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From a kinetic equation to a diffusion under an anomalous scaling

Giada Basile[1]

Università di Roma La Sapienza, Piazzale Aldo Moro 5, 00185 Roma, Italy. E-mail: basile@mat.uniroma1.it

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Abstract. A linear Boltzmann equation is interpreted as the forward equation for the probability density of a Markov process $(K(t), i(t), Y(t))$ on $(\mathbb{T}^2 \times \{1, 2\} \times \mathbb{R}^2)$, where $\mathbb{T}^2$ is the two-dimensional torus. Here $(K(t), i(t))$ is an autonomous reversible jump process, with waiting times between two jumps with finite expectation value but infinite variance. $Y(t)$ is an additive functional of K, defined as $\int_0^t v(K(s))\,\mathrm{d}s$, where $|v| \sim 1$ for small k. We prove that the rescaled process $(N \ln N)^{-1/2} Y(Nt)$ converges in distribution to a two-dimensional Brownian motion. As a consequence, the appropriately rescaled solution of the Boltzmann equation converges to the solution of a diffusion equation.

Résumé. Une équation de Boltzmann linéaire est interprétée comme équation de Fokker–Planck associée à la densité de probabilité d'un processus de Markov $(K(t), i(t), Y(t))$ sur $(\mathbb{T}^2 \times \{1, 2\} \times \mathbb{R}^2)$, où $\mathbb{T}^2$ est le tore bidimensionnel. Le processus Markovien $(K(t), i(t))$ est ici un processus de sauts réversible avec des temps d'attente entre deux sauts à moyenne finie mais variance infinie. $Y(t)$ est une fonctionnelle additive de K, définie par $Y(t) = \int_0^t v(K(s))\,\mathrm{d}s$, où $|v| \sim 1$ pour k petit. Nous prouvons que le processus $(N \ln N)^{-1/2} Y(Nt)$ converge en distribution vers un mouvement brownien bidimensionnel. En conséquence, et moyennant un changement d'échelle approprié, la solution de l'équation de Boltzmann converge vers celle d' une équation de diffusion.

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1. Introduction

One of the most interesting aspects of the problem of energy transport in a solid is an anomalous thermal conduction observed in low dimensional materials (see [8,19] for a general review; see also [16] for experimental data for graphene materials). So far very few results are obtained by a rigorous analysis of microscopic dynamics, and even crucial points, such as the exponent of the divergence of thermal conductivity in dimension one, are still debated.

The theoretical approach proposed by Peierls [26] intended to compute thermal conductivity in analogy with the kinetic theory of gases, conforming to the idea that at low temperatures the lattice vibrations, responsible of energy transport, can be described as a gas of interacting particles (phonons). The time-dependent distribution function of phonons solves a Boltzmann type equation, and an explicit expression for the thermal conductivity is obtained, which is of the form of the kinetic theory $\kappa = \int \mathrm{d}k C_k v_k^2 \tau_k$. Here C_k is the heat capacity of phonons with wave number k, v_k is their velocity and τ_k is the average time between two collisions. A goal of the kinetic approach is the prediction that the mean free path $\lambda_k = v_k \tau_k$ and thus thermal conductivity are infinite in dimension one when the phonon momentum is conserved.

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Over the last years, several papers are devoted to achieve phononic Boltzmann-type equations from microscopic dynamics (see [30] for main ideas and tools). In [2,18,22,27] a kinetic limit is performed for chains of an-harmonic oscillators, and in [21] a linear Boltzmann equation is rigorously derived for the harmonic chain of oscillators with random masses. In [5] the authors consider a system of harmonic oscillators in d dimensions, perturbed by a weak conservative stochastic noise. The following linear Boltzmann-type equation is deduced for the energy density distribution, over the space $\mathbb{R}^d$, of the phonons, characterized by a vector valued wave-number $k \in \mathbb{T}^d$ (d-dimensional torus)

$$\begin{aligned} &\partial_t u_\alpha(t,r,k) + v(k)\cdot\nabla u_\alpha(t,r,k) \\ &\quad = \frac{1}{d-1}\sum_{\beta\neq\alpha}\int_{\mathbb{T}^d} \mathrm{d}k' R\bigl(k,k'\bigr)\bigl[u_\beta\bigl(t,r,k'\bigr) - u_\alpha(t,r,k)\bigr], \end{aligned} \tag{1}$$

$\alpha = 1,\dots,d$, $d \geq 2$. Equation in dimension one is similar, except for the mixing of the components. The kernel R is not negative and symmetric. Despite the exact expressions of R and v (the velocity), the crucial features are that v is finite for small k, i.e. $|v| \to 1$ as $|k| \to 0$, while R behaves like $|k|^2$ for small k, and like $|k'|^2$ for small k'. Naïvely, it means that phonons with small wave numbers travel with finite velocity, but they have low probability to be scattered, thus one expects that the their mean free paths have a macroscopic length (ballistic transport). This is in accordance with rigorous results showing that thermal conductivity is infinite in dimension one and two for a system of harmonic oscillators perturbed by a conservative noise ([4,5]).

A probabilistic interpretation of (1) provides an exact statement of that intuition. The equation describes the evolution of the probability density of a Markov process $(K(t), i(t), Y(t))$ on $(\mathbb{T}^d \times \{1,\dots,d\} \times \mathbb{R}^d)$, where $(K(t), i(t))$ is a reversible jump process and $Y(t)$ is a vector-valued additive functional of K, namely $Y(t) = \int_0^t \mathrm{d}s\, v(K_s)$. K and i can be interpreted, respectively, as the wave number and the "polarization" of a phonon, while $Y(t)$ denotes its position. In order to investigate the property of the process $Y(t)$, one can look at the Markov chain $\{X_i\}$ on $\mathbb{T}^d$ given by the sequence of states visited by $K(t)$, and at the waiting times $\{\tau(X_i)\}$, where $\tau(X_i)$ is the (random) time that the process spends at the ith visited state. The vector-valued function $S_n = \sum_{i=1}^n \tau(X_i)v(X_i)$ gives the value of Y at the time of the nth jump $T_n = \sum_{i=1}^n \tau(X_i)$, then $Y(t)$ is just the piecewise interpolation of S_n at the random times T_n.

The behaviour of the rate R implies that the stationary distribution of the chain is of the form $\pi(\mathrm{d}k) \sim |k|^2\,\mathrm{d}k$ for k small, and since the average of $\tau(k)$ goes like $|k|^{-2}$ for $k \ll 1$, the tail distribution of the random variables $\{\tau(X_i)v(X_i)\}$ behaves like

$$\pi\bigl[\bigl|\tau(X_i)v(X_i)\bigr| > \lambda\bigr] \sim \frac{1}{\lambda^{1+d/2}} \quad \forall d \geq 1. \tag{2}$$

Therefore, in dimension one and two the variables $\tau(X_i)v(X_i)$ have infinite variance with respect to the stationary measure. We remark that the variance has the same expression of the thermal conductivity obtained in [5].

The one dimensional case is discussed in [3], where the authors prove that the rescaled process $N^{-2/3}Y(N\cdot)$ converges in distribution to a symmetric Lévy process, stable with index 3/2. Convergence of finite dimensional marginals has been proven earlier in [15]. Here we consider the other critical case $d = 2$. S_n is now a sum of variables with tail distribution $\sim \frac{1}{\lambda^2}$, which means that if they were independent, they would be in the domain of attraction of a multivariate normal distribution. Looking at the behaviour of the variance

$$\pi\bigl[\bigl(\tau(X_i)v_\alpha(X_i)\bigr)^2 \mathbf{1}_{\{|\tau(X_i)v_\alpha(X_i)| \leq \sqrt{\lambda}\}}\bigr] \sim \ln\lambda, \quad \alpha \in \{1,2\},$$

it turns out that the proper scaling contains an extra factor $(\ln n)^{1/2}$. The rescaled process $(n\ln n)^{-1/2}S_{nt}$ has a central part, given by the sum of truncated variables $\tau(X_i)v_\alpha(X_i)\mathbf{1}_{\{|\tau(X_i)v_\alpha(X_i)| \leq \sqrt{n}\}}$, with finite variance and an extremal part that goes to zero in probability, due to the extra term $(\ln n)^{-1/2}$. This is a standard argument used for sums of i.i.d. random variables with tail distribution (2), introduced for the first time by Kolmogorov and Gnedenko in [14], that we adapt to the case of dependent variables.

Then we are reduced to the problem of convergence of a sum of centered, dependent, bounded random variables to a Wiener process. We propose two different approaches. In Section 5.1, we will use an abstract theorem due to Durrett and Resnick [9], based on the invariance principle for martingale difference arrays with bounded variables

(Freedman, [12] and [13]), together with a random change of time (see, for example, Helland [17] and Billingsley [7]). The underlying central limit theorem for martingale difference arrays can be found in Dvoretzky [10,11] (see also [17,23] and references therein). The alternative proof, in Section 6, is based on the convergence of the moments to the moments of a Brownian motion, under some asymptotic factorization conditions, and it uses combinatorial techniques. In this case we will only show convergence of the finite dimensional marginals. The multidimensional generalization is based a Cramér–Wold argument (see for example [1,7,17,29]).

Convergence of $(n\ln n)^{-1/2}S_{n\cdot}$ to a two-dimensional Wiener process is in the Skorokhod J_1-topology. Moreover, since the random times T_n are sums of positive variables with finite expectation, one can prove, using the arguments in [3], that $(n\ln n)^{-1/2}Y(n\cdot)$ converges to a two dimensional Wiener process in the uniform topology.

Finally we show that the properly rescaled solution of the linear Boltzmann equation in dimension two converges to diffusion. The proof includes a result on the algebraic L^2-convergence rate of the semi-group (Section 4.4). The key point is the derivation of a Nash type inequality which provides an estimate for convergence rates slower than exponential ([6,20,28]). The diffusion coefficient is given by an infrared regularization of the thermal conductivity obtained in [4,5], with a proper renormalization (13).

Convergence of solutions of linear kinetic equations to a diffusion under an anomalous scaling was also proved by Mellet et al. [24], using an analytical approach. We remark that they assume a collision frequency strictly positive, while in our case it is zero in $k=0$.

The case $d\geq 3$ can be easily treated with the same strategy. In particular the rescaled solution of the Boltzmann equation converges to a diffusion equation, with a diffusion coefficient given by the thermal conductivity obtained in [4,5].

2. The model

We consider Eq. (1) in dimension two, namely

$$\begin{aligned}&\partial_t u_\alpha(t,r,k)+v(k)\cdot\nabla u_\alpha(t,r,k)\\&\quad=\sum_{\beta\neq\alpha}\int_{\mathbb{T}^d}\mathrm{d}k'\,R\big(k,k'\big)\big[u_\beta\big(t,r,k'\big)-u_\alpha(t,r,k)\big],\end{aligned}\tag{3}$$

$\forall\alpha=1,2$, $t\geq 0$, $x\in\mathbb{R}^2$, $k\in\mathbb{T}^2$, with a (vector valued) velocity v and a scattering kernel R given by:

$$v_\alpha(k)=\frac{\sin(\pi k_\alpha)\cos(\pi k_\alpha)}{(\sum_{\beta=1}^2\sin^2(\pi k_\beta))^{1/2}},\quad \forall k\in\mathbb{T}^2,\forall\alpha\in\{1,2\},\tag{4}$$

$$R\big(k,k'\big)=16\sum_{\alpha=1}^2\sin^2(\pi k_\alpha)\sin^2\big(\pi k'_\alpha\big),\quad \forall k,k'\in\mathbb{T}^2.\tag{5}$$

We denote with $(K(t),i(t))$ the jump process with values in $\mathbb{T}^2\times\{1,2\}$, defined by the generator

$$\mathcal{L}f(\alpha,k)=\sum_{\beta\neq\alpha}\int_{\mathbb{T}^2}\mathrm{d}k'\,R\big(k,k'\big)\big[f\big(\beta,k'\big)-f(\alpha,k)\big],\tag{6}$$

with $f:\{1,2\}\times\mathbb{T}^2\to\mathbb{R}$ continuous on $\mathbb{T}^2$. The process waits in the state (k,i) an exponential random time τ with parameter $\Phi(k,i)$

$$\Phi(k,i)=\sum_{j=1}^2(1-\delta_{i,j})\int_{\mathbb{T}^d}\mathrm{d}k'\,R\big(k,k'\big)=8\sum_{\alpha=1}^2\sin^2(\pi k_\alpha),\tag{7}$$

then it jumps to another state (j,k') with probability $\nu[i,k;\,j,\mathrm{d}k']=(1-\delta_{i,j})P(k,\mathrm{d}k')$, where

$$P\big(k,\mathrm{d}k'\big):=\Phi(k)^{-1}R\big(k,k'\big)\,\mathrm{d}k'=\frac{2\sum_\alpha\sin^2(\pi k_\alpha)\sin^2(\pi k'_\alpha)}{\sum_\beta\sin^2(\pi k_\beta)}\,\mathrm{d}k'.\tag{8}$$

Observe that the two processes $K(t)$ and $i(t)$ are independent. Disregarding the time, the stochastic sequence $\{X_n\}_{n\geq 0}$ of states visited by $K(t)$ is a Markov chain with value in $\mathbb{T}^2$, with probability kernel $P(k, \mathrm{d}k')$, which is strictly positive. Moreover, there exists a probability measure λ on $\mathbb{T}^2$, strictly positive on open sets, such that for any $k \in \mathbb{T}^2$ it holds $P(k, \cdot) \geq c_0\lambda(\cdot)$ for some $c_0 > 0$. This implies the Doeblin condition for kernel P. In view of [25], Thm. 16.0.2, the discrete time Markov chain $\{X_n\}_{n\geq 0}$ is uniform ergodic. That is there exists a probability π on $\mathbb{T}^2$ such that $P^n(k, \cdot)$ converges to π in total variation uniformly with respect to the initial condition k. Moreover, π is strictly positive on open sets. By direct computation $\pi(\mathrm{d}k) = \frac{1}{8}\Phi(k)\,\mathrm{d}k$.

The process $Y(t)$, with value in $\mathbb{R}^2$, is an additive functional of $K(t)$

$$Y(t) = Y(0) + \int_0^t \mathrm{d}s\, v(K_s)\,\mathrm{d}s. \tag{9}$$

We choose $Y(0) = 0$. In order to investigate its properties, we define two functions of the Markov chain $\{X_n\}_{n\geq 0}$, the clock, T_n, with values in $\mathbb{R}_+$ and the position, S_n, with values in $\mathbb{R}^2$

$$T_n = \sum_{\ell=0}^{n-1} e_\ell \Phi(X_\ell)^{-1}, \qquad S_n = \sum_{\ell=0}^{n-1} e_\ell v(X_\ell)\Phi(X_\ell)^{-1}.$$

Here $\{e_\ell\}_{\ell\geq 0}$ are i.i.d. exponential random variables with parameter 1, and we take $S_0 = 0$. The clock T_n is the time of the n – the jump of the process $K(t)$ and it is a sum of positive random variables with finite expectation with respect to the invariant measure, i.e. $\mathbb{E}_\pi[e_1\Phi(X_1)^{-1}] = 1$. S_n is a two-components vector which gives the value of $Y(t)$ at time T_n, i.e. $S_n = Y(T_n)$. It is a sum of centered random vectors whose components show a tail behavior given in (2). Moreover, the covariance matrix of each of these vectors is diagonal. By denoting with T^{-1} the right-continuous inverse function of T_n, i.e. $T^{-1}(t) := \inf\{n : T_n \geq t\}$, we can represent process $Y(t)$ as follows:

$$Y(t) = S_{\lfloor T^{-1}(t)-1\rfloor} + v(X_{\lfloor T^{-1}(t)-1\rfloor})(t - T_{\lfloor T^{-1}(t)-1\rfloor}),$$

where $\lfloor\cdot\rfloor$ denotes the lower integer part. In particular, $Y(t)$ is the (vector valued) function defined by linear interpolation between its values S_n at the random points T_n.

3. Main results

For every $N \geq 2$, $t \geq 0$, we define the rescaled processes

$$T_N(t) = \frac{1}{N} T_{\lfloor Nt\rfloor}, \qquad T_N^{-1}(t) = \frac{1}{N} T^{-1}(Nt), \tag{10}$$

$$Z_N(t) = \frac{1}{\sqrt{N\ln N}} S_{\lfloor Nt\rfloor} + \big(Nt - \lfloor Nt\rfloor\big)\frac{1}{\sqrt{N\ln N}} v(X_{\lfloor Nt\rfloor -1}). \tag{11}$$

Observe that Z_N is a two-dimensional continuous vector defined by linear interpolation between its values $\frac{1}{\sqrt{N\ln N}} S_n$ at the points n/N.

We assume that the initial distribution μ of the process K_t is not concentrated in $k = 0$, namely $\forall\varepsilon > 0$ exists δ such that

$$\mu\big[|k| < \delta\big] < \varepsilon. \tag{12}$$

This includes all the absolutely continuous measures w.r.t. Lebesgue measure and delta distributions $\delta_{k_0}(\mathrm{d}k)$, with $k_0 \in \mathbb{T}^2/\{0\}$.

Let us denote with

$$\sigma^2 := \lim_{N\to\infty} \frac{1}{\ln N}\mathbb{E}_\pi\left[\left|\frac{e_1 v_1(X_1)}{\Phi(X_1)}\right|^2 \mathbf{1}_{\{|e_1 v_1(X_1)/\Phi(X_1)|\leq\sqrt{N}\}}\right]. \tag{13}$$

We remark that this limit exists and one can prove by direct computation that it is equal to $\frac{1}{64}\frac{1}{2\pi}$. By symmetry, in this definition we can replace $v_1(X_1)$ with $v_2(X_1)$. We use the notation $\bar{W}_\sigma$ for the vector valued process $\bar{W}_\sigma = (W_\sigma^1, W_\sigma^2)$, where W_σ^1 and W_σ^2 are independent Wiener processes with marginal distribution $W_\sigma^\alpha(t) - W_\sigma^\alpha(s) \sim \mathcal{N}(0, \sigma^2(t-s))$ $\forall 0 \le s < t$, $\forall \alpha = 1, 2$.

Theorem 3.1. *Let Z_N be the process defined in (11). Then for any $0 < \mathcal{T} < \infty$, $\{Z_N(t)\}_{0\le t\le \mathcal{T}}$ converges to the two-dimensional Wiener process $\{\bar{W}_\sigma(t)\}_{0\le t\le \mathcal{T}}$. Convergence is in distribution on the space of continuous functions $C([0,\mathcal{T}],\mathbb{R}^2)$ equipped with the uniform topology.*

Then we will prove that $\{T_N^{-1}(t)\}_{t\in[0,\mathcal{T}]}$ converges in distribution to the function t. Combining these two results, we can show that $Z_N \circ T_N^{-1}$ converges in distribution to $\bar{W}_\sigma$. Observing that $Z_N \circ T_N^{-1}$ is the process

$$Y_N(t) = \frac{1}{(N\ln N)^{1/2}} \int_0^{Nt} \mathrm{d}s\, v(K_s),$$

this implies our main theorem.

Theorem 3.2. *For any $0 < \mathcal{T} < \infty$, $\{Y_N(t)\}_{0\le \mathcal{T}}$ converges to the two-dimensional Wiener process $\{\bar{W}_\sigma(t)\}_{0\le t\le t\le \mathcal{T}}$. Convergence is in distribution on the space of continuous functions $C([0,\mathcal{T}],\mathbb{R}^2)$ equipped with the uniform topology.*

Finally, we will use the previous result to show that the rescaled solution of the Boltzmann equation converges to a diffusion. We denote with u^N the two dimensional vector-valued measure defined as

$$u^N(t,k,x) := u\big(Nt, k, (N\ln N)^{1/2}x\big), \quad \forall t \ge 0, \forall k \in \mathbb{T}^2, \forall x \in \mathbb{R}^2,$$

where u is solution of (3) in $d = 2$ with initial condition $u(0,k,x) = u_0(k,(N\ln N)^{-1/2}x)$. Given a function $f \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{T}^2)$ – the Schwarz space, for any $a \ge 1$ we define the norm

$$\|f\|_{\mathcal{A}_a} = \left(\int_{\mathbb{R}^2\times\mathbb{T}^2} \mathrm{d}p\,\mathrm{d}k \big|\hat{f}(p,k)\big|^a\right)^{1/a},$$

where $\hat{f}$ is the Fourier transform of f in the first variable. We denote with $\mathcal{A}_a$ the completion of $\mathcal{S}$ in the norm $\|\cdot\|_{\mathcal{A}_a}$. Observe that $\mathcal{A}_2 = L^2(\mathbb{R}^2 \times \mathbb{T}^2)$.

Theorem 3.3. *Assume that $u_0 \in L^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2) \cap \mathcal{A}_a$, with $a > 2$. Then, $\forall t \in (0, \mathcal{T}]$, $u^N(t,\cdot,\cdot)$ converges in $L^2(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2)$ – weak to $\bar{u}(t,\cdot)$, which solves the following diffusion equation*

$$\begin{aligned} &\partial_t \bar{u}(t,r) = \frac{1}{2}\sigma^2 \Delta \bar{u}(t,r), \\ &\bar{u}^\alpha(0,r) = \frac{1}{2}\sum_{\beta=1,2} \int_{\mathbb{T}^2} \mathrm{d}k u_0^\beta(r,k) \qquad \forall \alpha \in 1,2, \forall r \in \mathbb{R}^2. \end{aligned} \tag{14}$$

4. Sketch of the proof

We present an outline of the proof of the main theorems. Details are postponed in Section 5.

4.1. *Theorem 3.1*

Define the two-dimensional random vector

$$\psi_n := \Phi(X_n)^{-1} v(X_n), \quad n \in \mathbb{N}_0. \tag{15}$$

We will denote with ψ_n^α, $\alpha = 1, 2$, the α-component of ψ_n.

We decompose Z_N, defined in (11) in two parts, i.e. $Z_N = Z_N^> + Z_N^<$, where $\forall t \geq 0$, $\forall \alpha = 1, 2$

$$\begin{aligned} Z_N^{\alpha>}(t) &= (N \ln N)^{-1/2} \sum_{n=0}^{\lfloor Nt \rfloor - 1} e_n \psi_n^\alpha \mathbf{1}_{\{e_n|\psi_n^\alpha| > \sqrt{N}\}} \\ &\quad + (N \ln N)^{-1/2} e_{\lfloor Nt \rfloor} \psi_{\lfloor Nt \rfloor}^\alpha \mathbf{1}_{\{e_{\lfloor Nt \rfloor}|\psi_{\lfloor Nt \rfloor}^\alpha| > \sqrt{N}\}} \big(Nt - \lfloor Nt \rfloor\big), \\ Z_N^{\alpha<}(t) &= (N \ln N)^{-1/2} \sum_{n=0}^{\lfloor Nt \rfloor - 1} e_n \psi_n^\alpha \mathbf{1}_{\{e_n|\psi_n^\alpha| \leq \sqrt{N}\}} \\ &\quad + (N \ln N)^{-1/2} e_{\lfloor Nt \rfloor} \psi_{\lfloor Nt \rfloor}^\alpha \mathbf{1}_{\{e_{\lfloor Nt \rfloor}|\psi_{\lfloor Nt \rfloor}^\alpha| \leq \sqrt{N}\}} \big(Nt - \lfloor Nt \rfloor\big). \end{aligned}$$

At first we will show that $Z_N^> \xrightarrow{P} 0$ when $N \to \infty$. It is enough to show that for every unitary vector $\lambda := (\lambda_1, \lambda_2)$

$$\lambda_1 Z_N^{1>} + \lambda_2 Z_N^{2>} \xrightarrow{P} 0, \quad N \to \infty.$$

This is stated in the next lemma.

Lemma 4.1. *For every* $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}\Big[\sup_{t\in[0,\mathcal{T}]} \big|\lambda_1 Z_N^{1>}(t) + \lambda_2 Z_N^{2>}(t)\big| > \delta\Big] = 0, \tag{16}$$

$\forall \lambda \in \mathbb{R}^2$ *such that* $|\lambda| = 1$.

Proof. For every $\lambda \in \mathbb{R}^2$ with $|\lambda| = 1$

$$\begin{aligned} &\mathbb{P}\Big[\sup_{t\in[0,\mathcal{T}]} \big|\lambda_1 Z_N^{1>}(t) + \lambda_2 Z_N^{2>}(t)\big| > \delta\Big] \\ &\quad \leq \mathbb{P}\Big[\sup_{t\in[0,\mathcal{T}]} \big\{\big|Z_N^{1>}(t)\big| + \big|Z_N^{2>}(t)\big|\big\} > \delta\Big] \\ &\quad \leq \sum_{\alpha=1,2} \mathbb{P}\bigg[\sup_{t\in[0,\mathcal{T}]} \big|Z_N^{\alpha>}(t)\big| > \frac{\delta}{2}\bigg]. \end{aligned}$$

For every $t \in [0, \mathcal{T}]$, $\forall \alpha = 1, 2$

$$\big|Z_N^{\alpha>}(t)\big| \leq \frac{1}{\sqrt{N \ln N}} \sum_{n=0}^{\lfloor N\mathcal{T} \rfloor - 1} e_n \big|\psi_n^\alpha\big| \mathbf{1}_{\{e_n|\psi_n^\alpha| > \sqrt{N}\}}.$$

Then, by Chebyshev's inequality

$$\begin{aligned} &\mathbb{P}\bigg[\sup_{t\in[0,\mathcal{T}]} \big|Z_N^{\alpha>}(t)\big| > \frac{\delta}{2}\bigg] \\ &\quad \leq \frac{2}{\delta} \frac{1}{\sqrt{N \ln N}} \sum_{n=0}^{\lfloor N\mathcal{T} \rfloor - 1} \mathbb{E}\big[e_n \big|\psi_n^\alpha\big| \mathbf{1}_{\{e_n|\psi_n^\alpha| > \sqrt{N}\}}\big] \\ &\quad \leq \frac{2}{\delta} \frac{1}{\sqrt{\ln N}} C_0 \mathcal{T}, \end{aligned}$$

where in the last inequality we used the fact that $\forall n \geq 0$, $\forall \alpha = 1, 2$

$$\mathbb{E}\big[e_n \big|\psi_n^\alpha\big| \mathbf{1}_{\{e_n|\psi_n^\alpha| > \sqrt{N}\}}\big] \leq C_0 \frac{1}{\sqrt{N}},$$

as one can easily compute, using the upper bound for P^m (29) and the fact that $|k|^2|\psi^\alpha(k)|$ is finite for every $k \in \mathbb{T}^2$, $\forall \alpha = 1, 2$. $\square$

Let us consider $Z_N^<$. As first step, we will prove that for every unitary vector $\lambda \in \mathbb{R}^2$, $\langle Z_N^<, \lambda\rangle := \lambda_1 Z_N^{1<} + \lambda_2 Z_N^{2<} \Rightarrow W_\sigma$, where W_σ is a one dimensional Wiener process such that $W_\sigma(t) - W_\sigma(s) \sim \mathcal{N}(0, \sigma^2(t-s))$. This is stated in the following proposition, the proof is postponed to the next section.

Proposition 4.2. *Fix $\mathcal{T} > 0$. Then as $N \to \infty$, for every $\lambda \in \mathbb{R}^2$, with $|\lambda| = 1$, $\langle Z_N^<, \lambda\rangle$ converges weakly to the one dimensional Wiener process W_σ. Convergence is in distribution on the space of continuous functions on $[0, \mathcal{T}]$ equipped with the uniform topology.*

Now we have to show that $Z_N^<$ converges to $\bar{W}_\sigma$. We follow the approach of [29] (see the proof of Lemma 4). The tightness of the sequence $\{Z_N^<\}_{N\geq 1}$ follows from the tightness of the sequence $\{\langle Z_N^<, \lambda\rangle\}_{N\geq 1}$, for every unitary vector λ. Thus we only have to prove the convergence of the finite dimensional distribution. In particular, we have to show the following:

(i) $Z_N^<(t) - Z_N^<(s) \Rightarrow \bar{W}_\sigma(t) - \bar{W}_\sigma(s)$, $\forall 0 \leq s \leq t \leq \mathcal{T}$;
(ii) $Z_N^<(s)$ and $(Z_N^<(t) - Z_N^<(s))$ are independent, as $N \to \infty$, $\forall 0 \leq s \leq t \leq \mathcal{T}$.

In order to verify the first condition, we observe that the convergence of the process $\langle Z_N^<(\cdot), \lambda\rangle$ to $W_\sigma(\cdot)$ implies that $(\langle Z_N^<(s), \lambda\rangle, \langle Z_N^<(t), \lambda\rangle) \Rightarrow (W_\sigma(s), W_\sigma(t))$, for every $s, t \geq 0$. But $(W_\sigma(s), W_\sigma(t))$ has the same law of $(\langle \bar{W}_\sigma(s), \lambda\rangle, \langle \bar{W}_\sigma(t), \lambda\rangle)$, then

$$\big\langle Z_N^<(t), \lambda\big\rangle - \big\langle Z_N^<(s), \lambda\big\rangle \Rightarrow \big\langle \bar{W}_\sigma(t), \lambda\big\rangle - \big\langle \bar{W}_\sigma(s), \lambda\big\rangle$$

for all $\forall 0 \leq s \leq t \leq \mathcal{T}$, $\forall \lambda \in \mathbb{R}^2$ with $|\lambda| = 1$, and this implies (i).

In order to verify condition (ii) it is sufficient to prove that $Z_N^<(s)$ and $Z_N^<(t) - Z_N^<(s)$ are asymptotically jointly Gaussian and uncorrelated. This is stated in the next lemma.

Lemma 4.3. *For all $\lambda, \mu \in \mathbb{R}^2$*

$$\big\langle Z_N^<(s), \lambda\big\rangle + \big\langle \big(Z_N^<(t) - Z_N^<(s)\big), \mu\big\rangle \Rightarrow \mathcal{N}\big(0, \sigma^2\big\{|\lambda|^2 s + |\mu|^2(t-s)\big\}\big), \tag{17}$$

$\forall 0 \leq s < t \leq \mathcal{T}$.

We postpone the proof in Section 5.2.

4.2. *Proof of Theorem* 3.2

Converge in probability of T_N^{-1} to the function χ, where $\chi(t) = t$, in a compact $[0, \mathcal{T}]$, is proved as in [3], see Lemma 8.1 and Proposition 8.2. Then

$$\big(Z_N, T_N^{-1}\big) \Rightarrow (\bar{W}_\sigma, \chi)$$

(Theorem 3.9 in Billingsley [7]) and therefore $Z_N \circ T_N^{-1} \Rightarrow \bar{W}_\sigma \circ \chi$ (Billingsley [7], Lemma p. 151).

4.3. *Proof of Theorem* 3.3

Given a vector valued, real function $J \in \mathcal{S}(\mathbb{R}^2; C(\mathbb{T}^2))$, we define the Fourier transform in the first variable

$$\hat{J}(p,k) = \int_{\mathbb{R}^2} \mathrm{d}u \mathrm{e}^{-\mathrm{i}p\cdot u} J(u,k), \quad \forall p \in \mathbb{R}^2, k \in \mathbb{T}^2,$$

and we introduce the norm on $\mathcal{S}(\mathbb{R}^2; C(\mathbb{T}^2))$

$$\|J\|_{\mathcal{B}_2}^2 = \int_{\mathbb{R}^2} \mathrm{d}p \Big(\sup_{k\in\mathbb{T}^2} \big|\hat{J}(p,k)\big|\Big)^2.$$

We use a probabilistic representation of the solution of the rescaled Boltzmann equation, namely

$$\begin{aligned}&\langle J, u^N(t)\rangle \\ &\quad = \sum_{\alpha=1,2} \int_{\mathbb{R}^2\times\mathbb{T}^2} \mathrm{d}p\,\mathrm{d}k\,\hat{J}_\alpha(p,k)^* \mathbb{E}_{(\alpha,k)}\big[\hat{u}_0(p,\alpha_{(Nt)},K_{(Nt)})\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t)}\big],\end{aligned}$$

where $\mathbb{E}_{(\alpha,k)}[\cdot]$ is the expectation starting from the state (α,k), and $\hat{F}(p,\beta,k) := \hat{F}_\beta(p,k)$. The measure $\tilde{\pi}$ on $\{1,2\}\times\mathbb{T}^2$, given by $\tilde{\pi}(\alpha,\mathrm{d}k) = \frac{1}{2}\,\mathrm{d}k$, is invariant for the (reversible) process $\{(\alpha(t),K(t)), t\geq 0\}$ on $(\{1,2\}\times\mathbb{T}^2)$.

Let us choose a sequence of real numbers $\{\theta_N\}_{N\geq 1}$ such that $\theta_N\to\infty$ for $N\uparrow\infty$ and $\frac{\theta_N}{\sqrt{N\ln N}}\to 0$. We show that we can replace $Y_N(t)$ with $Y_N(t-\theta_N t/N)$. Fix $R>0$. Then

$$\begin{aligned}&\bigg|\sum_{\alpha=1,2}\int_{\mathbb{R}^2\times\mathbb{T}^2}\mathrm{d}p\,\mathrm{d}k\,\hat{J}_\alpha(p,k)^* \\ &\qquad \times \mathbb{E}_{(\alpha,k)}\big[\hat{u}_0(p,\alpha_{(Nt)},K_{(Nt)})\big(\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t)} - \mathrm{e}^{-\mathrm{i}p\cdot Y_N(t-(\theta_N/N)t)}\big)\big]\bigg| \\ &\quad \leq \int_{\mathbb{R}^2}\mathrm{d}p \sup_{k\in\mathbb{T}^2}\big|\hat{J}(p,k)\big|\mathbf{1}_{\{|p|\leq R\}} \\ &\qquad \times \int_{\mathbb{T}^2}\mathrm{d}k\big|\mathbb{E}_{(\alpha,k)}\big[\hat{u}_0(p,\alpha_{(Nt)},K_{(Nt)})\big(\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t)} - \mathrm{e}^{-\mathrm{i}p\cdot Y_N(t-(\theta_N/N)t)}\big)\big]\big| \\ &\qquad + 2\int_{\mathbb{R}^2}\mathrm{d}p \sup_{k\in\mathbb{T}^2}\big|\hat{J}(p,k)\big|\mathbf{1}_{\{|p|> R\}}\int_{\mathbb{T}^2}\mathrm{d}k\mathbb{E}_{(\alpha,k)}\big[\big|\hat{u}_0(p,\alpha_{(Nt)},K_{(Nt)})\big|\big].\end{aligned} \tag{18}$$

Since

$$\big|\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t)} - \mathrm{e}^{-\mathrm{i}p\cdot Y_N(t-(\theta_N/N)t)}\big| \leq C_0 \frac{\theta_N}{\sqrt{N\ln N}}|p|\mathcal{T},$$

using Cauchy–Schwarz we have that the r.h.s. of (18) is bounded by

$$C_0 R\frac{\theta_N}{\sqrt{N\ln N}}\mathcal{T}\|J\|_{\mathcal{B}_2}\|u_0\|_{\mathcal{A}_2} + C_1\|J\|_{\mathcal{B}_2}\bigg(\int_{\mathbb{R}^2\times\mathbb{T}^2}\mathrm{d}p\,\mathrm{d}k|\hat{u}_0|^2\mathbf{1}_{\{|p|>R\}}\bigg)^{1/2}.$$

We send $N\to\infty$ and then $R\to\infty$.

Denoting with $\hat{\mathcal{U}}_p(\alpha_t,K_t) = \hat{u}_0(p,\alpha_t,K_t) - \tilde{\pi}[\hat{u}_0](p)$, $\forall p\in\mathbb{R}^2$, $\forall t>0$, we have

$$\begin{aligned}&\mathbb{E}_{(\alpha,k)}\big[\big(\hat{u}_0(p,\alpha_{(Nt)},K_{(Nt)}) - \tilde{\pi}[\hat{u}_0](p)\big)\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t-(\theta_N/N)t)}\big] \\ &\quad = \mathbb{E}_{(\alpha,k)}\big[\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t-(\theta_N/N)t)} S_{\theta_N t}\hat{\mathcal{U}}_p(\alpha_{t-\theta_N t},K_{t-\theta_N t})\big],\end{aligned}$$

where $\{S_t\}_{t\geq 0}$ is the semigroup associated to the generator (6). Thus, using Cauchy–Schwarz,

$$\begin{aligned}
&\bigg|\sum_{\alpha=1,2}\int_{\mathbb{R}^2\times\mathbb{T}^2}\mathrm{d}p\,\mathrm{d}k\,\hat{J}_\alpha(p,k)^* \\
&\quad\times\mathbb{E}_{(\alpha,k)}\big[\big(\hat{u}_0(p,\alpha_{(Nt)},K_{(Nt)})-\tilde{\pi}[\hat{u}_0](p)\big)\mathrm{e}^{-\mathrm{i}p\cdot Y_N(t-(\theta_N/N)t)}\big]\bigg| \\
&\leq 2\|J\|_{\mathcal{A}_2}\bigg(\int_{\mathbb{R}^2}\mathrm{d}p\|S_{\theta_N t}\hat{\mathcal{U}}_p\|^2_{L^2_{\tilde{\pi}}}\bigg)^{1/2}.
\end{aligned}\tag{19}$$

In order to prove that the last expression converges to zero, we use the following lemma on the L^2-convergence.

Lemma 4.4. *For every $f\in L^2_{\tilde{\pi}}$ with $\tilde{\pi}[f]=0$ the following inequality holds*:

$$\|S_t f\|^2_{L^2_{\tilde{\pi}}}\leq C\|f\|^2_{L^q_{\tilde{\pi}}}\frac{1}{t^{1-2/q}},\quad q>2,\tag{20}$$

for every $t\geq 0$.

We postpone the proof in Section 4.4. Then

$$\int_{\mathbb{R}^2}\mathrm{d}p\|S_{\theta_N t}\hat{\mathcal{U}}_p\|^2_{L^2_{\tilde{\pi}}}\leq C\frac{1}{(\theta_N t)^{1-2/q}}\int_{\mathbb{R}^2}\mathrm{d}p\|\hat{\mathcal{U}}_p\|^2_{L^q_{\tilde{\pi}}}$$

and the r.h.s. of (19) is bounded by

$$C_1\|J\|_{\mathcal{A}_2}\|u_0\|_{\mathcal{A}_q}\frac{1}{(\theta_N t)^{(q-2)/(2q)}},\quad q>2,$$

which converges to zero for $N\to\infty$. Finally, we can replace $\mathbb{E}_{(\alpha,k)}[\mathrm{e}^{-\mathrm{i}pY_N(t)}]$ with $\exp\{-\frac{1}{2}|p|^2\sigma^2 t\}$. We have

$$\begin{aligned}
&\bigg|\sum_{\alpha}\int_{\mathbb{R}^2\times\mathbb{T}^2}\mathrm{d}p\,\mathrm{d}k\,\hat{J}_\alpha(p,k)\tilde{\pi}\big[\hat{u}_0(p)\big]\mathbb{E}_{(\alpha,k)}\big[\mathrm{e}^{-\mathrm{i}pY_N(t)}-\mathrm{e}^{-(1/2)|p|^2\sigma^2 t}\big]\bigg| \\
&\leq C_0\|J\|_{\mathcal{B}_2}\bigg(\int_{\mathbb{R}^2}\mathrm{d}p\big|\tilde{\pi}\big[\hat{u}_0(p)\big]\big|^2\mathbf{1}_{\{|p|\geq R\}}\bigg)^{1/2} \\
&\quad+\int_{\mathbb{R}^2}\mathrm{d}p\sup_{k\in\mathbb{T}^2}\big|\hat{J}(p,k)\big|\big|\tilde{\pi}\big[\hat{u}_0(p)\big]\big|\mathbf{1}_{\{|p|\leq R\}} \\
&\quad\times\int_{\mathbb{T}^2}\mathrm{d}k\big|\mathbb{E}_{(\alpha,k)}\big[\mathrm{e}^{-\mathrm{i}pY_N(t)}-\mathrm{e}^{-(1/2)|p|^2\sigma^2 t}\big]\big|,
\end{aligned}\tag{21}$$

for any $R>0$. By Theorem 3.2, the second integral on the r.h.s. converges to zero for $N\to\infty$, $\forall t\in[0,\mathcal{T}]$, then we send $R\to\infty$.

We conclude the proof by observing that, since

$$\big\|S_t u^N(t)\big\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^2)}\leq\|u_0\|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^2)},\quad\forall N\geq 1,\forall t\geq 0,$$

then there exists $\tilde{u}(t)\in L^2(\mathbb{R}^2\times\mathbb{T}^2)$ such that $u^N(t)$ weakly converges to $\tilde{u}(t)$ as $N\to\infty$. Moreover, we have just proved that for every $J\in\mathcal{S}$ $\langle J,u^N(t)\rangle\to\langle J,\bar{u}(t)\rangle$ as $N\to\infty$, for any $t>0$, where $\bar{u}(t)$ is solution of (14). Therefore, using the fact that the Schwarz space $\mathcal{S}$ is dense in L^2, we have $u^N(t)\to\bar{u}(t)$ weakly in $L^2(\mathbb{R}^2\times\mathbb{T}^2)$.

4.4. *Algebraic convergence rate*

Suppose that, for every $f \in L^2_{\tilde{\pi}}$ such that $\tilde{\pi}[f] = 0$, the following weak Poincaré inequality holds:

$$\|f\|^2_{L^2_{\tilde{\pi}}} \le \frac{C_0}{r^{a-1}} \mathcal{E}(f, f) + r\|f\|^2_{L^q_{\tilde{\pi}}}, \quad a > 1, q > 2, \forall r > 0, \tag{22}$$

where $\mathcal{E}(f, f)$ is the Dirichelet form. By optimizing on r, one gets the following Nash type inequality:

$$\|f\|^2_{L^2_{\tilde{\pi}}} \le C\big[\mathcal{E}(f, f)\big]^{1/a}\big(\|f\|^2_{L^q_{\tilde{\pi}}}\big)^{1-1/a}, \quad q > 2, a > 1.$$

The $L^q_{\tilde{\pi}}$ norm is defined in a dense subset of $L^2_{\tilde{\pi}}$. Moreover, the $L^q_{\tilde{\pi}}$ norm is monotone under the semi-group $\{S_t\}_{t\ge 0}$, namely $\|S_t f\|^2_{L^q_{\tilde{\pi}}} \le \|f\|^2_{L^q_{\tilde{\pi}}}$ $\forall t \ge 0$, for every $q \ge 1$ (contractivity property of a Markov semi-group). Therefore, we can apply Theorem 2.2 of [20] (see also [28] and [6]) and we get the following algebraic rate of convergence

$$\|S_t f\|^2_{L^2_{\tilde{\pi}}} \le C\|f\|^2_{L^q_{\tilde{\pi}}} \frac{1}{t^{1/(a-1)}}, \quad q > 2,$$

which holds for every $f \in L^2_{\tilde{\pi}}$. Then, in order to prove Lemma 4.4, it suffices to show that (22) holds.

The Dirichelet form has the following expression:

$$\begin{aligned}\mathcal{E}(f, f) &= \frac{1}{2}\sum_{\alpha=1,2}\sum_{\beta\neq\alpha}\int_{\mathbb{T}^2} \mathrm{d}k f(\alpha, k)\int_{\mathbb{T}^2}\mathrm{d}k' R\big(k, k'\big)\big[f\big(\beta, k'\big) - f(\alpha, k)\big] \\ &= \frac{1}{2}\sum_{\alpha=1,2}\int_{\mathbb{T}^2\times\mathbb{T}^2}\mathrm{d}k \Phi(k) f(\alpha, k)[1 - P]f(\alpha, k),\end{aligned}$$

where P is the operator acting the vector-valued functions $f : \mathbb{T}^2 \to \mathbb{R}^2$

$$Pf(\alpha, k) = \sum_{\beta\neq\alpha}\int_{\mathbb{T}^2} P\big(k, \mathrm{d}k'\big) f\big(\beta, k'\big), \quad \forall\alpha = 1, 2.$$

Here $P(k, \mathrm{d}k')$ is the probability kernel defined in (8). The corresponding invariant measure is $\pi(\alpha, \mathrm{d}k) = \frac{1}{16}\Phi(k)\,\mathrm{d}k$. Since the operator P is compact with a positive kernel $P(k, \mathrm{d}k')$, using the same arguments of [15], Lemma 3.2, one can show that 0 is a simple eigenvalue for $1 - P$, and therefore the following gap estimate is obtained

$$\mathcal{E}(f, f) \ge c\sum_{\alpha=1,2}\int_{\mathbb{T}^2}\mathrm{d}k \Phi(k)\big|f(\alpha, k) - \pi[f]\big|^2, \tag{23}$$

with $c > 0$ and $\pi[f]$ the expectation value with respect to the measure $\pi(\alpha, \mathrm{d}k)$. We define the set $A_\delta = \{k \in \mathbb{T}^2 : |k| > \delta\}$, with $\delta \in (0, 1)$, and we denote by A^c_δ its complement. Then the r.h.s. of (23) is bounded from below by

$$\begin{aligned}&c\sum_{\alpha=1,2}\int_{\mathbb{T}^2}\mathrm{d}k \Phi(k)\mathbf{1}_{\{A_\delta\}}\big|f(\alpha, k) - \pi[f]\big|^2 \\ &\quad\ge c_1 \inf_{\{k\in A_\delta\}}\Phi(k)\sum_{\alpha=1,2}\int_{A_\delta}\mathrm{d}k\big|f(\alpha, k) - \pi[f]\big|^2.\end{aligned}$$

We observe that

$$\begin{aligned}\sum_{\alpha=1,2}\int_{A_\delta}\mathrm{d}k\big|f(\alpha, k) - \pi[f]\big|^2 &\ge \|f\mathbf{1}_{\{A_\delta\}}\|^2_{L^2_{\tilde{\pi}}} - 2\pi[f]\tilde{\pi}[f\mathbf{1}_{\{A_\delta\}}] \\ &= \|f\mathbf{1}_{\{A_\delta\}}\|^2_{L^2_{\tilde{\pi}}} + 2\pi[f]\tilde{\pi}[f\mathbf{1}_{\{A^c_\delta\}}]\end{aligned}$$

where in the last equality we use the fact that $\tilde{\pi}[f]=0$. Since $\inf_{\{k\in A_\delta\}}\Phi(k)=c_1\delta^2$, we obtain

$$\begin{aligned}\|f\mathbf{1}_{\{A_\delta\}}\|^2_{L^2_{\tilde{\pi}}} &\le \frac{C}{\delta^2}\mathcal{E}(f,f)-2\pi[f]\tilde{\pi}[f\mathbf{1}_{\{A^c_\delta\}}]\\ &\le \frac{C}{\delta^2}\mathcal{E}(f,f)+C'\|f\|^2_{L^p_{\tilde{\pi}}}\big(\tilde{\pi}\big[A^c_\delta\big]\big)^{1-1/p},\end{aligned} \tag{24}$$

with $p>1$. Now we observe that

$$\begin{aligned}\|f\|^2_{L^2_{\tilde{\pi}}} &= \|f\mathbf{1}_{\{A_\delta\}}\|^2_{L^2_{\tilde{\pi}}}+\|f\mathbf{1}_{\{A^c_\delta\}}\|^2_{L^2_{\tilde{\pi}}}\\ &\le \|f\mathbf{1}_{\{A_\delta\}}\|^2_{L^2_{\tilde{\pi}}}+\|f\|^2_{L^{2b}_{\tilde{\pi}}}\big(\tilde{\pi}\big[A^c_\delta\big]\big)^{1-1/b},\quad b>1\end{aligned}$$

and since $\tilde{\pi}[A^c_\delta]=\delta^2$, finally we get

$$\|f\|^2_{L^2_{\tilde{\pi}}}\le \frac{C}{\delta^2}\mathcal{E}(f,f)+C'\big(\delta^2\big)^{1-1/b}\|f\|^2_{L^{2b}_{\tilde{\pi}}},\quad b>1.$$

Setting $r=C'\delta^{2(1-1/b)}$ and $q=2b$, we get the weak Poincaré inequality (22) with $a-1=\frac{q}{q-2}$.

Remark. *We can extend this proof to the general case of the process in d-dimensions. We get the following algebraic convergence rate*:

$$\|S_t f\|^2_{L^2_{\tilde{\pi}}}\le C\|f\|^2_{L^p_{\tilde{\pi}}}\frac{1}{t^{(d/2)(1-2/q)}},\quad q>2,\forall d\ge 1. \tag{25}$$

5. Details

We start with some preliminary results on P^m, the mth convolution integral of P, the probability kernel defined in (8). By direct computation

$$P^m\big(k,\mathrm{d}k'\big)=\frac{2}{\sum_{\gamma=1}^2\sin^2(\pi k_\gamma)}\sum_{\alpha=1}^2\sum_{\beta=1}^2\sin^2(\pi k_\alpha)A^{(m)}_{\alpha,\beta}\sin^2\big(\pi k'_\beta\big)\,\mathrm{d}k', \tag{26}$$

where, $\forall\alpha,\beta\in\{1,2\}$,

$$A^{(1)}_{\alpha,\beta}=\delta_{\alpha,\beta},\qquad A^{(m+1)}_{\alpha,\beta}=\big[a^m\big]_{\alpha,\beta}\qquad\forall m\ge 1. \tag{27}$$

Here a is a 2×2 real matrix with elements

$$\begin{aligned}a_{11}=a_{22}&=2\int_{\mathbb{T}^2}\mathrm{d}k\frac{\sin^4(\pi k_1)}{\sum_\alpha\sin^2(\pi k_\alpha)},\\ a_{12}=a_{21}&=2\int_{\mathbb{T}^2}\mathrm{d}k\frac{\sin^2(\pi k_1)\sin^2(\pi k_2)}{\sum_\alpha\sin^2(\pi k_\alpha)}.\end{aligned}$$

Observe that the condition

$$\int_{\mathbb{T}^2}P^m\big(k,\mathrm{d}k'\big)=1\quad\forall m\ge 1,$$

implies

$$\sum_{\beta=1}^2 A^{(m)}_{\alpha,\beta}=1,\quad\forall\alpha=1,2,\forall m\ge 1, \tag{28}$$

and thus

$$P^m\left(k, \mathrm{d}k'\right) \le 2 \sum_{\beta=1,2} \sin^2\left(\pi k'_\beta\right) \mathrm{d}k', \quad \forall k \in \mathbb{T}^2, \forall m \ge 1. \tag{29}$$

5.1. *Proof of Proposition* 4.2

Fix $\lambda := (\lambda_1, \lambda_2)$ with $\lambda_1^2 + \lambda_2^2 = 1$. We will follow the strategy of Durrett and Resnick [9] to prove that $\langle Z_N^<, \lambda\rangle := \lambda_1 Z_N^{1<} + \lambda_2 Z_N^{2<}$ converges weakly to a Wiener process W_c. They use a result of Freedman [12], pp. 89–93, on martingale difference arrays with uniformly bounded variables. We start with the following

Definition 5.1. *A collection of random variables $\{\xi_{N,i}\}$, $N \ge 1$, $i \ge 1$ and σ-fields $\mathcal{F}_{N,i}$, $i \ge 0$, $N \ge 1$ is a martingale difference array if*

(i) *for all $N \ge 1$, $\mathcal{F}_{N,i}$, $i \ge 0$ is a nondecreasing sequence of σ-fields*;
(ii) *for all $N \ge 1$, $i \ge 1$, $\xi_{N,i}$ is $\mathcal{F}_{N,i}$ measurable*;
(iii) *for all $N \ge 1$, $E[\xi_{N,i}|\mathcal{F}_{N,i-1}] = 0$ a.s.*

We introduce the following notations:

$$\begin{aligned}\langle\lambda, \bar{\Psi}_{N,m}\rangle := \lambda_1 \frac{e_m \psi_m^1}{\sqrt{N \ln N}} \mathbf{1}_{\{e_m|\psi_m^1| \le \sqrt{N}\}} \\ + \lambda_2 \frac{e_m \psi_m^2}{\sqrt{N \ln N}} \mathbf{1}_{\{e_m|\psi_m^2| \le \sqrt{N}\}},\end{aligned} \tag{30}$$

$\forall N \ge 2, m \ge 0$, and, for $N = 1$, $m \ge 0$

$$\langle\lambda, \bar{\Psi}_{1,m}\rangle = \lambda_1 e_m \psi_m^1 \mathbf{1}_{\{e_m|\psi_m^1| \le 1\}} + \lambda_2 e_m \psi_m^2 \mathbf{1}_{\{e_m|\psi_m^2| \le 1\}}.$$

For all $N \ge 1, m \ge 0$, we denote with $\mathcal{F}_{N,m}$ the σ-field generated by $\{X_0, \dots, X_m\} \times \{e_0, \dots, e_m\}$, where $\{X_m\}_{m \ge 0}$ is the Markov chain with value in $\mathbb{T}^2$. Then we observe that $\{\langle\lambda, \bar{\Psi}_{N,m}\rangle, \mathcal{F}_{N,m}\}_{N \ge 1, m \ge 1}$ is a martingale difference array. In particular, condition (iii) of Definition 5.1 can be easily checked using the explicit form of probability kernel $P[k, \mathrm{d}k']$.

By definition, the variables $\langle\lambda, \bar{\Psi}_{N,m}\rangle$ are uniformly bounded in m, i.e. for all $N \ge 1$ $|\langle\lambda, \bar{\Psi}_{N,m}\rangle| \le \varepsilon_N$, $\forall m \ge 0$, where $\varepsilon_N = \frac{2}{\sqrt{\ln N}}$ if $N \ge 2$, and $\varepsilon_1 = 2$. In particular $\varepsilon_N \downarrow 0$ when $N \to \infty$.

For every $N \ge 1$, $j \ge 1$, let us define

$$\langle\lambda, S_{N,j}\rangle = \sum_{m=1}^{j} \langle\lambda, \bar{\Psi}_{N,m}\rangle, \tag{31}$$

$$\langle\lambda, V_{N,j}\rangle = \sum_{m=1}^{j} E\left[\langle\lambda, \bar{\Psi}_{N,m}\rangle^2 | \mathcal{F}_{N,m-1}\right]. \tag{32}$$

We will prove in Lemma 5.2 that $\mathbb{P}[\lim_{j\to\infty}\langle\lambda, V_{N,j}\rangle = \infty] = 1$, for all $N \ge 1$, i.e. the martingale difference array $\{\langle\lambda, \bar{\Psi}_{N,m}\rangle, \mathcal{F}_{N,m}\}_{N \ge 1, m \ge 0}$ satisfies the hypotheses of Theorem 2.1 in [9]. Thus, setting

$$j_{N,\lambda}(t) = \sup\left\{j | \langle\lambda, V_{N,j}\rangle \le t\right\},$$

we get that $\langle\lambda, S_{N, j_{N,\lambda}(\cdot)}\rangle$ converges weakly as a sequence of random elements of $D[0, \mathcal{T}]$ to a standard Wiener process W.

Now let $\phi_{N,\lambda}(t) = \langle\lambda, V_{N,\lfloor N\theta\rfloor}\rangle$, $\forall t \in [0, \mathcal{T}]$. By definition

$$j_{N,\lambda} \circ \phi_{N,\lambda}(t) = \lfloor Nt\rfloor.$$

In order to prove that $\phi_{N,\lambda}$ converges in probability to the function $\phi:\phi(t)=\sigma^2 t$, it suffices to show that $\phi_{N,\lambda}(t)\xrightarrow{P}\sigma^2 t$, $\forall t\in[0,\mathcal{T}]$, since ϕ is continuous and $\phi_{N,\lambda}$ is monotone. That will be proved in Lemma 5.2. Then

$$\big(\langle\lambda, S_{N,j_{N,\lambda}}\rangle,\phi_{N,\lambda}\big)\Rightarrow(W,\phi)$$

(Billingsley [7], Theorem 3.9), and therefore

$$\langle\lambda, S_{N,j_{N,\lambda}}\rangle\circ\phi_{N,\lambda}\Rightarrow W\circ\phi$$

(Billingsley [7], Lemma p. 151).

Finally,

$$\langle\lambda, S_{N,\lfloor N\cdot\rfloor}\rangle=\langle\lambda, S_{N,j_N(\phi_N(\cdot))}\rangle\Rightarrow W^2_\sigma,$$

where convergence is in distribution on the space $D[0,\mathcal{T}]$ equipped with the Skorokhod J_1-topology.

The process $\langle\lambda,\tilde S_N(t)\rangle:=\sum_{m=0}^{\lfloor Nt\rfloor-1}\langle\lambda,\bar\Psi_{N,m}\rangle$ converges also to W_σ. For every $N\ge 2$, $\langle Z_N^<,\lambda\rangle=\lambda_1 Z_N^{1<}+\lambda_2 Z_N^{2<}$ is the continuous function defined by linear interpolation between its values $\langle\lambda,\tilde S_N(m/N)\rangle$ at points m/N. The two sequences $\{\langle\lambda,\tilde S_N(t)\rangle, 0\le t\le\mathcal{T}\}$ and $\{\langle Z_N(\theta),\lambda\rangle\ 0\le t\le\mathcal{T}\}$ are asymptotically equivalent, i.e. if either converges in distribution as $N\to\infty$, then so does the other. Convergence of $\langle Z_N^<,\lambda\rangle$ to W_σ is in distribution on the space of continuous functions equipped with the uniform topology.

We conclude this subsection with the main lemma.

Lemma 5.2. *For every $N\ge 1$, for every unitary vector $\lambda\in\mathbb{R}^2$,*

$$\mathbb{P}\Big[\lim_{j\to\infty}\langle\lambda, V_{N,j}\rangle=\infty\Big]=1. \tag{33}$$

Moreover, for every $\delta>0$, for every unitary vector $\lambda\in\mathbb{R}^2$,

$$\lim_{N\to\infty}\mathbb{P}\big[\big|\langle\lambda, V_{N,\lfloor N\theta\rfloor}\rangle-\sigma^2\theta\big|>\delta\big]=0, \tag{34}$$

$\forall\theta\in[0,\mathcal{T}]$.

Proof. Fix $\lambda\in\mathbb{R}^2$, with $|\lambda|^2=1$. $\forall N\ge 2$, we define $f_N:\mathbb{T}^2\to\mathbb{R}^2$

$$f_N(k)=\int_0^\infty \mathrm{d}z\mathrm{e}^{-z}\times\int_{\mathbb{T}^2}P\big(k,\mathrm{d}k'\big)\bigg(\sum_{\alpha=1,2}\lambda_\alpha\frac{z\psi^\alpha(k')}{\sqrt{N\ln N}}\mathbf{1}_{\{z|\psi^\alpha(k')|\le\sqrt{N}\}}\bigg)^2. \tag{35}$$

Using (26), we get $f_N(k)\ge C_0/N$, with $0\le C_0<\infty$. Since

$$f_N(X_m)=\mathbb{E}\big[\langle\bar\Psi_{N,m+1},\lambda\rangle^2|\mathcal{F}_m\big],\qquad\forall m\ge 0$$

then, for all $N\ge 1$, $\langle\lambda, V_{N,j}\rangle\ge jC_0N^{-1}$ which goes to infinity for $j\to\infty$, a.s.

Now we focus on (34). By Chebychev inequality, for every $N\ge 1$

$$\begin{aligned}&\mathbb{P}\big[\big|\langle\lambda, V_{N,\lfloor Nt\rfloor}\rangle-\sigma^2 t\big|>\delta\big]\\&\quad\le\mathbb{P}\Bigg[\Bigg|\sum_{n=1}^{\lfloor Nt\rfloor}\bigg(\mathbb{E}\big[\langle\bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]-\frac{\sigma^2}{N}\bigg)\Bigg|>\delta-\frac{1}{N}\Bigg]\\&\quad\le\frac{1}{\tilde\delta_N^2}\sum_{n=1}^{\lfloor Nt\rfloor}\mathbb{E}\bigg[\bigg(\mathbb{E}\big[\langle\bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]-\frac{\sigma^2}{N}\bigg)^2\bigg]\end{aligned}$$

$$+\frac{1}{\tilde{\delta}_N^2}\sum_{n=1}^{\lfloor Nt\rfloor}\sum_{m\neq n}\mathbb{E}\bigg[\bigg(\mathbb{E}\big[\langle\bar{\Psi}_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]-\frac{\sigma^2}{N}\bigg)$$

$$\times\bigg(\mathbb{E}\big[\langle\bar{\Psi}_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\big]-\frac{\sigma^2}{N}\bigg)\bigg], \tag{36}$$

where $\tilde{\delta}_N = \delta - N^{-1}$. By (29), we get

$$\mathbb{E}\big[\langle\bar{\Psi}_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\big] = f_N(X_{m-1}) \le \frac{C_0}{N}, \tag{37}$$

thus the first sum on the r.h.s. of (36) is bounded by $\tilde{\delta}_N^{-2}C_1T/N$, with C_1 finite. Let us consider the second sum on the r.h.s. of (36). For $n > m$

$$\mathbb{E}\big[\mathbb{E}\big[\langle\bar{\Psi}_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]\mathbb{E}\big[\langle\bar{\Psi}_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\big]\big]$$
$$= \mathbb{E}\big[\mathbb{E}\big[\langle\bar{\Psi}_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\big]\mathbb{E}\big[\mathbb{E}\big[\langle\bar{\Psi}_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]|\mathcal{F}_{m-1}\big]\big].$$

We set

$$g_N^{n-m}(X_{m-1}) := \mathbb{E}\big[\mathbb{E}\big[\langle\bar{\Psi}_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]|\mathcal{F}_{m-1}\big],$$

where, for every $l \ge 1$, $N \ge 1$, the function $g : \mathbb{T}^2 \to \mathbb{R}^2$ is given by

$$g_N^l(k) = \int_{\mathbb{T}^2} \mathrm{d}k'\, P^l\big(k, \mathrm{d}k'\big) f_N\big(k'\big),$$

with f_N defined in (35). By (29) and (37) we get

$$g_N^l(k) \le \frac{C_0}{N}, \quad \forall k \in \mathbb{T}^2, \forall l \ge 1. \tag{38}$$

We fix M, $1 \le M < N$ and we get

$$\sum_{n=1}^{\lfloor Nt\rfloor}\sum_{m\neq n}\mathbb{E}\big[\mathbb{E}\big[\langle\bar{\Psi}_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]\mathbb{E}\big[\langle\bar{\Psi}_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\big]\big]$$
$$= 2\sum_{m=1}^{M}\sum_{n=m+1}^{\lfloor Nt\rfloor}\mathbb{E}\big[f_N(X_{m-1})g_N^{n-m}(X_{m-1})\big]$$
$$+ 2\sum_{m=M+1}^{\lfloor Nt\rfloor}\sum_{n=m+1}^{m+M}\mathbb{E}\big[f_N(X_{m-1})g_N^{n-m}(X_{m-1})\big]$$
$$+ 2\sum_{m=M+1}^{\lfloor Nt\rfloor}\sum_{n=m+M+1}^{\lfloor Nt\rfloor}\mathbb{E}\big[f_N(X_{m-1})g_N^{n-m}(X_{m-1})\big].$$

By (38), the first and the second sum on the r.h.s. are bounded form above by $C\mathcal{T}M/N$, with C finite. We denote by μP^{m-1} the convolution integral of the initial measure μ and the probability P^{m-1}. For every $l \ge 1$,

$$\mathbb{E}\big[f_N(X_{m-1})g_N^l(X_{m-1})\big] = \mathbb{E}_\pi\big[f_N(X_{m-1})g_N^l(X_{m-1})\big]$$
$$+ \int_{\mathbb{T}^2}\big[\mu P^{m-1}(\mathrm{d}k) - \pi(\mathrm{d}k)\big] f_N(k) g_N^l(k)$$

where the last term is bounded by $C'N^{-2}\int_{\mathbb{T}^2}|\mu P^{m-1}(\mathrm{d}k)-\pi(\mathrm{d}k)|$. Moreover, for every $l\ge 1$

$$\begin{aligned}
&\mathbb{E}_\pi\big[f_N(X_{m-1})g_N^l(X_{m-1})\big]\\
&\quad=\int_{\mathbb{T}^2}\pi(\mathrm{d}k)f_N(k)\int_{\mathbb{T}^2}\mathrm{d}k'P^l\big(k,\mathrm{d}k'\big)f_N\big(k'\big)\\
&\quad\le\left(\int_{\mathbb{T}^2}\pi(\mathrm{d}k)f_N(k)\right)^2+\frac{C'}{N^2}\int_{\mathbb{T}^2}\big|\mu P^{m-1}(\mathrm{d}k)-\pi(\mathrm{d}k)\big|.
\end{aligned}$$

We get

$$\begin{aligned}
&\sum_{n=1}^{\lfloor Nt\rfloor}\sum_{m\ne n}\mathbb{E}\big[\mathbb{E}\big[\langle\bar\Psi_{N,n},\lambda\rangle^2|\mathcal{F}_{n-1}\big]\mathbb{E}\big[\langle\bar\Psi_{N,m},\lambda\rangle^2|\mathcal{F}_{m-1}\big]\big]\\
&\quad\le\lfloor Nt\rfloor\big(\lfloor Nt\rfloor-1\big)\big(\mathbb{E}_\pi\big[\langle\bar\Psi_{N,1},\lambda\rangle^2\big]\big)^2\\
&\qquad+C\mathcal{T}\frac{M}{N}+C'\mathcal{T}\int_{\mathbb{T}^2}\big|\mu P^M(\mathrm{d}k)-\pi(\mathrm{d}k)\big|,
\end{aligned}$$

with C and C' finite. In the same way one can prove that

$$\begin{aligned}
\sum_{n=1}^{\lfloor Nt\rfloor}\mathbb{E}\big[\langle\bar\Psi_{N,n},\lambda\rangle^2\big]&\le\lfloor Nt\rfloor\mathbb{E}_\pi\big[\langle\bar\Psi_{N,n},\lambda\rangle^2\big]\\
&\quad+C\mathcal{T}\frac{M}{N}+C'\mathcal{T}\int_{\mathbb{T}^2}\big|\mu P^M(\mathrm{d}k)-\pi(\mathrm{d}k)\big|,
\end{aligned}$$

with some C, C' finite, and finally we get

$$\begin{aligned}
\mathbb{P}\big[\big|\langle\lambda,V_{N,\lfloor Nt\rfloor}\rangle-\sigma^2t\big|>\delta\big]&\le\frac{1}{\tilde\delta_N^2}C\mathcal{T}\frac{M}{N}\\
&\quad+\frac{1}{\tilde\delta_N^2}C'\mathcal{T}\int_{\mathbb{T}^2}\big|\mu P^M(\mathrm{d}k)-\pi(\mathrm{d}k)\big|,
\end{aligned}$$

where C, C' are finite. (34) is proved by sending $M,N\to\infty$ in such a way that $M/N\to 0$. □

5.2. *Proof of Lemma* 4.3

We use the central limit theorem for martingale difference array ([10], Theorem 1; see also [11,17]) which states the follows: fix $t>0$, and let $\{\xi_{N,i},\mathcal{F}_{N,i}\}_{N\ge1,i\ge0}$ be a martingale difference array such that

(i) $$\sum_{i=1}^{\lfloor Nt\rfloor}\mathbb{E}\big[\xi_{N,i}^2|\mathcal{F}_{N,i-1}\big]\xrightarrow{P}ct,\quad N\uparrow\infty;$$

(ii) $$\sum_{i=1}^{\lfloor Nt\rfloor}\mathbb{E}\big[\xi_{N,i}^2\mathbf{1}_{\{|\xi_{N,i}|>\varepsilon\}}|\mathcal{F}_{N,i-1}\big]\xrightarrow{P}0,\quad N\uparrow\infty,\forall\varepsilon>0.$$

Then

$$\sum_{i=1}^{\lfloor Nt\rfloor}\xi_{N,i}\Rightarrow\mathcal{N}(0,ct).$$

By definition of $Z_N^<$, $\forall\lambda \in \mathbb{R}^2$

$$\big\langle \lambda, Z_N^<(t)\big\rangle = \langle \lambda, S_{N,\lfloor Nt\rfloor}\rangle + \big(Nt - \lfloor Nt\rfloor\big)\langle \lambda, \bar{\Psi}_{\lfloor Nt\rfloor}\rangle, \tag{39}$$

$\forall t \in [0, \mathcal{T}]$, where $\langle \lambda, S_{N,\cdot}\rangle$ is defined in (31). The rightmost term in (39) goes to zero in probability by Chebyshev's inequality. We fix $\lambda, \mu \in \mathbb{R}^2$ and $0 \le s < t \le \mathcal{T}$, and we define the following array of variables:

$$\tilde{\xi}_{N,i} = \begin{cases} \langle \lambda, \bar{\Psi}_{N,i}\rangle & \text{if } 0 \le i \le \lfloor Ns\rfloor - 1, \\ \langle \mu, \bar{\Psi}_{N,i}\rangle & \text{if } \lfloor Ns\rfloor \le i, \forall N \ge 1. \end{cases}$$

We denote with $\mathcal{F}_{N,i}$ the σ-algebra generated by $(X_0, \dots, X_i) \times (e_0, \dots, e_i)$, $\forall N \ge 1$, $i \ge 0$. Then $\{\tilde{\xi}_{N,i}, \mathcal{F}_{N,i}\}_{N\ge 1, i\ge 0}$ is a martingale difference array. In particular, since $|\langle \nu, \bar{\Psi}_{N,i}\rangle| \le 2(\ln N)^{-1/2}$ for every $i \ge 1$, for every unitary vector $\nu \in \mathbb{R}^2$, it follows that $\forall \varepsilon > 0$, there exists $\bar{N}$ such that $|\tilde{\xi}_{N,i}| < \varepsilon$, $\forall N \ge \bar{N}$, $\forall i \ge 1$. Therefore condition (ii) is satisfied.

Moreover, with similar arguments of the proof of (34), one can prove that

$$\sum_{i=1}^{\lfloor Nt\rfloor} \mathbb{E}\big[\tilde{\xi}_{N,i}^2 | \mathcal{F}_{N,i-1}\big] \xrightarrow{P} \sigma^2|\lambda|^2 s + \sigma^2|\mu|^2(t-s),$$

with σ^2 defined in (13). Thus

$$\begin{aligned} \sum_{i=1}^{\lfloor Ns\rfloor - 1} \langle \lambda, \bar{\Psi}_{N,i}\rangle + \sum_{i=\lfloor Ns\rfloor}^{\lfloor Nt\rfloor - 1} \langle \mu, \bar{\Psi}_{N,i}\rangle &= \sum_{i=1}^{\lfloor Nt\rfloor} \tilde{\xi}_{N,i} \\ &\Rightarrow \mathcal{N}\big(0, \sigma^2\big\{|\lambda|^2 s + |\mu|^2(t-s)\big\}\big). \end{aligned}$$

6. An invariance principle for centered, bounded random variables

In this section we present an alternative proof of Proposition 4.2. We start with a CLT for arrays of centered, uniformly bounded random variables, based on the convergence of the moments to the moments of a normal distribution. Some asymptotic factorization conditions, holding on average, are required. Then we will use it to show that for every unitary vector $\lambda \in \mathbb{R}^2$, $\langle \lambda, Z_N^<(t)\rangle = \lambda_1 Z_N^{1<}(t) + \lambda_2 Z_N^{2<}(t) \Rightarrow W_\sigma(t)$, $\forall t \in [0\mathcal{T}]$.

Proposition 6.1 (CLT). *Let $\{\bar{X}_{n,i} i = 1, \dots, n, n \ge 1\}$ be an array of centered random variables and suppose that exists $\varepsilon_n \downarrow 0$ such that $|\bar{X}_{n,i}| \le \varepsilon_n$, for all n and i. Let $\bar{S}_n = \sum_{i=1}^n \bar{X}_{n,i}$. Then $\bar{S}_n \Rightarrow \mathcal{N}(0, c)$, if the following conditions hold*:

(i) $\forall \ell \ge 1$, *for every sequence of positive integers $\{p_1, \dots, p_\ell\}$ such that $\exists p_j = 1$, $j \in \{1, \dots, \ell\}$*

$$\sum_{i_1 \ne i_2 \ne \cdots \ne i_\ell}^{n} \mathbb{E}\big[(\bar{X}_{n,i_1})^{p_1} \cdots (\bar{X}_{n,i_\ell})^{p_\ell}\big] \xrightarrow{n\uparrow\infty} 0$$

(ii) $\forall \ell \ge 1$

$$\sum_{i_1 \ne i_2 \ne \cdots \ne i_\ell}^{n} \mathbb{E}\big[(\bar{X}_{n,i_1})^{2} \cdots (\bar{X}_{n,i_\ell})^{2}\big] \xrightarrow{n\uparrow\infty} c^\ell .$$

Proof. The proof is based on the convergence of the moments of $\bar{S}_n$. Of course $\mathbb{E}[\bar{S}_n] = 0$, while for the second moment we have

$$\mathbb{E}\big[(\bar{S}_n)^2\big] = \sum_{i=1}^n \mathbb{E}\big[(\bar{X}_{i,n})^2\big] + \sum_{i\ne j}^n \mathbb{E}[\bar{X}_{i,n}\bar{X}_{j,n}] \to c,$$

since the second sum goes to zero for condition (i).

Now let us compute the third moment:

$$\mathbb{E}\big[(\bar{S}_n)^3\big]=\sum_{i=1}^{n}\mathbb{E}\big[(\bar{X}_{i,n})^3\big]+3\sum_{i\neq j}^{n}\mathbb{E}\big[(\bar{X}_{i,n})^2\bar{X}_{j,n}\big]+\sum_{i\neq j\neq k}^{n}\mathbb{E}[\bar{X}_{i,n}\bar{X}_{j,n}\bar{X}_{k,n}].$$

The last two sums go to zero for condition (i). For the first sum we have

$$\left|\sum_{i=1}^{n}\mathbb{E}\big[(\bar{X}_{i,n})^3\big]\right|\le\sum_{i=1}^{n}\mathbb{E}\big[(\bar{X}_{i,n})^2|\bar{X}_{i,n}|\big]\le\varepsilon_n\sum_{i=1}^{n}\mathbb{E}\big[(\bar{X}_{i,n})^2\big]\sim\varepsilon_n c\overset{n\to\infty}{\longrightarrow}0\,.$$

In the general case, the mth moment $\mathbb{E}[(\bar{S}_n)^m]$ is made up of terms of the form

$$A(p_1,\dots,p_\ell)\sum_{i_1\neq i_2\cdots\neq i_\ell}^{n}\mathbb{E}\big[(\bar{X}_{i_1,n})^{p_1}\cdots(\bar{X}_{i_\ell,n})^{p_\ell}\big],\quad 1\le\ell\le m$$

with $\{p_i, i=1,\dots,\ell\}$ positive integers such that $p_1+p_2+\cdots+p_\ell=m$. Here $A(p_1,\dots,p_\ell)$ is the number of all possible partitions of m objects in ℓ subsets made up of $p_1,\dots,p_\ell$ objects. Since all sums containing a singleton (i.e. there is a $p_i=1$) go asymptotically to zero, we consider just the cases with $p_i\ge 2$, $\forall i=1,\dots,\ell$. Observe that this implies in particular that $\ell\le m/2$. In this case

$$\begin{aligned}\left|\sum_{i_1\neq i_2\cdots\neq i_\ell}^{n}\mathbb{E}\big[(\bar{X}_{i_1,n})^{p_1}\cdots(\bar{X}_{i_\ell,n})^{p_\ell}\big]\right|&\le\varepsilon_n^{m-2\ell}\sum_{i_1\neq i_2\cdots\neq i_\ell}^{n}\mathbb{E}\big[(\bar{X}_{i_1,n})^{2}\cdots(\bar{X}_{i_\ell,n})^{2}\big]\\&\sim\varepsilon_n^{m-2\ell}c^\ell,\end{aligned}$$

which goes to zero if $\ell\neq m/2$. Therefore all odd moments are asymptotically negligible, while for even moments asymptotically

$$\mathbb{E}\big[(\bar{S}_n)^{2k}\big]\sim A_k\sum_{i_1\neq\cdots\neq i_k}^{n}\mathbb{E}\big[(\bar{X}_{i_1,n})^{2}\cdots(\bar{X}_{i_k,n})^{2}\big]\to A_k c^k,$$

where A_k is the number of all possible pairings of $2k$ objects, namely

$$A_k=(2k-1)(2k-3)\cdots 1=(2k-1)!!.$$

Finally

$$\mathbb{E}\big[(\bar{S}_n)^m\big]\overset{n\to\infty}{\longrightarrow}\begin{cases}0 & m\text{ odd}\\(m-1)!!c^{m/2} & m\text{ even,}\end{cases}$$

which are the moments of a Gaussian variable $\mathcal{N}(0,c)$. □

Let us consider the array of variables $\{\langle\lambda,\bar{\Psi}_{N,m}\rangle, N\ge 2, m\ge 0\}$ defined in (30), (15), with $\lambda\in\mathbb{R}^2$ unitary vector. We have

$$\big\langle\lambda,Z_N^<(t)\big\rangle=\sum_{m=0}^{\lfloor Nt\rfloor-1}\langle\lambda,\bar{\Psi}_{N,m}\rangle+\big(Nt-\lfloor Nt\rfloor\langle\lambda,\bar{\Psi}_{N,\lfloor Nt\rfloor}\rangle\big),$$

$\forall t\in[0,\mathcal{T}]$, $\forall N\ge 2$, where the rightmost term goes to zero in probability by Chebyshev's inequality. By definition, $\langle\lambda,\bar{\Psi}_{N,m}\rangle\le\frac{2}{\sqrt{\ln N}}$ for every $m\ge 0$, $\forall N\ge 2$. Moreover, since $\psi(k)$ is an odd function, and the probability kernel

$P(k, \mathrm{d}k')$ has a density which is even in both k and k', the array satisfies condition (i). In order to check condition (ii), we will use the following lemma.

Lemma 6.2. *For every $\ell \geq 1$, for every sequence $(m_1, \dots, m_\ell)$ such that $m_1 \geq 0$, $m_i \geq 1$, for every $N \geq 2$*

$$\mathbb{E}\big[\langle\lambda, \bar{\Psi}_{N,m_1}\rangle^2 \cdots \langle\lambda, \bar{\Psi}_{N,m_1+\cdots+m_\ell}\rangle^2\big] \leq \frac{c_0^\ell}{N^\ell} \tag{40}$$

with c_0 finite, $\forall t \in [0, \mathcal{T}]$.

Proof. By definition

$$\begin{aligned}
&\mathbb{E}\big[\langle\lambda, \bar{\Psi}_{N,m_1}\rangle^2 \cdots \langle\lambda, \bar{\Psi}_{N,m_1+\cdots+m_\ell}\rangle^2\big] \\
&\quad = \int_0^\infty \mathrm{d}z_1 \mathrm{e}^{-z_1} \int_{\mathbb{T}^2} \mu P^{m_1}(\mathrm{d}k_1)\big\langle\lambda, \bar{\Psi}_N(k_1, z_1)\big\rangle^2 \int \cdots \\
&\quad\quad \times \int_0^\infty \mathrm{d}z_m \mathrm{e}^{-z_m} \int_{\mathbb{T}^2} P^{m_\ell}(k_{m-1}, k_m)\big\langle\lambda, \bar{\Psi}_N(k_m, z_m)\big\rangle^2 \\
&\quad \leq 2^\ell \left(\int_0^\infty \mathrm{d}z \mathrm{e}^{-z} \int_{\mathbb{T}^2} \pi(k)\big\langle\lambda, \bar{\Psi}_N(k, z)\big\rangle^2\right)^\ell,
\end{aligned}$$

where in the last inequality we used (29). We conclude the proof by observing that

$$\lim_{N\to\infty} N \int_0^\infty \mathrm{d}z \mathrm{e}^{-z} \int_{\mathbb{T}^2} \pi(k)\big\langle\lambda, \bar{\Psi}_N(k, z)\big\rangle^2 = \sigma^2,$$

with σ defined in (13). □

We observe that

$$\begin{aligned}
&\sum_{\substack{i_1\neq i_2\neq\cdots\neq i_\ell \\ \in\{0,\dots,\lfloor Nt\rfloor-1\}}} \mathbb{E}\big[\langle\lambda, \bar{\Psi}_{N,i_1}\rangle^2 \cdots \langle\lambda, \bar{\Psi}_{N,i_\ell}\rangle^2\big] \\
&\quad = \ell! \sum_{m_1\geq 0} \sum_{\substack{m_2,\dots,m_\ell\geq 1 \\ m_1+\cdots+m_\ell\leq\lfloor Nt\rfloor-1}} \mathbb{E}\big[\langle\lambda, \bar{\Psi}_{N,m_1}\rangle^2 \cdots \langle\lambda, \bar{\Psi}_{N,m_1+\cdots+m_\ell}\rangle^2\big].
\end{aligned}$$

We split the sum on m_1 in two part, namely $\sum_{m_1=0}^{M-1} + \sum_{m_1\geq M}$, with $0 < M < \lfloor Nt\rfloor - 1$. Using (40) and the relation

$$\lim_{N\to\infty} \sum_{\substack{m_1,\dots,m_k\geq 1 \\ m_1+\cdots+m_k\leq N}} N^{-k} = \frac{1}{k!},$$

we get that for every $\ell \geq 1$, $N \geq 2$, $\forall t \in [0, \mathcal{T}]$

$$\begin{aligned}
&\ell! \sum_{m_1=0}^{M-1} \sum_{\substack{m_2,\dots,m_\ell\geq 1 \\ m_1+\cdots+m_\ell\leq\lfloor Nt\rfloor-1}} \mathbb{E}\big[\langle\lambda, \bar{\Psi}_{N,m_1}\rangle^2 \cdots \langle\lambda, \bar{\Psi}_{N,m_1+\cdots+m_\ell}\rangle^2\big] \\
&\quad \leq C_\ell \mathcal{T}^{\ell-1} \frac{M}{N}.
\end{aligned}$$

By repeating this procedure for all the sums, we have

$$\sum_{\substack{i_1\neq i_2\neq\cdots\neq i_\ell\\ \in\{0,\ldots,\lfloor Nt\rfloor-1\}}} \mathbb{E}\big[\langle\lambda,\bar{\Psi}_{N,i_1}\rangle^2\cdots\langle\lambda,\bar{\Psi}_{N,i_\ell}\rangle^2\big]$$
$$=\ell!\sum_{\substack{m_1,\ldots,m_\ell\geq M\\ m_1+\cdots+m_\ell\leq\lfloor Nt\rfloor-1}} \mathbb{E}\big[\langle\lambda,\bar{\Psi}_{N,m_1}\rangle^2\cdots\langle\lambda,\bar{\Psi}_{N,m_1+..+m_\ell}\rangle^2\big]+\mathcal{E}_\ell(M,N), \tag{41}$$

with $\mathcal{E}_\ell(M,N)\leq\tilde{C}_\ell\mathcal{T}^{\ell-1}M/N$, $\forall\ell\geq 1$.

Observe that for every $m\geq 2$

$$\int_{\mathbb{T}^2}P^m\big(k,\mathrm{d}k'\big)\big\langle\lambda,\bar{\Psi}_N\big(k',z\big)\big\rangle^2=\int_{\mathbb{T}^2}\pi\big(\mathrm{d}k'\big)\big\langle\lambda,\bar{\Psi}_N\big(k',z\big)\big\rangle^2$$
$$+\int_{\mathbb{T}^2}\big[P^{m-1}(k,\mathrm{d}\tilde{k})-\pi(\mathrm{d}\tilde{k})\big]\int_{\mathbb{T}^2}P\big(\tilde{k},\mathrm{d}k'\big)\big\langle\lambda,\bar{\Psi}_N\big(k',z\big)\big\rangle^2,$$

where, using (29),

$$\sup_{k\in\mathbb{T}^2}\int_{\mathbb{T}^2}\big|P^{m-1}(k,\mathrm{d}\tilde{k})-\pi(\mathrm{d}\tilde{k})\big|\int_{\mathbb{T}^2}P\big(\tilde{k},\mathrm{d}k'\big)\big\langle\lambda,\bar{\Psi}_N\big(k',z\big)\big\rangle^2$$
$$\leq\frac{C_0}{N}\sup_{k\in\mathbb{T}^2}\int_{\mathbb{T}^2}\big|P^{m-1}(k,\mathrm{d}\tilde{k})-\pi(\mathrm{d}\tilde{k})\big|.$$

Thus, thanks to (40), for every $(m_1,\ldots,m_\ell)$ with $m_i\geq M$, $i=1,\ldots,\ell$,

$$\mathbb{E}\big[\langle\lambda,\bar{\Psi}_{N,m_1}\rangle^2\cdots\langle\lambda,\bar{\Psi}_{N,m_1+\cdots+m_\ell}\rangle^2\big]$$
$$=\bigg(\int_0^\infty\mathrm{d}z\mathrm{e}^{-z}\int_{\mathbb{T}^2}\pi\big(\mathrm{d}k'\big)\big\langle\lambda,\bar{\Psi}_N\big(k',z\big)\big\rangle^2\bigg)^\ell+\tilde{e}_\ell(M,N), \tag{42}$$

where

$$\tilde{e}_\ell(M,N)\leq\ell\frac{C_0}{N^\ell}\sup_{m\geq M-1}\sup_{k\in\mathbb{T}^2}\int_{\mathbb{T}^2}\big|P^m(k,\mathrm{d}\tilde{k})-\pi(\mathrm{d}\tilde{k})\big|.$$

Finally, by (41) and (42) we get

$$\sum_{\substack{i_1\neq i_2\neq\cdots\neq i_\ell\\ \in\{0,\ldots,\lfloor Nt\rfloor-1\}}} \mathbb{E}\big[\langle\lambda,\bar{\Psi}_{N,i_1}\rangle^2\cdots\langle\lambda,\bar{\Psi}_{N,i_\ell}\rangle^2\big]$$
$$=\ell!\sum_{\substack{m_1,\ldots,m_\ell\geq M\\ m_1+\cdots+m_\ell\leq\lfloor Nt\rfloor-1}}\big(\mathbb{E}_\pi\big[\langle\lambda,\bar{\Psi}_{N,1}\rangle^2\big]\big)^\ell+\mathcal{R}_\ell(M,N),$$

where

$$\mathcal{R}_\ell(M,N)\leq C_\ell\mathcal{T}^\ell\bigg(\frac{M}{N}+\sup_{m\geq M-1}\sup_{k\in\mathbb{T}^2}\int_{\mathbb{T}^2}\big|P^m(k,\mathrm{d}\tilde{k})-\pi(\mathrm{d}\tilde{k})\big|\bigg). \tag{43}$$

In the limit $M,N\to\infty$ such that $\frac{M}{N}\to 0$, $\mathcal{R}_\ell(M,N)\to 0$ and

$$\ell!\sum_{\substack{m_1,\ldots,m_\ell\geq M\\ m_1+\cdots+m_\ell\leq\lfloor Nt\rfloor-1}}\big(\mathbb{E}_\pi\big[\langle\lambda,\bar{\Psi}_{N,1}\rangle^2\big]\big)^\ell\to\big(\sigma^2\big)^\ell t^\ell,$$

with σ defined in (13). Thus the array of variables $\{\langle\lambda, \bar{\Psi}_{N,m}\rangle, N \geq 2, m \geq 0\}$ satisfies also condition (ii), and we get

$$\bar{S}_N(t) := \sum_{n=0}^{\lfloor Nt\rfloor-1} \langle\lambda, \bar{\Psi}_{N,n}\rangle \overset{N\uparrow\infty}{\to} \mathcal{N}\big(0, \sigma^2 t\big),$$

$\forall t \in [0, \mathcal{T}]$, $\forall\lambda \in \mathbb{R}^2$ such that $|\lambda| = 1$.

We can easily adapt the proof and show that $\forall 0 \leq s < t \leq \mathcal{T}$

$$\bar{S}_N(t) - \bar{S}_N(s) \to \mathcal{N}\big(0, \sigma^2(t-s)\big).$$

In order to prove the convergence of the finite dimensional marginal to the Wiener process W_σ, we have to show that $\forall n \geq 2$, for every partition $0 \leq t_1 < \cdots < t_n \leq \mathcal{T}$ the variables $\bar{S}_N(t_1)$, $\bar{S}_N(t_2) - \bar{S}_N(t_1), \dots, \bar{S}_N(t_n) - \bar{S}_N(t_{n-1})$ are asymptotically jointly Gaussian and uncorrelated. This is stated in the next lemma.

Lemma 6.3. *For every $n \geq 1$, $\forall\underline{\alpha}(n) := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that $|\underline{\alpha}(n)| = 1$*

$$\sum_{k=1}^{n} \alpha_k\big(\bar{S}_N(t_k) - \bar{S}_N(t_{k-1})\big) \Rightarrow \mathcal{N}\left(0, \sigma^2 \sum_{k=1}^{n} \alpha_k^2(t_k - t_{k-1})\right), \tag{44}$$

$\forall 0 = t_0 < t_1 < \cdots < t_n \leq \mathcal{T}$.

Proof. The case $n = 1$ is proved. Let us consider the case $n = 2$. Fixed $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, with $\alpha_1^2 + \alpha_2^2 = 1$, we consider the following array of variables

$$\xi_{N,m} = (\alpha_1 \mathbf{1}_{\{m \leq \lfloor Nt_1\rfloor - 1\}} + \alpha_2 \mathbf{1}_{\{m \geq \lfloor Nt_1\rfloor\}})\langle\lambda, \bar{\Psi}_{N,m}\rangle, \quad \forall N \geq 2, \forall m \geq 0,$$

which are uniformly bounded by $\frac{2}{\sqrt{N}}$ and satisfy condition (i). Let us define, $\forall t \geq 0$, $m \geq 0$, $N \geq 2$,

$$a_{N,m}(t) := \alpha_1 \mathbf{1}_{\{m \leq \lfloor Nt\rfloor - 1\}} + \alpha_2 \mathbf{1}_{\{m \geq \lfloor Nt\rfloor\}},$$

which is uniformly bounded by 1. In order to check condition (ii), we repeat the steps done for $\bar{S}_N(t)$ and we get

$$\begin{aligned}
&\sum_{\substack{i_1 \neq i_2 \neq \cdots \neq i_\ell \\ \in\{0,\dots,\lfloor Nt_2\rfloor - 1\}}} \mathbb{E}\big[\xi_{N,i_1}^2 \cdots \xi_{N,i_\ell}^2\big] \\
&\quad = \ell! \sum_{0 \leq i_1 < \cdots < i_\ell \leq \lfloor Nt_2\rfloor - 1} a_{N,i_1}(t_1)^2 \cdots a_{N,i_\ell}(t_1)^2 \big(\mathbb{E}_\pi\big[\langle\lambda, \bar{\Psi}_{N,1}\rangle^2\big]\big)^\ell \\
&\qquad + \mathcal{R}_\ell(M, N),
\end{aligned}$$

with $\mathcal{R}_\ell(M, N)$ the same of (43). By direct computation

$$\begin{aligned}
&\ell! \sum_{0 \leq i_1 < \cdots < i_\ell \leq \lfloor Nt_2\rfloor - 1} a_{N,i_1}(t_1)^2 \cdots a_{N,i_\ell}(t_1)^2 \\
&\quad = \sum_{k=0}^{\ell} \ell! \sum_{1 \leq i_1 < \cdots < i_k \leq \lfloor Nt_1\rfloor} (\alpha_1)^{2k} \sum_{\lfloor Nt_1\rfloor < i_{k+1} < \cdots < i_\ell \leq \lfloor Nt_2\rfloor} (\alpha_2)^{2(\ell-k)},
\end{aligned}$$

then using

$$\sum_{1 \leq i_1 < \cdots < i_k \leq N} N^{-k} \overset{N\uparrow\infty}{\to} \frac{1}{k!}, \qquad N^\ell\big(\mathbb{E}_\pi\big[\langle\lambda, \bar{\Psi}_{N,1}\rangle^2\big]\big)^\ell \overset{N\uparrow\infty}{\to} \big(\sigma^2\big)^\ell,$$

with σ defined in (13), we get that condition (ii) is satisfied, i.e.

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{\substack{i_1\neq i_2\neq\cdots\neq i_\ell\\ \in\{0,\dots,\lfloor Nt_2\rfloor-1\}}} \mathbb{E}\big[\xi_{N,i_1}^2\cdots\xi_{N,i_\ell}^2\big]\\
&\quad= \big(\sigma^2\big)^\ell \sum_{k=0}^{\ell} \frac{\ell!}{k!(\ell-k)!}\alpha_1^{2k}t_1^k\alpha_2^{2(\ell-k)}(t_2-t_1)^{\ell-k}\\
&\quad= \big(\sigma^2\big)^\ell\big[\alpha_1^2 t_1+\alpha_2^2(t_2-t_1)\big]^\ell,
\end{aligned}$$

thus

$$\begin{aligned}
\alpha_1\bar S_N(t_1)+\alpha_2\big[\bar S_N(t_2)-\bar S_N(t_1)\big] &= \sum_{m=0}^{\lfloor Nt_2\rfloor-1}\xi_{N,m}\\
&\to \mathcal{N}\big(0,\big(\sigma^2\big)\big[\alpha_1^2 t_1+\alpha_2^2(t_2-t_1)\big]\big).
\end{aligned}$$

The proof can be repeated for $n\geq 3$, in that case we find the multinomial formula for a polynomial with n terms to the power ℓ. □

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