n-1}\nabla\rho_\varepsilon(a_{i_1,\varepsilon}, \ldots, a_{i_k,\varepsilon}) \to 0$ we need a more careful estimates of the rests in Propositions 3.2 and 3.3 below.

The main ingredients of the proof of the Theorem 1.3 are a fine blow-up analysis, on the one hand, and a very delicate expansion of ∇J near infinity, on the other hand.

Step 2. We prove the following result.

THEOREM 1.5. – *For* $n \geqslant 3$, *let* $C_0 > 0$ *and let* $(x_1, x_2, \ldots, x_k) \in A_\varepsilon^k$ *such that*

$$d^{n-2}\rho_\varepsilon(x_1, \ldots, x_k) \to 0 \quad \textit{when } \varepsilon \to 0 \textit{ and } |x_i - x_j| \leqslant C_0 d,\ \forall i, j,$$

where $d = \min\{d(x_i, \partial A_\varepsilon) \mid 1 \leqslant i \leqslant k\}$. *Then*

$$d^{n-1}\nabla\rho_\varepsilon(x_1, \ldots, x_k) \not\to 0 \quad \textit{when } \varepsilon \to 0.$$

We notice that Theorem 1.1 is an easy consequence of Theorems 1.3 and 1.5.

The remainder of the present paper is organised as follows. Section 2 is devoted to the proof of the first part of Theorem 1.3, while the second part of Theorem 1.3 is proved in Section 3. In Section 4 we give the proof of Theorem 1.5. Lastly, we give in Appendix A some technical lemmas needed in Section 3.

2. Asymptotic behavior of solutions with bounded energy

In this section, we will study the asymptotic behavior of solutions u_ε of P_ε when ε is small enough and their energy is bounded. Thus, in the remainder, we assume that $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \leqslant C$, where C is a positive constant independent of ε. We begin by proving the following lemma.

LEMMA 2.1. – *We have the following claim*

$$\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \not\to 0, \quad M_\varepsilon \to +\infty, \quad \textit{when } \varepsilon \to 0,\ \textit{where } M_\varepsilon = |u_\varepsilon|_{L^\infty(A_\varepsilon)}.$$

Proof. – On the one hand, since u_ε is a solution of P_ε, we have

$$\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{A_\varepsilon} u_\varepsilon^{\frac{2n}{n-2}}.$$

On the other hand, we have

$$\Bigl(\int_{A_\varepsilon} |u_\varepsilon|^{\frac{2n}{n-2}}\Bigr)^{\frac{n-2}{n}} \leqslant \frac{1}{S}\int_{A_\varepsilon} |\nabla u_\varepsilon|^2$$

where S denotes the Sobolev constant defined in (1.6). Thus

$$S_n = S^{\frac{n}{2}} \leqslant \int_{A_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{A_\varepsilon} u_\varepsilon^{\frac{2n}{n-2}} \leqslant c\varepsilon M_\varepsilon^{\frac{2n}{n-2}}$$

and our lemma follows. □

LEMMA 2.2. – *There exists a positive constant c such that for ε small enough, we have*

$$\varepsilon M_\varepsilon^{\frac{2}{n-2}} \geqslant c, \quad \textit{where } M_\varepsilon = |u_\varepsilon|_{L^\infty(A_\varepsilon)}.$$

Proof. – On the one hand, we have

$$\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{A_\varepsilon} u_\varepsilon^{\frac{2n}{n-2}} \leqslant M_\varepsilon^{\frac{4}{n-2}} \int_{A_\varepsilon} u_\varepsilon^2(x)\,\mathrm{d}x.$$

On the other hand, we have

$$\int_{A_\varepsilon} u_\varepsilon^2(x)\,\mathrm{d}x = \varepsilon^n \int_{B_\varepsilon} v_\varepsilon^2(X)\,\mathrm{d}X$$

where $v_\varepsilon(X) = u_\varepsilon(\varepsilon X)$ and where $B_\varepsilon = \varphi(A_\varepsilon)$, with $\varphi : x \mapsto \varphi(x) = \varepsilon^{-1}x$. Observe that

$$\varepsilon^n \int_{B_\varepsilon} v_\varepsilon^2(X)\,\mathrm{d}X \leqslant \frac{\varepsilon^n}{c_\varepsilon}\int_{B_\varepsilon} |\nabla v_\varepsilon(X)|^2\,\mathrm{d}X = \frac{\varepsilon^2}{c_\varepsilon}\int_{A_\varepsilon} |\nabla u_\varepsilon(x)|^2\,\mathrm{d}x.$$

Thus

$$c_\varepsilon \leqslant \varepsilon^2 M_\varepsilon^{\frac{4}{n-2}}.$$

According to Lin [12], we have $\lim_{\varepsilon\to 0} c_\varepsilon = c > 0$, therefore our lemma follows. □

Now let $\widetilde{A}_\varepsilon = M_\varepsilon^{\frac{2}{n-2}}(A_\varepsilon - a_{1,\varepsilon})$, where $a_{1,\varepsilon} \in A_\varepsilon$ such that $M_\varepsilon = u_\varepsilon(a_{1,\varepsilon})$, and we denote by v_ε the function defined on $\widetilde{A}_\varepsilon$ by

$$v_\varepsilon(X) = M_\varepsilon^{-1} u_\varepsilon\bigl(a_{1,\varepsilon} + M_\varepsilon^{\frac{-2}{n-2}} X\bigr). \tag{2.1}$$

It is easy to see that v_ε satisfies

$$\begin{cases} -\Delta v_\varepsilon = v_\varepsilon^{\frac{n+2}{n-2}}, \quad 0 < v_\varepsilon \leqslant 1 & \text{in } \widetilde{A}_\varepsilon, \\ v_\varepsilon(0) = 1, \quad v_\varepsilon = 0 & \text{on } \partial\widetilde{A}_\varepsilon. \end{cases} \tag{2.2}$$

Observe that

$$\int_{\widetilde{A}_\varepsilon} |\nabla v_\varepsilon|^2 = \int_{A_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\widetilde{A}_\varepsilon} v_\varepsilon^{\frac{2n}{n-2}} = \int_{A_\varepsilon} u_\varepsilon^{\frac{2n}{n-2}} \leqslant C.$$

Let us prove the following lemma.

LEMMA 2.3. – *We have the following claim*

$$M_\varepsilon^{\frac{2}{n-2}} d(a_{1,\varepsilon}, \partial A_\varepsilon) \to +\infty, \quad \textit{when } \varepsilon \to 0.$$

Proof. – Let $l = \lim_{\varepsilon\to 0} M_\varepsilon^{2/(n-2)} d(a_{1,\varepsilon}, \partial A_\varepsilon)$. First we will prove $l \neq 0$. A similar result has been proved by Harrabi, Rebhi and Selmi [11]. We will adapt their proof to our case. Let $\bar{a}_{1,\varepsilon} \in \partial A_\varepsilon$ such that $d_\varepsilon := d(a_{1,\varepsilon}, \partial A_\varepsilon) = |\bar{a}_{1,\varepsilon} - a_{1,\varepsilon}|$. We may assume without loss of generality that the unit outward normal to ∂A_ε at $\bar{a}_{1,\varepsilon}$ is e_n, where e_n is the last element of a canonical basic of $\mathbb{R}^n$. We see that $d_\varepsilon = a_{1,\varepsilon}.e_n = a_{1,\varepsilon}^n$, where $a_{1,\varepsilon} = (a_{1,\varepsilon}^1, \dots, a_{1,\varepsilon}^n)$. v_ε is well defined in

$$B\big(0, (2/3)\varepsilon M_\varepsilon^{2/(n-2)}\big) \cap \big\{\big(x^1, \dots, x^n\big)/ - d_\varepsilon M_\varepsilon^{2/(n-2)} < x^n < (2\varepsilon/3 - d_\varepsilon) M_\varepsilon^{2/(n-2)}\big\}.$$

Let $z_\varepsilon = (0, \dots, -d_\varepsilon M_\varepsilon^{2/(n-2)})$ and let $\tilde{v}_\varepsilon(X) = v_\varepsilon(X + z_\varepsilon)$, for

$$X \in B\big(-z_\varepsilon, (2\varepsilon/3) M_\varepsilon^{2/(n-2)}\big) \cap \big\{0 < x^n < (2\varepsilon/3) M_\varepsilon^{2/(n-2)}\big\}.$$

We suppose, arguing by contradiction, that $|z_\varepsilon| = l_\varepsilon \to 0$ as $\varepsilon \to 0$. Then $-z_\varepsilon \in B^+(0,\theta) = B(0,\theta) \cap \{(x^1, \dots, x^n)/x^n > 0\}$, where θ is a fixed positive real small enough choosen below. Let $R > 0$ be such that

$$B^+(0,\theta) \subset B^+(0,R) \subset B\big(-z_\varepsilon, (2/3)\varepsilon M_\varepsilon^{\frac{2}{n-2}}\big) \cap \big\{0 < x^n < (2/3)\varepsilon M_\varepsilon^{\frac{2}{n-2}}\big\}$$

for ε small enough. We consider the following equation

$$\begin{cases} -\Delta\omega_\varepsilon = 0 & \text{in } B^+(0,R), \\ \omega_\varepsilon = \tilde{v}_\varepsilon & \text{on } \partial B^+(0,R). \end{cases}$$

Then $\omega_\varepsilon \in C^2(B^+(0,R)) \cap C^0(\overline{B^+(0,R)})$. We derive that

$$|\tilde{v}_\varepsilon - \omega_\varepsilon|_{W^{2,q}(B^+(0,R))} \leqslant c, \quad \forall q < \infty.$$

In particular, $\operatorname{grad}(\tilde{v}_\varepsilon - \omega_\varepsilon)$ is bounded in $B^+(0,R)$, and we have

$$\frac{\partial \tilde{v}_\varepsilon}{\partial x^n} < c + \frac{\partial \omega_\varepsilon}{\partial x^n}.$$

Observe that

$$\omega_\varepsilon(x) = -\int_{\partial B^+(0,R)} \frac{\partial G_{B^+}}{\partial \nu} \omega_\varepsilon(y)\,\mathrm{d}y \quad \text{in } B^+(0,\theta).$$

Thus

$$\frac{\partial \omega_\varepsilon}{\partial x^n}(x) = -\int_{\partial B^+(0,R)} \frac{\partial}{\partial x^n} \frac{\partial G_{B^+}}{\partial \nu} \omega_\varepsilon(y)\, \mathrm{d}y.$$

According to Lemma 2.6 [10], we have

$$\exists \theta > 0,\ \exists c > 0 \quad 0 < -\frac{\partial}{\partial x^n} \frac{\partial G_{B^+}}{\partial \nu}(x, y) \leqslant c, \quad \forall y \in \partial B^+(0, R),\ |x'| < \frac{1}{2},\ 0 < x^n < \theta.$$

Since $\omega_\varepsilon \leqslant 1$, we derive that $\partial \omega_\varepsilon / \partial x^n$ is bounded for $0 < x^n < \theta$. Thus $\partial \tilde{v}_\varepsilon / \partial x^n < c'$. Let $f_\varepsilon(t) = \tilde{v}_\varepsilon(te_n)$, then we have

$$f_\varepsilon(l_\varepsilon) - f_\varepsilon(0) = f'_\varepsilon(c_\varepsilon) l_\varepsilon = \frac{\partial \tilde{v}_\varepsilon}{\partial x^n}(c_\varepsilon e_n) l_\varepsilon \leqslant c' l_\varepsilon$$

where $l_\varepsilon = M_\varepsilon^{\frac{2}{n-2}} d_\varepsilon$. Hence $f_\varepsilon(l_\varepsilon) - f_\varepsilon(0) = \tilde{v}_\varepsilon(0) - \tilde{v}_\varepsilon(z_\varepsilon) = 1 \leqslant c' l_\varepsilon$ which contradicts the assumption $l_\varepsilon \to 0$. We derive that $l \neq 0$. We suppose, arguing by contradiction, $l < \infty$. Then it follows from (2.2) and standard elliptic theories that there exists some positive function v, such that (after passing to subsequence), $v_\varepsilon \to v$ in $C^1_{\mathrm{loc}}(\Omega)$, where Ω is a half space or a strip of $\mathbb{R}^n$, and v satisfies

$$\begin{cases} -\Delta v = v^{\frac{n+2}{n-2}}, \quad v > 0 & \text{in } \Omega, \\ v(0) = 1, \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

But if Ω is a half space or a strip of $\mathbb{R}^n$, by Pohozaev Identity (see Theorem III.1.3 [15]), then v must vanish identically. Thus we derive a contradiction and our lemma follows. □

From Lemma 2.3, we derive that there exists some positive function v, such that (after passing to a subsequence), $v_\varepsilon \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, and v satisfies

$$\begin{cases} -\Delta v = v^{\frac{n+2}{n-2}}, \quad v > 0 & \text{in } \mathbb{R}^n, \\ v(0) = 1, \quad \nabla v(0) = 0. \end{cases} \tag{2.3}$$

It follows from Cafferalli, Gidas and Spruck [7]

$$v(X) = \delta_{(0,\alpha_n)}(X), \quad \text{with } \alpha_n = \big(n(n-2)\big)^{-1/2}.$$

Hence

$$M_\varepsilon^{-1} u_\varepsilon\big(a_{1,\varepsilon} + M_\varepsilon^{\frac{-2}{n-2}} X\big) - \delta_{(0,\alpha_n)}(X) \to 0 \quad \text{in } C^1_{\mathrm{loc}}\big(\mathbb{R}^n\big),\ \text{when } \varepsilon \to 0.$$

Observe that

$$M_\varepsilon^{-1} u_\varepsilon\big(a_{1,\varepsilon} + M_\varepsilon^{\frac{-2}{n-2}} X\big) - \delta_{(0,\alpha_n)}(X) = M_\varepsilon^{-1}\big(u_\varepsilon(x) - \delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})}(x)\big)$$

where $\lambda_{1,\varepsilon} = \alpha_n M_\varepsilon^{\frac{2}{n-2}}$.

In the sequel, we denote by u_ε^1 the function defined on A_ε by

$$u_\varepsilon^1(x) = u_\varepsilon(x) - P_\varepsilon\delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})}(x). \tag{2.4}$$

Notice that $\lambda_{1,\varepsilon} \to +\infty$ and $\lambda_{1,\varepsilon} d(a_{1,\varepsilon}, \partial A_\varepsilon) \to +\infty$ when $\varepsilon \to 0$.

LEMMA 2.4. – *Let u_ε^1 be defined by* (2.4). *Then we have*

(i) $-\Delta u_\varepsilon^1 = |u_\varepsilon^1|^{\frac{4}{n-2}} u_\varepsilon^1 + g_\varepsilon$, *with* $|g_\varepsilon|_{H^{-1}_{(A_\varepsilon)}} \to 0$, *when* $\varepsilon \to 0$.

(ii) $\int_{A_\varepsilon} |\nabla u_\varepsilon^1|^2 = \int_{A_\varepsilon} |\nabla u_\varepsilon|^2 - S_n + \mathrm{o}(1)$.

(iii) $\int_{A_\varepsilon} |u_\varepsilon^1|^{\frac{2n}{n-2}} = \int_{A_\varepsilon} |u_\varepsilon|^{\frac{2n}{n-2}} - S_n + \mathrm{o}(1)$ *where* $S_n = S^{\frac{n}{2}}$.

Proof. –

(i) We have

$$-\Delta u_\varepsilon^1 = -\Delta u_\varepsilon + \Delta P_\varepsilon \delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})} = u_\varepsilon^{\frac{n+2}{n-2}} - \delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})}^{\frac{n+2}{n-2}} = \left|u_\varepsilon^1\right|^{\frac{4}{n-2}} u_\varepsilon^1 + g_\varepsilon$$

where

$$g_\varepsilon = u_\varepsilon^{\frac{n+2}{n-2}} - \delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})}^{\frac{n+2}{n-2}} - |u_\varepsilon^1|^{\frac{4}{n-2}} u_\varepsilon^1.$$

Observe that

$$g_\varepsilon = \mathrm{O}\big(|P_\varepsilon\delta|^{\frac{4}{n-2}} |u_\varepsilon - P_\varepsilon\delta| + |u_\varepsilon - P_\varepsilon\delta|^{\frac{4}{n-2}} P_\varepsilon\delta\big) + \mathrm{O}\big(\delta_\varepsilon^{\frac{4}{n-2}}(\delta_\varepsilon - P_\varepsilon\delta)\big)$$

where $P_\varepsilon\delta = P_\varepsilon\delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})}$ and $\delta_\varepsilon = \delta_{(a_{1,\varepsilon},\lambda_{1,\varepsilon})}$. Since $L^{\frac{2n}{n+2}} \hookrightarrow H^{-1}$, it is sufficient to prove that

$$\int_{A_\varepsilon} \delta_\varepsilon^{\frac{8n}{n^2-4}} |u_\varepsilon - P_\varepsilon\delta|^{\frac{2n}{n+2}} \to 0 \quad \text{and} \quad \int_{A_\varepsilon} \delta_\varepsilon^{\frac{2n}{n+2}} |u_\varepsilon - P_\varepsilon\delta|^{\frac{8n}{n^2-4}} \to 0, \quad \text{when } \varepsilon \to 0.$$

Observe that

$$\begin{aligned}
\int_{A_\varepsilon} \delta_\varepsilon^{\frac{8n}{n^2-4}} |u_\varepsilon - P_\varepsilon\delta|^{\frac{2n}{n+2}} &\leqslant c \int_{A_\varepsilon} \delta_\varepsilon^{\frac{8n}{n^2-4}} |u_\varepsilon - \delta_\varepsilon|^{\frac{2n}{n+2}} + c \int_{A_\varepsilon} \delta_\varepsilon^{\frac{8n}{n^2-4}} |\delta_\varepsilon - P_\varepsilon\delta|^{\frac{2n}{n+2}} \\
&\leqslant c \int_{\widetilde{A}_\varepsilon} \delta_{(0,\alpha_n)}^{\frac{8n}{n^2-4}}(X) \left|v_\varepsilon(X) - M_\varepsilon^{-1} \delta_\varepsilon\big(a_{1,\varepsilon} + M_\varepsilon^{\frac{-2}{n-2}} X\big)\right|^{\frac{2n}{n+2}} \mathrm{d}X \\
&\quad + \mathrm{O}\big(|\delta_\varepsilon - P_\varepsilon\delta|_{L^{\frac{2n}{n-2}}}^{\frac{2n}{n+2}}\big) \\
&= c \int_{B(0,R)} + c \int_{\widetilde{A}_\varepsilon \setminus B(0,R)} + \mathrm{o}(1)
\end{aligned}$$

where R is a large enough positive constant such that $\int_{\mathbb{R}^n \setminus B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} = \mathrm{o}(1)$.

Now we are going to estimate the 2nd integral

$$\int_{\widetilde{A}_\varepsilon\setminus B(0,R)} \delta_{(0,\alpha_n)}^{\frac{8n}{n^2-4}}(X)\big|v_\varepsilon(X)-\delta_{(0,\alpha_n)}(X)\big|^{\frac{2n}{n+2}}\,\mathrm{d}X$$
$$\leqslant\Bigg(\int_{\widetilde{A}_\varepsilon\setminus B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}(X)\Bigg)^{\frac{4}{n+2}}\Bigg(\int_{\widetilde{A}_\varepsilon\setminus B(0,R)} \big|v_\varepsilon(X)-\delta_{(0,\alpha_n)}(X)\big|^{\frac{2n}{n-2}}\Bigg)^{\frac{n-2}{n+2}}$$
$$\leqslant c_1\Bigg(\int_{\mathbb{R}^n\setminus B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}(X)\Bigg)^{\frac{4}{n+2}}$$

indeed $\int_{\widetilde{A}_\varepsilon} v^{\frac{2n}{n-2}}\leqslant C$. For the first integral, we have

$$\int_{B(0,R)} \delta_{(0,\alpha_n)}^{\frac{8n}{n^2-4}}(X)\big|v_\varepsilon(X)-\delta_{(0,\alpha_n)}(X)\big|^{\frac{2n}{n+2}}\,\mathrm{d}X\leqslant C\int_{B(0,R)}\big|v_\varepsilon(X)-\delta_{(0,\alpha_n)}(X)\big|^{\frac{2n}{n+2}}\,\mathrm{d}X\to 0,$$

when $\varepsilon\to 0$, indeed $v_\varepsilon-\delta_{(0,\alpha_n)}\to 0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. In the same way, we prove that

$$\int_{A_\varepsilon}\delta_\varepsilon^{\frac{2n}{n+2}}|u_\varepsilon-P_\varepsilon\delta|^{\frac{8n}{n^2-4}}\to 0,\quad \text{when } \varepsilon\to 0.$$

(ii) We also have

$$\int_{A_\varepsilon}|\nabla u_\varepsilon^1|^2=\int_{A_\varepsilon}|\nabla u_\varepsilon|^2+\int_{A_\varepsilon}|\nabla P_\varepsilon\delta|^2-2\int_{A_\varepsilon}\nabla u_\varepsilon\nabla P_\varepsilon\delta.$$

Observe that

$$\int_{A_\varepsilon}|\nabla P_\varepsilon\delta|^2=\int_{A_\varepsilon}\delta_\varepsilon^{\frac{n+2}{n-2}}P_\varepsilon\delta=\int_{A_\varepsilon}\delta_\varepsilon^{\frac{2n}{n-2}}-\int_{A_\varepsilon}\delta_\varepsilon^{\frac{n+2}{n-2}}(\delta_\varepsilon-P_\varepsilon\delta)$$
$$=\int_{\widetilde{A}_\varepsilon}\delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}-\int_{A_\varepsilon}\delta_\varepsilon^{\frac{n+2}{n-2}}(\delta_\varepsilon-P_\varepsilon\delta)$$
$$=S_n-\int_{\mathbb{R}^n\setminus\widetilde{A}_\varepsilon}\delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}-\int_{A_\varepsilon}\delta_\varepsilon^{\frac{n+2}{n-2}}(\delta_\varepsilon-P_\varepsilon\delta).$$

For the 2nd integral, we have

$$\int_{A_\varepsilon}\delta_\varepsilon^{\frac{n+2}{n-2}}(\delta_\varepsilon-P_\varepsilon\delta)\leqslant C|\delta_\varepsilon-P_\varepsilon\delta|_{L^{\frac{2n}{n-2}}(A_\varepsilon)}\leqslant c(\lambda_{1,\varepsilon}d_{1,\varepsilon})^{\frac{2-n}{2}}\to 0,\quad \text{when } \varepsilon\to 0$$

where $d_{1,\varepsilon}=d(a_{1,\varepsilon},\partial A_\varepsilon)$.

For the first integral, we have

$$\int_{\mathbb{R}^n\setminus\widetilde{A}_\varepsilon}\delta_{(0,\alpha_n)}^{\frac{2n}{n-2}}=\mathrm{o}(1)$$

indeed $\widetilde{A}_\varepsilon \to \mathbb{R}^n$ and $\delta_{(0,\alpha_n)} \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. Then

$$\int_{A_\varepsilon} |\nabla P_\varepsilon \delta|^2 = S_n + \mathrm{o}(1).$$

We also have

$$\int_{A_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon \delta = \int_{A_\varepsilon} \nabla (u_\varepsilon - P_\varepsilon \delta) \nabla P_\varepsilon \delta + \int_{A_\varepsilon} |\nabla P_\varepsilon \delta|^2.$$

Observe that

$$\begin{aligned}
\int_{A_\varepsilon} \nabla (u_\varepsilon - P_\varepsilon \delta) \nabla P_\varepsilon \delta &= \int_{A_\varepsilon} (u_\varepsilon - P_\varepsilon \delta) \delta_\varepsilon^{\frac{n+2}{n-2}} \\
&= \int_{A_\varepsilon} (u_\varepsilon - \delta_\varepsilon) \delta_\varepsilon^{\frac{n+2}{n-2}} + \int_{A_\varepsilon} (\delta_\varepsilon - P_\varepsilon \delta) \delta_\varepsilon^{\frac{n+2}{n-2}} \\
&= \int_{\widetilde{A}_\varepsilon} (v_\varepsilon - \delta_{(0,\alpha_n)}) \delta_{(0,\alpha_n)}^{\frac{n+2}{n-2}} + \mathrm{o}(1) \\
&\leqslant \int_{B(0,R)} (v_\varepsilon - \delta_{(0,\alpha_n)}) \delta_{(0,\alpha_n)}^{\frac{n+2}{n-2}} + \int_{\mathbb{R}^n \setminus B(0,R)} (v_\varepsilon - \delta_{(0,\alpha_n)}) \delta_{(0,\alpha_n)}^{\frac{n+2}{n-2}} + \mathrm{o}(1).
\end{aligned}$$

Notice that, on the one hand

$$\int_{\mathbb{R}^n \setminus B(0,R)} (v_\varepsilon - \delta_{(0,\alpha_n)}) \delta_{(0,\alpha_n)}^{\frac{n+2}{n-2}} \leqslant C \Bigg(\int_{\mathbb{R}^n \setminus B(0,R)} \delta_{(0,\alpha_n)}^{\frac{2n}{n-2}} \Bigg)^{\frac{n+2}{2n}} = \mathrm{o}(1).$$

On the other hand

$$\int_{B(0,R)} (v_\varepsilon - \delta_{(0,\alpha_n)}) \delta_{(0,\alpha_n)}^{\frac{n+2}{n-2}} = \mathrm{o}(1), \quad \text{since } v_\varepsilon \to \delta_{(0,\alpha_n)} \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Then

$$\int_{A_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon \delta = S_n + \mathrm{o}(1).$$

Thus (ii) of Lemma 2.4 follows.

(iii) The proof of (iii) in Lemma 2.4 is similar to the proof of (ii), so we will omit it. □

Now, we distinguish two cases

(i) $\int_{A_\varepsilon} |\nabla u_\varepsilon^1|^2 \to 0$ when $\varepsilon \to 0$.

(ii) $\int_{A_\varepsilon} |\nabla u_\varepsilon^1|^2 \not\to 0$ when $\varepsilon \to 0$.

If $\int_{A_\varepsilon} |\nabla u_\varepsilon^1|^2 \to 0$, the proof of (i) (Theorem 1.3) is finished.

In the sequel, we consider the second case, that is $\int_{A_\varepsilon} |\nabla u_\varepsilon^1|^2 \nrightarrow 0$, when $\varepsilon \to 0$ and we are going to look for a second point of blow up of u_ε.

In order to simplify the notations, in remainder we often omit the index ε of a_ε and λ_ε.

Let us introduce the following notations

$$u_\varepsilon(a_2) := \lambda_2^{\frac{n-2}{2}} = \max_{(A_\varepsilon \setminus B(a_1,\varepsilon))} u_\varepsilon(x), \tag{2.5}$$

$$h_\varepsilon = \max_{B(a_1,2\varepsilon)} |x-a_1|^{\frac{n-2}{2}} u_\varepsilon(x) = |a_1-a_3|^{\frac{n-2}{2}} u_\varepsilon(a_3) = |a_1-a_3|^{\frac{n-2}{2}} \lambda_3^{\frac{n-2}{2}}. \tag{2.6}$$

We distinguish two cases.

Case 1. $h_\varepsilon \to +\infty$ when $\varepsilon \to 0$.

Case 2. $h_\varepsilon \leqslant c$, when $\varepsilon \to 0$.

Now we study the first case, that is $h_\varepsilon \to \infty$ when $\varepsilon \to 0$. Let

$$\lambda_4 = \max(\lambda_2, \lambda_3) := u_\varepsilon^{\frac{2}{n-2}}(a_4).$$

For $X \in B(0, \frac{\lambda_4}{2}|a_1-a_4|) \cap D_\varepsilon$, we set

$$w_\varepsilon(X) = \lambda_4^{\frac{-(n-2)}{2}} u_\varepsilon\bigl(a_4 + \lambda_4^{-1} X\bigr), \quad \text{with } D_\varepsilon = \lambda_4(A_\varepsilon - a_4).$$

It is easy to check the following claims

$$\lambda_4|a_1-a_4| \geqslant (1/2)\lambda_3|a_1-a_3|, \quad \text{and} \quad \lambda_4\varepsilon \geqslant (1/2)\lambda_3|a_1-a_3|.$$

Thus

$$\lambda_4|a_1-a_4| \to +\infty \quad \text{and} \quad \lambda_4\varepsilon \to +\infty \quad \text{as } \varepsilon \to 0.$$

We also have

$$w_\varepsilon(X) \leqslant c, \quad \forall X \in B\bigl(0, (1/2)\lambda_4|a_1-a_4|\bigr) \cap D_\varepsilon.$$

As in Lemma 2.3, we can prove

$$\lambda_4 d(a_4, \partial A_\varepsilon) \to +\infty, \quad \text{as } \varepsilon \to 0.$$

Thus, there exist $b \in \mathbb{R}^n$ and $\lambda > 0$ such that $w_\varepsilon \to \delta_{(b,\lambda)}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore we have found a second point of blow up $\bar{a}_2$ of u_ε with the concentration $\bar{\lambda}_2$ in this case ($\bar{a}_2 = a_4 + b/\lambda_4$ and $\bar{\lambda}_2 = \lambda\lambda_4$).

Next we study the second case, that is h_ε remains bounded when $\varepsilon \to 0$, where h_ε is defined in (2.6). In this case we consider two subcases.

(i) $\int_{B(a_1,2\varepsilon)} |u_\varepsilon^1|^{\frac{2n}{n-2}} \to 0$ as $\varepsilon \to 0$.

(ii) $\int_{B(a_1,2\varepsilon)} |u_\varepsilon^1|^{\frac{2n}{n-2}} \nrightarrow 0$ as $\varepsilon \to 0$.

Let us consider the first case, that is h_ε remains bounded and $\int_{B(a_1,2\varepsilon)} |u_\varepsilon^1|^{\frac{2n}{n-2}} \to 0$ as $\varepsilon \to 0$. Thus there exists $c > 0$ such that

$$0 < c \leqslant \int_{A_\varepsilon \setminus B(a_1,2\varepsilon)} |u_\varepsilon^1|^{\frac{2n}{n-2}} \leqslant c\lambda_2^2 \int_{A_\varepsilon} u_\varepsilon^2 \leqslant c(\varepsilon\lambda_2)^2.$$

Hence, there exists $\bar{c} > 0$ such that

$$\lambda_2|a_1 - a_2| \geqslant \lambda_2\varepsilon \geqslant 2\bar{c}.$$

Now, for $X \in E_\varepsilon = \lambda_2(A_\varepsilon - a_2)$, we introduce the following function

$$U_\varepsilon(X) = \lambda_2^{\frac{2-n}{2}} u_\varepsilon\left(a_2 + \lambda_2^{-1}X\right).$$

As in Lemma 2.3, we can prove $\lambda_2 d(a_2, \partial A_\varepsilon) \to +\infty$. It is easy to see that U_ε satisfies

$$U_\varepsilon \leqslant 1, \quad \text{in } B\left(0, (1/2)\lambda_2|a_1 - a_2|\right).$$

Thus, there exists $b \in \mathbb{R}^n$ and $\lambda > 0$ such that $U_\varepsilon \to \delta_{(b,\lambda)}$ in $C^1_{\text{loc}}(\mathbb{R}^n)$. Therefore we have also found a second point of blow up $\bar{a}_2$ of u_ε with the concentration $\bar{\lambda}_2$ in this case ($\bar{a}_2 = a_2 + b/\lambda_2$ and $\bar{\lambda}_2 = \lambda\lambda_2$).

Now, we study the second case, that is h_ε remains bounded and $\int_{B(a_1,2\varepsilon)} |u_\varepsilon^1|^{\frac{2n}{n-2}} \nrightarrow 0$ as $\varepsilon \to 0$. We introduce the following function defined on $F_\varepsilon = \varepsilon^{-1}(A_\varepsilon - a_1)$ by

$$W_\varepsilon(X) = \varepsilon^{\frac{n-2}{2}} u_\varepsilon^1(a_1 + \varepsilon X).$$

Observe that F_ε "converges" to a strip of $\mathbb{R}^n$ when $\varepsilon \to 0$. We notice that W_ε satisfies

$$\begin{cases} -\Delta W_\varepsilon = |W_\varepsilon|^{\frac{4}{n-2}} W_\varepsilon + f_\varepsilon & \text{in } F_\varepsilon, \\ W_\varepsilon = 0 & \text{on } \partial F_\varepsilon, \end{cases}$$

with $|f_\varepsilon|_{H^{-1}(F_\varepsilon)} \to 0$ as $\varepsilon \to 0$. We also have

$$\int_{B(0,2)\cap F_\varepsilon} |W_\varepsilon|^{\frac{2n}{n-2}} = \int_{B(a_1,2\varepsilon)\cap A_\varepsilon} \left|u_\varepsilon^1\right|^{\frac{2n}{n-2}} \nrightarrow 0, \quad \text{as } \varepsilon \to 0$$

and

$$\int_{F_\varepsilon} |\nabla W_\varepsilon|^2 = \int_{A_\varepsilon} \left|\nabla u_\varepsilon^1\right|^2 \leqslant C.$$

It is easy to check that there exists some fixed domain $F \subset B(0,2) \cap F_\varepsilon$ such that $|W_\varepsilon|^{\frac{2n}{n-2}} \to 0$ almost everywhere and $|W_\varepsilon|^{\frac{2n}{n-2}} \nrightarrow 0$ in $L^1(F)$. From Dunford–Pettis Lemma [5], we have

$$\exists\delta_0 > 0,\ \exists\alpha_\varepsilon > 0,\ \alpha_\varepsilon \to 0,\ \exists b_\varepsilon \in F \text{ s.t.} \int_{B(b_\varepsilon,\alpha_\varepsilon)\cap F_\varepsilon} |W_\varepsilon|^{\frac{2n}{n-2}} \geqslant \delta_0. \tag{2.7}$$

We can choose b_ε and α_ε such that α_ε is minimum and $\int_{B(b_\varepsilon,\alpha_\varepsilon)\cap F_\varepsilon} |W_\varepsilon|^{\frac{2n}{n-2}} = \delta_0$.

LEMMA 2.5. – *Let* $(\alpha_\varepsilon, b_\varepsilon)$ *be defined by* (2.7) *and let* $\bar\lambda_2 = (\varepsilon\alpha_\varepsilon)^{-1}$, *and* $\bar a_2 = a_1 + \varepsilon b_\varepsilon$. *Then we have*

$$\frac{\lambda_1}{\bar\lambda_2} \to +\infty \quad or \quad \frac{\bar\lambda_2}{\lambda_1} \to +\infty \quad or \quad \lambda_1\bar\lambda_2|a_1 - \bar a_2|^2 \to +\infty \quad when\ \varepsilon \to 0$$

where $\lambda_1 = M_\varepsilon^{\frac{2}{n-2}}$.

Proof. – We argue by contradiction. Let us suppose that $\lambda_1/\bar\lambda_2$, $\bar\lambda_2/\lambda_1$ and $\lambda_1\bar\lambda_2|a_1 - \bar a_2|^2$ are bounded when $\varepsilon \to 0$.

For $X \in \widetilde{A}_\varepsilon := \lambda_1(A_\varepsilon - a_1)$, we introduce ω_ε defined by

$$\omega_\varepsilon(X) = M_\varepsilon^{-1} u_\varepsilon^1\big(a_1 + \lambda_1^{-1}X\big). \tag{2.8}$$

Observe that, on the one hand

$$\int_{B(\lambda_1(\bar a_2 - a_1),\lambda_1/\bar\lambda_2)\cap\widetilde{A}_\varepsilon} |\omega_\varepsilon(X)|^{\frac{2n}{n-2}}\,\mathrm{d}X = \int_{B(\bar a_2,1/\bar\lambda_2)\cap A_\varepsilon} \big|u_\varepsilon^1(x)\big|^{\frac{2n}{n-2}}\,\mathrm{d}x$$
$$= \int_{B(b_\varepsilon,\alpha_\varepsilon)\cap F_\varepsilon} |W_\varepsilon(X)|^{\frac{2n}{n-2}}\,\mathrm{d}X = \delta_0 > 0.$$

On the other hand, since $\lambda_1|\bar a_2 - a_1|$ and $\lambda_1/\bar\lambda_2$ are bounded, we have

$$\exists R > 0 \text{ such that} \quad B\big(\lambda_1(\bar a_2 - a_1), \lambda_1/\bar\lambda_2\big) \subset B(0, R).$$

Thus

$$\int_{B(\lambda_1(\bar a_2 - a_1),\lambda_1/\bar\lambda_2)\cap\widetilde{A}_\varepsilon} |\omega_\varepsilon(X)|^{\frac{2n}{n-2}}\,\mathrm{d}X$$
$$\leqslant \int_{B(0,R)\cap\widetilde{A}_\varepsilon} |\omega_\varepsilon(X)|^{\frac{2n}{n-2}}\,\mathrm{d}X$$
$$= \int_{B(0,R)\cap\widetilde{A}_\varepsilon} \big|M_\varepsilon^{-1}u_\varepsilon\big(a_1 + \lambda_1^{-1}X\big) - M_\varepsilon^{-1}P_\varepsilon\delta_{(a_1,\lambda_1)}\big(a_1 + \lambda_1^{-1}X\big)\big|^{\frac{2n}{n-2}}\,\mathrm{d}X$$
$$\leqslant c\int_{B(0,R)} \big|v_\varepsilon - \delta_{(0,\alpha_n)}(X)\big|^{\frac{2n}{n-2}} + c\int_{A_\varepsilon} \big|\delta_\varepsilon(x) - P_\varepsilon\delta(x)\big|^{\frac{2n}{n-2}}.$$

Thus

$$\int_{B(\lambda_1(\bar a_2 - a_1),\lambda_1\bar\lambda_2)\cap\widetilde{A}_\varepsilon} |W_\varepsilon(X)|^{\frac{2n}{n-2}}\,\mathrm{d}X \to 0 \quad \text{when } \varepsilon \to 0$$

which yields a contradiction and our lemma follows. □

Now we set $\widetilde{\widetilde{A}}_\varepsilon = \bar\lambda_2(A_\varepsilon - \bar a_2)$ and we introduce the function V_ε defined by

$$V_\varepsilon(X) = \bar\lambda_2^{\frac{(2-n)}{2}} u_\varepsilon^1\bigl(\bar a_2 + \bar\lambda_2^{-1}X\bigr).$$

Observe that

$$\int\limits_{B(0,1)\cap\widetilde{\widetilde{A}}_\varepsilon} |V_\varepsilon|^{\frac{2n}{n-2}} = \int\limits_{B(\bar a_2,1/\bar\lambda_2)\cap A_\varepsilon} |u_\varepsilon^1|^{\frac{2n}{n-2}} = \delta_0 > 0 \tag{2.9}$$

and we also have

$$\int |\nabla V_\varepsilon|^2 \leqslant C, \qquad \int |V_\varepsilon|^{\frac{2n}{n-2}} \leqslant C.$$

It is easy to see that there exists some functions V such that (after passing to a subsequence), $V_\varepsilon \to V$ in $H^1_{\mathrm{loc}}(\Omega)$ and V satisfies

$$\begin{cases} -\Delta V = |V|^{\frac{4}{n-2}} V \quad \text{in } \Omega, \quad V = 0 \quad \text{on } \partial\Omega, \\ \int_\Omega |\nabla V|^2 \leqslant C, \quad \int_\Omega |V|^{\frac{2n}{n-2}} \leqslant C, \end{cases} \tag{2.10}$$

where Ω is a half space or a strip or a $\mathbb{R}^n$.

From (2.9), it is easy to see that $V \neq 0$.

LEMMA 2.6. – *Let V be defined by* (2.10). *Then, we have* $V \geqslant 0$.

Proof. – We have

$$\begin{aligned} V_\varepsilon(X) = {} & \bar\lambda_2^{\frac{(2-n)}{2}} u_\varepsilon\bigl(\bar a_2 + \bar\lambda_2^{-1}X\bigr) - \bar\lambda_2^{\frac{(2-n)}{2}} \delta_{(a_1,\lambda_1)}\bigl(\bar a_2 + \bar\lambda_2^{-1}X\bigr) \\ & + \bar\lambda_2^{\frac{(2-n)}{2}}\bigl(\delta_{(a_1,\lambda_1)}\bigl(\bar a_2 + \bar\lambda_2^{-1}X\bigr) - P_\varepsilon\delta_{(a_1,\lambda_1)}\bigl(\bar a_2 + \bar\lambda_2^{-1}X\bigr)\bigr). \end{aligned} \tag{2.11}$$

Thus, it is sufficient to prove that

$$\bar\lambda_2^{\frac{(2-n)}{2}} \delta_{(a_1,\lambda_1)}\bigl(\bar a_2 + \bar\lambda_2^{-1}X\bigr) \to 0 \quad \text{in } H^1_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr).$$

Observe that

$$\begin{aligned} I_\varepsilon := {} & \int\limits_{B(0,R)} \left(\bar\lambda_2^{\frac{(2-n)}{2}} \frac{\lambda_1^{\frac{n-2}{2}}}{(1+\lambda_1^2|\bar a_2 + \bar\lambda_2^{-1}X - a_1|^2)^{\frac{n-2}{2}}}\right)^{\frac{n+2}{n-2}} \mathrm{d}X \\ = {} & \left(\frac{\lambda_1}{\bar\lambda_2}\right)^{\frac{n+2}{2}} \int\limits_{B(0,R)} \frac{\mathrm{d}X}{(1+(\lambda_1/\bar\lambda_2)^2|X - \bar\lambda_2(a_1-\bar a_2)|^2)^{\frac{n+2}{2}}}. \end{aligned} \tag{2.12}$$

If $\lambda_1/\bar\lambda_2 \to 0$ it is clear that $I_\varepsilon \to 0$ when $\varepsilon \to 0$. If $\lambda_1\bar\lambda_2 \to +\infty$, let $y = (\lambda_1/\bar\lambda_2)X$. Thus

$$I_\varepsilon \leqslant \left(\frac{\bar\lambda_2}{\lambda_1}\right)^{\frac{n-2}{2}} \int\limits_{\mathbb{R}^n} \left(\frac{1}{1+|y-\lambda_1(a_1-\bar a_2)|^2}\right)^{\frac{n+2}{2}} \mathrm{d}y \to 0 \quad \text{when } \varepsilon \to 0.$$

Lastly if $\lambda_1/\bar\lambda_2 \not\to +\infty$ and $\bar\lambda_2/\lambda_1 \not\to +\infty$, then, by Lemma 2.5, we have

$$\lambda_1\bar\lambda_2|a_1-\bar a_2|^2 \to +\infty, \quad \text{when } \varepsilon \to 0.$$

Observe that for $X \in B(0,R)$, we have

$$\left|X-\bar\lambda_2(a_1-\bar a_2)\right| \geqslant \left|\bar\lambda_2(a_1-\bar a_2)\right| - |X| \geqslant c\left|\bar\lambda_2(a_1-\bar a_2)\right|.$$

Therefore

$$\begin{aligned} I_\varepsilon &\leqslant \left(\frac{\lambda_1}{\bar\lambda_2}\right)^{\frac{n+2}{2}} \int\limits_{B(0,R)} \frac{1}{(c(\frac{\lambda_1}{\bar\lambda_2})^2|\bar\lambda_2(a_1-\bar a_2)|^2)^{\frac{n+2}{2}}} \\ &\leqslant \frac{1}{(\lambda_1\bar\lambda_2|a_1-\bar a_2|^2)^{\frac{n+2}{2}}} C(R) \to 0, \quad \text{when } \varepsilon \to 0. \end{aligned}$$

Then, our lemma follows. □

Now, from Theorem III 1.3 [15], we derive $\Omega = \mathbb{R}^n$. Thus, using (2.10) and Lemma 2.6, we also obtain a second point of blow up of u_ε in this case. Thus in all cases we have built a second point $a_{2,\varepsilon}$ of blow up of u_ε with the concentration $\lambda_{2,\varepsilon}$ such that $\lambda_{2,\varepsilon} \to +\infty$ and $\lambda_{2,\varepsilon} d(a_{2,\varepsilon}, \partial A_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. It is clear that we can proceed by inductions. Thus, we obtain a sequence $(u_\varepsilon^k)_k$ such that

$$\int\limits_{A_\varepsilon} \left|\nabla u_\varepsilon^k\right|^2 = \int\limits_{A_\varepsilon} \left|\nabla u_\varepsilon^{k-1}\right|^2 - S_n + \mathrm{o}(1) = \int\limits_{A_\varepsilon} |\nabla u_\varepsilon|^2 - kS_n + \mathrm{o}(1).$$

Thus

$$0 \leqslant \int\limits_{A_\varepsilon} \left|\nabla u_\varepsilon^k\right|^2 = \int\limits_{A_\varepsilon} |\nabla u_\varepsilon|^2 - kS_n + \mathrm{o}(1) \leqslant C - kS_n + \mathrm{o}(1). \tag{2.13}$$

Since the later term in (2.13) will be negative for large k, the induction will terminate after some index $p \in \mathbb{N}^*$. Moreover, for this index, we have

$$\left|\nabla\left(u_\varepsilon - \sum_{i=1}^{p} P_\varepsilon\delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})}\right)\right|_{L^2(A_\varepsilon)} \to 0, \quad \lambda_{i,\varepsilon} \to +\infty,$$

$\lambda_{i,\varepsilon} d(a_{i,\varepsilon}, \partial A_\varepsilon) \to +\infty$,

$$\forall i \neq j, \quad \varepsilon_{ij} = \left(\frac{\lambda_{i,\varepsilon}}{\lambda_{j,\varepsilon}} + \frac{\lambda_{j,\varepsilon}}{\lambda_{i,\varepsilon}} + \lambda_{i,\varepsilon}\lambda_{j,\varepsilon}|a_{i,\varepsilon} - a_{j,\varepsilon}|^2\right)^{-\frac{n-2}{2}} \to 0$$

as desired in the first part of Theorem 1.3.

3. Proof of the second part of Theorem 1.3

Let, for $p \in \mathbb{N}^*$ and $\eta > 0$ given

$$V_\varepsilon(p,\eta) = \Bigg\{ u \in \Sigma^+(A_\varepsilon) \text{ s.t. } \exists x_1,\ldots,x_p \in A_\varepsilon,\ \exists \lambda_1,\ldots,\lambda_p > 0 \text{ with}$$
$$\left| u - C(p)\sum_{i=1}^{p} P_\varepsilon \delta_{(x_i,\lambda_i)} \right|_{H_0^1(A_\varepsilon)} < \eta,\ \forall i\ \lambda_i > \frac{1}{\eta},\ \lambda_i d(x_i,\partial A_\varepsilon) \geqslant \frac{1}{\eta},$$
$$\forall i \neq j \quad \varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j |x_i - x_j|^2 \right)^{-\frac{n-2}{2}} < \eta \Bigg\}$$

where $\Sigma^+(A_\varepsilon) = \{u \in H_0^1(A_\varepsilon) \mid u \geqslant 0, |u|_{H_0^1(A_\varepsilon)} = 1\}$.

If a function u belongs to $V_\varepsilon(p,\eta)$, then, for $\eta > 0$ small enough, the minimization problem

$$\min_{\alpha_i,\lambda_i>0,\ x_i\in A_\varepsilon} \left| u - \sum_{i=1}^{p} \alpha_i P_\varepsilon \delta_{(x_i,\lambda_i)} \right|_{H_0^1(A_\varepsilon)} \tag{3.1}$$

has a unique solution, up to permutation (see Lemma A.2 in [3]).

Therefore, for $\varepsilon > 0$ sufficiently small, Section 2 implies that u_ε(solution of P_ε) can be uniquely writen as

$$\tilde{u}_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|_{H_0^1}} = \sum \alpha_{i,\varepsilon} P_\varepsilon \delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})} + v_\varepsilon \tag{3.2}$$

where v_ε satisfies the following conditions:

$$(V_0) \quad (v_\varepsilon, P_\varepsilon\delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})})_{H_0^1} = \left(v_\varepsilon, \frac{\partial P_\varepsilon \delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})}}{\partial \lambda_i} \right)_{H_0^1}$$
$$= \left(v_\varepsilon, \frac{\partial P_\varepsilon \delta_{(a_{i,\varepsilon},\lambda_{i,\varepsilon})}}{\partial a_i} \right)_{H_0^1} = 0$$

and $\alpha_{i,\varepsilon}$ satisfies:

$$\big(J(u_\varepsilon)\big)^{\frac{n}{n-2}} \alpha_j^{\frac{4}{n-2}} = 1 + \mathrm{o}(1), \quad \forall j.$$

In order to simplify the notations, in the remainder, we write $\alpha_i, a_i, \lambda_i, \delta_i$ and $P\delta_i$ instead of $\alpha_{i,\varepsilon}, a_{i,\varepsilon}, \lambda_{i,\varepsilon}, \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}}$ and $P\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}}$ respectively and we also write u_ε instead of $\tilde{u}_\varepsilon$.

As usual in this type of problems, we first deal with the v_ε-part of u_ε.

PROPOSITION 3.1. – *Let v_ε be defined by* (3.2). *Then, we have the following estimate*

$$|v_\varepsilon|_{H_0^1(A_\varepsilon)} \leqslant C \begin{cases} \sum_i \frac{1}{(\lambda_i d_i)^{n-2}} + \sum \varepsilon_{ij} \left(\mathrm{Log}(\varepsilon_{ij}^{-1})\right)^{\frac{n-2}{n}} & \text{if } n < 6, \\ \sum_i \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}} + \sum \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} \left(\mathrm{Log}(\varepsilon_{ij}^{-1})\right)^{\frac{n+2}{2n}} & \text{if } n \geqslant 6. \end{cases}$$

Proof. – From (3.2), we derive

$$\begin{aligned}
-\Delta v_\varepsilon &= -\Delta u_\varepsilon + \Delta\Big(\sum \alpha_i P\delta_i\Big)\\
&= \big(J(u_\varepsilon)\big)^{\frac{n}{n-2}}\Big(\sum \alpha_i P\delta_i + v_\varepsilon\Big)^{\frac{n+2}{n-2}} - \sum \alpha_i \delta_i^{\frac{n+2}{n-2}}\\
&= \big(J(u_\varepsilon)\big)^{\frac{n}{n-2}}\bigg[\Big(\sum \alpha_i P\delta_i\Big)^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\Big(\sum \alpha_i P\delta_i\Big)^{\frac{4}{n-2}} v_\varepsilon + \mathrm{O}\big(|v_\varepsilon|^{\frac{n+2}{n-2}}\big)\\
&\quad + \mathrm{O}\Big(\mathrm{Sup}\Big(\sum \alpha_i P\delta_i, v_\varepsilon\Big)^{\frac{4}{n-2}-1}|v_\varepsilon|^2\Big)\bigg] - \sum \alpha_i \delta_i^{\frac{n+2}{n-2}}.
\end{aligned}$$

Thus, since $J(u_\varepsilon)$ is bounded,

$$\int_{A_\varepsilon} |\nabla v_\varepsilon|^2 = \big(J(u_\varepsilon)\big)^{\frac{n}{n-2}}\bigg[\langle f, v\rangle + \frac{n+2}{n-2}\int_{A_\varepsilon}\Big(\sum \alpha_i P\delta_i\Big)^{\frac{4}{n-2}} v_\varepsilon^2\bigg] + \mathrm{O}\big(|v_\varepsilon|_{H_0^1}^{\inf(3,2n/(n-2)}\big)$$

where

$$\langle f, v\rangle = \int_{A_\varepsilon}\Big(\sum \alpha_i P\delta_i\Big)^{\frac{n+2}{n-2}} v. \tag{3.3}$$

Then

$$Q(v_\varepsilon, v_\varepsilon) = \big(J(u_\varepsilon)\big)^{\frac{n}{n-2}}\langle f, v_\varepsilon\rangle + \mathrm{O}\big(|v_\varepsilon|_{H_0^1}^{\inf(3,2n/(n-2))}\big),$$

where

$$Q(v_\varepsilon, v_\varepsilon) = \int_{A_\varepsilon} |\nabla v_\varepsilon|^2 - \frac{n+2}{n-2}\big(J(u_\varepsilon)\big)^{\frac{n}{n-2}}\int_{A_\varepsilon}\Big(\sum \alpha_i P\delta_i\Big)^{\frac{4}{n-2}} v_\varepsilon^2.$$

Observe that, since $J(u_\varepsilon)^{\frac{n}{n-2}}\alpha_i^{\frac{4}{n-2}} = 1 + \mathrm{o}(1)$, then $Q(v, v)$ is close to

$$\int_{A_\varepsilon} |\nabla v|^2 - \frac{n+2}{n-2}\sum_i \int_{A_\varepsilon} P\delta_i^{\frac{4}{n-2}} v^2$$

and therefore Q is a positive definite quadratic form on v (see [2]). Thus

$$\begin{aligned}
\alpha_0 |v_\varepsilon|_{H_0^1}^2 &\leqslant Q(v_\varepsilon, v_\varepsilon) + \mathrm{O}\big(|v_\varepsilon|_{H_0^1}^{\inf(3,2n/n-2)}\big)\\
&= \big(J(u_\varepsilon)\big)^{\frac{n}{n-2}}\langle f, v_\varepsilon\rangle\\
&\leqslant C|f||v_\varepsilon|_{H_0^1}.
\end{aligned}$$

Hence

$$|v_\varepsilon|_{H_0^1} \leqslant C'|f|.$$

Now we estimate $|f|$. We have

$$\langle f, v_\varepsilon\rangle = \sum \alpha_i^{\frac{n+2}{n-2}}\int_{A_\varepsilon} P\delta_i^{\frac{n+2}{n-2}} v_\varepsilon + \mathrm{O}\bigg(\sum \int_{P\delta_j \leqslant P\delta_i} P\delta_i^{\frac{4}{n-2}} P\delta_j |v_\varepsilon|\bigg).$$

Observe that

$$\int_{A_\varepsilon} P\delta_i^{\frac{n+2}{n-2}} v_\varepsilon = \int_{A_\varepsilon} v_\varepsilon \delta_i^{\frac{n+2}{n-2}} + \mathrm{O}\biggl(\int_{A_\varepsilon} \delta_i^{\frac{4}{n-2}}(\delta_i - P\delta_i)|v_\varepsilon|\biggr)$$

$$= \mathrm{O}\biggl(\int_{B(a_i,d_i)} \delta_i^{\frac{4}{n-2}}(\delta_i - P\delta_i)|v_\varepsilon|\biggr) + \mathrm{O}\biggl(\int_{\mathbb{R}^n\setminus B(a_i,d_i)} \delta_i^{\frac{n+2}{n-2}}|v_\varepsilon|\biggr)$$

where $d_i = d(a_i, \partial A_\varepsilon)$. Thus

$$\int_{A_\varepsilon} P\delta_i^{\frac{n+2}{n-2}} v_\varepsilon = \mathrm{O}\biggl(|v_\varepsilon|_{H_0^1}\biggl[|\delta_i - P\delta_i|_{L^\infty}\biggl(\int_{B(a_i,d_i)} \delta_i^{\frac{8n}{n^2-4}}\biggr)^{\frac{n+2}{2n}} + \biggl(\int_{B^c(a_i,d_i)} \delta_i^{\frac{2n}{n-2}}\biggr)^{\frac{n+2}{2n}}\biggr]\biggr). \tag{3.4}$$

Notice that

$$\int_{\mathbb{R}^n\setminus B(a_i,d_i)} \delta_i^{\frac{2n}{n-2}} = \mathrm{O}\biggl(\frac{1}{(\lambda_i d_i)^n}\biggr)$$

and

$$\biggl(\int_{B(a_i,d_i)} \delta_i^{\frac{8n}{n^2-4}}\biggr)^{\frac{n+2}{2n}} = \mathrm{O}\biggl(\frac{d_i^{\frac{n-6}{2}}}{\lambda_i^2}(\text{if } n>6) + \frac{\mathrm{Log}(\lambda_i d_i)}{\lambda_i^2}(\text{if } n=6) + \frac{1}{\lambda_i^{\frac{n-2}{2}}}(\text{if } n<6)\biggr). \tag{3.5}$$

Therefore

$$\int_{A_\varepsilon} P\delta_i^{\frac{n+2}{n-2}} v_\varepsilon = \mathrm{O}\biggl(|v_\varepsilon|_{H_0^1}\biggl[\frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}} + (\text{if } n=6)\frac{\mathrm{Log}(\lambda_i d_i)}{(\lambda_i d_i)^4} + (\text{if } n<5)\frac{1}{(\lambda_i d_i)^{n-2}}\biggr]\biggr). \tag{3.6}$$

We also have

$$\int_{P\delta_j\leqslant P\delta_i} P\delta_i^{\frac{4}{n-2}} P\delta_j|v_\varepsilon| \leqslant |v_\varepsilon|\biggl[\int_{P\delta_j\leqslant P\delta_i} \bigl((P\delta_i)^{\frac{4}{n-2}} P\delta_j\bigr)^{\frac{2n}{n+2}}\biggr]^{\frac{n+2}{2n}}. \tag{3.7}$$

If $n \geqslant 6$, we have $\frac{2n}{n+2} \geqslant \frac{n}{n-2}$ and thus

$$\int_{P\delta_j\leqslant P\delta_i} \bigl(P\delta_i^{\frac{4}{n-2}} P\delta_j\bigr)^{\frac{2n}{n+2}} \leqslant \int_{P\delta_j\leqslant P\delta_i} (P\delta_i P\delta_j)^{\frac{n}{n-2}} = \mathrm{O}\bigl(\varepsilon_{ij}^{\frac{n}{n-2}}\,\mathrm{Log}\bigl(\varepsilon_{ij}^{-1}\bigr)\bigr). \tag{3.8}$$

If $n \leqslant 5$, we have $1 < \frac{4}{n-2}$, thus

$$\int_{P\delta_j\leqslant P\delta_i}\left(P\delta_i^{\frac{4}{n-2}}P\delta_j\right)^{\frac{2n}{n+2}}\leqslant c\left(\int_{P\delta_j\leqslant P\delta_i}(\delta_i\delta_j)^{\frac{n}{n-2}}\right)^{\frac{2(n-2)}{n+2}}\leqslant c\varepsilon_{ij}^{\frac{2n}{n+2}}\left(\mathrm{Log}\left(\varepsilon_{ij}^{-1}\right)\right)^{\frac{2(n-2)}{n+2}}. \tag{3.9}$$

Using (3.6), (3.7), (3.8) and (3.9) we conclude that

$$|f|\leqslant\begin{cases}\sum\frac{1}{(\lambda_i d_i)^{n-2}}+\sum\varepsilon_{ij}\left(\mathrm{Log}(\varepsilon_{ij}^{-1})\right)^{\frac{(n-2)}{n}} & \text{if } n<6,\\ \sum\frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}+\sum\varepsilon_{ij}^{\frac{(n+2)}{2(n-2)}}\left(\mathrm{Log}(\varepsilon_{ij}^{-1})\right)^{\frac{(n+2)}{2n}} & \text{if } n\geqslant 6\end{cases}$$

and Proposition 3.1 follows. □

PROPOSITION 3.2. – *For $n\geqslant 4$, we have the following expansion*

$$\begin{aligned}\left(\nabla J(u_\varepsilon),\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right)_{H_0^1(A_\varepsilon)}&=2J(u_\varepsilon)c_1\left[-\frac{(n-2)}{2}\alpha_i\frac{H_\varepsilon(a_i,a_i)}{\lambda_i^{n-2}}\left(1+\mathrm{o}(1)\right)\right.\\&\left.-\sum\alpha_j\left(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}+\frac{(n-2)}{2}\frac{H_\varepsilon(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\right)\left(1+\mathrm{o}(1)\right)+R\right],\end{aligned}$$

where

$$R=\mathrm{O}\left(\sum_k\frac{\mathrm{Log}(\lambda_k d_k)}{(\lambda_k d_k)^n}+\sum_{i\neq j}\varepsilon_{ij}^{\frac{n}{n-2}}\mathrm{Log}\left(\varepsilon_{ij}^{-1}\right)\right).$$

Proof. – We have

$$\nabla J(u_\varepsilon)=2J(u_\varepsilon)\left[u_\varepsilon+J(u_\varepsilon)^{\frac{n}{n-2}}\Delta^{-1}\left(u_\varepsilon^{\frac{n+2}{n-2}}\right)\right]. \tag{3.10}$$

Thus

$$\begin{aligned}\left(\nabla J(u_\varepsilon),\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right)&=2J(u_\varepsilon)\left[\sum\alpha_j\left(P\delta_j,\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right)\right.\\&-J(u_\varepsilon)^{\frac{n}{n-2}}\left[\lambda_i\int\left(\sum\alpha_jP\delta_j\right)^{\frac{n+2}{n-2}}\frac{\partial P\delta_i}{\partial\lambda_i}\right.\\&+\frac{n+2}{n-2}\int\left(\sum\alpha_jP\delta_j\right)^{\frac{4}{n-2}}v_\varepsilon\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\\&\left.\left.+\mathrm{O}\left(|v_\varepsilon|_{H_0^1}^{\inf(2,(n+2)/(n-2))}\right)\right]\right].\end{aligned} \tag{3.11}$$

Notice that if $n\geqslant 4$, we have

$$\begin{aligned}\left(\sum\alpha_jP\delta_j\right)^{\frac{n+2}{n-2}}&=\sum(\alpha_jP\delta_j)^{\frac{n+2}{n-2}}+\frac{n+2}{n-2}\sum_{i\neq j}(\alpha_iP\delta_i)^{\frac{4}{n-2}}\alpha_jP\delta_j\\&+\mathrm{O}\left(\sum_{i\neq j}(\alpha_iP\delta_i)^{\frac{2}{n-2}}(\alpha_jP\delta_j)^{\frac{n}{n-2}}\right)\\&+\mathrm{O}\left(\sum_{k\notin\{i,j\},i\neq j}(\alpha_jP\delta_j)^{\frac{4}{n-2}}(\alpha_kP\delta_k)\right)\end{aligned} \tag{3.12}$$

and

$$\Big(\sum \alpha_j P\delta_j\Big)^{\frac{4}{n-2}} = (\alpha_i P\delta_i)^{\frac{4}{n-2}} + \mathrm{O}\Big(\sum_{j\neq i} P\delta_i^{\frac{4}{n-2}-1} P\delta_j + P\delta_j^{\frac{4}{n-2}}\Big). \tag{3.13}$$

Combining (3.12), (3.13) and (3.11) and using the fact that $|\lambda_i \partial P\delta_i/\partial\lambda_i| \leqslant c\delta_i$ and $P\delta_k \leqslant \delta_k$, we need to estimate the following integrals

$$\begin{aligned}\int_{A_\varepsilon} \delta_j \delta_i^{\frac{4}{n-2}} |v_\varepsilon| &\leqslant \int_{\delta_j \leqslant |v_\varepsilon|} + \int_{\delta_i \leqslant |v_\varepsilon|} + \int_{|v_\varepsilon| \leqslant \inf(\delta_i,\delta_j)} \\ &\leqslant \mathrm{O}\big(|v_\varepsilon|_{H_0^1}^{\frac{n+2}{n-2}}\big) + \mathrm{O}\big(|v_\varepsilon|_{H_0^1}^2\big) + \mathrm{O}\Big(\int_{A_\varepsilon} (\delta_j\delta_i)^{\frac{n}{n-2}}\Big),\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\int_{B(a_i,d_i)} P\delta_i^{\frac{4}{n-2}} v_\varepsilon \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \\ &= -\int_{B_i} P\delta_i^{\frac{4}{n-2}} v_\varepsilon \lambda_i \frac{\partial(\delta_i - P\delta_i)}{\partial\lambda_i} + \int_{B_i} P\delta_i^{\frac{4}{n-2}} v_\varepsilon \lambda_i \frac{\partial \delta_i}{\partial\lambda_i} \\ &\leqslant \frac{|v_\varepsilon|_{H_0^1}}{(\lambda_i d_i^2)^{\frac{n-2}{2}}} \Big(\int \delta^{\frac{8n}{n^2-4}}\Big)^{\frac{n+2}{2n}} + \int_{B_i} \delta_i^{\frac{4}{n-2}} v_\varepsilon \lambda_i \frac{\partial\delta_i}{\partial\lambda_i} + \mathrm{O}\Big(\int_{B_i} \delta_i^{\frac{4}{n-2}} v_\varepsilon (\delta_i - P\delta_i)\Big) \\ &= \mathrm{O}\big(|v_\varepsilon|_{H_0^1}\big)\Big(\frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}} + (\text{if } n=6)\frac{\mathrm{Log}(\lambda_i d_i)}{(\lambda_i d_i)^4} + (\text{if } n \leqslant 5)\frac{1}{(\lambda_i d_i)^{n-2}}\Big),\end{aligned} \tag{3.15}$$

$$\int_{\mathbb{R}^n \setminus B(a_i,d_i)} P\delta_i^{\frac{4}{n-2}} v_\varepsilon \lambda_i \frac{\partial P\delta_i}{\partial\lambda_i} = \mathrm{O}\Big(|v_\varepsilon|_{H_0^1} \frac{1}{(\lambda_i d_i)^{\frac{n+2}{2}}}\Big). \tag{3.16}$$

Now, using Proposition 3.1, Lemmas A.5–A.9 in Appendix A, and the fact that

$$J(u_\varepsilon)^{\frac{n}{n-2}} \alpha_j^{\frac{4}{n-2}} = 1 + \mathrm{o}(1),$$

Proposition 3.2 follows. □

PROPOSITION 3.3. – *For $n \geqslant 4$, we have the following expansion*

$$\begin{aligned}\Big(\nabla J(u_\varepsilon), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big)_{H_0^1} = J(u)c_1\Big[&\frac{\alpha_i}{\lambda_i^{n-1}} \frac{\partial H_\varepsilon(a_i,a_i)}{\partial a_i}\big(1+\mathrm{o}(1)\big) \\ &- 2\sum \alpha_j \Big(\frac{1}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{1}{\lambda_i(\lambda_i\lambda_j)^{\frac{n-2}{2}}} \frac{\partial H_\varepsilon(a_i,a_j)}{\partial a_i}\Big)\big(1+\mathrm{o}(1)\big) \\ &+ \mathrm{O}\Big(R + \sum \lambda_j |a_i - a_j| \varepsilon_{ij}^{\frac{n+1}{n-2}}\Big)\Big],\end{aligned}$$

where R is defined in Proposition 3.2.

Proof. – As in the proof of Proposition 3.2, we obtain (3.11) but with $\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}$ instead of $\lambda_i\frac{\partial P\delta_i}{\partial \lambda_i}$ and using Lemmas A.10–A.14 in Appendix A our proposition follows. □

Next we are going to give the proof of the second part of Theorem 1.3. From Proposition 3.2 we easily derive that $p \geqslant 2$. Now for $i \in \{1, \dots, p\}$, we introduce the following condition

$$2^{-p-1}\sum_{k\neq i}\varepsilon_{ki} \leqslant \sum_{j=1}^{p} H_\varepsilon(a_i, a_j)(\lambda_i\lambda_j)^{\frac{2-n}{2}}. \tag{3.17}$$

We divide the set $\{1, \dots, p\}$ into $T_1 \cup T_2$ with

$$T_1 = \{i \mid \text{ s.t. } i \text{ satisfies (3.17)}\},$$
$$T_2 = \{i \mid \text{ s.t. } i \text{ does not satisfy (3.17)}\}.$$

In T_2 we order the $\lambda_i' s$: $\lambda_{i_1} \leqslant \lambda_{i_2} \leqslant \cdots \leqslant \lambda_{i_s}$. We begin by proving the following lemma

LEMMA 3.4. – *For $n \geqslant 4$, we have the following estimate*

$$\sum_{j\in T_2, j\neq i}\left(\varepsilon_{ij} + (\lambda_i d_i)^{2-n}\right) = R_1$$

where $R_1 = \mathrm{O}[\sum_{k\in T_1}(\frac{\mathrm{Log}\,\lambda_k d_k}{(\lambda_k d_k)^n} + \sum_{r\neq k, r\in T_1}\varepsilon_{kr}^{\frac{n}{n-2}}\mathrm{Log}(\varepsilon_{kr}^{-1}))]$.

Proof. – Using Proposition 3.2, we derive

$$\begin{aligned}
0 &= \sum_{k=1}^{s} 2^k\alpha_{i_k}\left(\nabla J(u_\varepsilon), \lambda_{i_k}\frac{\partial P\delta_{i_k}}{\partial\lambda_{i_k}}\right)\\
&= 2J(u_\varepsilon)c_1\sum_{k=1}^{s}\Bigg[-\sum_{j\neq i_k}2^k\alpha_j\alpha_{i_k}\lambda_{i_k}\frac{\partial\varepsilon_{j i_k}}{\partial\lambda_{i_k}}\big(1+\mathrm{o}(1)\big)\\
&\quad - \frac{n-2}{2}\sum_{j=1}^{p}2^k\alpha_j\alpha_{i_k}\frac{H_\varepsilon(a_j, a_{i_k})}{(\lambda_j\lambda_{i_k})^{\frac{n-2}{2}}}\big(1+\mathrm{o}(1)\big) + R\Bigg].
\end{aligned} \tag{3.18}$$

Notice that

$$-\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = \frac{n-2}{2}\varepsilon_{ij}\left(1 - \frac{2\lambda_j}{\lambda_i}\varepsilon_{ij}^{\frac{2}{n-2}}\right). \tag{3.19}$$

Thus, if $\lambda_i \geqslant \lambda_j$ and $i, j \in T_2$, we have

$$-2\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} - \lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} \geqslant -\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \geqslant \frac{n-2}{4}\varepsilon_{ij}. \tag{3.20}$$

For $j \in T_1$ and $i \in T_2$, two cases may occur.

(i) $\frac{1}{2}d_j \leqslant d_i \leqslant 2d_j$. Using in this case the fact that j satisfies (3.17) and $H_\varepsilon(a_i, a_k) \leqslant (d_i d_k)^{\frac{-(n-2)}{2}}$, we obtain

$$\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-2}{2}}\varepsilon_{ij} \leqslant \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-2}{2}} c\sum_{k=1}^{p}\frac{H_\varepsilon(a_k, a_j)}{(\lambda_j\lambda_k)^{\frac{n-2}{2}}} \leqslant c\sum_{k=1}^{p}\big((\lambda_i d_i)(\lambda_k d_k)\big)^{\frac{-(n-2)}{2}}.$$

We deduce that

$$\frac{\lambda_i}{\lambda_j}\varepsilon_{ij}^{\frac{2}{n-2}} = \mathrm{O}\left(\frac{1}{(\lambda_1 d_1)^2}\right). \tag{3.21}$$

(ii) In other cases, we have $|a_i - a_j| \geqslant \frac{1}{2}\max(d_i, d_j)$, then

$$\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}^{\frac{2}{n-2}} \leqslant \frac{\lambda_j}{\lambda_i}\left(\lambda_i\lambda_j|a_i - a_j|^2\right)^{-1} \leqslant \left(\lambda_i|a_i - a_j|\right)^{-2} = \mathrm{O}\left(\frac{1}{(\lambda_1 d_1)^2}\right)$$

and (3.21) follows in this case. Using (3.18)–(3.20), and (3.21), we see that

$$0 \geqslant \left((n-2)/4\right)\sum_{i\in T_2}\left(\sum_{j\neq i}\varepsilon_{ij} - 2^p\sum_{j=1}^{p} H_\varepsilon(a_i, a_j)(\lambda_i\lambda_j)^{\frac{2-n}{2}} + R\right).$$

Since $i \in T_2$ and $H_\varepsilon(a_i; a_i) \sim c/d_i^{n-2}$ for ε small enough (see [1]), then

$$0 \geqslant c\sum_{i\in T_2}\left(\sum_{j\neq i}\varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}}\right)\left(1 + \mathrm{o}(1)\right) + R_1.$$

Therefore our lemma follows. □

Now, in T_1 we order all the $\lambda_i d_i$: $\lambda_{j_1} d_{j_1} \leqslant \lambda_{j_2} d_{j_2} \leqslant \cdots \leqslant \lambda_{j_q} d_{j_q}$. In order to simplify the notations, we suppose that $T_1 = \{1, 2, \ldots, q\}$ and $\lambda_1 d_1 \leqslant \lambda_2 d_2 \leqslant \cdots \leqslant \lambda_q d_q$.

Let us introduce the following sets:

$$K_0 = \Big\{ i \in T_1 \mid \exists k_1, \ldots, k_m \in T_1 \text{ s.t. } k_1 = i, \ldots, k_m = 1 \text{ and } \frac{|a_{k_j} - a_{k_{j+1}}|}{\inf(d_{k_j}, d_{k_{j+1}})} \leqslant C_0 \Big\}, \tag{3.22}$$

$$B = K_0 \cap \{1, \ldots, l\} \tag{3.23}$$

where $l = \max\{i \in T_1 \mid \lambda_i d_i / \lambda_{i-1} d_{i-1} \leqslant C_1\}$ and C_0 and C_1 are positive constants choosen later.

LEMMA 3.5. – *Let B be defined by* (3.23). *Then, there exists $i \in T_1$ such that $i \in B$ and $i \neq 1$.*

Proof. – We argue by contradiction. We assume that $B = \{1\}$. Using Proposition 3.2, and the fact that $H_\varepsilon(a_i; a_i) \sim c/d_i^{n-2}$, we derive

$$\begin{aligned} 0 &= \left(\nabla J(u_\varepsilon), \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}\right) \\ &= 2J(u)c_1\left[-\frac{(n-2)}{2}\alpha_1\frac{H_\varepsilon(a_1, a_1)}{\lambda_1^{n-2}}\left(1 + \mathrm{o}(1)\right) + \mathrm{O}\left(\sum_{k\neq 1}\varepsilon_{k1}\right)\right]. \end{aligned}$$

Thus

$$0 \leqslant -c(\lambda_1 d_1)^{2-n} + \mathrm{O}\left(\sum_{k\neq 1}\varepsilon_{1k}\right). \tag{3.24}$$

Observe that:

– for $k \in T_2$, we have by Lemma 3.1, $\varepsilon_{1k} = \mathrm{O}(R_1)$;

– for $k \in T_1$, two cases may occur.

If $k > l$, then

$$\varepsilon_{1k} \leqslant 2^{p+1} \sum_{j=1}^{p} \frac{H_\varepsilon(a_k, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \leqslant c \left(\frac{1}{(\lambda_k d_k)(\lambda_1 d_1)} \right)^{\frac{n-2}{2}} \leqslant \frac{1}{C_1^{\frac{n-2}{2}}} \frac{1}{((\lambda_l d_l)(\lambda_1 d_1))^{\frac{n-2}{2}}}.$$

Thus $\varepsilon_{1k} = \mathrm{o}((\lambda_1 d_1)^{2-n})$ if we choose C_1 large enough.

In the other case, we have $k \notin K_0$, then $|a_1 - a_k| \geqslant C_0 \inf(d_1, d_k)$, then

$$\varepsilon_{1k} \leqslant \left(\frac{1}{\lambda_1 \lambda_k |a_1 - a_k|^2} \right)^{\frac{n-2}{2}} \leqslant \frac{1}{C_0^{n-2}} \frac{1}{((\lambda_1 d_1)(\lambda_k d_k))^{\frac{n-2}{2}}} = \mathrm{o}\left(\frac{1}{(\lambda_1 d_1)^{n-2}} \right)$$

if we choose C_0 large enough. Thus (3.24) yields a contradiction and our lemma follows. □

In order to finish the proof of the second part of Theorem 1.3, it is sufficient to prove the following lemma.

LEMMA 3.6. – *For $n \geqslant 4$, we have $d^{n-2}\rho_B \to 0$ and $d^{n-1}\nabla\rho_B \to 0$, when $\varepsilon \to 0$, where $d = \inf_{i \in B} d(a_i, \partial A_\varepsilon)$ and $\rho_B = \rho(a_{i_1}, \ldots, a_{i_m})$, with $B = \{i_1, \ldots, i_m\}$ the set defined by* (3.23).

Before giving the proof of this lemma, we begin by studying the vector Λ defined by

$$\Lambda = \left(\lambda_{i_1}^{\frac{2-n}{2}}, \ldots, \lambda_{i_m}^{\frac{2-n}{2}}\right). \tag{3.25}$$

We denote by e the eingenvector associated to ρ_B. We know that all components of e are strictly positive (see [4]). Let $\eta > 0$ be such that for any γ belongs to a neighborhood $C(e, \eta)$ of e, we have

$$^T\gamma M_B \gamma - \rho_B |\gamma|^2 \leqslant \frac{c_2}{d^{n-2}} |\gamma|^2 \quad \text{and} \quad ^T\gamma \frac{\partial M_B}{\partial a_i} \gamma = \left(\frac{\partial \rho_B}{\partial a_i} + \mathrm{o}\left(\frac{1}{d^{n-1}} \right) \right) |\gamma|^2 \tag{3.26}$$

and for $\gamma \in (\mathbb{R}^+)^m \setminus C(e, \eta)$, we have

$$^T\gamma M_B \gamma - \rho_B |\gamma|^2 \geqslant \frac{c_3 |\gamma|^2}{d^{n-2}} \tag{3.27}$$

where

$$C(e, \eta) \subset \left\{ y \in (\mathbb{R}_+^*)^m \text{ s.t. } \quad \left| \frac{y}{|y|} - e \right| < \eta \right\},$$

m = card B_1 and $M_B = M(a_i,\ i \in B)$ the matrix defined by (1.3).

LEMMA 3.7. – *Let* Λ *be defined by* (3.25). *Then* $\Lambda \in C(e,\eta)$.

Proof of Lemma 3.7. – We argue by contradiction. We assume that $\Lambda \in (\mathbb{R}_+^*)^m \setminus C(e,\eta)$. Let

$$\Lambda(t) = |\Lambda| \frac{(1-t)\Lambda + t|\Lambda|e}{|(1-t)\Lambda + t|\Lambda|e|} := \frac{y(t)}{|y(t)|}.$$

From Proposition 3.2, we derive

$$\big(\nabla J(u_\varepsilon), Z\big)\big|_{t=0} = -c\frac{d}{dt}\big({}^T\Lambda(t) M_B \Lambda(t)\big) + O\Big(\sum_{i\in B,\, j\in(T_1\setminus B)\cup T_2} \varepsilon_{ij}\Big) + R + o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big)$$

where Z is the vector field defined on the variables λ along the flow line defined by $\Lambda(t)$. Observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big({}^T\Lambda(t) M_B \Lambda(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{{}^T\Lambda(t) M_B \Lambda(t)}{|\Lambda(t)|^2}|\Lambda(0)|^2\Big)\\
&= |\Lambda(0)|^2 \frac{\mathrm{d}}{\mathrm{d}t}\Big(\rho_B + \frac{(1-t)^2}{|y(t)|^2}\big({}^T\Lambda(0) M_B \Lambda(0) - \rho_B|\Lambda(0)|^2\big)\Big)\\
&= |\Lambda(0)|^2 \Big(\frac{2(1-t)}{|y(t)|^4}\big({}^T\Lambda(0) M_B \Lambda(0) - \rho_B|\Lambda(0)|^2\big)\\
&\quad \times \big(-(1-t)|\Lambda(0)|\langle e, \Lambda(0)\rangle - t|\Lambda|^2\big)\Big).
\end{aligned}$$

Thus

$$\big(\nabla J(u_\varepsilon), Z\big)\big|_{t=0} = -\frac{2c}{|\Lambda|^2}\big({}^T\Lambda M_B \Lambda - \rho_B|\Lambda|^2\big)\big(-|\Lambda|\langle e, \Lambda(0)\rangle\big) + o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big) + O\Big(\sum_{i\in B,\, j\in(T_1\setminus B)\cup T_2} \varepsilon_{ij}\Big).$$

Since $|e| = 1$, then there exists k such that $e_k \geqslant \frac{1}{m}$. Thus

$$\langle e, \Lambda(0)\rangle = \sum_i e_i \Lambda_i \geqslant \frac{1}{m}\Lambda_k.$$

Using (3.27), we obtain

$$\begin{aligned}
\big(\nabla J(u_\varepsilon), Z\big)\big|_{t=0} &\geqslant \frac{cc_3}{d^{n-2}}|\Lambda|\Lambda_k + o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big) + O\Big(\sum_{i\in B,\, j\in(T_1\setminus B)} \varepsilon_{ij}\Big)\\
&\geqslant \frac{c}{(\lambda_1 d_1 \lambda_k d_k)^{\frac{n-2}{2}}} + o\Big(\frac{1}{(\lambda_1 d_1)^{n-2}}\Big) + O\Big(\sum_{i\in B,\, j\in T_1\setminus B} \varepsilon_{i,j}\Big).
\end{aligned}$$

Observe that

– if $j > l$, since $j \in T_1$, we have

$$\varepsilon_{ij} \leqslant \frac{c}{(\lambda_1 d_1)^{\frac{n-2}{2}}} \cdot \frac{1}{(\lambda_j d_j)^{\frac{n-2}{2}}} \leqslant \frac{c}{C_1^{\frac{n-2}{2}} (\lambda_1 d_1 \lambda_l d_l)^{\frac{n-2}{2}}} = \mathrm{o}\left(\frac{1}{(\lambda_1 d_1 \lambda_k d_k)^{\frac{n-2}{2}}}\right);$$

– if $j \notin K_0$

$$\varepsilon_{ij} \leqslant \left(\frac{1}{(\lambda_i \lambda_j |a_i - a_j|^2)}\right)^{\frac{n-2}{2}} \leqslant \frac{C_0^{2-n}}{(\lambda_1 d_1 \lambda_j d_j)^{\frac{n-2}{2}}}$$
$$\leqslant \frac{C_0^{2-n} C_1^{m-1}}{(\lambda_1 d_1 \lambda_k d_k)^{\frac{n-2}{2}}} = \mathrm{o}\left(\frac{1}{(\lambda_1 d_1 \lambda_k d_k)^{\frac{n-2}{2}}}\right)$$

if we chose $C_0 \gg C_1$.

Thus

$$0 \geqslant \left(\frac{c}{(\lambda_1 d_1 \lambda_k d_k)^{\frac{n-2}{2}}}\right) + \mathrm{o}\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) \geqslant \left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right)\left(\frac{c}{(C_1)^{\frac{n-2}{2}}} + \mathrm{o}(1)\right) > 0.$$

This yields a contradiction and our lemma follows. □

Proof of Lemma 3.6. – Using Proposition 3.2, we have

$$\begin{aligned} 0 &= \sum_{i \in B} \left(\nabla J(u_\varepsilon), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right) \\ &= \sum_{i \in B} \left[\frac{H_\varepsilon(a_i, a_i)}{\lambda_i^{n-2}}(1 + \mathrm{o}(1)) - \sum_{j \neq i,\, j \in B} \left(\varepsilon_{ij} - \frac{H_\varepsilon(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}}\right)(1 + \mathrm{o}(1)) \right. \\ &\quad \left. + \mathrm{O}\left(\sum_{j \in (T_1 \setminus B) \cup T_2} \varepsilon_{ij}\right) + R\right] \\ &= {}^T \Lambda M_B \Lambda + \mathrm{o}\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) + R_1 + \mathrm{O}\left(\sum_{j \in (T_1 \setminus B),\ i \in B} \varepsilon_{ij}\right). \end{aligned}$$

Observe that, for $i \in B$ and $j \in T_1 \setminus B$, we have, as in the proof of Lemma 3.5, $\varepsilon_{ij} = \mathrm{o}((\lambda_1 d_1)^{2-n})$, for C_0 and C_1 large enough. Thus

$$0 = {}^T \Lambda M_B \Lambda + \mathrm{o}\big((\lambda_1 d_1)^{2-n}\big). \tag{3.28}$$

We assume, arguing by contradiction, that $d^{n-2} \rho_B \not\to 0$, when $\varepsilon \to 0$. Therefore, there exists $C_4 > 0$ such that $|\rho_B d^{n-2}| \geqslant C_4$.

Now, we distinguish two cases.

1st *case*: $\rho_B > 0$. In this case, we derive from (3.28)

$$0 \geqslant \rho_B |\Lambda|^2 + \mathrm{o}\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) \geqslant C_2 \frac{|\Lambda|^2}{d^{n-2}} + \mathrm{o}\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) > 0.$$

This yields a conttradiction and we derive that $d^{n-2} \rho_B \to 0$ in this case.

2nd *case*: $\rho_B \leqslant 0$. In this case, we derive from (3.26) and (3.28),

$$\begin{aligned}0 &\leqslant \rho_B|\Lambda|^2 + \frac{c_2|\Lambda|^2}{d^{n-2}} + o\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right)\\ &\leqslant \frac{|\Lambda|^2}{d^{n-2}}\left(\rho_B d^{n-2} + C_2\right) + o\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right)\\ &\leqslant \frac{|\Lambda|^2}{d^{n-2}}(-C_4 + C_2) + o\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right).\end{aligned}$$

If we choose $C_2 \leqslant \frac{1}{2}C_4$, we obtain a contradiction. Then $d^{n-2}\rho_B \to 0$, when $\varepsilon \to 0$ also in this case.

Observe that, since $d^{n-2}\rho_B \to 0$, then there exists $C_5 > 0$ such that $|a_i - a_j| \geqslant C_5 d$, for any $i, j \in B$ and $i \neq j$.

We assume, arguing by contradiction, that $d^{n-1}\nabla\rho_B \not\to 0$ when $\varepsilon \to 0$.
Using Proposition 3.3, we derive

$$\begin{aligned}0 = {}^T\Lambda \frac{\partial M_B}{\partial a_i}\Lambda + O\left(\sum_{j\in T_1\setminus B}\left(\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\frac{\partial H_\varepsilon}{\partial a_i}(a_i, a_j)\right)\right)\\ + o\left(\frac{1}{d_i}\frac{1}{(\lambda_1 d_1)^{n-2}}\right) + \lambda_i R + O\left(\sum_{j\in T_1}\lambda_i\lambda_j|a_i - a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}}\right) + O\left(\sum_{j\in T_2}\lambda_i\varepsilon_{ij}\right).\end{aligned}$$

Observe that:

- for $j \in T_2$, we have by Lemma 3.1 $\lambda_i\varepsilon_{ij} = O(\lambda_i R_1)$;
- for $j \notin K_0$, we have

$$\begin{aligned}&\left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right| + \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\left|\frac{\partial H_\varepsilon}{\partial a_i}(a_i, a_j)\right|\\ &\quad\leqslant \frac{c}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\left(\frac{1}{|a_i - a_j|^{n-1}} + \frac{1}{d_i|a_i - a_j|^{n-2}}\right)\\ &\quad\leqslant \frac{c}{C_0^{n-2}}\left(\frac{1}{(\lambda_i d_i \lambda_j d_j)^{\frac{n-2}{2}} d_i}\right)\left(\frac{1}{C_0} + 1\right)\\ &\quad\leqslant \frac{2c}{C_0^{n-2}}\frac{1}{d}\frac{1}{(\lambda_1 d_1)^{n-2}} = o\left(\frac{1}{d(\lambda_1 d_1)^{n-2}}\right);\end{aligned}$$

- for $j > l$, we have

$$\begin{aligned}&\left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right| + \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\left|\frac{\partial H_\varepsilon}{\partial a_i}(a_i, a_j)\right|\\ &\quad\leqslant \frac{c}{C_1^{\frac{n-2}{2}}}\left(\frac{1}{(\lambda_1 d_1)^{n-2} d}\right) = o\left(\frac{1}{d(\lambda_1 d_1)^{n-2}}\right);\end{aligned}$$

- for $j \in T_1$, we have

$$\lambda_i\lambda_j|a_i - a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}} = o\left(d(\lambda_1 d_1)^{1-n}\right).$$

Therefore, by (3.26) we have

$$0 = ^T\Lambda \frac{\partial M_B}{\partial a_i}\Lambda + \mathrm{o}\left(\frac{1}{d(\lambda_1 d_1)^{n-2}}\right) = \left(\frac{\partial \rho_B}{\partial a_i} d^{n-1} + \mathrm{o}(1)\right)\frac{|\Lambda|^2}{d^{n-1}} + \mathrm{o}\left(\frac{1}{d(\lambda_1 d_1)^{n-2}}\right).$$

Thus

$$0 \geqslant \left(\left|\frac{\partial \rho}{\partial a}\right| d^{n-1} + \mathrm{o}(1)\right)\frac{|\Lambda|^2}{d^{n-1}} + \mathrm{o}\left(\frac{1}{d(\lambda_1 d_1)^{n-2}}\right) \geqslant C_6 \frac{|\Lambda|^2}{d^{n-1}} + \mathrm{o}\left(\frac{1}{d(\lambda_1 d_1)^{n-2}}\right) > 0.$$

This yields a contradiction and our lemma follows. □

4. Proof of Theorem 1.5

Let $(x_1, \ldots, x_k) \in A_\varepsilon$ such that

$$d^{n-2}\rho_\varepsilon(x_1, \ldots, x_k) \to 0 \quad \text{when } \varepsilon \to 0 \text{ and } |x_i - x_j| \leqslant C_0 d,\ \forall i, j, \tag{4.1}$$

where C_0 is a fixed positive constant and $d = \min_{1\leqslant i\leqslant k} d(x_i, \partial A_\varepsilon)$.

We may assume, without loss of generality, that $d_1 = \inf_{1\leqslant i\leqslant k} d_i$.

Now we introduce the map

$$A_\varepsilon \to \widetilde{A}_\varepsilon, \quad x \mapsto \tilde{x} = d_1^{-1}(x - x_1).$$

According to [1], we have

$$\rho_\varepsilon(x_1, \ldots, x_k) = d_1^{2-n}\tilde{\rho}_\varepsilon(0, \tilde{x}_2, \ldots, \tilde{x}_k) \tag{4.2}$$

where $\tilde{\rho}_\varepsilon$ is the function defined, replacing A_ε^k by $\widetilde{A}_\varepsilon^k$ in (1.3), and $\widetilde{A_\varepsilon}$ converges in the C^1-topology on every compact set to Ω, where Ω is a half-space or a strip.

Observe that $|\tilde{x}_i| \leqslant C_0,\ \forall i \in \{2, \ldots, k\}$.

Now, we have the following Lemmas.

LEMMA 4.1. – *For $\varepsilon > 0$, let*

$$F_k(\varepsilon) = \{(X_1, \ldots, X_k) \in \widetilde{A}_\varepsilon^k \mid \exists i \neq j \ s.t.\ X_i = X_j\}.$$

Then $\tilde{\rho}_\varepsilon$ converges in the C^1-topology to ρ_Ω, when $\varepsilon \to 0$, on every compact set that does not intersect V, where V is any neighborhood of $F_k(\varepsilon)$ and ρ_Ω is the function defined, replacing A_ε^k by Ω^k in (1.3).

The proof of Lemma 4.1 is similar to the proof of Lemma 4.1 in [1].

LEMMA 4.2. – *Let ρ_Ω the function defined replacing A_ε^k by Ω^k in* (3). *Then the map*

$$]0, 1] \to \mathbb{R}, \quad t \mapsto t^{n-2}\rho_\Omega(0, tX_2, \ldots, tX_k)$$

decreases when t decreases for any $X_2, \ldots, X_k \in \Omega$.

The same arguments in the proof of Lemma 4.5 in [1] prove easily our lemma. From (4.1), (4.2) and Lemmas 4.1 and 4.2, we easily deduce our theorem.

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Appendix A

In this appendix, we collect the estimates of the different integral quantities which occur in the paper. For the proof of these estimates, we refer the interested readers to [2] and [14]. In the sequel, we assume that ε is small enough.

LEMMA A.1. – *We have the following estimate*

$$\int_{A_\varepsilon} |\nabla P\delta_{(a,\lambda)}|^2 = S_n - c_1 \frac{H_\varepsilon(a,a)}{\lambda^{n-2}} + \mathrm{O}\left(\frac{\operatorname{Log}\lambda d}{(\lambda d)^n}\right)$$

where $c_1 > 0$ *and* $d = d(a, \partial A_\varepsilon)$. *Futhermore* c_1 *and* O *are independent of* ε.

LEMMA A.2. – *For* $n \geqslant 4$, *we have the following estimate*

$$\int_{A_\varepsilon} P\delta^{\frac{2n}{n-2}} = S_n - \frac{2n}{n-2} c_1 \frac{H_\varepsilon(a,a)}{\lambda^{n-2}} + \mathrm{O}\left(\frac{\log\lambda d}{(\lambda d)^n}\right)$$

where c_1 *and* O *are independent of* ε.

LEMMA A.3. – *For* $i \neq j$, *we have*

$$(P\delta_i, P\delta_j)_{H_0^1(A_\varepsilon)} = c_1\left(\varepsilon_{ij} - \frac{H_\varepsilon(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\right) + \mathrm{O}\left(\varepsilon_{ij}^{\frac{n}{n-2}} \operatorname{Log}(\varepsilon_{ij}^{-1}) + \sum_{k\in(i,j)} \frac{\operatorname{Log}(\lambda_k d_k)}{(\lambda_k d_k)^n}\right)$$

where c_1 *and* O *are independent of* ε.

LEMMA A.4. – *For* $n \geqslant 4$, *we have the following estimate*

$$\int_{A_\varepsilon} P\delta_i^{\frac{n+2}{n-2}} P\delta_j = (P\delta_i, P\delta_j)_{H_0^1} + R$$

where $R = \mathrm{O}(\varepsilon_{ij}^{\frac{n}{n-2}} \operatorname{Log}(\varepsilon_{ij}^{-1}) + \sum_{k\in(i,j)} \frac{\operatorname{Log}(\lambda_k d_k)}{(\lambda_k d_k)^n})$.

LEMMA A.5. – *We have*

$$\left(P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}\right)_{H_0^1} = \frac{n-2}{2} c_1 \frac{H_\varepsilon(a,a)}{\lambda^{n-2}} + \mathrm{O}\left(\frac{\operatorname{Log}(\lambda d)}{(\lambda d)^n}\right)$$

where c_1 *and* O *are independent of* ε.

LEMMA A.6. – *For $n \geqslant 4$, we have*

$$\int_{A_\varepsilon} P\delta^{\frac{n+2}{n-2}} \lambda \frac{\partial P\delta}{\partial \lambda} = 2\left(P\delta, \lambda \frac{\partial P\delta}{\partial \lambda}\right)_{H_0^1} + O\left(\frac{\operatorname{Log}(\lambda d)}{(\lambda d)^n}\right).$$

LEMMA A.7. – *For $i \neq j$, we have*

$$\left(P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right)_{H_0^1} = c_1\left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H_\varepsilon(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}}\right) + R$$

where R is defined in Lemma A.4.

LEMMA A.8. – *For $n \geqslant 4$ and $i \neq j$, we have*

$$\int_{A_\varepsilon} P\delta_j^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \left(P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right)_{H_0^1} + R$$

where R is defined in Lemma A.4.

LEMMA A.9. – *For $n \geqslant 4$ and $i \neq j$, we have*

$$\frac{n+2}{n-2} \int_{A_\varepsilon} P\delta_j \left(P\delta_i^{\frac{4}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right) = \left(P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right)_{H_0^1} + O\left(\sum_{k \in (i,j)} \frac{\operatorname{Log}(\lambda_k d_k)}{(\lambda_k d_k)^n}\right).$$

LEMMA A.10. – *We have*

$$\left(P\delta_i, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial \lambda_i}\right)_{H_0^1} = -\frac{1}{2} \frac{c_1}{\lambda_i^{n-1}} \frac{\partial H_\varepsilon}{\partial a_i}(a_i, a_i) + O\left(\frac{1}{(\lambda d)^n}\right).$$

LEMMA A.11. – *For $n \geqslant 4$, we have*

$$\int_{A_\varepsilon} P\delta^{\frac{n+2}{n-2}} \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} = 2\left(P\delta, \frac{1}{\lambda} \frac{\partial P\delta}{\partial a}\right) + O\left(\frac{\operatorname{Log}(\lambda d)}{(\lambda d)^n}\right).$$

LEMMA A.12. – *For $i \neq j$, we have*

$$\left(P\delta_j, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i}\right)_{H_0^1} = -\frac{c_1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}} \lambda_i} \frac{\partial H_\varepsilon}{\partial a_i}(a_i, a_j) + \frac{c_1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} + O\left(\sum_{k \in i,j} \frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n+1}{n-2}} \lambda_j |a_i - a_j|\right).$$

LEMMA A.13. – *For $i \neq j$ and $n \geqslant 4$, we have*

$$\int_{A_\varepsilon} P\delta_j^{\frac{n+2}{n-2}} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} = \left(P\delta_j, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i}\right)_{H_0^1} + O\left(\sum_{k \in i,j} \frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}} \operatorname{Log}(\varepsilon_{ij}^{-1})\right).$$

LEMMA A.14. – *For $n \geqslant 4$ and $i \neq j$, we have*

$$\frac{n+2}{n-2}\int_{A_\varepsilon} P\delta_j P\delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = \left(P\delta_i, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right)_{H_0^1} + \mathrm{O}\left(\sum_{k\in\{i,j\}} \frac{1}{(\lambda_k d_k)^n}\right) + \mathrm{O}\left(\varepsilon_{ij}^{\frac{n}{n-2}} \operatorname{Log}\left(+\varepsilon_{ij}^{-1}\right)\right).$$

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