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Phase transition for the vacant set left by random walk on the giant component of a random graph

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Abstract. We study the simple random walk on the giant component of a supercritical Erdős–Rényi random graph on n vertices, in particular the so-called vacant set at level u, the complement of the trajectory of the random walk run up to a time proportional to u and n. We show that the component structure of the vacant set exhibits a phase transition at a critical parameter $u_\star$: For $u < u_\star$ the vacant set has with high probability a unique giant component of order n and all other components small, of order at most $\log^7 n$, whereas for $u > u_\star$ it has with high probability all components small. Moreover, we show that $u_\star$ coincides with the critical parameter of random interlacements on a Poisson–Galton–Watson tree, which was identified in (*Electron. Commun. Probab.* **15** (2010) 562–571).

Résumé. Nous étudions la marche aléatoire sur la composante principale d'un graphe aléatoire d'Erdős–Rényi avec n sommets, en particulier l'ensemble vacant au niveau u, le complément de la trajectoire de la marche aléatoire jusqu'à un moment proportionnel à u et n. Nous prouvons que la structure de composant montre une transition de phase à un valeur critique $u_\star$: Pour $u < u_\star$ l'ensemble vacant se compose, avec une forte probabilité quand n croît, d'une seule composante principale avec volume d'ordre n et des composantes petites d'ordre au plus $\log^7 n$, alors que pour $u > u_\star$ tous les composants sont petits. En outre nous montrons que $u_\star$ coïncide avec le paramètre critique des entrelacs aléatoires sur un arbre de Poisson–Galton–Watson identifié en (*Electron. Commun. Probab.* **15** (2010) 562–571).

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1. Introduction

Recently, several authors have been studying percolative properties of the vacant set left by random walk on finite graphs and the connections of this problem to the random interlacements model introduced in [21]. The topic was initiated with the study of random walk on the d-dimensional discrete torus in [6], which was further investigated in [24]. [8,10] and [11] studied random walk on the random regular graph, and [11] also studied random walk on the Erdős–Rényi random graph above the connectivity threshold.

In this work we consider the supercritical Erdős–Rényi random graph below the connectivity threshold. We prove a phase transition in the component structure of the vacant set left by random walk on the giant component of this graph, and we identify the critical point of this phase transition with the critical parameter of random interlacements on a Poisson–Galton–Watson tree.

We start by introducing some notation to precisely state the result. Let $\mathbb{P}_{n,p}$ be the law of an Erdős–Rényi random graph, i.e. a random graph G such that every possible edge is present independently with probability $p = \frac{\rho}{n}$, defined on the space $\mathcal{G}(n)$ of graphs with vertex set $\{1, 2, \dots, n\}$ endowed with the σ-algebra $\mathbb{G}_n$ of all subsets. It is well known that the component structure of G varies with the parameter ρ (see e.g. [7,12,13,16]). We will in this paper consider such a random graph for a fixed constant $\rho > 1$. In this case, with probability tending to 1 as $n \to \infty$, the graph G is supercritical: There exists a unique largest connected component $\mathcal{C}_1(G)$ of size approximately ξn, the so-called giant component. Here, ξ is the unique solution in $(0, 1)$ of $\mathrm{e}^{-\rho\xi} = 1 - \xi$.

For a graph G on n vertices and its largest connected component $\mathcal{C}_1 = \mathcal{C}_1(G)$ (determined by some arbitrary tie-breaking rule), let $P^{\mathcal{C}_1}$ be the law of the simple discrete-time random walk $(X_k)_{k\geq 0}$ on $\mathcal{C}_1$ started from its stationary distribution, defined on the space $\{1, 2, \dots, n\}^{\mathbb{N}_0}$ of trajectories on n vertices endowed with the cylinder-σ-algebra $\mathbb{F}_n$. Let $\Omega_n = \mathcal{G}(n) \times \{1, 2, \dots, n\}^{\mathbb{N}_0}$ endowed with the product σ-algebra $\mathbb{G}_n \otimes \mathbb{F}_n$, and define the "annealed" measure by

$$\mathbf{P}_n(A \times B) = \sum_{G\in A} \mathbb{P}_{n,p}(G) P^{\mathcal{C}_1(G)}(B) \quad \text{for } A \in \mathbb{G}_n, B \in \mathbb{F}_n. \tag{1.1}$$

On the product space Ω_n we define the vacant set of the random walk at level u as

$$\mathcal{V}^u = \mathcal{C}_1 \setminus \big\{X_k \colon 0 \leq k \leq u\rho(2 - \xi)\xi n\big\}. \tag{1.2}$$

We refer to Remark 1.2 for an explanation of this somewhat unusual time scaling. Let $\mathcal{C}_1(\mathcal{V}^u)$ and $\mathcal{C}_2(\mathcal{V}^u)$ be the largest and second largest connected components of the subgraph induced by $\mathcal{V}^u$.

Theorem 1.1. *The component structure of the subgraph induced by $\mathcal{V}^u$ exhibits a phase transition at a critical value $u_\star$:*

- *For $u < u_\star$, there are positive constants $\zeta(u, \rho) \in (0, 1)$, $C < \infty$, such that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbf{P}_n\left[\left|\frac{|\mathcal{C}_1(\mathcal{V}^u)|}{n} - \zeta(u, \rho)\right| \leq \varepsilon\right] = 1, \tag{1.3}$$

$$\lim_{n\to\infty} \mathbf{P}_n\left[\frac{|\mathcal{C}_2(\mathcal{V}^u)|}{\log^7 n} \leq C\right] = 1. \tag{1.4}$$

- *For $u > u_\star$, there is a positive constant $C < \infty$, such that*

$$\lim_{n\to\infty} \mathbf{P}_n\left[\frac{|\mathcal{C}_1(\mathcal{V}^u)|}{\log^7 n} \leq C\right] = 1. \tag{1.5}$$

The critical parameter $u_\star$ is the same as the critical parameter of random interlacements on a Poisson(ρ)–Galton–Watson tree conditioned on non-extinction, which is by [22] *given as the solution of a certain equation.*

We refer to Section 2.3 for a short summary of the used results on random interlacements and its critical parameter, and the derivation of the characterizing equation (2.15) for $u_\star$. The constant $\zeta(u, \rho)$ is given as the solution of equation (5.2).

Theorem 1.1 confirms the following general principle: The vacant set of random walk on a sufficiently fast mixing graph exhibits a phase transition and the critical point is related to the critical value of random interlacements on the corresponding infinite volume limit.

This principle has been investigated recently in several other situations. Results that are more detailed than Theorem 1.1 are known to hold for random walk on a random d-regular graph on n vertices run up to time un: [8] and with different methods [11] proved the phase transition in the component structure of the vacant graph, [8] identified the critical parameter $u_\star$ with the critical value of random interlacements on the infinite d-regular tree, and [10] showed that there is a critical window of width $n^{-1/3}$ around $u_\star$ in which the largest component is of order $n^{2/3}$. [11] used their methods to also prove a phase transition for random walk on the Erdős–Rényi random graph above the connectivity threshold ($\rho \gg \log n$). Weaker statements are known for random walk run up to time uN^d on the discrete d-dimensional torus of sidelength N, see [6] and [24]. The statements in this case are proved for u small or large enough respectively, but it is only conjectured that there is indeed a phase transition at a critical parameter $u_\star$ that coincides with the critical value of random interlacements on $\mathbb{Z}^d$ (cf. Conjecture 2.6 in [9]). We believe that in our case, as in [10] for the random regular graph, it should be possible to prove the existence of a critical window around the critical point. We did not further investigate this.

The main difficulties in proving Theorem 1.1 compared to previous results are that our graph, i.e. the giant component of an Erdős–Rényi random graph, is of random size and non-regular. The proof consists of three main steps. The key idea of the first step is the following "spatial Markov property" of random walk on a random graph. Instead of sampling a random graph and performing random walk on the fixed graph, one can consider sites unvisited by the random walk as not yet sampled sites of the random graph. Then the unvisited or vacant part of the graph has the law of some random graph, depending on the random graph model. In the case of a connected Erdős–Rényi random graph the vacant part is again an Erdős–Rényi random graph, this was used to prove the phase transition in [11]. In the case of a random regular graph the vacant part is a random graph with a given degree sequence, a well-studied object (see e.g. [14]). This was used to prove the phase transition in [11] and the critical behaviour in [10].

The situation in our case is more involved, because we consider random walk only on the giant component of a not connected Erdős–Rényi random graph. This random walk cannot satisfy such a spatial Markov property, since the graph must be fixed in advance for the giant component to be known. To be able to still use the idea, we introduce in Algorithm 4.1 a process $\bar{X} = (\bar{X}_k)_{k\geq 0}$ on an Erdős–Rényi random graph that behaves like a random walk but jumps to another component after having covered a component. In Lemma 4.2 we make precise the aforementioned spatial Markov property for this process $\bar{X}$, namely that the vacant graph left by $\bar{X}$ still has the law of an Erdős–Rényi random graph, but with different parameters. The classical results on random graphs imply a phase transition for this vacant graph.

In a second step we translate this phase transition to the vacant graph left by the simple random walk $X = (X_k)_{k\geq 0}$ on the giant component. To this end, we introduce in Proposition 4.3 a coupling of X and $\bar{X}$ where the two processes are with high probability identified in a certain time interval. This can be done because the process $\bar{X}$ will typically "find" the giant component after a short time and then stay on it long enough.

The third step, requiring most of the technical work, is the identification of the critical point of the phase transition. From Lemma 4.2 it is clear that the crucial quantity deciding the critical point is the size of the vacant set left by $\bar{X}$. The coupling of X and $\bar{X}$ has the property that the sizes of the vacant sets of X and $\bar{X}$ are closely related (Lemma 4.4), which allows to reduce the problem to the investigation of the size of the vacant set left by X. The first part of this paper, Section 3, is devoted to this investigation. In Proposition 3.1 we will on one hand compute the expectation of the size of the vacant set left by X, and on the other hand we will show that the size of the vacant set left by X is concentrated around its expectation.

We close the introduction with a remark on the connection to random interlacements and a heuristic explanation of the time scaling $u\rho(2-\xi)\xi n$ that appears in the definition (1.2) of $\mathcal{V}^u$. For readers unfamiliar with random interlacements and the notation, we refer to Section 2, in particular Section 2.3.

Remark 1.2. *In the giant component $\mathcal{C}_1$ of an Erdős–Rényi random graph the balls $B(x,r)$ around a vertex x with radius r of order $\log n$ typically look like balls around the root $\varnothing$ in a Poisson(ρ)–Galton–Watson tree $\mathcal{T}$ conditioned on non-extinction. One expects that random interlacements on $\mathcal{T}$ give a good description of the trace of random walk on $\mathcal{C}_1$ locally in such balls, where the intensity u of random interlacements is proportional to the running time of the walk. To determine the proportionality factor, we compare the probability that a vertex $x \in \mathcal{C}_1$ has not been visited by the random walk on $\mathcal{C}_1$ up to time t with the probability that the root $\varnothing \in \mathcal{T}$ is in the vacant set of random interlacements on $\mathcal{T}$ at level u.*

Note first that the probability that the random walk on $\mathcal{C}_1$ started at x leaves a ball of large radius around x before returning to x is approximately the same as the probability that the random walk on $\mathcal{T}$ started at the root never returns to the root,

$$P_x^{\mathcal{C}_1}[\tilde{H}_x > H_{B(x,r)^c}] \approx P_\varnothing^{\mathcal{T}}[\tilde{H}_\varnothing = \infty]. \tag{1.6}$$

The main task of Section 3 will be rigorous proof of the following approximation for the random walk on $\mathcal{C}_1$,

$$P^{\mathcal{C}_1}[x \text{ is vacant at time } t] \approx \mathrm{e}^{-t P_x^{\mathcal{C}_1}[\tilde{H}_x > H_{B(x,r)^c}]\pi(x)}. \tag{1.7}$$

We will also show that the average degree of a vertex in $\mathcal{C}_1$ is $\rho(2-\xi)$, and so the stationary distribution π of the random walk on $\mathcal{C}_1$ is $\pi(x) \approx \frac{\deg(x)}{\rho(2-\xi)\xi n}$. On the other hand, according to [23], the law Q^u of the vacant set of random *interlacements on the infinite graph $\mathcal{T}$ at level u satisfies*

$$Q^u[\varnothing \text{ is vacant}] = \mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}, \tag{1.8}$$

where the capacity is here $\mathrm{cap}_{\mathcal{T}}(\varnothing) = \deg(\varnothing) P_{\varnothing}^{\mathcal{T}}[\tilde{H}_{\varnothing} = \infty]$. *As argued above, random interlacements describe the random walk locally, so the probabilities* (1.7) *and* (1.8) *should be approximately equal for the time t corresponding to random interlacements at level u. The approximation of $\pi(x)$ together with* (1.6) *leads to $t = u\rho(2-\xi)\xi n$ if the parameter u in both models should be the same.*

Compared to the time scalings uN^d and un in the discussions of random walk on the torus [6,24] *and random regular graphs* [8,10] *respectively, where only the size of the graph (in our case the factor ξn) appears in the time scaling, the additional factor $\rho(2-\xi)$ for the average degree might be surprising. It is however only a consequence of how one defines the uniform edge-weight on the underlying graph, which scales the capacity by a constant. For the aforementioned $2d$-regular graphs the weight chosen is $\frac{1}{2d}$. For non-regular graphs it is the canonical choice to define edge weights as* 1, *as is done in* [23] *and* [22], *and we stick to this definition.*

The paper is structured as follows. In Section 2 we introduce some further notation and recall some facts on random graphs, random walks, and random interlacements. In Section 3 we investigate the size of the vacant set left by the simple random walk X on the giant component. In Section 4 we introduce the process $\bar{X}$ and compare it to the random walk X. Finally, we gather all intermediate results to prove Theorem 1.1 in Section 5.

2. Notations and preliminaries

We will denote by c, c', c'' positive finite constants with values changing from place to place. ε will always denote a small positive constant with value changing from place to place. All these constants may depend on u and ρ, but not on any other object. We will tacitly assume that values like $u\rho(2-\xi)\xi n$, $\log^5 n$, n^{ε} etc. are integers, omitting to take integer parts to ease the notation.

We use the standard o- and O-notation: Given a positive function $g(n)$, a function $f(n)$ is $\mathrm{o}(g)$ if $\lim_{n\to\infty} f/g = 0$, and it is $\mathrm{O}(g)$ if $\limsup_{n\to\infty} |f|/g < \infty$. We extend this notation to random variables in the following way. For a random variable A_n on a space (Ω_n, Q_n) we use the notation "$A_n = f(n) + \mathrm{o}(g)$ Q_n-asymptotically almost surely" meaning "$\forall \varepsilon > 0$, $Q_n[|A_n - f(n)| \le \varepsilon g(n)] \to 1$ as $n \to \infty$," and "$A_n = \mathrm{O}(g)$ Q_n-asymptotically almost surely" meaning "$\exists C > 0$ such that $Q_n[|A_n| \le C g(n)] \to 1$ as $n \to \infty$."

2.1. (*Random*) *graphs*

For a non-oriented graph we use the notation G to denote the set of vertices in the graph as well as the graph itself, consisting of vertex-set and edge-set. For vertices $x, y \in G$, $x \sim y$ means that x and y are neighbours, i.e. $\{x, y\}$ is an edge of G. We denote by $\deg(x)$ the number of neighbours of x in G, and by $\Delta_G = \max_{x\in G} \deg(x)$ the maximum degree. By $\mathrm{dist}(x, y)$ we denote the usual graph distance, and for $r \in \mathbb{N}$, $B(x, r)$ is the set of vertices y with $\mathrm{dist}(x, y) \le r$. For a subset $A \subset G$, denote its complement $A^c = G \setminus A$ and its (interiour) boundary $\partial A = \{x \in A\colon \exists y \in A^c, x \sim y\}$.

We denote by $\mathcal{C}_i(G)$ the ith largest connected component of a graph G. If there are equally large components, we order these arbitrarily. The subgraph induced by a vertex-set $V \subset G$ is defined as the graph with vertices V and edges $\{x, y\}$ if and only if $x, y \in V$ and $x \sim y$ in G. Again we use the notation $\mathcal{C}_i(G)$ for the set of vertices as well as for the induced subgraph. Usually (but not necessarily) $\mathcal{C}_1 = \mathcal{C}_1(G)$ will be the unique giant component. A graph or graph component is called "simple" if it is connected and has at most one cycle, i.e. the number of edges is at most equal to the number of vertices.

Recall from the introduction that $\mathbb{P}_{n,p}$ denotes the law of an Erdős–Rényi random graph, i.e. a random graph on n vertices such that every edge is present independently with probability $p = \frac{\rho}{n}$. Let $\mathbb{E}_{n,p}$ be the corresponding expectation. An event is said to hold "asymptotically almost surely" (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$ (cf. the above defined o- and O-notation). Throughout this work $\rho > 1$ is a fixed constant. It is well known that the following properties then hold $\mathbb{P}_{n,p}$-a.a.s.

The graph G has a unique giant component $\mathcal{C}_1$ of size $|\mathcal{C}_1|$ satisfying $||\mathcal{C}_1| - \xi n| \le n^{3/4}$, where ξ is the unique solution in $(0, 1)$ of $\mathrm{e}^{-\rho\xi} = 1 - \xi$. All other components are simple and of size smaller than $C \log n$, for some fixed constant C. (2.1)

The spectral gap $\lambda_{\mathcal{C}_1}$ of the random walk on the giant component (cf. (2.12)) satisfies $\lambda_{\mathcal{C}_1} \geq \frac{c}{\log^2 n}$ for some fixed constant c. (2.2)

The maximum degree Δ_G satisfies $\Delta_G \leq \log n$. (2.3)

(2.1) and (2.3) are classical results (see e.g. [7,13,16] or [12]), and (2.2) follows from [18, Theorem 12.4] with the $\mathrm{O}(\log^2 n)$ bound on the mixing time of the random walk on the giant component proved in [5]. We use the terminology "typical graphs" for graphs G on n vertices satisfying (2.1), (2.2) and (2.3). We will usually prove our statements for typical graphs only, since we are interested in a.a.s.-behaviour.

For a quantitative version of the first statement in (2.1) see [15, Theorem 4.8], which states that

$$\mathbb{P}_{n,p}\big[\big||\mathcal{C}_1| - \xi n\big| > n^{3/4}\big] \leq cn^{-c'}. \tag{2.4}$$

The choice of the constant $3/4$ is arbitrary, any $\nu \in (\frac{1}{2}, 1)$ would work.

We will also need a quantitative version of (2.3), we therefore briefly present a proof. Fix a vertex $x \in G$ and denote all other vertices by y_i, $i = 1, \dots, n-1$. Let $\mathcal{E}_i = \mathbf{1}_{\{\{x,y_i\} \text{ is an edge}\}}$. Then the $\mathcal{E}_i$ are i.i.d. Bernoulli(p) random variables, $\deg(x) = \sum_{i=1}^{n-1} \mathcal{E}_i$, and for any fixed $\alpha > 0$ by the exponential Chebyshev inequality,

$$\mathbb{P}_{n,p}\big[\deg(x) > \log n\big] \leq n^{-\alpha}\mathbb{E}_{n,p}\big[\mathrm{e}^{\alpha \sum \mathcal{E}_i}\big] = n^{-\alpha}\bigg(1 + \frac{\rho}{n}\big(\mathrm{e}^{\alpha} - 1\big)\bigg)^{n-1} \leq cn^{-\alpha},$$

where the constant c depends on α. We choose $\alpha = 4$, this will be suitable for our purposes. Then a union bound implies

$$\mathbb{P}_{n,p}[\Delta_G > \log n] \leq n\mathbb{P}_{n,p}\big[\deg(x) > \log n\big] \leq cn^{1-\alpha} = cn^{-3}. \tag{2.5}$$

2.2. *Random walks*

Let $P^{\mathcal{C}_1}$ be the law and $E^{\mathcal{C}_1}$ the corresponding expectation of the simple discrete-time random walk $X = (X_k)_{k\geq 0}$ on the component $\mathcal{C}_1$ started stationary, i.e. the law of the Markov chain with state space $\mathcal{C}_1$, transition probabilities $p_{xy} = \frac{1}{\deg(x)}\mathbf{1}_{\{x\sim y\}}$ and $X_0 \sim \pi$, where π is the stationary distribution, $\pi(x) = \frac{\deg(x)}{\sum_{y\in\mathcal{C}_1}\deg(y)}$. (2.1) and the a.a.s. upper bound (2.3) on the maximum degree Δ_G imply the following bounds on π. $\mathbb{P}_{n,p}$-a.a.s.

$$\pi(x) = \frac{\deg(x)}{\sum_{v\in\mathcal{C}_1}\deg(v)} \leq \frac{c\log n}{n}, \tag{2.6}$$

$$\pi(x) = \frac{\deg(x)}{\sum_{v\in\mathcal{C}_1}\deg(v)} \geq \frac{c}{n\log n}. \tag{2.7}$$

For real numbers $0 \leq s \leq r$ denote by $X_{[s,r]} = \{X_k\colon s \leq k \leq r\}$ the set of vertices visited by X between times s and r. We let the random walk X run up to time t and denote by $\mathcal{V}(t) = \mathcal{C}_1 \setminus X_{[0,t]}$ the vacant set left by the random walk at time t, and again we use the notation $\mathcal{V}(t)$ to also denote the subgraph of $\mathcal{C}_1$ induced by these vertices. As defined in (1.2), we will use the short notation $\mathcal{V}^u$ for $\mathcal{V}(u\rho(2-\xi)\xi n)$.

We will, where it is clear in the context, drop the superscript from $P^{\mathcal{C}_1}$ and $E^{\mathcal{C}_1}$. The notation P_x is then used to denote the law of the random walk on $\mathcal{C}_1$ started at vertex x, E_x is the corresponding expectation. For a set $A \subset \mathcal{C}_1$ we denote by

$$H_A = \inf\{t \geq 0\colon X_t \in A\}, \qquad \tilde{H}_A = \inf\{t \geq 1\colon X_t \in A\}$$

the entrance time and hitting time respectively of A, and we write H_x and $\tilde{H}_x$ if $A = \{x\}$. From [1, Lemma 2] or [2, Chapter 3, Proposition 21] together with (2.6) we get the following bound on $E[H_x]$. $\mathbb{P}_{n,p}$-a.a.s. for all $x \in \mathcal{C}_1$,

$$E[H_x] \geq \frac{(1-\pi(x))^2}{\pi(x)} \geq \frac{cn}{\log n}. \tag{2.8}$$

For all real valued functions f and g on $\mathcal{C}_1$ define the Dirichlet form

$$\mathcal{D}(f,g)=\frac{1}{2}\sum_{x,y\in\mathcal{C}_1}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\pi(x)p_{xy}. \tag{2.9}$$

A function f on $\mathcal{C}_1$ is harmonic on $A\subset\mathcal{C}_1$ if $\sum_y p_{xy}f(y)=f(x)$ for $x\in A$. For $x\in\mathcal{C}_1$ and $r\in\mathbb{N}$ define the equilibrium potential $g^\star:\mathcal{C}_1\to\mathbb{R}$ as the unique function harmonic on $B(x,r)\setminus\{x\}$, 1 on $\{x\}$ and 0 on $B(x,r)^c$. The dependence of $g^\star$ on x and r is kept implicit. Then it is well known that

$$g^\star(y)=P_y[H_x<H_{B(x,r)^c}], \tag{2.10}$$

$$\mathcal{D}\big(g^\star,g^\star\big)=P_x[\tilde{H}_x>H_{B(x,r)^c}]\pi(x). \tag{2.11}$$

The spectral gap of the random walk on $\mathcal{C}_1$ is given by

$$\lambda_{\mathcal{C}_1}=\min\big\{\mathcal{D}(f,f):\ \pi\big(f^2\big)=1,\pi(f)=0\big\}. \tag{2.12}$$

The relevance of the bound (2.2) on $\lambda_{\mathcal{C}_1}$ is in the speed of mixing of the random walk on $\mathcal{C}_1$. From [18, Theorem 12.3 and Lemma 6.13] it follows that for all $t\in\mathbb{N}$

$$\max_{x,y\in\mathcal{C}_1}\big|P_x[X_t=y]-\pi(y)\big|\le\frac{1}{\min_{z\in\mathcal{C}_1}\pi(z)}\mathrm{e}^{-\lambda_{\mathcal{C}_1}t}. \tag{2.13}$$

2.3. *Random interlacements*

Random interlacements were introduced in [21] on $\mathbb{Z}^d$ as a model to describe the local structure of the trace of a random walk on a large discrete torus, and in [23] the model was generalized to arbitrary transient graphs. It is a special dependent site-percolation model where the occupied vertices on a graph are constructed as the trace left by a Poisson point process on the space of doubly infinite trajectories modulo time shift. The density of this Poisson point process is determined by a parameter $u>0$. The critical value $u_\star$ is the infimum over the u for which almost surely all connected components of non-occupied vertices are finite.

In [22] it is shown that for random Galton–Watson trees the critical value $u_\star$ is almost surely constant with respect to the tree measure and is implicitly given as the solution of a certain equation. Except for the identification of the critical parameter of Theorem 1.1 with this $u_\star$ as the solution of the same equation, we will not use any results on random interlacements. We refer to the lecture notes [9] for an introduction to random interlacements and many more references.

We quote the result from [22] to derive the characterizing equation for $u_\star$ in the case of a Poisson–Galton–Watson tree. This requires some more notation. Denote by $\mathbb{P}_\mathcal{T}$ the law of the supercritical Poisson(ρ)–Galton–Watson rooted tree conditioned on non-extinction and by $\mathbb{E}_\mathcal{T}$ the corresponding conditional expectation. Let $f(s)=\mathrm{e}^{\rho(s-1)}$ be the probability generating function of the Poisson(ρ) distribution, and denote by q the extinction probability of a (unconditioned) Poisson(ρ)–Galton–Watson tree. It is well known that q is the unique solution in (0, 1) of the equation $f(s)=s$, and hence $q=1-\xi$, where ξ is as in (2.1). Let

$$\tilde{f}(s)=\frac{f((1-q)s+q)-q}{1-q}. \tag{2.14}$$

This is in fact the probability generating function of the offspring in the subtree of vertices with infinite line of descent (see e.g. [19, Proposition 5.26]).

Consider the simple discrete-time random walk $(X_k)_{k\ge0}$ on the rooted tree $\mathcal{T}$ started at the root $\varnothing$, whose law we denote by $P_\varnothing^\mathcal{T}$, and let $\tilde{H}_\varnothing=\inf\{t\ge1:\ X_t=\varnothing\}$ be the hitting time of the root. Define the capacity of the root by $\mathrm{cap}_\mathcal{T}(\varnothing)=\deg(\varnothing)P_\varnothing^\mathcal{T}[\tilde{H}_\varnothing=\infty]$.

By [22, Theorem 1], the critical parameter $u_\star$ of random interlacements on the Galton–Watson tree conditioned on non-extinction is $\mathbb{P}_\mathcal{T}$-a.s. constant and given as the unique solution in $(0,\infty)$ of the equation

$$\big(\tilde{f}^{-1}\big)'\big(\mathbb{E}_\mathcal{T}\big[\mathrm{e}^{-u\,\mathrm{cap}_\mathcal{T}(\varnothing)}\big]\big)=1.$$

In particular for the Poisson(ρ)–Galton–Watson tree,

$$\big(\tilde{f}^{-1}\big)'(t) = \frac{1}{\rho\xi t + \rho(1-\xi)},$$

and $u_\star$ is the solution of

$$\rho\xi\mathbb{E}_{\mathcal{T}}\big[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\big] + \rho(1-\xi) = 1. \tag{2.15}$$

3. Size of the vacant set

In this section we investigate the size of the vacant set $\mathcal{V}^u$ left by the random walk X on the giant component $\mathcal{C}_1$. As already mentioned we omit the superscripts from $P^{\mathcal{C}_1}$ and $E^{\mathcal{C}_1}$. Recall the definition (1.1) of the annealed measure $\mathbf{P}_n$.

Proposition 3.1.

1. $E[|\mathcal{V}^u|]$ *can asymptotically be approximated in terms of a Poisson*(ρ)*–Galton–Watson tree conditioned on non-extinction*:

$$E\big[\big|\mathcal{V}^u\big|\big] = \xi n\mathbb{E}_{\mathcal{T}}\big[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\big] + \mathrm{o}(n), \quad \mathbb{P}_{n,p}\text{-}a.a.s.$$

2. *The random variable* $|\mathcal{V}^u|$ *is concentrated around its mean*:

$$\big|\mathcal{V}^u\big| = E\big[\big|\mathcal{V}^u\big|\big] + \mathrm{o}(n), \quad \mathbf{P}_n\text{-}a.a.s.$$

3.1. *Expectation of the size of the vacant set*

The proof of part (1) of Proposition 3.1 is split up into several steps. We first quote and extend [17, Proposition 11.2]. It formalizes the well known fact that an Erdős–Rényi random graph locally looks like a Galton–Watson tree. Here, by locally we mean balls of radius of order $\log n$. More precisely, fix some $\gamma > 0$ such that $6\gamma\log\rho < 1$, and set

$$r = \gamma\log n. \tag{3.1}$$

For a graph G, a vertex $x \in G$ and a tree $\mathcal{T}$ with root $\varnothing$, define the event

$$\mathcal{I}_x(G,\mathcal{T}) = \left\{\begin{array}{l} B(x,r+1)\subset G \text{ is isomorphic to } B(\varnothing,r+1)\subset\mathcal{T},\ \text{with the} \\ \text{isomorphism sending } x \text{ to } \varnothing \end{array}\right\}. \tag{3.2}$$

Denote by $\mathbb{P}^0_{\mathcal{T}}$ the law of the unconditioned Poisson(ρ)–Galton–Watson tree $\mathcal{T}$, and by $\{|\mathcal{T}|<\infty\}$, $\{|\mathcal{T}|=\infty\}$ the events of extinction and non-extinction respectively of the tree $\mathcal{T}$.

Proposition 3.2.

1. *Given an arbitrary fixed vertex* $x \in \{1,2,\dots,n\}$, *there is a coupling* Q_x *of* G *under* $\mathbb{P}_{n,p}$ *and a tree* $\mathcal{T}$ *under* $\mathbb{P}^0_{\mathcal{T}}$, *such that for n large enough*

$$Q_x\big[\mathcal{I}_x(G,\mathcal{T})\big] \ge 1 - cn^{3\gamma\log\rho-1}. \tag{3.3}$$

For n large enough, *this coupling satisfies*

$$Q_x\big[x\in\mathcal{C}_1, |\mathcal{T}|<\infty\big] \le cn^{-c'}, \tag{3.4}$$

$$Q_x\big[x\notin\mathcal{C}_1, |\mathcal{T}|=\infty\big] \le cn^{-c'}. \tag{3.5}$$

2. *For an arbitrary point $x \in G$, with r as in (3.1),*

$$\mathbb{P}_{n,p}\big[\big|B(x,r)\big| \ge n^{3\gamma \log \rho}\big] \le cn^{3\gamma \log \rho - 1}. \tag{3.6}$$

3. *Given two arbitrary fixed vertices $x \neq y$, there is a coupling $Q_{x,y}$ of G under $\mathbb{P}_{n,p}$ and two trees $\mathcal{T}_x$ and $\mathcal{T}_y$, each having law $\mathbb{P}^0_{\mathcal{T}}$, such that $\mathcal{T}_x$ and $\mathcal{T}_y$ are independent and for n large enough*

$$Q_{x,y}\big[\mathcal{I}_x(G,\mathcal{T}_x) \text{ and } \mathcal{I}_y(G,\mathcal{T}_y)\big] \ge 1 - cn^{6\gamma \log \rho - 1}, \tag{3.7}$$

and statements (3.4) and (3.5) hold under $Q_{x,y}$ for x, $\mathcal{T}_x$ and y, $\mathcal{T}_y$, respectively.

Proof. (3.3) is, up to the enlargement of the radius by 1, the statement of [17, Proposition 11.2], and (3.6) is [17, Corollary 11.3]. Note that, in contrary to the actual statement, [17, Proposition 11.2] is proved for an a priori fixed vertex and not a randomly chosen one.

For part (1) it remains to show the properties (3.4) and (3.5). For simplicity write $B_x = B(x,r) \subset G$ and $B_\varnothing = B(\varnothing, r) \subset \mathcal{T}$. Denote by $\{z \leftrightarrow B_z^c\}$ the event that z is connected to the complement of B_z, or equivalently that ∂B_z is non-empty, and by $\{z \not\leftrightarrow B_z^c\}$ its complement. To prove (3.4), we first claim that

$$\mathbb{P}_{n,p}\big[x \in \mathcal{C}_1, x \not\leftrightarrow B_x^c\big] \le cn^{-c'}. \tag{3.8}$$

To see this, note that if $x \in \mathcal{C}_1$ and $x \not\leftrightarrow B_x^c$, then $B_x = \mathcal{C}_1$. But by (3.6), B_x is unlikely to be large: For every small $\varepsilon > 0$, $\mathbb{P}_{n,p}[|B_x| \ge n^{1-\varepsilon}] \le cn^{-c'}$. However, if B_x is smaller than $n^{1-\varepsilon}$ and $B_x = \mathcal{C}_1$, then $\mathcal{C}_1$ is smaller than $n^{1-\varepsilon}$, but this happens with probability smaller than $cn^{-c'}$ by (2.4), and (3.8) follows.

Note that if the coupling succeeds, i.e. the balls of radius $r+1$ are isomorphic, then $\{x \leftrightarrow B_x^c\} = \{\varnothing \leftrightarrow B_\varnothing^c\}$. This happens with probability $\ge 1 - cn^{-c'}$ by (3.3), so together with (3.8),

$$\begin{aligned} Q_x\big[x \in \mathcal{C}_1, |\mathcal{T}| < \infty\big] &\le Q_x\big[x \leftrightarrow B_x^c, |\mathcal{T}| < \infty\big] + cn^{-c'} \\ &\le Q_x\big[\varnothing \leftrightarrow B_\varnothing^c, |\mathcal{T}| < \infty\big] + cn^{-c'} = \mathbb{P}^0_{\mathcal{T}}\big[\varnothing \leftrightarrow B_\varnothing^c, |\mathcal{T}| < \infty\big] + cn^{-c'}. \end{aligned}$$

The tree $\mathcal{T}$ conditioned on extinction has the law of a subcritical Galton–Watson tree with mean offspring number $m < 1$ (see e.g. [19, Proposition 5.26]). If q is the extinction probability and Z_k denotes the size of the kth generation of the tree, we can use the Markov inequality to get

$$\begin{aligned} \mathbb{P}^0_{\mathcal{T}}\big[\varnothing \leftrightarrow B_\varnothing^c, |\mathcal{T}| < \infty\big] &= \mathbb{P}^0_{\mathcal{T}}\big[Z_r \ge 1 | |\mathcal{T}| < \infty\big] q \\ &\le \mathbb{E}^0_{\mathcal{T}}\big[Z_r | |\mathcal{T}| < \infty\big] q = q m^{\gamma \log n} = cn^{-c'}, \end{aligned}$$

which proves (3.4).

For (3.5), let C_x be the component of G containing x. Let $M > 0$ be such that $M\gamma > (\rho - 1 - \log \rho)^{-1}$. Then, by e.g. [12, Theorem 2.6.4], $\mathbb{P}_{n,p}[x \notin \mathcal{C}_1, |C_x| > M\gamma \log n] \le cn^{-c'}$. Using this on the first line and (3.3) on the second, it follows that

$$\begin{aligned} Q_x\big[x \notin \mathcal{C}_1, |\mathcal{T}| = \infty\big] &\le Q_x\big[|C_x| \le M\gamma \log n, |\mathcal{T}| = \infty\big] + cn^{-c'} \\ &\le Q_x\big[|B_\varnothing| \le M\gamma \log n, |\mathcal{T}| = \infty\big] + cn^{-c'}. \end{aligned}$$

To bound this latter probability that the ball of radius $r = \gamma \log n$ in a surviving Poisson(ρ)–Galton–Watson tree is smaller than Mr, let again Z_r be the size of the rth generation and denote by $Z_r^\star$ the number of particles in the rth generation with infinite line of descent. Then

$$\begin{aligned} Q_x\big[|B_\varnothing| \le Mr, |\mathcal{T}| = \infty\big] &\le \mathbb{P}^0_{\mathcal{T}}\big[Z_r \le Mr | |\mathcal{T}| = \infty\big] \mathbb{P}^0_{\mathcal{T}}\big[|\mathcal{T}| = \infty\big] \\ &\le \mathbb{P}^0_{\mathcal{T}}\big[Z_r^\star \le Mr | |\mathcal{T}| = \infty\big] \xi. \end{aligned}$$

By e.g. [19, Proposition 5.26] or [4, Theorem I.12.1]

$$\mathbb{P}^0_{\mathcal{T}}\big[Z^\star_r \le Mr \,\big|\, |\mathcal{T}| = \infty\big] = \tilde{\mathbb{P}}_{\mathcal{T}}[\tilde{Z}_r \le Mr],$$

where $\tilde{Z}_r$ under $\tilde{\mathbb{P}}_{\mathcal{T}}$ is the rth generation size of a Galton–Watson tree with offspring distribution defined by the probability generating function $\tilde{f}$ as in (2.14), a tree with extinction probability $\tilde{q} = 0$. Let $\kappa = \tilde{f}'(0) = f'(q)$. Since f, the probability generating function of Poisson(ρ), is strictly convex and increasing, and by definition of $q = 1 - \xi$, we have $0 < \kappa < 1$. Let $\tilde{f}_r$ be the rth iterate of $\tilde{f}$, which is in fact the probability generating function of $\tilde{Z}_r$. From [4, Corollary I.11.1] we know that

$$\lim_{r\to\infty} \kappa^{-r} \tilde{f}_r(s) = Q(s) \in (0, \infty) \quad \text{exists for } 0 \le s < 1.$$

It follows that

$$\tilde{f}_r(s) \le \big(Q(s) + \varepsilon\big)\kappa^r$$

for $r \ge r_0(s, \varepsilon)$. Using this, for any $\lambda > 0$ we obtain for $r \ge r_0(\mathrm{e}^{-\lambda}, \varepsilon)$

$$\begin{aligned}\tilde{\mathbb{P}}_{\mathcal{T}}[\tilde{Z}_r \le Mr] &\le \tilde{\mathbb{P}}_{\mathcal{T}}\big[\mathrm{e}^{-\lambda\tilde{Z}_r} \ge \mathrm{e}^{-\lambda Mr}\big] \le \mathrm{e}^{\lambda Mr}\tilde{f}_r\big(\mathrm{e}^{-\lambda}\big)\\ &\le \big(Q(s) + \varepsilon\big)\mathrm{e}^{\lambda Mr + r\log\kappa}.\end{aligned}$$

By choosing $\lambda < -\frac{\log\kappa}{M}$ we can make this smaller than $c\mathrm{e}^{-c'r}$, and (3.5) follows since $r = \gamma\log n$. This finishes the proof of part (1) of the proposition.

We now prove part (3). Define the coupling $Q_{x,y}$ as follows. By using part (1) of the proposition, we can find a coupling of two independent graphs G_x and G_y, both with vertex set $x, y, 3, \dots, n$, and two independent Poisson(ρ)–Galton–Watson trees $\mathcal{T}_x$ and $\mathcal{T}_y$, such that with probability larger than $1 - 2cn^{3\gamma\log\rho - 1}$ both $\mathcal{I}_x(G_x, \mathcal{T}_x)$ and $\mathcal{I}_y(G_y, \mathcal{T}_y)$ hold.

We then construct a graph G with the same vertex set $x, y, 3, \dots, n$ in the following way. We first explore the ball $B(x, r+1) \subset G$ by determining the state of all possible edges with at least one adjacent vertex in $B(x, r) \subset G_x$ according to their state in G_x, i.e. setting them present or absent. In a second step we determine the ball $B(y, r+1) \subset G$ in the same way by G_y, only that we do not change the state of already determined edges. The remaining edges in G are set present independently with probability p and absent otherwise.

By construction this graph G has law $\mathbb{P}_{n,p}$. If both $\mathcal{I}_x(G_x, \mathcal{T}_x)$ and $\mathcal{I}_y(G_y, \mathcal{T}_y)$ hold and there is no collision in the second step, i.e. we never want to set an edge present that is already set absent or vice versa, then both $\mathcal{I}_x(G, \mathcal{T}_x)$ and $\mathcal{I}_y(G, \mathcal{T}_y)$ hold, and the coupling succeeds. It thus remains to bound the probability of such a collision.

Note that if there is a collision, then the sets of vertices $B(x, r+1)$ and $B(y, r+1)$ must have non-empty intersection: If $B(x, r+1) \cap B(y, r+1) = \varnothing$, the only edges possibly causing a collision are edges $\{u, v\}$ with $u \in B(x, r)$ and $v \in B(y, r)$, but these edges must be set absent by both G_x and G_y, or else $u \in B(y, r+1)$ or $v \in B(x, r+1)$.

The sets $B(x, r+1)$ and $B(y, r+1)$ are smaller than $n^{3\gamma\log\rho}$ with probability larger than $1 - cn^{3\gamma\log\rho - 1}$ by (3.6), and they are by construction random subsets of $\{x, y, 3, \dots, n\}$. But the probability that two random subsets of $\{x, y, 3, \dots, n\}$ of size k intersect is smaller than $\frac{k^2}{n}$, so the probability of a collision is smaller than

$$Q_{x,y}\big[B(x, r+1) \cap B(y, r+1) \ne \varnothing\big] \le 2cn^{3\gamma\log\rho - 1} + \frac{1}{n}n^{6\gamma\log\rho} \le cn^{6\gamma\log\rho - 1}.$$

This proves (3.7). By construction it is clear that statements (3.4) and (3.5) hold analogously under $Q_{x,y}$. □

We will denote by $\mathbb{E}_{Q_x}$ and $\mathbb{E}_{Q_{x,y}}$ the expectations corresponding to the couplings Q_x and $Q_{x,y}$. For easier use later we now define some events and estimate their probabilities. Let $\mathcal{B}_x$ on the space of the coupling Q_x be the event

$$\mathcal{B}_x = \mathcal{I}_x(G, \mathcal{T}) \cap \big(\{x \in \mathcal{C}_1, |\mathcal{T}| = \infty\} \cup \{x \notin \mathcal{C}_1, |\mathcal{T}| < \infty\}\big). \tag{3.9}$$

This event can canonically also be defined on the space of the coupling $Q_{x,y}$ when replacing $\mathcal{T}$ by $\mathcal{T}_x$. Then define on the space of $Q_{x,y}$ the event

$$\mathcal{B}_{x,y} = \mathcal{B}_x \cap \mathcal{B}_y. \tag{3.10}$$

From Proposition 3.2 it is immediate that

$$Q_x[\mathcal{B}_x] \geq 1 - cn^{-c'}, \tag{3.11}$$

$$Q_{x,y}[\mathcal{B}_{x,y}] \geq 1 - cn^{-c'}. \tag{3.12}$$

On the space of the coupling Q_x, and similarly on the space of $Q_{x,y}$, we further define the event

$$\{x \text{ good}\} = \{x \in \mathcal{C}_1\} \cap \big\{|\mathcal{T}| = \infty\big\} \cap \mathcal{I}_x(G, \mathcal{T}) = \mathcal{B}_x \cap \{x \in \mathcal{C}_1\}. \tag{3.13}$$

Since $\mathbb{P}^0_{\mathcal{T}}[|\mathcal{T}| = \infty] = \xi$ and $\mathbf{1}_{\{x \text{ good}\}} = \mathbf{1}_{\{|\mathcal{T}|=\infty\}} - \mathbf{1}_{\{|\mathcal{T}|=\infty, x \notin \mathcal{C}_1\}} - \mathbf{1}_{\{|\mathcal{T}|=\infty, x\in\mathcal{C}_1, \mathcal{I}_x(G,\mathcal{T})^c\}}$, it follows with (3.3) and (3.5) that

$$Q_x[x \text{ good}] = \xi + \mathrm{o}(1) \quad \text{as } n \to \infty.$$

Note that the probability of x being good is bounded away from zero, so every graph property holding $\mathbb{P}_{n,p}$-a.a.s., as well as every property of a ball of radius r in a Galton–Watson tree holding $\mathbb{P}^0_{\mathcal{T}}$-a.a.s. as $r \to \infty$ will also hold $Q_x[\cdot|x \text{ good}]$-a.a.s.

As a first application of Proposition 3.2 we prove a law of large numbers for the sum of degrees of vertices in the giant component, which leads to an approximation of the stationary measure π. This result may be well known, we did however not find it in the literature. The technique of the proof will be used again later.

Lemma 3.3.

$$\sum_{x\in\mathcal{C}_1} \deg(x) = \sum_{x\in G} \mathbf{1}_{\{x\in\mathcal{C}_1\}} \deg(x) = \rho(2-\xi)\xi n + \mathrm{o}(n), \quad \mathbb{P}_{n,p}\text{-}a.a.s.$$

Proof. Every vertex in the random graph G has Binomial($n-1, \frac{\rho}{n}$) neighbours, but on $\mathcal{C}_1$ their degree is above average and there is some dependency. For $x \in G$ denote

$$Z_x = \mathbf{1}_{\{x\in\mathcal{C}_1\}} \deg(x),$$

$$\tilde{Z}_x = \mathbf{1}_{\{|\mathcal{T}|=\infty\}} \deg(\varnothing),$$

where the tree $\mathcal{T}$ is defined by the coupling Q_x from Proposition 3.2, and $\varnothing$ is the root of $\mathcal{T}$. We will approximate $\mathbb{E}_{n,p}[Z_x] = \mathbb{E}_{Q_x}[Z_x]$ by $\mathbb{E}_{Q_x}[\tilde{Z}_x]$ and show that the sum of the Z_x is concentrated around its expectation using the second moment method.

Let us first compute the expectation of $\tilde{Z}_x$. Recall that $\mathbb{P}_{\mathcal{T}}$ denotes the law of the Poisson(ρ)–Galton–Watson tree conditioned on non-extinction, and $\mathbb{E}_{\mathcal{T}}$ the corresponding conditional expectation. Then

$$\mathbb{E}_{Q_x}[\tilde{Z}_x] = \mathbb{E}^0_{\mathcal{T}}\big[\deg(\varnothing) \big| |\mathcal{T}| = \infty\big] \mathbb{P}^0_{\mathcal{T}}\big[|\mathcal{T}| = \infty\big] = \mathbb{E}_{\mathcal{T}}\big[\deg(\varnothing)\big]\xi. \tag{3.14}$$

Using the same technique as in the proof of [19, Proposition 5.26], it is straightforward to see that the expected offspring in a Galton–Watson tree conditioned on non-extinction is

$$\mathbb{E}_{\mathcal{T}}\big[\deg(\varnothing)\big] = \frac{1}{1-q}\big(f'(1) - qf'(q)\big),$$

where f is the probability generating function of the offspring distribution. Here, the offspring is Poisson(ρ), so $q = 1-\xi$, $f'(1) = \rho$ and $f'(q) = \rho(1-\xi)$, which leads to

$$\mathbb{E}_{\mathcal{T}}\big[\deg(\varnothing)\big] = \frac{1}{\xi}\big(\rho - \rho(1-\xi)^2\big) = \rho(2-\xi). \tag{3.15}$$

We now approximate $\mathbb{E}_{Q_x}[Z_x]$ by $\mathbb{E}_{Q_x}[\tilde{Z}_x]$. Because $\tilde{Z}_x$ is unbounded, we will truncate it by $\log n$. By definition $\tilde{Z}_x$ is stochastically dominated by a Poisson(ρ)-random variable Λ, in particular it has finite mean, and therefore $\mathbb{E}_{Q_x}[\tilde{Z}_x \mathbf{1}_{\{\tilde{Z}_x < \log n\}}] \nearrow \mathbb{E}_{Q_x}[\tilde{Z}_x]$ as $n \to \infty$. Using $E[\mathrm{e}^{t\Lambda}] = \mathrm{e}^{\rho(\mathrm{e}^t - 1)}$ we have $P[\Lambda \geq \log n] = P[\mathrm{e}^{t\Lambda} \geq n^t] \leq \mathrm{e}^{\rho(\mathrm{e}^t-1)} n^{-t} = cn^{-c'}$. It follows that

$$\begin{aligned}\mathbb{E}_{Q_x}[\tilde{Z}_x \wedge \log n] &= \mathbb{E}_{Q_x}[\tilde{Z}_x \mathbf{1}_{\{\tilde{Z}_x < \log n\}}] + \log n\, Q_x[\tilde{Z}_x \geq \log n] \\ &= \mathbb{E}_{Q_x}[\tilde{Z}_x] + \mathrm{o}(1) \quad \text{as } n \to \infty.\end{aligned}$$

Recall from (3.9) the definition of the event $\mathcal{B}_x$, on which $Z_x = \tilde{Z}_x$, and $Z_x = \tilde{Z}_x \wedge \log n$ if $\Delta_G \leq \log n$. With (3.11) and (2.5) we can bound

$$\bigl|\mathbb{E}_{Q_x}[Z_x] - \mathbb{E}_{Q_x}[\tilde{Z}_x \wedge \log n]\bigr| \leq n Q_x[\Delta_G > \log n] + \log n\, Q_x\bigl[\mathcal{B}_x^c\bigr] \leq cn^{-c'}. \tag{3.16}$$

With (3.14) and (3.15) it follows that

$$\mathbb{E}_{n,p}\Biggl[\sum_{x \in G} Z_x\Biggr] = n\mathbb{E}_{Q_x}[Z_x] = \rho(2-\xi)\xi n + \mathrm{o}(n) \quad \text{as } n \to \infty.$$

It remains to show that the sum of the Z_x is concentrated. Take $x \neq y$ arbitrary vertices in G and consider the coupling $Q_{x,y}$ from Proposition 3.2. Recall from (3.10) the definition of the event $\mathcal{B}_{x,y}$. On $\mathcal{B}_{x,y}$ we have $Z_x = \tilde{Z}_x$ and $Z_y = \tilde{Z}_y$, so with (3.12) and (2.5) we get

$$\begin{aligned}&\bigl|\mathbb{E}_{Q_{x,y}}[Z_x Z_y] - \mathbb{E}_{Q_{x,y}}\bigl[(\tilde{Z}_x \wedge \log n)(\tilde{Z}_y \wedge \log n)\bigr]\bigr| \\ &\quad \leq n^2 Q_{x,y}[\Delta_G > \log n] + \log^2 n\, Q_{x,y}\bigl[\mathcal{B}_{x,y}^c\bigr] \leq cn^{-c'}.\end{aligned} \tag{3.17}$$

The trees $\mathcal{T}_x$ and $\mathcal{T}_y$ are independent, so $\tilde{Z}_x \wedge \log n$ and $\tilde{Z}_y \wedge \log n$ are independent. Therefore, from (3.16) and (3.17) we conclude that for two arbitrary vertices $x \neq y$,

$$\mathbb{E}_{n,p}[Z_x Z_y] = \mathbb{E}_{n,p}[Z_x]\mathbb{E}_{n,p}[Z_y] + \mathrm{o}(1) \quad \text{as } n \to \infty.$$

Denote $Z = \sum_{x \in G} Z_x$. It follows from the above, together with (2.5), that

$$\begin{aligned}\mathbb{E}_{n,p}\bigl[Z^2\bigr] &= \sum_{x \in G} \mathbb{E}_{n,p}\bigl[Z_x^2\bigr] + \sum_{x \neq y}\bigl(\mathbb{E}_{n,p}[Z_x]\mathbb{E}_{n,p}[Z_y] + \mathrm{o}(1)\bigr) \\ &= \mathrm{O}\bigl(n \log^2 n\bigr) + \mathrm{O}\bigl(n^3\bigr)\mathbb{P}_{n,p}[\Delta_G > \log n] + \mathbb{E}_{n,p}[Z]^2 - n\mathbb{E}_{n,p}[Z_x]^2 + \mathrm{o}\bigl(n^2\bigr) \\ &= \mathbb{E}_{n,p}[Z]^2 + \mathrm{o}\bigl(n^2\bigr) \quad \text{as } n \to \infty.\end{aligned}$$

Thus $\operatorname{Var} Z = \mathrm{o}(n^2)$ and the Chebyshev inequality implies for any $\varepsilon > 0$

$$\mathbb{P}_{n,p}\bigl[\bigl|Z - \mathbb{E}_{n,p}[Z]\bigr| > \varepsilon n\bigr] = \mathrm{o}(1) \quad \text{as } n \to \infty.$$

This finishes the proof of the lemma. □

We proceed with the proof of part (1) of Proposition 3.1, i.e. the computation of $E[|\mathcal{V}^u|]$. First observe that

$$E\bigl[\bigl|\mathcal{V}^u\bigr|\bigr] = \sum_{x \in \mathcal{C}_1} P\bigl[x \text{ is vacant at time } u\rho(2-\xi)\xi n\bigr] = \sum_{x \in \mathcal{C}_1} P\bigl[H_x > u\rho(2-\xi)\xi n\bigr].$$

The task is therefore to approximate the probabilities $P[H_x > u\rho(2-\xi)\xi n]$.

Assume that the random walk X is the discrete skeleton of a simple continuous-time random walk X^c, i.e. the times between jumps of X^c are i.i.d. Exponential(1). Denote by H_x^c the entrance time of x for this continuous-time walk and by S_k the time of the kth jump. It is clear that $E[S_k]=k$ and $E[H_x^c]=E[H_x]$. From [1] or [2, Chapter 3, Proposition 23] we know that the distribution of the entrance time of such a continuous-time walk can be approximated by an exponential distribution, namely for all $t>0$

$$\left|P\left[H_x^c>t\right]-\mathrm{e}^{-t/E[H_x]}\right|\le\frac{1}{\lambda_{\mathcal{C}_1}E[H_x]}. \tag{3.18}$$

If $k=k(n)\to\infty$ as $n\to\infty$, by the law of large numbers $P[|S_k-k|>\varepsilon k]=\mathrm{o}(1)$ as $n\to\infty$ for all $\varepsilon>0$. This implies

$$\begin{aligned}P[H_x>k]&=P\left[H_x^c>S_k\right]=P\left[H_x^c>S_k,S_k\ge(1-\varepsilon)k\right]+P\left[H_x^c>S_k,S_k<(1-\varepsilon)k\right]\\&\le P\left[H_x^c>(1-\varepsilon)k\right]+\mathrm{o}(1)\quad\text{as } n\to\infty\text{ for all }\varepsilon>0,\end{aligned}$$

and similarly

$$P[H_x>k]\ge P\left[H_x^c>(1+\varepsilon)k\right]+\mathrm{o}(1)\quad\text{as } n\to\infty\text{ for all }\varepsilon>0.$$

We obtain $P[H_x>k]=P[H_x^c>k]+\mathrm{o}(1)$ as $n\to\infty$, and together with the bounds (2.2) for $\lambda_{\mathcal{C}_1}$ and (2.8) for $E[H_x]$ it follows from (3.18) that $\mathbb{P}_{n,p}$-a.a.s.

$$\left|P\left[H_x>u\rho(2-\xi)\xi n\right]-\mathrm{e}^{-u\rho(2-\xi)\xi n/E[H_x]}\right|=\mathrm{o}(1). \tag{3.19}$$

Approximating the probabilities $P[H_x>u\rho(2-\xi)\xi n]$ therefore reduces to the investigation of $E[H_x]$. We will use Proposition 3.2 from [8], which states that $E[H_x]$ can be approximated in terms of the Dirichlet form of the equilibrium potential $g^\star$ (cf. (2.10) and (2.11)).

Proposition 3.4 [8, Proposition 3.2].

$$\mathcal{D}\left(g^\star,g^\star\right)\left(1-2\sup_{y\in B(x,r)^c}\left|f^\star(y)\right|\right)\le\frac{1}{E[H_x]}\le\mathcal{D}\left(g^\star,g^\star\right)\frac{1}{\pi(B(x,r)^c)^2}, \tag{3.20}$$

where $f^\star(y)=1-\frac{E_y[H_x]}{E[H_x]}$.

To use this result, we need to control the function $f^\star$. To this end, we give in the next lemma a bound on the probability that the random walk on $\mathcal{C}_1$ started outside $B(x,r)$ hits x before some time T. Recall the coupling Q_x from Proposition 3.2, the definition (3.13) of the event $\{x\text{ good}\}$, and the definition (3.1) of the radius r.

Lemma 3.5. *There is a constant c, such that, for $T\in\mathbb{N}$ possibly depending on n,*

$$Q_x\left[\sup_{y\in B(x,r)^c}P_y[H_x\le T]\le T\mathrm{e}^{-cr}\,\middle|\,x\text{ good}\right]\to 1\quad\text{as } n\to\infty.$$

Proof. For x good let $\mathcal{T}$ be the infinite Poisson(ρ)–Galton–Watson tree defined by the coupling Q_x to which the neighbourhood of x is isomorphic. Let $P_w^{\mathcal{T}}$ be the law of the simple random walk on the tree $\mathcal{T}$ started at $w\in\mathcal{T}$. To bound the escape probability of random walk on a Galton–Watson tree we use [17, Proposition 11.5], which states that

$$\sup_{w\in\partial B(\varnothing,r)}P_w^{\mathcal{T}}[H_\varnothing<\infty]\le\mathrm{e}^{-cr},\quad\mathbb{P}_{\mathcal{T}}^0\text{-a.a.s. as } r\to\infty.$$

Since $P_w^{\mathcal{T}}[H_\varnothing<\infty]\ge P_w^{\mathcal{T}}[H_\varnothing<H_{B(\varnothing,r)^c}]$, this implies

$$\sup_{w\in\partial B(\varnothing,r)}P_w^{\mathcal{T}}[H_\varnothing<H_{B(\varnothing,r)^c}]\le\mathrm{e}^{-cr},\quad\mathbb{P}_{\mathcal{T}}^0\text{-a.a.s. as } r\to\infty.$$

As argued before, since $Q_x[x \text{ good}]$ is bounded away from zero, this also holds $Q_x[\cdot|x \text{ good}]$-a.a.s. For x good, $P_w^{\mathcal{T}}[H_\varnothing < H_{B(\varnothing,r)^c}] = P_z[H_x < H_{B(x,r)^c}]$, where $z \in \partial B(x,r)$ is the image of w under the isomorphism between $B(x,r+1) \subset G$ and $B(\varnothing, r+1) \subset \mathcal{T}$. It follows that

$$Q_x\Big[\sup_{z\in\partial B(x,r)} P_z[H_x < H_{B(x,r)^c}] \le \mathrm{e}^{-cr} \Big| x \text{ good}\Big] \to 1 \quad \text{as } n \to \infty.$$

On the way from $y \in B(x,r)^c$ to x, the random walk on $\mathcal{C}_1$ must visit some $z \in \partial B(x,r)$. From there it either reaches x or leaves $B(x,r)$ again. The probability of the first event is $Q_x[\cdot|x \text{ good}]$-a.a.s. bounded by e^{-cr}, and if the second event occurs, we can repeat the previous reasoning. But in time T, this procedure can be repeated at most T times, leading to the required bound on $P_y[H_x \le T]$. □

With Lemma 3.5 we can give a bound on $\sup_{y\in B(x,r)^c} |f^\star(y)|$ on the left hand side of (3.20).

Lemma 3.6. *There are constants c, c', such that*

$$Q_x\Big[\sup_{y\in B(x,r)^c}\Big|1 - \frac{E_y[H_x]}{E[H_x]}\Big| \le cn^{-c'} \Big| x \text{ good}\Big] \to 1 \quad \text{as } n \to \infty. \tag{3.21}$$

Proof. Note first that by the general $\mathrm{O}(k^3)$-bound on the expected cover time C_G of a graph G on k vertices (see e.g. [3]), we have

$$\sup_{z\in\mathcal{C}_1} E_z[H_x] \le C_{\mathcal{C}_1} \le n^3. \tag{3.22}$$

Before considering the expectation of H_x with the random walk started from $y \in B(x,r)^c$, we consider the expectation of H_x starting from X_T for some time T where the walk is well mixed. Set $T = \log^4 n$. With (2.13), (2.2), (2.7) and (3.22) we get $\mathbb{P}_{n,p}$-a.a.s. for all $z \in \mathcal{C}_1$

$$\begin{aligned}
\big|E_z\big[E_{X_T}[H_x]\big] - E[H_x]\big| &\le \sum_{z'\in\mathcal{C}_1} \big|P_z\big[X_T = z'\big] - \pi\big(z'\big)\big| E_{z'}[H_x] \\
&\le \sum_{z'\in\mathcal{C}_1} \frac{1}{\min_{v\in\mathcal{C}_1}\pi(v)} \mathrm{e}^{-\lambda_{\mathcal{C}_1} T} E_{z'}[H_x] \\
&\le cn^5 \log n \mathrm{e}^{-c'\log^2 n} \le cn^{-c'}.
\end{aligned} \tag{3.23}$$

By the Markov property at time T and using (3.23), $\mathbb{P}_{n,p}$-a.a.s.

$$E_z[H_x] \le T + E_z\big[E_{X_T}[H_x]\big] \le T + E[H_x] + cn^{-c'}. \tag{3.24}$$

With (2.8) it follows that $\mathbb{P}_{n,p}$-a.a.s. for all $z \in \mathcal{C}_1$

$$\frac{E_z[H_x]}{E[H_x]} - 1 \le \big(T + cn^{-c'}\big)\frac{1}{E[H_x]} \le cn^{-c'}. \tag{3.25}$$

Since everything holding $\mathbb{P}_{n,p}$-a.a.s. also holds $Q_x[\cdot|x \text{ good}]$-a.a.s., (3.25) is enough for one side of (3.21).

For the other side take now $y \in B(x,r)^c$ and apply the Markov property at time T, use (3.23) on the first line and (3.24) for the supremum on the second line to get $\mathbb{P}_{n,p}$-a.a.s.

$$\begin{aligned}
E_y[H_x] &\ge E_y\big[\mathbf{1}_{\{H_x>T\}} E_{X_T}[H_x]\big] = E_y\big[E_{X_T}[H_x]\big] - E_y\big[\mathbf{1}_{\{H_x\le T\}} E_{X_T}[H_x]\big] \\
&\ge E[H_x] - cn^{-c'} - P_y[H_x \le T] \sup_{z\in\mathcal{C}_1} E_z[H_x] \\
&\ge E[H_x] - 2cn^{-c'} - P_y[H_x \le T]\big(T + E[H_x]\big).
\end{aligned}$$

This holds $\mathbb{P}_{n,p}$-a.a.s., so as argued before it also holds $Q_x[\cdot|x \text{ good}]$-a.a.s. With the bound (2.8) and using Lemma 3.5, where we note that $e^{-cr} = n^{-c'}$ by (3.1), it follows that $Q_x[\cdot|x \text{ good}]$-a.a.s.

$$\frac{E_y[H_x]}{E[H_x]} - 1 \geq -cn^{-1-c'} \log n - \log^4 n e^{-c''r} \left(\frac{c''' \log^5 n}{n} + 1 \right) \geq -cn^{-c'}.$$

Together with (3.25) this proves the lemma. □

Applying Lemma 3.6 in (3.20) and using Lemma 3.3, we obtain the following approximation of the probabilities $P[H_x > u\rho(2-\xi)\xi n]$.

Lemma 3.7. *For any fixed $u > 0$ and every $\varepsilon > 0$,*

$$Q_x\bigl[\bigl|P\bigl[H_x > u\rho(2-\xi)\xi n\bigr] - e^{-uP_\varnothing^{\mathcal{T}}[\tilde{H}_\varnothing > H_{B(\varnothing,r)^c}]\deg(\varnothing)}\bigr| \leq \varepsilon | x \text{ good}\bigr] \to 1 \quad \text{as } n \to \infty.$$

Proof. First recall (2.11) and use (2.6) to get $\mathbb{P}_{n,p}$-a.a.s.

$$\mathcal{D}\bigl(g^\star, g^\star\bigr) = P_x[\tilde{H}_x > H_{B(x,r)^c}]\pi(x) \leq \frac{c \log n}{n}. \tag{3.26}$$

For the left hand approximation in (3.20), Lemma 3.6 and (3.26) imply that $Q_x[\cdot|x \text{ good}]$-a.a.s.

$$\frac{1}{E[H_x]} \geq \mathcal{D}\bigl(g^\star, g^\star\bigr) - cn^{-1-c'}. \tag{3.27}$$

For the right hand approximation in (3.20), first recall that by (3.6) $\mathbb{P}_{n,p}$-a.a.s., $|B(x,r)| \leq n^{1-\varepsilon}$ for some $\varepsilon > 0$. Together with (2.6) we get $\mathbb{P}_{n,p}$-a.a.s.

$$\pi\bigl(B(x,r)^c\bigr) \geq 1 - \bigl|B(x,r)\bigr| \max_{v \in \mathcal{C}_1} \pi(v) \geq 1 - cn^{-\varepsilon} \log n.$$

Using this and (3.26) in (3.20) yields $\mathbb{P}_{n,p}$-a.a.s.

$$\begin{aligned} \frac{1}{E[H_x]} &\leq \mathcal{D}\bigl(g^\star, g^\star\bigr) \frac{1}{(1 - cn^{-\varepsilon} \log n)^2} \leq \mathcal{D}\bigl(g^\star, g^\star\bigr)\bigl(1 + cn^{-\varepsilon} \log n\bigr) \\ &\leq \mathcal{D}\bigl(g^\star, g^\star\bigr) + cn^{-1-\varepsilon} \log^2 n \leq \mathcal{D}\bigl(g^\star, g^\star\bigr) + cn^{-1-c'}. \end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28) we obtain that $Q_x[\cdot|x \text{ good}]$-a.a.s.

$$e^{-u\rho(2-\xi)\xi n / E[H_x]} = e^{-u\rho(2-\xi)\xi n(\mathcal{D}(g^\star, g^\star) + o(n^{-1}))} = e^{-u\rho(2-\xi)\xi n \mathcal{D}(g^\star, g^\star)} + o(1).$$

Together with (3.19) it follows that

$$Q_x\bigl[\bigl|P\bigl[H_x > u\rho(2-\xi)\xi n\bigr] - e^{-u\rho(2-\xi)\xi n \mathcal{D}(g^\star, g^\star)}\bigr| \leq \varepsilon | x \text{ good}\bigr] \to 1 \quad \text{as } n \to \infty. \tag{3.29}$$

Lemma 3.3 implies that $\mathbb{P}_{n,p}$-a.a.s. for $x \in \mathcal{C}_1$, $\pi(x) = \frac{\deg(x)}{\rho(2-\xi)\xi n}(1 + o(1))$. Recalling (2.11), this implies that $\mathbb{P}_{n,p}$-a.a.s.

$$u\rho(2-\xi)\xi n \mathcal{D}\bigl(g^\star, g^\star\bigr) = uP_x[\tilde{H}_x > H_{B(x,r)^c}]\deg(x) + o(1).$$

Using this in (3.29), and noting that if x is good,

$$e^{-uP_x[\tilde{H}_x > H_{B(x,r)^c}]\deg(x)} = e^{-uP_\varnothing^{\mathcal{T}}[\tilde{H}_\varnothing > H_{B(\varnothing,r)^c}]\deg(\varnothing)},$$

finishes the proof of the lemma. □

Proof of part (1) of Proposition 3.1. We use the same technique as in the proof of Lemma 3.3: We compute the expectation of $E[|\mathcal{V}^u|]$ under $\mathbb{P}_{n,p}$ and then show that $E[|\mathcal{V}^u|]$ is concentrated. Define the random variables

$$W_x = \mathbf{1}_{\{x\in\mathcal{C}_1\}} P\big[H_x > u\rho(2-\xi)\xi n\big],$$
$$\tilde{W}_x = \mathbf{1}_{\{|\mathcal{T}|=\infty\}} \mathrm{e}^{-uP_\varnothing^{\mathcal{T}}[\tilde{H}_\varnothing > H_{B(\varnothing,r)^c}]\deg(\varnothing)},$$

where the tree $\mathcal{T}$ is defined by the coupling Q_x from Proposition 3.2, and $\varnothing$ is the root of $\mathcal{T}$.

Let us first compute the expectation of $\tilde{W}_x$ as $n\to\infty$. Since $r\to\infty$ as $n\to\infty$, and the tree $\mathcal{T}$ has law $\mathbb{P}^0_{\mathcal{T}}$,

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}_{Q_x}[\tilde{W}_x] &= \lim_{n\to\infty}\mathbb{E}^0_{\mathcal{T}}\big[\mathrm{e}^{-uP_\varnothing^{\mathcal{T}}[\tilde{H}_\varnothing > H_{B(\varnothing,r)^c}]\deg(\varnothing)}\big||\mathcal{T}|=\infty\big]\mathbb{P}^0_{\mathcal{T}}\big[|\mathcal{T}|=\infty\big]\\ &= \mathbb{E}_{\mathcal{T}}\big[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\big]\xi.\end{aligned} \tag{3.30}$$

For $\varepsilon>0$, define on the space of the coupling Q_x the event

$$\mathcal{A}_{x,\varepsilon} = \big\{|W_x - \tilde{W}_x| \le \varepsilon\big\}. \tag{3.31}$$

By definitions (3.9) and (3.13) of the events $\mathcal{B}_x$ and $\{x \text{ good}\}$, on $\mathcal{B}_x$ either $W_x = \tilde{W}_x = 0$ or x is good, i.e. $\mathcal{A}^c_{x,\varepsilon}\cap\mathcal{B}_x = \mathcal{A}^c_{x,\varepsilon}\cap\{x \text{ good}\}$. With Lemma 3.7 and (3.11) it follows that

$$\begin{aligned}Q_x\big[\mathcal{A}^c_{x,\varepsilon}\big] &\le Q_x\big[\mathcal{A}^c_{x,\varepsilon},\mathcal{B}_x\big] + Q_x\big[\mathcal{B}^c_x\big]\\ &\le Q_x\big[\mathcal{A}^c_{x,\varepsilon}|x\text{ good}\big]Q_x[x\text{ good}] + Q_x\big[\mathcal{B}^c_x\big] = \mathrm{o}(1)\quad\text{as } n\to\infty.\end{aligned} \tag{3.32}$$

Since W_x and $\tilde{W}_x$ are bounded by 1, this implies

$$\big|\mathbb{E}_{Q_x}[W_x] - \mathbb{E}_{Q_x}[\tilde{W}_x]\big| \le 2\varepsilon + Q_x\big[\mathcal{A}^c_{x,\varepsilon}\big]\quad\text{for any } \varepsilon>0,$$

and thus

$$\mathbb{E}_{Q_x}[W_x] = \mathbb{E}_{Q_x}[\tilde{W}_x] + \mathrm{o}(1)\quad\text{as } n\to\infty. \tag{3.33}$$

With (3.30) we conclude that

$$\mathbb{E}_{n,p}\big[E\big[|\mathcal{V}^u|\big]\big] = \mathbb{E}_{n,p}\Big[\sum_{x\in G} W_x\Big] = n\mathbb{E}_{Q_x}[W_x] = \xi n\mathbb{E}_{\mathcal{T}}\big[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\big] + \mathrm{o}(n)\quad\text{as } n\to\infty.$$

For the concentration of $E[|\mathcal{V}^u|]$ we use again the second moment method. Consider the coupling $Q_{x,y}$ from Proposition 3.2 for two fixed vertices $x\neq y$. The random variable $\tilde{W}_z$ as well as the event $\mathcal{A}_{z,\varepsilon}$ for $z\in\{x,y\}$ are canonically also defined on the space of $Q_{x,y}$ when replacing $\mathcal{T}$ by $\mathcal{T}_z$ in the definition of $\tilde{W}_z$. Let $\mathcal{A}_{x,y,\varepsilon} = \mathcal{A}_{x,\varepsilon}\cap\mathcal{A}_{y,\varepsilon}$, and recall the definition (3.10) of the set $\mathcal{B}_{x,y}$, on which either $W_z = \tilde{W}_z = 0$ or z is good, for both $z\in\{x,y\}$. Note that the statement of Lemma 3.7 also holds on the space of $Q_{x,y}$ when replacing $\mathcal{T}$ by $\mathcal{T}_z$ for both $z\in\{x,y\}$ respectively. As in (3.32), with Lemma 3.7 and (3.12) we obtain

$$\begin{aligned}Q_{x,y}\big[\mathcal{A}^c_{x,y,\varepsilon}\big] &\le Q_{x,y}\big[\mathcal{A}^c_{x,\varepsilon},\mathcal{B}_{x,y}\big] + Q_{x,y}\big[\mathcal{A}^c_{y,\varepsilon},\mathcal{B}_{x,y}\big] + Q_{x,y}\big[\mathcal{B}^c_{x,y}\big]\\ &\le Q_{x,y}\big[\mathcal{A}^c_{x,\varepsilon}|x\text{ good}\big] + Q_{x,y}\big[\mathcal{A}^c_{y,\varepsilon}|y\text{ good}\big] + Q_{x,y}\big[\mathcal{B}^c_{x,y}\big] = \mathrm{o}(1)\quad\text{as } n\to\infty.\end{aligned}$$

Since the W_z and $\tilde{W}_z$ are bounded by 1, it follows that

$$\big|\mathbb{E}_{Q_{x,y}}[W_xW_y] - \mathbb{E}_{Q_{x,y}}[\tilde{W}_x\tilde{W}_y]\big| \le \varepsilon + Q_{x,y}\big[\mathcal{A}^c_{x,y,\varepsilon}\big]\quad\text{for any } 1>\varepsilon>0,$$

and thus

$$\mathbb{E}_{Q_{x,y}}[W_xW_y] = \mathbb{E}_{Q_{x,y}}[\tilde{W}_x\tilde{W}_y] + \mathrm{o}(1)\quad\text{as } n\to\infty. \tag{3.34}$$

The trees $\mathcal{T}_x$ and $\mathcal{T}_y$ are independent, so the random variables $\tilde{W}_x$ and $\tilde{W}_y$ are independent. Therefore, (3.33) and (3.34) imply that for arbitrary vertices $x \neq y$

$$\mathbb{E}_{n,p}[W_x W_y] = \mathbb{E}_{n,p}[W_x]\mathbb{E}_{n,p}[W_y] + \mathrm{o}(1) \quad \text{as } n \to \infty.$$

Recall that $E[|\mathcal{V}^u|] = \sum_{x\in G} W_x$. By the boundedness of the W_x, it follows directly from the above that

$$\mathbb{E}_{n,p}\bigl[E\bigl[|\mathcal{V}^u|\bigr]^2\bigr] = \mathbb{E}_{n,p}\bigl[E\bigl[|\mathcal{V}^u|\bigr]\bigr]^2 + \mathrm{o}\bigl(n^2\bigr) \quad \text{as } n \to \infty.$$

Thus $\operatorname{Var} E[|\mathcal{V}^u|] = \mathrm{o}(n^2)$ and the Chebyshev inequality implies for any $\varepsilon > 0$

$$\mathbb{P}_{n,p}\bigl[\bigl|E\bigl[|\mathcal{V}^u|\bigr] - \mathbb{E}_{n,p}\bigl[E\bigl[|\mathcal{V}^u|\bigr]\bigr]\bigr| > \varepsilon n\bigr] = \mathrm{o}(1) \quad \text{as } n \to \infty.$$

This finishes the proof of the first part of Proposition 3.1. □

3.2. *Concentration of the size of the vacant set*

To prove part (2) of Proposition 3.1, we use similar techniques as in [8] and [10]. We define a sequence of i.i.d. stationary started random walk trajectories of length n^δ and glue them together at the endpoints to obtain a trajectory which is, by the fast mixing of the random walk, in distribution close to the random walk on $\mathcal{C}_1$ but has a different dependency structure, which allows to apply the following concentration result by [20].

Theorem 3.8 [20, Theorem 3.7]. *Let $W = (W_1, \dots, W_M)$ be a family of random variables W_k taking values in a set $\mathcal{A}_k$, and let f be a bounded real-valued function on $\prod \mathcal{A}_k$. Let μ denote the mean of $f(W)$. Define*

$$\begin{aligned} &r_k(y_1, \dots, y_{k-1}) \\ &\quad = \sup_{y,y'\in\mathcal{A}_k} \bigl|E\bigl[f(W)|W_k = y, W_i = y_i \ \forall i < k\bigr] - E\bigl[f(W)|W_k = y', W_i = y_i \ \forall i < k\bigr]\bigr|, \end{aligned}$$

and let

$$R^2 = \sup_{y_1,\dots,y_{M-1}} \sum_{k=1}^{M} r_k^2(y_1, \dots, y_{k-1}).$$

Then for any $t \geq 0$,

$$P\bigl[\bigl|f(W) - \mu\bigr| \geq t\bigr] \leq 2\mathrm{e}^{-t^2/R^2}.$$

Let us define precisely the above mentioned approximation of the random walk. Denote by P_x^L the restriction of P_x to $\mathcal{C}_1^{L+1}$, i.e. the law of the trajectory $(X_0, \dots, X_L)$ and by $P_{x,z}^L$ the law of the random walk bridge, that is P_x^L conditioned on $X_L = z$. Fix $\delta > 0$ and let $L = n^\delta$. For a given typical random graph G define on an auxiliary probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ the i.i.d. random variables $(Z^i)_{i\geq 0}$ as vertices of $\mathcal{C}_1$ chosen according to the stationary measure π. Given the collection (Z^i), let $(Y^i)_{i\geq 1}$ be conditionally independent elements of $\mathcal{C}_1^{L+1}$ such that each $(Y_k^i)_{k=0,\dots,L}$ is distributed according to the random walk bridge P_{Z^{i-1},Z^i}^L. We define the concatenation of the Y^i as

$$\mathcal{X}_t = Y_{t-(i-1)L}^i, \quad \text{when } (i-1)L \leq t < iL.$$

Denote by $\mathcal{P}^u$ the law of $\mathcal{X}$ on $\mathcal{C}_1^{u\rho(2-\xi)\xi n+1}$ and write P^u for $P^{u\rho(2-\xi)\xi n}$, that is P restricted to $\mathcal{C}_1^{u\rho(2-\xi)\xi n+1}$. The next lemma shows that $\mathcal{P}^u$ approximates P^u well if L is large enough.

Lemma 3.9. *$\mathbb{P}_{n,p}$-a.a.s. the measures $\mathcal{P}^u$ and P^u are equivalent, and for n large enough and constants c, c' depending on δ,*

$$\left|\frac{dP^u}{d\mathcal{P}^u} - 1\right| \le c\mathrm{e}^{-c'n^{\delta/2}}.$$

Proof. Let u' be the smallest number greater or equal to u such that $u'\rho(2-\xi)\xi n$ is an integer multiple of L and set $m = \frac{u'\rho(2-\xi)\xi n}{L}$. Since $\mathcal{P}^u$ and P^u are the restrictions of $\mathcal{P}^{u'}$ and $P^{u'}$ to $\mathcal{C}_1^{u\rho(2-\xi)\xi n+1}$, it is sufficient to prove the lemma for $\mathcal{P}^{u'}$ and $P^{u'}$. Let A be any measurable subset of $\mathcal{C}_1^{u\rho(2-\xi)\xi n+1}$. Then by the Markov property

$$\begin{aligned} P^{u'}[A] &= \sum_{x_0,\dots,x_m\in\mathcal{C}_1} P^{u'}[A|X_{iL}=x_i, 0\le i\le m]P^{u'}[X_{iL}=x_i, 0\le i\le m] \\ &= \sum_{x_0,\dots,x_m\in\mathcal{C}_1} P^{u'}[A|X_{iL}=x_i, 0\le i\le m]\pi(x_0)\prod_{k=0}^{m} P^L_{x_k}[X_L=x_{k+1}]. \end{aligned} \tag{3.35}$$

Next, note that $\mathcal{P}^{u'}[X_{iL}=x_i, 0\le i\le m]=0$ if and only if $P^{u'}[X_{iL}=x_i, 0\le i\le m]=0$: One can always choose the m x_i's, but there might not be any way to connect them by random walk bridges, whence the probability is zero. In this case, there is also no random walk trajectory going through this points. On the other hand, when there is no such trajectory, there are also no bridges.

From this and the construction of the measure $\mathcal{P}$ it follows that, whenever this is well-defined,

$$\begin{aligned} &\mathcal{P}^{u'}[A|X_{iL}=x_i, 0\le i\le m] = P^{u'}[A|X_{iL}=x_i, 0\le i\le m], \\ &\mathcal{P}^{u'}[X_{iL}=x_i, 0\le i\le m] = \prod_{k=0}^{m}\pi(x_k). \end{aligned} \tag{3.36}$$

Comparing (3.35) and (3.36), it remains to control the ratio $\frac{P^L_x[X_L=y]}{\pi(y)}$. We use (2.13), (2.7) and (2.2) to get $\mathbb{P}_{n,p}$-a.a.s.

$$\left|\frac{P^L_x[X_L=y]}{\pi(y)} - 1\right| \le \frac{1}{(\min_{z\in\mathcal{C}_1}\pi(z))^2}\mathrm{e}^{-\lambda_{\mathcal{C}_1}L} \le cn^2\log^2 n\mathrm{e}^{-(c'/\log^2 n)L}.$$

With $\frac{n^\delta}{\log^2 n} \ge cn^{\delta/2}$ for n large enough it follows that $\mathbb{P}_{n,p}$-a.a.s.

$$\left(1 - cn^2\log^2 n\mathrm{e}^{-c'n^{\delta/2}}\right)^m \le \frac{P^{u'}[A]}{\mathcal{P}^{u'}[A]} \le \left(1 + cn^2\log^2 n\mathrm{e}^{-c'n^{\delta/2}}\right)^m,$$

and hence $\mathbb{P}_{n,p}$-a.a.s. $\mathcal{P}^{u'}$ and $P^{u'}$ are equivalent, and the lemma follows by changing constants to accommodate the terms polynomial in n and $\log n$. □

Proof of part (2) of Proposition 3.1. We show that for any $\delta > 0$,

$$P\left[\left||\mathcal{V}^u| - E\left[|\mathcal{V}^u|\right]\right| \ge n^{1/2+\delta}\right] \le c\mathrm{e}^{-c'n^{\delta/2}}, \quad \mathbb{P}_{n,p}\text{-a.a.s.}, \tag{3.37}$$

which implies the statement of the proposition.

Set $m = \lfloor\frac{u\rho(2-\xi)\xi n}{L}\rfloor$ and $u' = \frac{mL}{\rho(2-\xi)\xi n}$. Then $u\rho(2-\xi)\xi n - u'\rho(2-\xi)\xi n \le L$, and so $||\mathcal{V}^u| - |\mathcal{V}^{u'}|| \le L$. It follows that for n large enough

$$\begin{aligned} P\left[\left||\mathcal{V}^u| - E\left[|\mathcal{V}^u|\right]\right| \ge n^{1/2+\delta}\right] &\le P\left[\left||\mathcal{V}^{u'}| - E\left[|\mathcal{V}^{u'}|\right]\right| \ge n^{1/2+\delta} - 2L\right] \\ &\le P\left[\left||\mathcal{V}^{u'}| - E\left[|\mathcal{V}^{u'}|\right]\right| \ge \frac{1}{2}n^{1/2+\delta}\right]. \end{aligned} \tag{3.38}$$

Let $\mathcal{U}^{u'} = \mathcal{C}_1 \setminus \mathcal{X}_{[0,mL]}$ be the vacant set left by the concatenation $\mathcal{X}$, and denote by $\mathcal{E}$ the expectation corresponding to $\mathcal{P}$. Lemma 3.9 implies that $\mathbb{P}_{n,p}$-a.a.s.

$$\big|P[\mathcal{V}^{u'} \in \cdot] - \mathcal{P}[\mathcal{U}^{u'} \in \cdot]\big| \le c\mathrm{e}^{-c'n^{\delta/2}},$$
$$\big|E[|\mathcal{V}^{u'}|] - \mathcal{E}[|\mathcal{U}^{u'}|]\big| \le cn\mathrm{e}^{-c'n^{\delta/2}} \le \frac{1}{4}n^{1/2+\delta}.$$

From this we obtain that $\mathbb{P}_{n,p}$-a.a.s.

$$P\Big[\big||\mathcal{V}^{u'}| - E[|\mathcal{V}^{u'}|]\big| \ge \frac{1}{2}n^{1/2+\delta}\Big] \le \mathcal{P}\Big[\big||\mathcal{U}^{u'}| - \mathcal{E}[|\mathcal{U}^{u'}|]\big| \ge \frac{1}{4}n^{1/2+\delta}\Big] + c\mathrm{e}^{-c'n^{\delta/2}}. \tag{3.39}$$

We now apply Theorem 3.8 with $M = m$, $\mathcal{A}_k = \mathcal{C}_1^{L+1}$, $W_k = Y^k$ and $f(W) = |\mathcal{U}^{u'}|$. We claim that

$$\begin{aligned} &r_k(y_1, \dots, y_{k-1}) \\ &\quad = \sup_{y,y' \in \mathcal{A}_k} \big|E[|\mathcal{U}^{u'}| \mid Y^k = y, Y^i = y_i\ \forall i < k] - E[|\mathcal{U}^{u'}| \mid Y^k = y', Y^i = y_i\ \forall i < k]\big| \\ &\quad \le 2L. \end{aligned}$$

Indeed, when conditioning additionally on $Y^{k+2}, \dots, Y^m$, the only two different segments Y^k and Y^{k+1} can change the size of the vacant set by at most the length of two segments, and the claim follows by integrating over all possible $Y^{k+2}, \dots, Y^m$.

Then $R^2 \le m(2L)^2 \le \frac{u\rho(2-\xi)\xi n}{L} 4L^2 = cn^{1+\delta}$, and Theorem 3.8 implies

$$\mathcal{P}\Big[\big||\mathcal{U}^{u'}| - \mathcal{E}[|\mathcal{U}^{u'}|]\big| \ge \frac{1}{4}n^{1/2+\delta}\Big] \le 2\mathrm{e}^{-2(n^{1+2\delta}/16)/(cn^{1+\delta})} = c\mathrm{e}^{-c'n^{\delta}}.$$

This together with (3.38) and (3.39) proves (3.37) and hence part (2) of Proposition 3.1. □

4. Coupling of processes

In this section we introduce a process $\bar{X}$ which satisfies the spatial Markov property described in the introduction. We derive a phase transition in the vacant set of this process, and we compare it with the simple random walk X on the giant component.

Consider the following algorithm defined on an auxiliary probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ which builds an element of $\Omega_n = \mathcal{G}(n) \times \{1, 2, \dots, n\}^{\mathbb{N}_0}$, that is a graph on n vertices and a random walk-like process on this graph. All the random choices made in the algorithm are independent variables defined on $\tilde{\Omega}$.

Algorithm 4.1. *At the beginning all n vertices are unvisited, and all $\binom{n}{2}$ possible edges are unexplored. When the algorithm (or the so defined process) passes an unvisited vertex, this vertex is marked visited. Edges adjacent to the vertex will be explored and become either open or closed. After being explored, the state of an edge does not change.*

(1) *Start at time 0 with a uniformly chosen vertex v_0 among all n vertices, mark it visited.*
(2) *Being at time $k \ge 0$ with current vertex v_k, check first if there are any unvisited vertices left:*

- *If there are, let any unexplored edge adjacent to v_k be explored and marked open with probability $p = \frac{\rho}{n}$ and closed otherwise. All vertices w such that the edge $\{v_k, w\}$ is open are called neighbours of v_k.*
- *If there are no unvisited vertices left, let $\{v_l\}_{l>k}$ be uniformly at random chosen vertices and terminate the algorithm (this choice of continuation of the process v_k is totally arbitrary and does not influence the reasoning below).*

(3) *If v_k has at least one neighbour, and if there are any unvisited vertices adjacent to explored edges, choose vertex v_{k+1} uniformly among all neighbours of v_k and mark v_{k+1} visited, go to step (2) and proceed with current vertex v_{k+1}.*

(4) *If v_k has no neighbours or if there are no unvisited vertices adjacent to explored edges, the current component is entirely covered. Then choose vertex v_{k+1} uniformly among all n vertices, mark it visited, go to step* (2) *and proceed with current vertex v_{k+1}.*

By construction, the law of the graph explored by this algorithm (edges present if they are marked open) is $\mathbb{P}_{n,p}$. Let $\bar{X}$ be the process defined by $\bar{X}_k = v_k$.

It will be helpful to have two different points of view on Algorithm 4.1. The first is to look at the picture at the end of the algorithm: There is a graph G and a trajectory of $\bar{X}$ covering all the vertices of the graph. Using this point of view, denote by $\bar{P}^G$ the law on $(\{1,2,\dots,n\}^{\mathbb{N}_0}, \mathbb{F}_n)$ of the process $\bar{X}$ under $\tilde{P}$ conditioned on the event that the graph explored by the algorithm is $G \in \mathcal{G}(n)$ (i.e. conditioned on the random choices in Algorithm 4.1 that determine the states of edges, but not on the random choices that determine the trajectory of $\bar{X}$). Under $\bar{P}^G$, the process $\bar{X}$ is, between two occurrences of step (4) of the algorithm, a simple random walk on the currently explored component, started with uniform distribution on this component. Define on $\Omega_n = \mathcal{G}(n) \times \{1,2,\dots,n\}^{\mathbb{N}_0}$ the annealed measure (cf. (1.1)) by

$$\bar{\mathbf{P}}_n(A \times B) = \sum_{G \in A} \mathbb{P}_{n,p}(G)\bar{P}^G(B) \quad \text{for } A \in \mathbb{G}_n, B \in \mathbb{F}_n.$$

The second point of view is to look at Algorithm 4.1 as building the graph G on-the-go. Having this in mind, the next lemma, which is crucial for the proof of Theorem 1.1, is straightforward (cf. [11, Lemma 6] for a similar statement). Let $\bar{\mathcal{V}}(t) = G \setminus \bar{X}_{[0,t]}$ be the vacant set left by the process $\bar{X}$ at time t, defined on $(\Omega_n, \bar{\mathbf{P}}_n)$. Once again we use the same notation $\bar{\mathcal{V}}(t)$ for the set of vertices as well as the induced subgraph of G.

Lemma 4.2. *Under $\bar{\mathbf{P}}_n$ conditioned on $|\bar{\mathcal{V}}(t)| = N$ the graph $\bar{\mathcal{V}}(t)$ has marginal law $\mathbb{P}_{N,p}$.*

Proof. By construction of Algorithm 4.1, the vacant graph $\bar{\mathcal{V}}(t)$ consists of the $|\bar{\mathcal{V}}(t)|$ unvisited vertices at time t. Edges possibly connecting $\bar{\mathcal{V}}(t)$ and the already visited vertices as well as all edges possibly connecting two already visited vertices are explored. So the edges eligible to be edges of $\bar{\mathcal{V}}(t)$ are exactly all unexplored edges at time t. Because their state has not yet been decided by the algorithm, all these edges are open with probability $\frac{\rho}{n}$, independently of what happened before, independently of each other. Therefore, the vacant graph $\bar{\mathcal{V}}(t)$ is a standard Erdős–Rényi random graph on $N = |\bar{\mathcal{V}}(t)|$ vertices, every edge present with probability $p = \frac{\rho}{n}$, and hence it has law $\mathbb{P}_{N,p}$. □

From Lemma 4.2 and the classical results on random graphs it follows directly that the component structure of the vacant graph $\bar{\mathcal{V}}(t)$ exhibits a phase transition at the time t for which $|\bar{\mathcal{V}}(t)|\frac{\rho}{n} = 1$. To translate this phase transition to the simple random walk X on the giant component $\mathcal{C}_1(G)$, we need to couple X to the process $\bar{X}$. We do this by first giving a coupling of X and $\bar{X}$ under $P^{\mathcal{C}_1}$ and $\bar{P}^G$ respectively on a fixed typical graph G. In Section 5 we will extend this coupling to an "annealed" coupling of X and $\bar{X}$ under $\mathbf{P}_n$ and $\bar{\mathbf{P}}_n$, respectively.

Proposition 4.3. *For n large enough, for every fixed typical graph $G \in \mathcal{G}(n)$ there exists a coupling Q^G of $\bar{X}$ under $\bar{P}^G$ and X under $P^{\mathcal{C}_1(G)}$ such that*

$$Q^G\big[\big\{X_k = \bar{X}_{k+2\log^5 n} \textit{ for all } k = 0,1,\dots,u\rho(2-\xi)\xi n\big\}^c\big] \le \frac{c}{n^{c'}}.$$

Proof. We first show that $\bar{X}$ typically is on the largest component $\mathcal{C}_1$ at time $\log^5 n$, that it mixes quickly, and then stays on $\mathcal{C}_1$ until time $u\rho(2-\xi)\xi n + 2\log^5 n$. This will allow us to identify X with $\bar{X}$ in this time interval on an event of high probability.

Let G be the typical graph (i.e. a graph satisfying (2.1), (2.2) and (2.3)) explored by Algorithm 4.1 and $\mathcal{C}_1$ its giant component, i.e. we look at the picture after completion of the algorithm. Define the probability distribution $\bar{\pi}$ on G as the distribution of $\bar{X}_{2\log^5 n}$, and view the stationary distribution π of the random walk on $\mathcal{C}_1$ as a distribution on the whole graph G by setting $\pi \equiv 0$ on $G \setminus \mathcal{C}_1$. Denote by $\|\cdot\|_{\mathrm{TV}}$ the total variation norm. Define $\tau = \min\{t \ge \log^5 n$:

step (4) of Algorithm 4.1 is performed}. τ is the first time after $\log^5 n$ where $\bar{X}$ does not behave like a random walk. We show that for n large enough the following properties hold for a typical graph G:

$$\bar{P}^G[\bar{X}_{\log^5 n} \notin \mathcal{C}_1] \le \frac{c}{n^{1+c'}}, \tag{4.1}$$

$$\bar{P}^G\big[\tau \le u\rho(2-\xi)\xi n + 2\log^5 n\big] \le \frac{c}{n^{1+c'}}, \tag{4.2}$$

$$\|\bar{\pi} - \pi\|_{\mathrm{TV}} \le \frac{c}{n^{c'}}. \tag{4.3}$$

Since G is typical, there is a giant component of size $||\mathcal{C}_1| - \xi n| \le n^{3/4}$, and all other components are simple (i.e. they have at most as many edges as vertices) and of size smaller than $C\log n$. For (4.1), since for n large enough the random walk cannot cover $\mathcal{C}_1$ in $\log^4 n$ steps,

$$\begin{aligned}\bar{P}^G[\bar{X}_{\log^4 n} \notin \mathcal{C}_1] \le \bar{P}^G\big[&\bar{X} \text{ starts on a small component and stays on small} \\ &\text{components for time longer than } \log^4 n\big].\end{aligned} \tag{4.4}$$

Let N_s be the number of small components that $\bar{X}$ visits before reaching the giant component. By construction and since by (2.1) $|\mathcal{C}_1| \ge (\xi - \varepsilon)n$ for some $\varepsilon > 0$, N_s is stochastically dominated by a Geometric($\xi - \varepsilon$) random variable, in particular it has a finite mean. Then by the Markov inequality

$$\bar{P}^G[N_s \ge \log n] \le \frac{c}{\log n}. \tag{4.5}$$

Let $C_s^{(i)}$ be the cover time of the ith small component covered by $\bar{X}$. The expected cover time of a graph on k vertices and m edges is bounded by $2m(k-1)$ (see e.g. [3]), so the expected cover time $\bar{E}^G[C_s^{(i)}]$ of a simple component of size smaller than $C\log n$ is bounded by $C'\log^2 n$. The Markov inequality implies

$$\bar{P}^G\left[\sum_{i=1}^{\log n} C_s^{(i)} \ge \log^4 n\right] \le \frac{\log n\,\bar{E}^G[C_s^{(i)}]}{\log^4 n} \le \frac{c}{\log n}. \tag{4.6}$$

From (4.5) and (4.6) it follows that the probability on the right hand side of (4.4) is smaller than $\frac{c}{\log n}$. Given $\bar{X}$ has not found $\mathcal{C}_1$ after $\log^4 n$ steps, some small components are partly or entirely covered, but one can use the same line of arguments as above for the next $\log^4 n$ steps to get

$$\bar{P}^G[\bar{X}_{2\log^4 n} \notin \mathcal{C}_1 | \bar{X}_{\log^4 n} \notin \mathcal{C}_1] \le \bar{P}^G[\bar{X}_{\log^4 n} \notin \mathcal{C}_1].$$

Using this, we have

$$\bar{P}^G[\bar{X}_{2\log^4 n} \notin \mathcal{C}_1] = \bar{P}^G[\bar{X}_{2\log^4 n} \notin \mathcal{C}_1 | \bar{X}_{\log^4 n} \notin \mathcal{C}_1]\bar{P}^G[\bar{X}_{\log^4 n} \notin \mathcal{C}_1] \le \bar{P}^G[\bar{X}_{\log^4 n} \notin \mathcal{C}_1]^2.$$

Since $\bar{X}$ cannot cover $\mathcal{C}_1$ in $\log^5 n$ steps we can iterate the above $\log n$ times, then

$$\bar{P}^G[\bar{X}_{\log^5 n} \notin \mathcal{C}_1] \le \left(\frac{c}{\log n}\right)^{\log n} \le \frac{c}{n^{1+c'}},$$

which proves (4.1).

To prove (4.2) first note that $P^{\mathcal{C}_1}[\cdot] = \sum_{z\in\mathcal{C}_1} \pi(z) P_z^{\mathcal{C}_1}[\cdot]$. With (2.7) it follows that

$$\sup_{z\in\mathcal{C}_1} P_z^{\mathcal{C}_1}[\cdot] \le \frac{1}{\min_{z\in\mathcal{C}_1}\pi(z)} P^{\mathcal{C}_1}[\cdot] \le cn\log n\, P^{\mathcal{C}_1}[\cdot]. \tag{4.7}$$

Using (4.1) we have

$$\begin{aligned}
&\bar P^G\big[\tau \le u\rho(2-\xi)\xi n + 2\log^5 n\big]\\
&\quad\le \sup_{z\in\mathcal{C}_1} P_z^{\mathcal{C}_1}\big[\text{cover time of } \mathcal{C}_1 \text{ is smaller than } u\rho(2-\xi)\xi n + 2\log^5 n\big] + \bar P^G[\bar X_{\log^5 n} \notin \mathcal{C}_1]\\
&\quad\le \sup_{z\in\mathcal{C}_1} P_z^{\mathcal{C}_1}\big[\text{vacant set } \mathcal{V}\big(u\rho(2-\xi)\xi n + 2\log^5 n\big) \text{ is empty}\big] + \frac{c}{n^{1+c'}}.
\end{aligned}$$

Since adding a trajectory of length $2\log^5 n$ can decrease the size of the vacant set by at most $2\log^5 n = \mathrm{o}(n)$, it follows that asymptotically $|\mathcal{V}(u\rho(2-\xi)\xi n + 2\log^5 n)| = |\mathcal{V}(u\rho(2-\xi)\xi n)| + \mathrm{o}(n)$. Using (4.7), from (3.37) and part (1) of Proposition 3.1 it follows that for a typical graph and ε small enough

$$\sup_{z\in\mathcal{C}_1} P_z^{\mathcal{C}_1}\big[\big|\mathcal{V}\big(u\rho(2-\xi)\xi n\big)\big| < \varepsilon n\big] \le cn\log n P^{\mathcal{C}_1}\big[\big|\mathcal{V}\big(u\rho(2-\xi)\xi n\big)\big| < \varepsilon n\big] \le c' n \log n \mathrm{e}^{-c'' n^{\delta/2}},$$

where $\delta > 0$ is the parameter defining the length of the random walk bridges in Section 3.2. For any choice of δ we can find constants such that the above expression is smaller than $\frac{c}{n^{1+c'}}$, and (4.2) follows.

For the proof of (4.3) let $P_\mu^{\mathcal{C}_1}$ denote the law of the random walk on $\mathcal{C}_1$ started at initial distribution μ. When $\bar X$ is on $\mathcal{C}_1$ at time $\log^5 n$, it has then some distribution μ and it cannot cover $\mathcal{C}_1$ in time $\log^5 n$. Using (2.13), we thus get for every $y \in \mathcal{C}_1$

$$\begin{aligned}
\big|\bar P^G[\bar X_{2\log^5 n} = y] - \pi(y)\big| &\le \bar P^G[\bar X_{\log^5 n} \notin \mathcal{C}_1] + \sup_\mu \big|P_\mu^{\mathcal{C}_1}[X_{\log^5 n} = y] - \pi(y)\big|\\
&\le \bar P^G[\bar X_{\log^5 n} \notin \mathcal{C}_1] + \frac{1}{\min_{v\in\mathcal{C}_1}\pi(v)} \mathrm{e}^{-\lambda_{\mathcal{C}_1}\log^5 n}.
\end{aligned}$$

With (4.1), (2.2) and (2.7), it follows for every $y \in \mathcal{C}_1$

$$\big|\bar P^G[\bar X_{2\log^5 n} = y] - \pi(y)\big| \le \frac{c}{n^{1+c'}}. \tag{4.8}$$

We set $\pi \equiv 0$ on $G\setminus\mathcal{C}_1$, and by (4.1) and (4.2), $\bar P^G[\bar X_{2\log^5 n} = y] \le \frac{c}{n^{1+c'}}$ for $y \in G\setminus\mathcal{C}_1$. Thus (4.8) holds for all $y \in G$. (4.3) follows from (4.8) since by e.g. [18, Proposition 4.2] we have $\|\bar\pi - \pi\|_{\mathrm{TV}} \le n \max_{y\in G} |\bar P^G[\bar X_{2\log^5 n} = y] - \pi(y)|$.

We can now define the coupling of X under $P^{\mathcal{C}_1}$ and $\bar X$ under $\bar P^G$. Consider again the (possibly enlarged) auxiliary probability space $(\tilde\Omega, \tilde{\mathcal{A}}, \tilde P)$, on which originally $\bar X$ was defined. On this auxiliary space we define a random variable Y on G with distribution π. Y depends on the graph G (i.e. it depends on the random choices in Algorithm 4.1 that determine the states of edges), and it may depend on the random choices that determine the trajectory of $\bar X$ up to time $2\log^5 n$, but it is independent of all the random choices that determine the trajectory of $\bar X$ at times $2\log^5 n + k$, $k \ge 1$. By e.g. [18, Proposition 4.7] we can choose Y such that $\tilde P[\bar X_{2\log^5 n} \ne Y] = \|\bar\pi - \pi\|_{\mathrm{TV}}$. By (4.3) it follows that

$$\tilde P[\bar X_{2\log^5 n} \ne Y] \le \frac{c}{n^{c'}}. \tag{4.9}$$

Moreover, we define on $\tilde\Omega$ a collection $\tilde X^z$, $z \in G$, of independent simple random walks on G started at z, independent of $\bar X$ and Y (i.e. depending only on the random choices in Algorithm 4.1 that determine the states of edges, but independent of the random choices that determine the trajectory of $\bar X$).

Define the process X using $\bar X$, Y and $\tilde X^z$ as follows,

$$\begin{aligned}
&\left.\begin{aligned} &X_k = \bar X_{k+2\log^5 n} \quad \text{for } 0 \le k \le \tau,\\ &X_k = \tilde X_k^{\bar X_\tau} \quad \text{for } k > \tau, \end{aligned}\right\} && \text{if } \bar X_{\log^5 n} \in \mathcal{C}_1 \text{ and } Y = \bar X_{2\log^5 n},\\
&X_k = \tilde X_k^Y \quad \text{for } k \ge 0, && \text{if } \bar X_{\log^5 n} \in \mathcal{C}_1 \text{ and } Y \ne \bar X_{2\log^5 n},\\
&X_k = \tilde X_k^Y \quad \text{for } k \ge 0, && \text{if } \bar X_{\log^5 n} \notin \mathcal{C}_1.
\end{aligned} \tag{4.10}$$

Let Q^G denote the joint law of $\bar{X}$ and X on $\{1,2,\dots,n\}^{2\mathbb{N}_0}$. Since Y has distribution π and $\bar{X}$ behaves like a random walk between occurrences of step (4) of Algorithm 4.1, in any case X is a simple random walk on $\mathcal{C}_1$ started stationary, so Q^G is indeed a coupling of simple random walk on the giant component and the process $\bar{X}$ from Algorithm 4.1 with marginal laws $P^{\mathcal{C}_1}$ and $\bar{P}^G$ respectively.

By (4.9), (4.1) and (4.2) the first case of the coupling (4.10) happens with probability $\geq 1-\frac{c}{n^{c'}}$, and by (4.2) also $\tau > u\rho(2-\xi)\xi n + 2\log^5 n$ with high probability, and the statement of the proposition follows. □

The coupling (4.10) defined in the proof of Proposition 4.3 will allow us to deduce the phase transition in the vacant set left by X from the phase transition in the vacant set left by $\bar{X}$. To apply the results of Section 3, we have to find the relation between the sizes of these vacant sets. This relation is given by the next lemma. Denote $\bar{\mathcal{V}}^u = \bar{\mathcal{V}}(u\rho(2-\xi)\xi n + 2\log^5 n) = G \setminus \bar{X}_{[0,u\rho(2-\xi)\xi n+2\log^5 n]}$ and as before $\mathcal{V}^u = \mathcal{V}(u\rho(2-\xi)\xi n) = \mathcal{C}_1 \setminus X_{[0,u\rho(2-\xi)\xi n]}$.

Lemma 4.4. *For a sequence of typical graphs G and any fixed $u>0$, with respect to the corresponding sequence of couplings Q^G, the random variables $|\bar{\mathcal{V}}^u|$ and $|\mathcal{V}^u|$ satisfy*

$$\left|\bar{\mathcal{V}}^u\right| = \left|\mathcal{V}^u\right| + (1-\xi)n + \mathrm{o}(n), \quad Q^G\text{-a.a.s.}$$

Proof. Denote $\bar{\mathcal{W}}^u = G\setminus \bar{X}_{[2\log^5 n, u\rho(2-\xi)\xi n+2\log^5 n]}$. Then $||\bar{\mathcal{V}}^u| - |\bar{\mathcal{W}}^u|| \leq 2\log^5 n$, and for any $\varepsilon>0$, $\varepsilon n - 2\log^5 n \geq \frac{\varepsilon}{2}n$ for n large enough, thus

$$Q^G\left[\left|\left|\bar{\mathcal{V}}^u\right| - \left|\mathcal{V}^u\right| - (1-\xi)n\right| > \varepsilon n\right] \leq Q^G\left[\left|\left|\bar{\mathcal{W}}^u\right| - \left|\mathcal{V}^u\right| - (1-\xi)n\right| > \frac{\varepsilon}{2}n\right].$$

By Proposition 4.3, Q^G-a.a.s. the sets $\bar{\mathcal{W}}^u$ and $\mathcal{V}^u$ differ only by the small components of the graph G, i.e. $\bar{\mathcal{W}}^u = \mathcal{V}^u \cup \bigcup_{i\geq 2}\mathcal{C}_i(G)$. By (2.1), in a typical graph G the total size of small components satisfies $|\bigcup_{i\geq 2}\mathcal{C}_i(G) - (1-\xi)n| \leq \frac{\varepsilon}{2}n$ for n large enough. Therefore, for every $\varepsilon>0$,

$$Q^G\left[\left|\left|\bar{\mathcal{W}}^u\right| - \left|\mathcal{V}^u\right| - (1-\xi)n\right| > \frac{\varepsilon}{2}n\right] \leq Q^G\left[\bar{\mathcal{W}}^u \neq \mathcal{V}^u \cup \bigcup_{i\geq 2}\mathcal{C}_i(G)\right] \to 0 \quad \text{as } n\to\infty.$$

This proves the lemma. □

5. Proof of main result

We first extend the coupling Q^G that was defined for typical graphs in Proposition 4.3. Let Q^G for a non-typical graph G be the joint law on $\{1,2,\dots,n\}^{2\mathbb{N}_0}$ of two independent processes X and $\bar{X}$ under $P^{\mathcal{C}_1(G)}$ and $\bar{P}^G$ respectively. We define the annealed coupling measure $\mathbf{Q}_n$ on the space $\Omega_n' = \mathcal{G}(n)\times\{1,2,\dots,n\}^{2\mathbb{N}_0}$ with the canonical coordinates G, $\bar{X}$, X as

$$\mathbf{Q}_n(A\times B) = \sum_{G\in A}\mathbb{P}_{n,p}(G)Q^G(B),$$

where $A\in\mathbb{G}_n$ and $B = B_1\times B_2$ with $B_i \in \mathbb{F}_n$ for $i=1,2$ (cf. (1.1) for the definition of the σ-algebras $\mathbb{G}_n$ and $\mathbb{F}_n$). Then $\mathbf{Q}_n$ is a coupling of the two processes X and $\bar{X}$, where X has marginal law $\mathbf{P}_n$ and $\bar{X}$ has marginal law $\bar{\mathbf{P}}_n$, and since every G is $\mathbb{P}_{n,p}$-a.a.s. a typical graph the statements of Proposition 4.3 and Lemma 4.4 hold $\mathbf{Q}_n$-a.a.s.

Proof of Theorem 1.1. For the proof we use the annealed coupling $\mathbf{Q}_n$ of X and $\bar{X}$. As a direct consequence of Proposition 3.1 and Lemma 4.4 we obtain that

$$\left|\bar{\mathcal{V}}^u\right| = \xi n\mathbb{E}_{\mathcal{T}}\left[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\right] + (1-\xi)n + \mathrm{o}(n), \quad \mathbf{Q}_n\text{-a.a.s.}$$

It follows from Lemma 4.2 and the classical results on random graphs that the graph $\bar{\mathcal{V}}^u = G \setminus \bar{X}_{[0,u\rho(2-\xi)\xi n+2\log^5 n]}$ exhibits a phase transition at the value u such that $\lim_{n\to\infty} |\bar{\mathcal{V}}^u| \frac{\rho}{n} = 1$. This value is the solution $u_\star$ of the equation

$$\rho\xi\mathbb{E}_{\mathcal{T}}\big[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\big] + \rho(1-\xi) = 1. \tag{5.1}$$

$\bar{\mathcal{V}}^u$ has therefore $\mathbf{Q}_n$-a.a.s. a unique giant component $\mathcal{C}_1(\bar{\mathcal{V}}^u)$ of size $\zeta(u,\rho)n + \mathrm{o}(n)$ and all other components of size smaller than $\bar{C}\log n$ if $u < u_\star$, and it has $\mathbf{Q}_n$-a.a.s. all components of size smaller than $\bar{C}\log n$ for $u > u_\star$, where $\bar{C} > 0$ is some fixed constant. For $u < u_\star$, the constant $\zeta(u,\rho)$ is given as the unique solution in $(0,1)$ of the equation

$$\exp\big\{-\zeta\big(\rho\xi\mathbb{E}_{\mathcal{T}}\big[\mathrm{e}^{-u\,\mathrm{cap}_{\mathcal{T}}(\varnothing)}\big] + \rho(1-\xi)\big)\big\} = 1-\zeta. \tag{5.2}$$

It remains to translate this phase transition to the vacant graph $\mathcal{V}^u$ of the random walk on the giant component.

Let us first translate the phase transition to the subgraph induced by the slightly enlarged set $\bar{\mathcal{V}}^u \cup \bar{X}_{[0,2\log^5 n]}$. Adding one vertex of degree d in G to the graph $\bar{\mathcal{V}}^u$ can merge at most d components of $\bar{\mathcal{V}}^u$. By (2.3) the degree d is $\mathbf{Q}_n$-a.a.s. bounded by $\log n$, so adding the vertices of $\bar{X}_{[0,2\log^5 n]}$ can $\mathbf{Q}_n$-a.a.s. merge at most $2\log^6 n$ components. It follows that $\mathbf{Q}_n$-a.a.s., by adding $\bar{X}_{[0,2\log^5 n]}$ to $\bar{\mathcal{V}}^u$, any component of size smaller than $\bar{C}\log n$ in $\bar{\mathcal{V}}^u$ can either merge with the giant component if there is one, or it can become a component of size at most $2\bar{C}\log^7 n$. Also, in the supercritical phase the giant component can $\mathbf{Q}_n$-a.a.s. grow by at most $2\bar{C}\log^7 n = \mathrm{o}(n)$. Therefore, the graph induced by $\bar{\mathcal{V}}^u \cup \bar{X}_{[0,2\log^5 n]}$ exhibits a phase transition at $u_\star$ with the same size $\zeta(u,\rho)n + \mathrm{o}(n)$ of the giant component for $u < u_\star$, and with the bound $2\bar{C}\log^7 n$ for the size of the second largest component for $u < u_\star$ and the largest component for $u > u_\star$.

Recall that $\bar{\mathcal{W}}^u$ denotes the set $G \setminus \bar{X}_{[2\log^5 n, u\rho(2-\xi)\xi n+2\log^5 n]}$ as well as the induced subgraph. We have the following inclusions of sets and induced subgraphs in G,

$$\bar{\mathcal{V}}^u \subset \bar{\mathcal{W}}^u \subset \bar{\mathcal{V}}^u \cup \bar{X}_{[0,2\log^5 n]}.$$

Consider the vacant set $\mathcal{V}^u \subset \mathcal{C}_1$ of the random walk X on the giant component. By Proposition 4.3 and since every graph is $\mathbb{P}_{n,p}$-a.a.s. a typical graph, we have $\mathbf{Q}_n$-a.a.s. $\bar{\mathcal{W}}^u = \mathcal{V}^u \cup \bigcup_{i\geq 2}\mathcal{C}_i(G)$. It follows that

$$\bar{\mathcal{V}}^u \subset \mathcal{V}^u \cup \bigcup_{i\geq 2}\mathcal{C}_i(G), \quad \mathbf{Q}_n\text{-a.a.s.}, \tag{5.3}$$

$$\mathcal{V}^u \subset \bar{\mathcal{V}}^u \cup \bar{X}_{[0,2\log^5 n]}, \quad \mathbf{Q}_n\text{-a.a.s.} \tag{5.4}$$

Note that $\mathbf{Q}_n$-a.a.s. the union $\bigcup_{i\geq 2}\mathcal{C}_i(G)$ of all components of G except the largest are exactly all small components of size smaller than $C\log n$. From this and (5.3) it follows that $|\mathcal{C}_1(\mathcal{V}^u)|$ is $\mathbf{Q}_n$-a.a.s. bounded from below by $|\mathcal{C}_1(\bar{\mathcal{V}}^u)|$ whenever $|\mathcal{C}_1(\bar{\mathcal{V}}^u)|$ is larger than of order $\log n$. From (5.4) it follows that $|\mathcal{C}_1(\mathcal{V}^u)|$ is $\mathbf{Q}_n$-a.a.s. bounded from above by $|\mathcal{C}_1(\bar{\mathcal{V}}^u \cup \bar{X}_{[0,2\log^5 n]})|$. The respective phase transitions in $\bar{\mathcal{V}}^u$ and $\bar{\mathcal{V}}^u \cup \bar{X}_{[0,2\log^5 n]}$ thus immediately imply the statements (1.3) and (1.5) of Theorem 1.1.

To prove (1.4), i.e. the uniqueness of the giant component in the supercritical phase, fix $u < u_\star$ and let $\mathcal{L}_n$ be the event that there are two distinct components $\mathcal{C}_a$ and $\mathcal{C}_b$ in $\mathcal{V}^u$ both of size strictly larger than $2\bar{C}\log^7 n$, with $\bar{C}$ as defined below (5.1). We show that $\mathbf{Q}_n[\mathcal{L}_n] \to 0$ as $n \to \infty$, which proves (1.4). First note that if $\mathcal{L}_n$ happens, then either $\mathcal{C}_a \cap \bar{\mathcal{V}}^u$ and $\mathcal{C}_b \cap \bar{\mathcal{V}}^u$ are distinct components in $\bar{\mathcal{V}}^u$ or the inclusion in (5.3) does not hold, which is unlikely, so

$$\mathbf{Q}_n[\mathcal{L}_n] \leq \mathbf{Q}_n\big[\mathcal{L}_n, \mathcal{C}_a \cap \bar{\mathcal{V}}^u \text{ and } \mathcal{C}_b \cap \bar{\mathcal{V}}^u \text{ are distinct components in } \bar{\mathcal{V}}^u\big] + \mathrm{o}(1) \quad \text{as } n \to \infty.$$

But if $\mathcal{C}_a \cap \bar{\mathcal{V}}^u$ and $\mathcal{C}_b \cap \bar{\mathcal{V}}^u$ are distinct components in $\bar{\mathcal{V}}^u$, at least one of $\mathcal{C}_a \cap \bar{\mathcal{V}}^u$ or $\mathcal{C}_b \cap \bar{\mathcal{V}}^u$ is subset of $\bigcup_{i\geq 2}\mathcal{C}_i(\bar{\mathcal{V}}^u)$, which is a union of components that are $\mathbf{Q}_n$-a.a.s. all of size smaller than $\bar{C}\log n$. On the other hand by (5.4), $\mathcal{C}_a \subset (\mathcal{C}_a \cap \bar{\mathcal{V}}^u) \cup \bar{X}_{[0,2\log^5 n]}$, and as discussed before this last union cannot be larger than $2\bar{C}\log^7 n$ if $\mathcal{C}_a \cap \bar{\mathcal{V}}^u$ consists only of components of size smaller than $\bar{C}\log n$. Thus

$$\begin{aligned}\mathbf{Q}_n[\mathcal{L}_n] &\le \mathbf{Q}_n\Big[\mathcal{L}_n, \mathcal{C}_a \cap \bar{\mathcal{V}}^u \text{ or } \mathcal{C}_b \cap \bar{\mathcal{V}}^u \text{ is subset of } \textstyle\bigcup_{i\ge 2} \mathcal{C}_i(\bar{\mathcal{V}}^u)\Big] + \mathrm{o}(1) \\ &\le \mathbf{Q}_n\big[\text{at least one of the } \mathcal{C}_i(\bar{\mathcal{V}}^u), i \ge 2, \text{ is larger than } \bar{C}\log n\big] + \mathrm{o}(1) \\ &= \mathrm{o}(1) \quad \text{as } n \to \infty.\end{aligned}$$

This proves (1.4).

To see that the critical parameter $u_\star$ coincides with the critical parameter $u_\star$ of random interlacements on a Poisson(ρ)–Galton–Watson tree conditioned on non-extinction, it suffices to notice that the characterizing equations (5.1) and (2.15) of these two parameters are the same. □

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