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Transitions on a noncompact Cantor set and random walks on its defining tree

Jun Kigami

Graduate School of Informatics, Kyoto University, Kyoto, 606-8501, Japan. E-mail: kigami@i.kyoto-u.ac.jp

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Abstract. First, noncompact Cantor sets along with their defining trees are introduced as a natural generalization of p-adic numbers. Secondly we construct a class of jump processes on a noncompact Cantor set from given pairs of eigenvalues and measures. At the same time, we have concrete expressions of the associated jump kernels and transition densities. Then we construct intrinsic metrics on noncompact Cantor set to obtain estimates of transition densities and jump kernels under some regularity conditions on eigenvalues and measures. Finally transient random walks on the defining tree are shown to induce a subclass of jump processes discussed in the second part.

Résumé. Nous commençons par introduire des ensembles de Cantor non-compacts, ainsi que leurs arbres associés. Ils peuvent être considerés comme une généralisation naturelle des nombres p-adiques. Nous construisons ensuite une classe de processus de saut sur un ensemble de Cantor non-compact, à l'aide d'un couple de valeurs propres et de mesures. De plus, nous obtenons des expressions concrètes pour les noyaux de la chaleurs associés à ces processus de saut et pour les probabilités de transition correspondantes. Sous certaines hypothèses de régularité sur les valeurs propres et les mesures, nous construisons ensuite des métriques intrinsèques sur cet ensemble de Cantor non-compact afin d'obtenir des estimations fines sur les noyaux de la chaleur et les probabilités de transitions. Finalement, nous montrons que les marches aléatoires sur l'arbre définissant l'ensemble de Cantor non-compact induisent une sous-classe des processus de saut discutés dans la seconde partie de l'article.

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1. Introduction

As is pointed out in the introduction of Albeverio and Karwowski [2], various theories of physics have been constructed on the collection of p-adic numbers $\mathbb{Q}_p$. Some of them took advantage of the algebraic structure and symmetries of $\mathbb{Q}_p$ and others made use of the hierarchical structure of it. See [9,12,19] for concise reviews and detailed references.

In this paper, we are going to study the construction and asymptotic behaviors of jump processes on noncompact Cantor sets, which are generalization of p-adic numbers $\mathbb{Q}_p$ from the view point of hierarchical structure. Recall that the p-adic numbers $\mathbb{Q}_p$ is the collection of "limit points" of the homogeneous tree with degree $p+1$ in a particular order. See Fig. 1 for the case $p=2$. We will generalize this respect of $\mathbb{Q}_p$ and define noncompact Cantor set as the collection of "limit points" of an ordered tree which is called the defining tree of a noncompact Cantor set. Such a generalization has been formulated by Albeverio and Karwowski in [2], where a noncompact Cantor set is called leaves of multibranching trees. Topologically, a noncompact Cantor set turns out to be merely the (ternary) Cantor set minus one point by Proposition 2.6. This is why we call those "limits points" of an ordered tree noncompact Cantor set. From physical point of view, we give a description of how stochastic particles are moving around on a limit of a hierarchical structure.

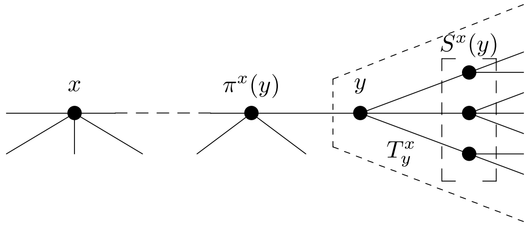


Fig. 1. Structures of tree with a vertex x as the root.

There are numeous studies of stochastic processes on totally disconnected state spaces. For example, Evans has considered Lévy processes on a totally disconnected group in [13]. In this direction, Aldous and Evans have constructed Dirichlet forms on general totally disconnected spaces in [5].

As for p-adic numbers $\mathbb{Q}_p$, a class of jump processes has been intensively studied by Albeverio, Karwowski and their coworkers in [1–4,17]. They first considered transition probabilities from balls to balls and then obtained a continuous time process and associated Dirichlet form as a limit in [1]. Moreover, the eigenfunctions of the associated self-adjoint operator have been shown to form a kind of Haar's wavelet basis on $\mathbb{Q}_p$. In [17], the construction of processes in [1] has been extended to cases with inhomogeneous underlying measures on $\mathbb{Q}_p$ and the asymptotic and spectral results as in [1] have been obtained for this broadened class in [4]. Furthermore, an exact expression of the heat kernel associated to a process constructed in [1] has been presented in [3] in relation with trace formula. In [2], most of the parts of those results have been generalized to a class of jump processes on leaves of multibranching trees, which we mentioned above.

On the other hand, regular Dirichlet forms on the Cantor set have been constructed as traces of random walks on trees in [18]. (A regular Dirichlet form naturally corresponds to a Hunt process, which is a jump process in our case. See [14] for example.) Note that the Martin boundary of a random walk on a tree is (homeomorphic to) the Cantor set by [10]. By this fact, if f is a real valued function on the Cantor set belonging to a suitable class like bounded measurable or L^1 for example, we have a harmonic function Hf on the tree with given boundary value f on the Cantor set as

$$(Hf)(x) = \int_{\Sigma} \frac{G(x, y)}{G(\phi, y)} f(y)\nu_\phi(\mathrm{d}y),$$

where ϕ is a fixed reference point, Σ is the Cantor set, G is the Green function of the random walk, ν_ϕ is the hitting distribution starting from ϕ. In [18], a regular Dirichlet form on the Cantor set was constructed as the energy of the harmonic function Hf associated with the random walk. It was shown that eigenfunctions of the associated self-adjoint operator formed a kind of Haar's wavelet on the Cantor set as in the case of p-adic numbers $\mathbb{Q}_p$ obtained in [1,2,17]. Moreover, an explicit expression and asymptotic behaviors of associated transition density have been obtained by introducing an intrinsic metric on the Cantor set. Since a noncompact Cantor set is the Cantor set minus one point, one may naturally expect to obtain a class of jump processes on a noncompact Cantor set from the processes constructed above on the Cantor set by ignoring the one point removed from the Cantor set. (See Section 11 for the exact meaning of "ignoring.") In fact, we are going to see that this is the case.

Consequently, we have two classes of stochastic processes (Hunt processes and/or regular Dirichlet forms to be exact) on a noncompact Cantor set. One of them has been constructed by Albeverio and Karwowski in [1] and the other has induced by random walks on the trees associated with the noncompact Cantor set. These two classes have one feature in common. Namely, the associated eigenfunctions form a kind of Haar's wavelet basis. However, one is not a subset of the other although they have non-empty intersection. In this paper, we introduce a natural class of regular Dirichlet forms on a noncompact Cantor set which includes both classes. More precisely, let us denote a noncompact Cantor set by Σ^+. We are going to construct a closed form $\mathcal{Q}$ on Σ^+ from a pair (λ, μ) of nonnegative function λ on the tree defining Σ^+ and a Radon measure μ on Σ^+ so that λ and the counterpart of Haar's wavelet basis associated with μ give the eigenvalues and the eigenfunctions respectively of the non-negative self-adjoint operator associated with the closed form $\mathcal{Q}$. Under a suitable condition, which is easily verified by the pair (λ, μ), the closed form $\mathcal{Q}$ is shown to give a regular Dirichlet form on $L^2(\Sigma^+, \mu)$ which has the following expression:

$$\mathcal{Q}(u, v) = \int_{\Sigma^+ \times \Sigma^+} J(\omega, \tau)\big(u(\omega) - u(\tau)\big)\big(v(\omega) - v(\tau)\big)\mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau) + \lambda_I(u, v)_\mu, \tag{1.1}$$

where $J(\cdot, \cdot)$ is a nonnegative kernel explicitly determined by (λ, μ) and λ_I is the infimum of λ. By Theorem 3.2.1 of [14], the first part of the above expression of $\mathcal{Q}$ regarding the integral kernel $J(\cdot, \cdot)$ corresponds to jumps and the latter part $\lambda_I(u, v)_\mu$ represents the decay of the total probability $P_x(X_t \in \Sigma^+)$, where $(\{X_t\}_{t>0}, \{P_\omega\}_{\omega \in \Sigma^+})$ is the Hunt process associated with the regular Dirichlet form $\mathcal{Q}$ on $L^2(\Sigma^+, \mu)$. If $\lambda_I = 0$, then $\mathcal{Q}$ is shown to be conservative in Theorem 3.7.

Our next interest is the existence and asymptotic behaviors of a transition density $p(t, \omega, \tau)$, which corresponds to a fundamental solution in the case of parabolic PDE's. In brief, $p(t, \omega, \tau)$ is a transition density of a regular Dirichlet form $\mathcal{Q}$, i.e. a Hunt process $(\{X_t\}_{t>0}, \{P_\omega\}_{\omega \in \Sigma^+})$ if

$$\big(\mathrm{e}^{-Lt} f\big)(\omega) = E_\omega\big(f(X_t)\big) = \int_{\Sigma^+} p(t, \omega, \tau) f(\tau)\mu(\mathrm{d}\tau)$$

for any bounded measurable function f and any $\omega \in \Sigma^+$, where L is the nonnegative self-adjoint operator associated with $\mathcal{Q}$ and E_ω is the expectation with respect to P_ω. Under a mild assumption, we will explicitly construct a transition density $p(t, \omega, \tau)$ in terms of λ and μ in Section 4.

To consider asymptotic behaviors of $p(t, \omega, \tau)$, the first question is to find the best metric, if it ever exists, for the purpose. Note that there exist plenty of metrics on Σ^+ which provide the topology of Σ^+ as the Cantor set minus one point. Among them, we need to search the one which yields a nice estimate of $p(t, \omega, \tau)$. In Section 5, we actually construct such an intrinsic metric d_λ from λ in an analogous way as in [18]. Roughly, it is defined so that the reciprocal of λ gives the diameters of balls. Assuming a kind of regularity of the decay of λ and μ, we have the following estimate of the transition density in terms of d_λ in Theorem 6.2: there exists $c > 0$ such that

$$p(t, \omega, \tau) \le c \min\left\{\frac{t}{\mu(B(\omega, d_\lambda(\omega, \tau)))d_\lambda(\omega, \tau)}, \frac{1}{\mu(B(\omega, t))}\right\}, \tag{1.2}$$

for any ω, τ, where $B(\omega, r) = \{\tau | d_\lambda(\omega, \tau) < r\}$. The right-hand side of (1.2) represents one of typical asymptotic behaviors of transition densities for jump processes. See [11] for example. We also have this type of transition density estimate in the case of the Cantor set in [18]. Note that the first term in the minimum of (1.2) is realized on $\{(t, \omega, \tau) | d(\omega, \tau) \ge t\}$, which is called the off-diagonal part, whereas the second term is realized on $\{(t, \omega, \tau) | d(\omega, \tau) \le t\}$, which is called the near-diagonal part. For the near-diagonal lower estimate, we will have

$$\frac{c}{\mu(B(\omega, t))} \le p(t, \omega, \tau)$$

if $d_\lambda(\omega, \tau) \le \varepsilon t$, where $c, \varepsilon > 0$ are some given constants which are independent of t, ω and τ. The off-diagonal lower estimate is a little tricky. In general, it holds on certain proportion of the whole space. To be exact, there exist a set $U \subseteq \Sigma^+ \times \Sigma^+$ and $\gamma > 0$ which satisfy

$$\mu\Big(\Big\{\tau | (\omega, \tau) \in U\Big\} \cap A(\omega, r_1, r_2)\Big) \ge \gamma\mu\big(A(\omega, r_1, r_2)\big)$$

for any ω and any $r_2 > r_1 > 0$, where $A(\omega, r_1, r_2)$ is an annulus defined as $B(\omega, r_2) \backslash B(\omega, r_1)$ and

$$c\frac{t}{\mu(B(\omega, d_\lambda(\omega, \tau)))d_\lambda(\omega, \tau)} \le p(t, \omega, \tau) \tag{1.3}$$

for any $(\omega, \tau) \in U$ if $d_\lambda(\omega, \tau) > \varepsilon t$, where c is a constant depending only on $\varepsilon > 0$. Moreover, this estimate is best possible in the sense that there exists an example in Section 8 where (1.3) cannot hold on the complement of U for any $\varepsilon > 0$. This kind of peculiar behavior of the transition density in the off-diagonal part has never been observed before. It is interesting to know how common such a phenomena is in general.

Apart from transition densities, we also study different kind of problem on the processes induced by the random walks on the tree defining the noncompact Cantor set. Recall that originally we have a process on the Cantor set and make it a process on the noncompact Cantor set by ignoring one point which is removed from the Cantor set. Intuitively, if the original random walk on the tree hits the removed point with positive probability, we may not just "ignore" this point. There must be some difference. This intuition is rationalized in Theorem 10.8, where the resulting process on the noncompact Cantor set is shown to be conservative, i.e. $\lambda_I = 0$ if and only if the hitting probability of the removed point is 0.

The organization of this paper is as follows. In Section 2, we give the basic notions regarding trees, ordered trees and the associated noncompact Cantor sets. In Section 3, we construct a closed form $(\mathcal{Q}, \mathcal{D})$, where $\mathcal{Q}$ is a form and $\mathcal{D}$ is its domain, on a noncompact Cantor set from a pair (λ, μ) and show that $(\mathcal{Q}, \mathcal{D})$ is a regular Dirichlet form under additional assumptions. At the same time, we give an explicit expression for the jump kernel $J(\cdot,\cdot)$ appearing in (1.1). We also prove that the Dirichlet forms introduced by Albeverio and Karwowski belong to the collection of Dirichlet forms $(\mathcal{Q}, \mathcal{D})$ given by (λ, μ). Section 4 is devoted to transition densities. As we mentioned above, we obtain an exact expression of transition densities under a mild assumption. An intrinsic metric for asymptotic estimates of a transition density is given in Section 5. Using the intrinsic metric, we present asymptotic estimates of a transition density explained above in Section 6. Section 7 is devoted to proving the asymptotic estimates given in Section 6. In Section 8, we give various examples in the case of 2-adic numbers $\mathbb{Q}_2$. From Section 9, we study the class of Dirichlet forms induced by random walks on the defining tree. In Section 9, we review the fundamental results on random walks on trees including the energy, transience, resistances, harmonic functions and the Martin boundary. Then we show relations between resistances and hitting distributions in Section 10. In Section 11, these relations help us to show that Dirichlet forms on a noncompact Cantor set induced by random walks on its defining tree are of the form of $(\mathcal{Q}, \mathcal{D})$ given in Section 3. Finally in Section 12, we consider the inverse problem: when is a Dirichlet form $(\mathcal{Q}, \mathcal{D})$ on noncompact Cantor set given in Section 3 derived from a random walk on its defining tree.

In this paper we use the following convention: Let f and g be real valued functions defined on a set A. We write $f(x) \asymp g(x)$ on A if and only if there exist positive constants c_1, c_2 such that $c_1 f(x) \le g(x) \le c_2 f(x)$ for any $x \in A$. Moreover, $B_d(x,r) = \{y | d(x,y) < r\}$ if (X,d) is a metric space.

2. Ordered trees and noncompact Cantor sets

In this section, we introduce the fundamental notions on an infinite (ordered) tree and associated noncompact Cantor set, which corresponds to the p-adic numbers as a special case. First we review the basics on an infinite tree and its boundary which consists of the ends of the tree. The boundary is (homeomorphic to) the Cantor set in general. Later, to obtain a noncompact Cantor set, we are going to choose an arbitrary point in the boundary and introduce an order on the tree associated with the chosen point.

Definition 2.1. *Let T be a countably infinite set and let $\mathcal{A}: T \times T \to \{0,1\}$ which satisfies $\mathcal{A}(x,y) = \mathcal{A}(y,x)$ and $\mathcal{A}(x,x) = 0$ for any $x, y \in T$. We call the pair $(T, \mathcal{A})$ a (non-directed) graph with the vertices T and the adjacent matrix $\mathcal{A}$.*

(1) *Define $V(x) = \{y | \mathcal{A}(x,y) = 1\}$ and call it the neighborhood of x. $(T, \mathcal{A})$ is said to be locally finite if $V(x)$ is a finite set for any $x \in T$.*

(2) *For $x_0, \dots, x_n \in T$, $(x_0, x_1, \dots, x_n)$ is called a path between x_0 and x_n if $\mathcal{A}(x_i, x_{i+1}) = 1$ for any $i = 0, 1, \dots n-1$. A path $(x_0, x_1, \dots, x_n)$ is called simple if and only if $x_i \neq x_j$ for any i, j with $0 \le i < j \le n$ and $(i,j) \neq (0,n)$.*

(3) *$(T, \mathcal{A})$ is called a (non-directed) tree if and only if there exists a unique simple path between x and y for any $x, y \in T$ with $x \neq y$.*

In this paper, $(T, \mathcal{A})$ is always a locally finite tree and the number of neighbors of any vertex is no less than 3. Namely, we assume the followings troughout this paper.

Assumption 2.2. $(T, \mathcal{A})$ *is a tree.* $3 \le \#(V(x)) < +\infty$ *for any* $x \in T$, *where* $\#(\cdot)$ *is the number of elements of a set.*

Even without the above assumption, most of the results in this paper essentially remain true. However the exact statements may become more complicated than those given in the present paper.

Next we define structures of a tree with a vertex x as the root. See Fig. 1.

Definition 2.3. *Let $(T, \mathcal{A})$ be a tree.*

(1) *The unique simple path between two vertices x and y is called the geodesic between x and y and denoted by $\overline{xy}$. We write $z \in \overline{xy}$ if $\overline{xy} = (x_0, x_1, \ldots, x_n)$ and $z \in x_i$ for some i.*

(2) *For $x, y \in T$, define $T^x_y = \{z|z \in T, y \in \overline{xz}\}$. We regard T^x_y as a tree with an adjacent matrix $\mathcal{A}|_{T^x_y \times T^x_y}$.*

(3) *For any $x \in T$, define $\pi^x : T \to T$ by*

$$\pi^x(y) = \begin{cases} x_{n-1} & \text{if } x \neq y \quad \text{and} \quad \overline{xy} = (x_0, x_1, \ldots, x_{n-1}, x_n), \\ x & \text{if } x = y. \end{cases}$$

Also set $S^x(y) = V(y)\backslash\{\pi^x(y)\}$.

(4) *$(x_0, x_1, \ldots)$ is called an infinite geodesic ray originated from x_0 if and only if $(x_0, \ldots, x_n) = \overline{x_0x_n}$ for any $n \geq 0$. Two infinite geodesic rays $(x_0, x_1, \ldots)$ and $(y_0, y_1, \ldots)$ are equivalent if and only if there exists $k \in \mathbb{Z}$ such that $x_{n+k} = y_n$ for sufficiently large n. An equivalent class of infinite geodesic rays is called an end of T. We use Σ to denote the collection of ends of T. Furthermore, we define $\widehat{T} = T \cup \Sigma$.*

(5) *Define Σ^x as the collection of infinite geodesic rays originated from $x \in T$. For any $y \in T$, Σ^x_y is defined as the collection of elements of Σ^x passing through y, namely*

$$\Sigma^x_y = \big\{(x, x_1, \ldots)|(x, x_1, \ldots) \in \Sigma^x, x_n = y \text{ for some } n \geq 1\big\}.$$

Two infinite geodesic rays $(x, x_1, \ldots), (x, y_1, \ldots) \in \Sigma^x$ are equivalent if and only if $(x, x_1, \ldots) = (x, y_1, \ldots)$. Thus, Σ is naturally identified with Σ^x.

Next we give a topology of $\widehat{T}$.

Proposition 2.4. *Define $\widehat{\mathcal{O}} = \{\{x\}|x \in T\} \cup \{T^x_y \cup \Sigma^x_y|x, y \in T\}$ and $\mathcal{O} = \{\bigcup_{O \in \mathcal{U}} O|\mathcal{U} \subseteq \widehat{\mathcal{O}}\}$. Then $\mathcal{O}$ satisfies the axiom of open sets and $(\widehat{T}, \mathcal{O})$ is a compact metrizable space. Moreover T is dense in $\widehat{T}$.*

See [22], Setion 9.B, or [21], Section 6.B, for the proof of the above proposition.

In light of the above proposition, $\widehat{T}$ is called the end compactification of T. $\Sigma = \widehat{T}\backslash T$ is the topological boundary of T in $\widehat{T}$. Under Assumption 2.2, Σ is a Cantor set with respect to the relative topology, i.e. it is compact, perfect and totally disconnected under the topology $\mathcal{O}_\Sigma = \{U \cap \Sigma|U \in \mathcal{O}\}$.

Next we fix $\phi_* \in \Sigma$ and introduce a partial order $\leq$ associated with ϕ_*. In other words, we are going to determine a natural direction of every (x, y) with $\mathcal{A}(x, y) = 1$ towards ϕ_*. Note that there exists a unique $(x, x^{(1)}, x^{(2)}, \ldots) \in \Sigma^x$ which is identified with ϕ_* for any $x \in T$. See Fig. 2 for the special case where $(T, \mathcal{A})$ is the homogeneous tree with degree 3.

Definition 2.5. *Fix $\phi_* \in \Sigma$.*

(1) *Define $\pi_{\phi_*} : T \to T$ by $\pi_{\phi_*}(x) = x^{(1)}$, where $\phi_* = (x, x^{(1)}, \ldots) \in \Sigma^x$. We use x^-_n to denote $(\pi_{\phi_*})^n(x)$ for $x \in T$ and $n \geq 0$, where $(\pi_{\phi_*})^n$ is the nth iteration of π_{ϕ_*}. For $x, y \in T$, we write $y \leq x$ if and only if $y = x^-_n$ for some $n \geq 0$. The triple $(T, \mathcal{A}, \phi_*)$ is called an ordered tree.*

(2) *An infinite geodesic ray $(x_0, x_1, x_2, \ldots)$ originated from x_0 is called an ascending ray if and only if $x_i \leq x_{i+1}$ for any $i = 0, 1, \ldots$. The collection of the equivalence class of ascending rays is denoted by Σ^+, which is called the noncompact Cantor set associated with an ordered tree $(T, \mathcal{A}, \phi_*)$. Conversely $(T, \mathcal{A}, \phi_*)$ is called the defining tree of the noncompact Cantor set Σ^+. Define Σ^+_x as the collection of ascending rays originated from x.*

(3) *Define $T^+_x = \{y|y \geq x\}$ and $S^+(x) = T^+_x \cap V(x)$.*

As a figure of speech, an ordered tree $(T, \mathcal{A}, \phi_*)$ represents a family tree of a species reproducing unisexually. If each vertex x represents an individual, then $S^+(x)$ is the direct descendants, T^+_x is the collection of all decendants

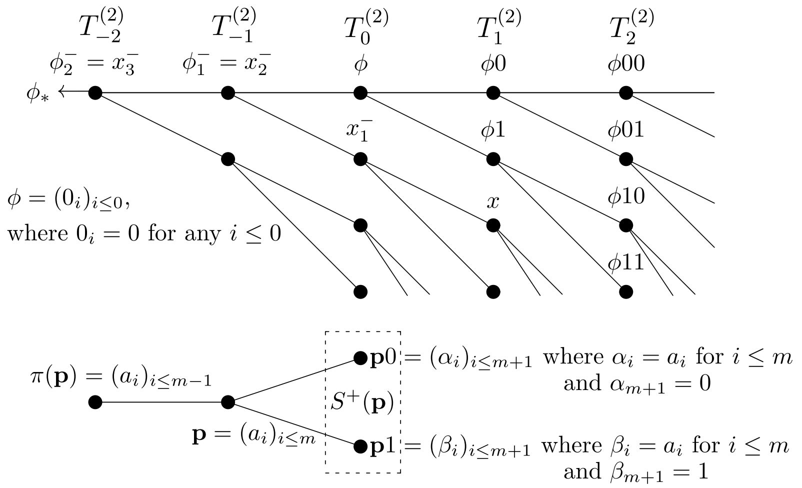


Fig. 2. $T^{(2)}$, homogeneous tree with degree 3 associated with $\mathbb{Q}_2$.

of x, $\pi_{\phi_*}(x)$ is the parent and $(x, x^-_1, x^-_2, \ldots)$ is the bloodline, i.e. the list of ancestors with x itself. The order $x < y$ means that x is an ancestor of y and that y is a descendant of x. The minus sign in x^-_m stands for "ancestors" and the plus sign in Σ^+, T^+_x and $S^+(x)$ stands for "decendants." Assumption 2.2 means that every individual has a least 2 direct descendants.

Note that a noncompact Cantor set Σ^+ depends on a choice of $\phi_* \in \Sigma$. In [2], $(T, \mathcal{A})$ and Σ^+ are called a multibranching tree and its leaves, respectively. The next proposition is immediate from the definitions. In particular, we identify noncompact Cantor set Σ^+ with $\Sigma \backslash \{\phi_*\}$.

Proposition 2.6.

(1) *The unique infinite geodesic ray identified with ϕ_* originated from x is $(x, x^-_1, x^-_2, \ldots)$.*

(2) $S^+(x) = \{y | y \in V(x), y \ge x\} = V(x) \backslash \{\pi_{\phi_*}(x)\}$. $T^+_x = T^{\pi_{\phi_*}(x)}_x$.

(3) $\Sigma^+ = \Sigma \backslash \{\phi_*\}$.

Choosing a reference point $\phi \in T$, we may introduce an (absolute) degree of a vertex $x \in T$. Fix $\phi \in T$. For any $x \in T$, since both $(\phi, \phi^-_1, \ldots)$ and $(x, x^-_1, \ldots)$ represent ϕ_*, we see that $x^-_n \in \{\phi^-_m | m \ge 0\}$ for sufficiently large n. Note that if $x^-_n = \phi^-_m$, then the value $n - m$ only depends on x.

Definition 2.7. *Define the degree $|x|$ of a vertex $x \in T$ by $|x| = n - m$ if $x^-_n = \phi^-_m$. Let $T_m = \{x | x \in T, |x| = m\}$ for any $m \in \mathbb{Z}$.*

Note that $|x|$ takes value in $\mathbb{Z}$. For example $|\phi^-_m| = -m$ for $m \ge 0$.

By the analogy using family tree, ϕ introduces an absolute scale of generations. More precisely, $|x|$ is the generation of a individual x and T_n is the collection of individuals in the nth generation.

For any $\omega \in \Sigma^+$, we may correspond two-sided infinite geodesic ray representing ω and ϕ_* in positive and negative directions respectively. Such a two-sided infinite geodesic ray is unique. More precisely, we have the following proposition.

Proposition 2.8.

(1) *For any $\omega \in \Sigma^+$, there exists a unique $(x_i)_{i \in \mathbb{Z}}$ such that $x_i \in T_i$ and $\pi_{\phi_*}(x_i) = x_{i-1}$ for any i and $(x_0, x_1, \ldots)$ is the infinite geodesic ray corresponding to ω. We use $[\omega]_m$ to denote x_m for any $m \in \mathbb{Z}$.*

(2) *For any $\omega \ne \tau \in \Sigma^+$, there exists unique $n \in \mathbb{Z}$ such that $[\omega]_m = [\tau]_m$ for any $m \le n$ and $[\omega]_{n+1} \ne [\tau]_{n+1}$. Define $\omega \wedge_{\phi_*} \tau = [\omega]_n$ and call it the confluent of ω and τ.*

If no confusion may occur, we always use $\omega \wedge \tau$ in place of $\omega \wedge_{\phi_*} \tau$ throughout this paper.

Example 2.9 **(*p-adic numbers*).** *Fix an integer $p \geq 2$. For $m \in \mathbb{Z}$, define*

$$T_m^{(p)} = \big\{(a_i)_{i \leq m} | a_i \in \{0, 1, \ldots, p-1\}, \text{there exists } N \in \mathbb{Z} \text{ such that } a_i = 0 \text{ for any } i < N\big\}$$

and let $T^{(p)} = \bigcup_{m \in \mathbb{Z}} T_m^{(p)}$. Define $\pi : T_m^{(p)} \to T_{m-1}^{(p)}$ by $\pi((a_i)_{i \leq m}) = (a_i)_{i \leq m-1}$ for $(a_i)_{i \leq m} \in T_m^{(p)}$. We may naturally regard π as a map from $T^{(p)}$ to itself. Define $\mathcal{A} : T^{(p)} \times T^{(p)} \to \{0, 1\}$ by

$$\mathcal{A}(x, y) = \begin{cases} 1 & \text{if } \pi(x) = y \text{ or } \pi(y) = x, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T^{(p)}, \mathcal{A})$ is a tree and $V(x) = \{\pi(x)\} \cup \pi^{-1}(x)$ for any $x \in T^{(p)}$. Note that $\#(V(x)) = p + 1$ for any $x \in T^{(p)}$. $(T^{(p)}, \mathcal{A})$ is called the homogeneous tree with degree $p + 1$. (In other terminology, $(T^{(p)}, \mathcal{A})$ is also called Bethe lattice with coordination number $p + 1$.) Now let $\phi_m = (\alpha_i)_{i \leq m} \in T_m^{(p)}$, where $\alpha_i = 0$ for all $i \leq m$ and let $\phi_* = (\phi_0, \phi_{-1}, \phi_{-2}, \ldots)$. Consider the ordered tree $(T, \mathcal{A}, \phi_*)$. We fix $\phi = \phi_0$ as the reference point. Then $\phi_n^- = \phi_{-n}$ and the collection of equivalence classes of ascending rays, Σ^+, is represented as

$$\big\{(\alpha_i)_{i \in \mathbb{Z}} | \alpha_i \in \{0, 1, \ldots, p-1\}, \text{there exists } N \in \mathbb{Z} \text{ such that } \alpha_i = 0 \text{ for any } i < N\big\}.$$

If p is a prime number, then Σ^+ is naturally identified with the p-adic numbers $\mathbb{Q}_p$ which is defined as

$$\mathbb{Q}_p = \left\{ \sum_{i \geq N} \alpha_i p^i \,\middle|\, N \in \mathbb{Z}, \alpha_i \in \{0, 1, \ldots, p-1\} \right\}.$$

Let $x = (\alpha_i)_{i \leq m} \in T^{(p)}$. Then $\pi_{\phi_*}(x) = \pi(x)$ and $S^+(x) = \pi^{-1}(x)$, $|x| = m$ and $x_n^- = (\alpha_i)_{i \leq m-n}$ for any $n \geq 0$. Moreover, if $\omega = (\alpha_i)_{i \in \mathbb{Z}} \in \Sigma^+$, then $[\omega]_m = (\alpha_i)_{i \leq m}$. Let $n_p(\cdot)$ be the p-adic norm defined by $n_p((\alpha)_{i \in \mathbb{Z}}) = p^{-I}$, where $I = \min\{i | i \in \mathbb{Z}, \alpha_i \neq 0\}$. Then $n_p(\omega - \tau) = p^{-|\omega \wedge \tau| - 1}$ for any $\omega \neq \tau \in \Sigma^+$. In particular $n_p(\omega) = p^{-|\omega \wedge 0| - 1}$, where $0 = (\ldots, 0, 0, 0, \ldots)$. The topology of $\mathbb{Q}_p$ induced by n_p coincides with the relative topology $\mathcal{O}_{\Sigma^+} = \{O \cap \Sigma^+ | O \in \mathcal{O}\}$.

3. Dirichlet forms on noncompact Cantor set

In this section, we are going to construct a family of Dirichlet forms on Σ^+ from a map $\lambda : T \to [0, \infty)$ and a Radon measure μ on Σ^+. This class of Dirichlet forms includes those on p-adic numbers (or, more generally, leaves of multibranching trees) studied in the series of papers, [1] and [2] for example, by Albeverio and Karwowski. See Definition 3.12 and Proposition 3.13.

Throughout this section, we fix a locally finite non-directed tree $(T, \mathcal{A})$, $\phi_* \in \Sigma$ and $\phi \in T$ which satisfies Assumption 2.2. Let $\mathcal{T} = (T, \mathcal{A}, \phi_*)$. We use π to denote π_{ϕ_*}.

Notation.

(1) *Let $\mathcal{M}(\Sigma^+)$ be the collection of Radon measures on Σ^+ which satisfies $\mu(\Sigma_x^+) > 0$ for any $x \in T$ and let $\ell^+(T) = \{\lambda | \lambda : T \to [0, \infty)\}$.*

(2) *Let μ be a Borel regular measure on Σ^+. We use $\mu(x)$ to denote $\mu(\Sigma_x^+)$ for $x \in T$.*

First we define a symmetric quadratic form on $L^2(\Sigma^+, \mu)$ from $(\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$.

Definition 3.1. *For $\Gamma = (\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$, define*

$$\mathcal{D}_\Gamma = \left\{ u \middle| u \in L^2(\Sigma^+, \mu), \sum_{x \in T} \frac{\lambda(x)}{2\mu(x)} \sum_{y,z \in S^+(x)} \mu(y)\mu(z)\big((u)_{y,\mu} - (u)_{z,\mu}\big)^2 < +\infty \right\}$$

and

$$\mathcal{Q}_\Gamma(u, v) = \sum_{x \in T} \frac{\lambda(x) - \lambda_I}{2\mu(x)} \sum_{y,z \in S^+(x)} \mu(y)\mu(z)\big((u)_{y,\mu} - (u)_{z,\mu}\big)\big((v)_{y,\mu} - (v)_{z,\mu}\big) + \lambda_I (u, v)_\mu$$

for any $u, v \in \mathcal{D}_\Gamma$, where $(u)_{x,\mu} = \mu(x)^{-1} \int_{\Sigma_x^+} u \, d\mu$ for any $x \in T$, $\lambda_I = \inf_{x \in T} \lambda(x)$ and $(u, v)_\mu$ is the inner product of $L^2(\Sigma^+, \mu)$.

The symmetric quadratic form $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ is shown to be a closed form in Theorem 3.4. In fact, we may describe associated eigenfunctions as follows.

Definition 3.2. *Let $\mu \in \mathcal{M}(\Sigma^+)$. Set $N(x) = \#(S^+(x)) = \#(V(x)) - 1$ for any $x \in T$. Define*

$$E_{x,\mu} = \left\{ f \middle| f = \sum_{y \in S^+(x)} a_y \chi_{\Sigma_y^+} \text{ where } a_y \in \mathbb{R} \text{ for any } y, \int_{\Sigma_x^+} f \, d\mu = 0 \right\},$$

where χ_A is the characteristic function of a set A. Let $(\varphi^\mu_{x,1}, \ldots, \varphi^\mu_{x,N(x)-1})$ be a complete orthonormal basis of $E_{x,\mu}$ with respect to $(\cdot,\cdot)_\mu$. Moreover, we use P^μ_x to denote the orthogonal projection from $L^2(\Sigma^+, \mu)$ to $E_{x,\mu}$.

Since $(T, \mathcal{A})$ is locally finite, $N(x)$ is finite. Moreover, $N(x) \geq 2$ for any $x \in T$ by Assumption 2.2. One can easily prove the following proposition.

Proposition 3.3. *Let $\mu \in \mathcal{M}(\Sigma^+)$. If $\mu(\Sigma^+) = +\infty$, then $(\varphi^\mu_{x,k} | x \in T, 1 \leq k \leq N(x) - 1)$ is a complete orthonormal system of $L^2(\Sigma^+, \mu)$. If $\mu(\Sigma^+) < +\infty$, then, $(\chi_{\Sigma^+}/\sqrt{\mu(\Sigma)}, \varphi^\mu_{x,k} | x \in T, 1 \leq k \leq N(x) - 1))$ is a complete orthonormal system of $L^2(\Sigma^+, \mu)$.*

The basis $(\varphi^\mu_{x,k} | x \in T, 1 \leq k \leq N(x) - 1)$ is a counterpart of the Haar's wavelet on $\mathbb{R}$. Now we show that $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ is closed.

Theorem 3.4. *Let $\Gamma = (\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$. Then $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ is a closed form on $L^2(\Sigma^+, \mu)$. Moreover,*

$$\mathcal{D}_\Gamma = \left\{ u \middle| u \in L^2(\Sigma^+, \mu), \sum_{x \in T} \lambda(x) \big(P^\mu_x u, P^\mu_x u\big)_\mu < +\infty \right\}$$

and

$$\mathcal{Q}_\Gamma(u, v) = \sum_{x \in T} \big(\lambda(x) - \lambda_I\big)\big(P^\mu_x u, P^\mu_x v\big)_\mu + \lambda_I (u, v)_\mu$$

for any $u, v \in \mathcal{D}$. In particular, if L_Γ is the non-negative self-adjoint operator associated with the closed form $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$, then $L_\Gamma u = \lambda(x) u$ for any $u \in E_{x,\mu}$ and any $x \in T$.

If no confusion may occur, we sometimes omit Γ in the notations and use $\mathcal{Q}$ and $\mathcal{D}$ instead of $\mathcal{Q}_\Gamma$ and $\mathcal{D}_\Gamma$ respectively.

Proof. For $x \in T$, define

$$\mathcal{Q}_x(u, v) = \frac{1}{2\mu(x)} \sum_{y,z \in S^+(x)} \mu(y)\mu(z)\big((u)_{y,\mu} - (u)_{z,\mu}\big)\big((v)_{y,\mu} - (v)_{z,\mu}\big)$$

for $u, v \in L^2(\Sigma^+, \mu)$. Then we see that $\mathcal{Q}_x(u, v) = (P_x^\mu u, P_x^\mu v)_\mu$. This fact immediately imply the desired statements. □

The next question is when $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ is a regular Dirichlet form. This problem is solved by finding a integral kernel of the closed form $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$. To obtain an integral kernel, we assume the following condition $(\lambda 1)$ on $\lambda \in \ell^+(T)$ in the rest of this section:

$(\lambda 1)$ $\sum_{m=0}^\infty |\lambda(\phi_m^-) - \lambda(\phi_{m+1}^-)| < +\infty$.

Remark. *Under the assumption* $(\lambda 1)$, $\lambda(\phi_m^-)$ *converges as* $m \to \infty$.

Proposition 3.5. *Let* $(\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$. *Under* $(\lambda 1)$,

$$\frac{1}{2} \sum_{m=0}^\infty \frac{\lambda(x_m^-) - \lambda(x_{m+1}^-)}{\mu(x_m^-)} \le \frac{1}{\mu(x)} \sum_{m=0}^\infty \big|\lambda\big(x_m^-\big) - \lambda\big(x_{m+1}^-\big)\big|. \tag{3.1}$$

More precisely, the infinite sums in the both sides of (3.1) *are absolutely convergent and the inequality* (3.1) *holds.*

Proof. Fix $x \in T$. Then

$$\left|\frac{\lambda(x_m^-) - \lambda(x_{m+1}^-)}{\mu(x_m^-)}\right| \le \frac{|\lambda(x_m^-) - \lambda(x_{m+1}^-)|}{\mu(x)}$$

for any $m \ge 0$. Since there exists $M \ge |x|$ such that $x_m^- = \phi_{|x|-m}^-$ for any $m \ge M$, $(\lambda 1)$ implies (3.1). □

The left-hand side of (3.1) is actually the integral kernel of $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$. We will show that $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ is a regular Dirichlet form if the integral kernel is non-negative in Theorem 3.7.

Definition 3.6.

(1) *Let* $\Gamma = (\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$ *which satisfies* $(\lambda 1)$. *We use* $J_\Gamma(x)$ *to denote the value of the infinite sum in the left-hand side of* (3.1). *Furthermore, abusing a notation, we define* $J_\Gamma(\omega, \tau) = J_\Gamma(\omega \wedge \tau)$ *for any* $\omega \ne \tau \in \Sigma^+$.
(2) *Define* $\Theta^+(\mathcal{T})$ *by*

$$\Theta^+(\mathcal{T}) = \big\{\Gamma \,|\, \Gamma = (\lambda, \mu) \in \ell^+(T) \times \mathcal{M}\big(\Sigma^+\big), \lambda \text{ satisfies } (\lambda 1) \text{ and } J_\Gamma(x) \ge 0 \text{ for any } x \in T\big\}.$$

By definition, $J_\Gamma(\omega, \tau) = J_\Gamma(\tau, \omega)$. As is the case of $\mathcal{D}$ and $\mathcal{Q}$, we use J in place of J_Γ if no confusion may occur.

Theorem 3.7. *Let* $\Gamma = (\lambda, \mu) \in \Theta^+(\mathcal{T})$. *Then* $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ *is a regular Dirichlet form on* $L^2(\Sigma^+, \mu)$,

$$\mathcal{D}_\Gamma = \left\{u \,\middle|\, u \in L^2\big(\Sigma^+, \mu\big), \int_{\Sigma^+ \times \Sigma^+} J_\Gamma(\omega, \tau)\big(u(\omega) - u(\tau)\big)^2 \mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau) < +\infty\right\}$$

and

$$\mathcal{Q}_\Gamma(u, v) = \mathcal{Q}_\Gamma^c(u, v) + \lambda_I(u, v)_\mu$$

for any $u, v \in \mathcal{D}_\Gamma$, where we define

$$\mathcal{Q}^c_\Gamma(u,v) = \int_{\Sigma^+\times\Sigma^+} J_\Gamma(\omega,\tau)\big(u(\omega)-u(\tau)\big)\big(v(\omega)-v(\tau)\big)\mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau).$$

Furthermore, $(\mathcal{Q}^c_\Gamma, \mathcal{D}_\Gamma)$ is a conservative regular Dirichlet form on $L^2(\Sigma^+,\mu)$.

Remark. *For any $(\lambda,\mu) \in \Theta^+(\mathcal{T})$, it follows from Lemma 3.16 that*

$$\lambda_I = \inf_{x\in T}\lambda(x) = \lim_{m\to\infty}\lambda\big(\phi^-_m\big). \tag{3.2}$$

We call $\mathcal{Q}^c_\Gamma$ the conservative part of $\mathcal{Q}_\Gamma$. The following proposition is immediate.

Proposition 3.8. *Let $\Gamma = (\lambda,\mu) \in \Theta^+(\mathcal{T})$. Define λ^c by $\lambda^c(x) = \lambda(x) - \lambda_I$ for any $x \in T$ and $\Gamma^c = (\lambda^c,\mu)$. Then $\Gamma^c \in \Theta^+(\mathcal{T})$, $\mathcal{D}_\Gamma = \mathcal{D}_{\Gamma^c}$, $J_\Gamma = J_{\Gamma^c}$ and $\mathcal{Q}^c_\Gamma = \mathcal{Q}_{\Gamma^c}$.*

We will give a proof of Theorem 3.7 at the end of this section. For the moment, we present two classes of (λ,μ) included in $\Theta^+(\mathcal{T})$. The first one is called the monotone class. The Dirichlet forms on Σ^+ induced by random walks on T are shown to belong to this class in Section 11.

Definition 3.9 (Monotone class). *$\lambda \in \ell^+(T)$ is said to be monotone if and only if $\lambda(\pi(x)) \le \lambda(x)$ for any $x \in T$, where $\pi = \pi_{\phi_*}$. Define $\ell^+_M(T) = \{\lambda|\lambda \in \ell^+(T), \lambda \text{ is monotone}\}$ and $\Theta^+_M(\mathcal{T}) = \ell^+_M(T) \times \mathcal{M}(\Sigma^+)$.*

Proposition 3.10. *$\Theta^+_M(\mathcal{T}) \subseteq \Theta^+(\mathcal{T})$.*

Proof. Let $(\lambda,\mu) \in \Theta^+_M(\mathcal{T})$. Then $\lambda(\phi^-_m) \ge \lambda(\phi^-_{m+1})$ for any $m \ge 0$. Hence $\{\lambda(\phi^-_m)\}_{m\ge 0}$ converges as $m \to \infty$. This shows

$$\sum_{m=0}^{\infty}\big|\lambda\big(\phi^-_m\big) - \lambda\big(\phi^-_{m+1}\big)\big| = \sum_{m=0}^{\infty}\big(\lambda\big(\phi^-_m\big) - \lambda\big(\phi^-_{m+1}\big)\big) = \lambda(\phi) - \lim_{m\to\infty}\lambda\big(\phi^-_m\big).$$

Therefore we have (λ1). Obviously $J(x) \ge 0$ for any $x \in T$. Thus $(\lambda,\mu) \in \Theta^+(\mathcal{T})$. □

The second class is the Albeverio–Karwowski class, AK class for short. Albeverio and Karwowski have constructed and studied the correspondent class of jump processes on p-adic numbers in [1] and on general noncompact Cantor set Σ^+ in [2]. We start from the construction of a Radon measure $\mu_\mathcal{T}$.

Proposition 3.11. *There exists a unique Radon measure $\mu_\mathcal{T}$ on Σ^+ which satisfies $\mu_\mathcal{T}(\Sigma^+_\phi) = 1$ and $\mu_\mathcal{T}(\Sigma^+_{\pi(x)}) = N(\pi(x))\mu_\mathcal{T}(\Sigma^+_x)$ for any $x \in T$.*

Note that $\mu_\mathcal{T}(\Sigma^+) = +\infty$ since $N(x) \ge 2$ for any $x \in T$ by Assumption 2.2.

Definition 3.12 (AK class). *For any $\eta : \mathbb{Z} \to [0,\infty)$, we define $\lambda_\eta : T \to \mathbb{R}$ by*

$$\lambda_\eta(x) = \frac{N(x)\eta(|x|) - \eta(|x|-1)}{N(x)-1}.$$

Moreover, define

$$\ell_{AK}(T) = \big\{\lambda_\eta|\eta : \mathbb{Z} \to [0,\infty), \eta(n) \le \eta(n+1) \text{ for any } n \in \mathbb{Z}\big\}$$

and $\Theta^+_{AK}(\mathcal{T}) = \ell_{AK}(T) \times \{\mu_\mathcal{T}\}$.

In [2], $\eta(\cdot)$ is denoted by $a(\cdot)$. More precisely, $a(n) = \eta(-n)$ for $n \in \mathbb{Z}$.
By the definition of λ_η, we see that $\ell_{AK}(T) \subseteq \ell^+(T)$.

Proposition 3.13. *$\Theta^+_{AK}(\mathcal{T}) \subseteq \Theta^+(\mathcal{T})$. Moreover, $\lambda_I = \lim_{m\to-\infty} \eta(m)$ for any $(\lambda_\eta, \mu_{\mathcal{T}}) \in \Theta^+_{AK}(\mathcal{T})$.*

Proof. In this proof, we write $\eta_m = \eta(m)$ for ease of notation. Let $\Gamma = (\lambda_\eta, \mu_{\mathcal{T}}) \in \Theta^+_{AK}(\mathcal{T})$. Set $\lambda = \lambda_\eta$. Then

$$\lambda\big(x^-_m\big) - \lambda\big(x^-_{m+1}\big) = \frac{N(x^-_m)}{N(x^-_m) - 1}(\eta_{|x|-m} - \eta_{|x|-m-1}) - \frac{1}{N(x^-_{m+1}) - 1}(\eta_{|x|-m-1} - \eta_{|x|-m-2})$$

for any $x \in T$ and any $m \geq 0$. Since $N(x) - 1 \geq 1$ and $N(x)/(N(x) - 1) \leq 2$ for any $x \in T$, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\big|\lambda\big(\phi^-_m\big) - \lambda\big(\phi^-_{m+1}\big)\big| &\leq \sum_{n\leq 0}\big(2(\eta_n - \eta_{n-1}) + (\eta_{n-1} - \eta_{n-2})\big)\\ &= 2\eta_0 + \eta_{-1} - 3\lim_{m\to-\infty}\eta_m.\end{aligned}$$

Hence $(\lambda 1)$ is satisfied. Now by a routine calculation, we obtain

$$J_\Gamma(x) = \frac{N(x)}{(N(x) - 1)\mu_{\mathcal{T}}(\Sigma^+_x)}(\eta_{|x|} - \eta_{|x|-1}) \geq 0. \tag{3.3}$$

Thus $(\lambda_\eta, \mu_{\mathcal{T}}) \in \Theta^+(\mathcal{T})$. The equality $\lambda_I = \lim_{m\to\infty}\lambda(\phi^-_m) = \lim_{m\to-\infty}\eta_m$ is immediate. □

There are examples of $(\lambda, \mu) \in \Theta^+_M(\mathcal{T})\backslash\Theta^+_{AK}(\mathcal{T})$ in Example 8.3 and $(\lambda, \mu) \in \Theta^+_{AK}(\mathcal{T})\backslash\Theta^+_M(\mathcal{T})$ in Example 8.4.

The rest of this section is devoted to proving Theorem 3.7. In the followings, $\Gamma = (\lambda, \mu) \in \Theta^+(T)$. We use the following notations. Define

$$\begin{aligned}&J_{u,v}(\omega, \tau) = J(\omega, \tau)\big(u(\omega) - u(\tau)\big)\big(v(\omega) - v(\tau)\big),\\ &\widetilde{\mathcal{D}} = \left\{u \,\middle|\, u \in L^2\big(\Sigma^+, \mu\big), \int_{\Sigma^+\times\Sigma^+} J_{u,u}(\omega, \tau)\mu(d\omega)\mu(d\tau) < +\infty\right\}\end{aligned}$$

and, for any $u, v \in \widetilde{\mathcal{D}}$,

$$\widetilde{\mathcal{Q}}(u, v) = \int_{\Sigma^+\times\Sigma^+} J_{u,v}(\omega, \tau)\mu(d\omega)\mu(d\tau).$$

Definition 3.14. *Define*

$$\mathcal{C} = \left\{\sum_{x\in Y}\alpha(x)\chi_{\Sigma^+_x} \,\middle|\, Y \text{ is a finite subset of } T \text{ and } \alpha : Y \to \mathbb{R}\right\}.$$

Lemma 3.15. *Define $\lambda_* = \lim_{m\to\infty}\lambda(\phi^-_m)$. If $x \in T$ and $\omega \in \Sigma^+_x$, then*

$$\begin{aligned}\int_{\Sigma^+\backslash\Sigma^+_x} J(\omega, \tau)\mu(d\tau) &= \sum_{m=1}^{\infty} J\big(x^-_m\big)\big(\mu\big(x^-_m\big) - \mu\big(x^-_{m-1}\big)\big)\\ &= \frac{1}{2}\big(\lambda\big(\pi(x)\big) - \lambda_*\big) - J\big(\pi(x)\big)\mu(x).\end{aligned}$$

Proof.

$$
\begin{aligned}
&\sum_{m=1}^{n} J\bigl(x_m^-\bigr)\bigl(\mu\bigl(x_m^-\bigr)-\mu\bigl(x_{m-1}^-\bigr)\bigr)\\
&\quad=-J\bigl(\pi(x)\bigr)\mu(x)+\sum_{m=1}^{n-1}\mu\bigl(x_m^-\bigr)\bigl(J\bigl(x_m^-\bigr)-J\bigl(x_{m+1}^-\bigr)\bigr)+J\bigl(x_n^-\bigr)\mu\bigl(x_n^-\bigr)\\
&\quad=-J\bigl(\pi(x)\bigr)\mu(x)+\frac{1}{2}\sum_{m=1}^{n-1}\bigl(\lambda\bigl(x_m^-\bigr)-\lambda\bigl(x_{m+1}^-\bigr)\bigr)+J\bigl(x_n^-\bigr)\mu\bigl(x_n^-\bigr)\\
&\quad=-J\bigl(\pi(x)\bigr)\mu(x)+\frac{1}{2}\bigl(\lambda\bigl(\pi(x)\bigr)-\lambda\bigl(x_n^-\bigr)\bigr)+J\bigl(x_n^-\bigr)\mu\bigl(x_n^-\bigr).
\end{aligned}
\tag{3.4}
$$

Since $x_n^-=\phi_{|x|-n}^-$ for sufficiently large n, we have $\lim_{n\to\infty}\lambda(x_n^-)=\lambda_*$. By (3.1),

$$
J\bigl(x_n^-\bigr)\mu\bigl(x_n^-\bigr)\le\sum_{m=n}^{\infty}\bigl|\lambda\bigl(x_m^-\bigr)-\lambda\bigl(x_{m+1}^-\bigr)\bigr|. \tag{3.5}
$$

This shows that $J(x_n^-)\mu(x_n^-)\to 0$ as $n\to\infty$. Hence (3.4) yields the desired equality. □

Lemma 3.16.

(1) *For any $\varphi\in\mathcal{C}$ and any $u\in L^1(\Sigma^+,\mu)\cup L^\infty(\Sigma^+,\mu)$, $J_{\varphi,u}$ is $\mu\times\mu$-integrable on $\Sigma^+\times\Sigma^+$. In particular, $\mathcal{C}\subseteq\widetilde{\mathcal{D}}$. Moreover, if $\varphi=\chi_{\Sigma_x^+}$, then*

$$
\begin{aligned}
&\int_{\Sigma^+\times\Sigma^+} J_{\varphi,u}(\omega,\tau)\mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau)\\
&\quad=\bigl(\lambda\bigl(\pi(x)\bigr)-\lambda_*-2J\bigl(\pi(x)\bigr)\mu(x)\bigr)\int_{\Sigma_x^+}u\,\mathrm{d}\mu-2\mu(x)\sum_{m=1}^{\infty}J\bigl(x_m^-\bigr)\int_{\Sigma^+_{x_m^-}\setminus\Sigma^+_{x_{m-1}^-}}u\,\mathrm{d}\mu.
\end{aligned}
\tag{3.6}
$$

(2) *$\widetilde{\mathcal{Q}}(\varphi,u)=(\lambda(x)-\lambda_*)(\varphi,u)_\mu$ for any $x\in T$, $\varphi\in E_{x,\mu}$ and $u\in\widetilde{\mathcal{D}}$. In particular, $\lambda_*=\lambda_I$.*

Proof.

(1) To prove the integrability, it is enough to choose $\varphi=\chi_{\Sigma_x^+}$ for $x\in T$. Unless $(\omega,\tau)\in(\Sigma_x^+\times(\Sigma^+\setminus\Sigma_x^+))\cup((\Sigma^+\setminus\Sigma_x^+)\times\Sigma_x^+)$, $J_{\varphi,u}(\omega,\tau)=0$. Let $\Sigma_m=\Sigma^+_{x_m^-}\setminus\Sigma^+_{x_{m-1}^-}$ for $m\ge 1$. Write $J_m=J(x_m^-)$ and $\lambda_m=\lambda(x_m^-)$. If $(\omega,\tau)\in\Sigma_x\times\Sigma_m$, then $J_{\varphi,u}(\omega,\tau)=J_m(u(\omega)-u(\tau))$. Hence

$$
\int_{\Sigma_x^+\times\Sigma_m}\bigl|J_{\varphi,u}(\omega,\tau)\bigr|\mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau)\le J_m\bigl(\mu\bigl(x_m^-\bigr)-\mu\bigl(x_{m-1}^-\bigr)\bigr)\int_{\Sigma_x^+}|u|\,\mathrm{d}\mu+\mu(x)J_m\int_{\Sigma_m}|u|\,\mathrm{d}\mu.
$$

Since $\Sigma^+\setminus\Sigma_x^+=\bigcup_{m=1}^\infty\Sigma_m$, we obtain

$$
\int_{\Sigma_x^+\times(\Sigma^+\setminus\Sigma_x^+)}\bigl|J_{\varphi,u}(\omega,\tau)\bigr|\mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau)\le\sum_{m=1}^{\infty}J_m\bigl(\mu\bigl(x_m^-\bigr)-\mu\bigl(x_{m-1}^-\bigr)\bigr)\int_{\Sigma_x^+}|u|\,\mathrm{d}\mu+\mu(x)\sum_{m=1}^{\infty}J_m\int_{\Sigma_m}|u|\,\mathrm{d}\mu.
$$

The first infinite sum of the right-hand side of above inequality is finite by Lemma 3.15. Note that $\mu(x)J_m \le \mu(x_m^-)J(x_m^-) \to 0$ as $m \to \infty$ by (3.5). Hence if $u \in L^1(\Sigma^+, \mu)$, then

$$\mu(x)\sum_{m=1}^{\infty} J_m \int_{\Sigma_m} |u|\,\mathrm{d}\mu \le \mu(x)\sup_{m\ge 1} J_m \int_{\Sigma^+\setminus\Sigma_x^+} |u|\,\mathrm{d}\mu < +\infty.$$

If $u \in L^\infty(\Sigma^+, \mu)$, then again using Lemma 3.15, we see

$$\mu(x)\sum_{m=1}^{\infty} J_m \int_{\Sigma_m} |u|\,\mathrm{d}\mu \le \mu(x)\|u\|_\infty \sum_{m=1}^{\infty} J_m\big(\mu(x_m^-) - \mu(x_{m-1}^-)\big) < +\infty.$$

Therefore, $J_{\varphi,u}$ is integrable. (3.6) follows by the similar arguments.

(2) Assume that $\varphi = \sum_{y\in S^+(x)} \alpha_y \chi_{\Sigma_y^+} \in E_{x,\mu}$. (3.6) along with the fact that $\int_{\Sigma^+} \varphi\,\mathrm{d}\mu = 0$ yields

$$\int_{\Sigma^+\times\Sigma^+} J_{\varphi,v}(\omega,\tau)\mu(\mathrm{d}\omega)\mu(\mathrm{d}\tau) = \big(\lambda(x) - \lambda_*\big)\int_{\Sigma^+} \varphi v\,\mathrm{d}\mu$$

for any $v \in L^\infty(\Sigma^+, \mu)$. Let $u \in \widetilde{\mathcal{D}}$ and define u_n for $n \ge 1$ by

$$u_n(\omega) = \begin{cases} n & \text{if } u(\omega) \ge n, \\ u(\omega) & \text{if } u(\omega) \in (-n, n), \\ -n & \text{if } u(\omega) \le -n. \end{cases}$$

Then $|u_n(\omega) - u_n(\tau)| \le |u(\omega) - u(\tau)|$ and $|(u(\omega) - u_n(\omega)) - (u(\tau) - u_n(\tau))| \le |u(\omega) - u(\tau)|$ for any $\omega, \tau \in \Sigma^+$. Hence Lebesgue's dominant convergence theorem implies that $u_n \in \widetilde{\mathcal{D}}$, $\widetilde{\mathcal{Q}}(u_n, u_n) \to \widetilde{\mathcal{Q}}(u, u)$ as $n \to \infty$ and $\widetilde{\mathcal{Q}}(u - u_n, u) \to 0$ as $n \to \infty$. Combining these, we have $\widetilde{\mathcal{Q}}(u - u_n, u - u_n) \to 0$ as $n \to \infty$. Now, as $u_n \in L^\infty(\Sigma^+, \mu)$, it follows that $\widetilde{\mathcal{Q}}(u_n, \varphi) = (\lambda(x) - \lambda_*)(u_n, \varphi)_\mu$. Letting $n \to \infty$, we see that $\widetilde{\mathcal{Q}}(u, \varphi) = (\lambda(x) - \lambda_*)(u, \varphi)_\mu$. Since $(\varphi, \varphi)_\mu > 0$ for some $\varphi \in E_{x,\mu}$, we have $\lambda(x) - \lambda_* \ge 0$ for any $x \in T$. Hence $\lambda_* = \inf_{x\in T}\lambda(x) = \lambda_I$. □

Proof of Theorem 3.7. First we show that $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{D}})$ is a regular Dirichlet form on $L^2(\Sigma^+, \mu)$. We will verify the conditions (j.1) and (j.2) in Example 1.2.4 of [14]. Define $\rho(\omega, \tau) = 2^{-|\omega\wedge\tau|}$ for $\omega \ne \tau \in \Sigma^+$. We let $\rho(\omega, \omega) = 0$. Then ρ is a metric on Σ^+. If $2^{-m} < r \le 2^{-m+1}$, then $B_\rho(\omega, r) = \Sigma^+_{[\omega]_m}$. Hence, if $j_m(\omega) = \int_{\Sigma^+\setminus\Sigma^+_{[\omega]_m}} J(\omega, \tau)\mu(\mathrm{d}\tau)$, then the condition (j.1) is equivalent to that $j_m(\omega)$ is locally integrable. By Lemma 3.15,

$$j_m(\omega) = \frac{1}{2}\big(\lambda\big([\omega]_{m-1}\big) - \lambda_I\big) - J\big([\omega]_{m-1}\big)\mu\big([\omega]_m\big).$$

This implies that $j_m = \sum_{x\in T_m}((\lambda(\pi(x)) - \lambda_I)/2 - J(\pi(x))\mu(x))\chi_{\Sigma_x}$. Hence j_m is locally integrable. The condition (j.2) is immediate from the fact that $J(\omega, \tau) = J(\tau, \omega)$. Moreover, since $\mathcal{C} \subseteq \widetilde{\mathcal{D}}$ by Lemma 3.16, $\widetilde{\mathcal{D}}$ is dense in $L^2(\Sigma^+, \mu)$ and the argument in Example 1.2.4 of [14] shows that $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{D}})$ is a Dirichlet form on $L^2(\Sigma^+, \mu)$. Moreover, Lemma 3.16 implies that $\varphi \in E_{x,\mu}$ is an eigenfunction with eigenvalue $\lambda(x) - \lambda_I$ of the non-negative self-adjoint operator on $L^2(\Sigma^+, \mu)$ associated with $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{D}})$. Hence by Theorem 3.4, we see that $(\mathcal{Q}, \mathcal{D}) = (\widetilde{\mathcal{Q}}, \widetilde{\mathcal{D}})$. Let $\mathcal{Q}^1(u, v) = \mathcal{Q}(u, v) + (u, v)_\mu$. Since $E_{x,\mu} \subseteq \mathcal{C}$, it follows that $\mathcal{C}$ is $\mathcal{Q}_1$-dense in $\mathcal{D}$. Moreover $\mathcal{C}$ is dense in $C_0(\Sigma^+)$ is the sense of supremum norm. Hence $\mathcal{C}$ is a core of the Dirichlet form $(\mathcal{Q}, \mathcal{D})$. Thus $(\mathcal{Q}, \mathcal{D})$ is a regular Dirichlet form on $L^2(\Sigma^+, \mu)$.

About $(\mathcal{Q}^c_\Gamma, \mathcal{D}_\Gamma)$, Proposition 3.8 shows that $(\mathcal{Q}^c_\Gamma, \mathcal{D}_\Gamma)$ is a regular Dirichlet form on $L^2(\Sigma^+, \mu)$. The rest is to verify that $(\mathcal{Q}^c_\Gamma, \mathcal{D}_\Gamma)$ is conservative. We may assume that $\lambda_I = 0$ and hence $\mathcal{Q}^c_\Gamma = \mathcal{Q}_\Gamma$ without loss of generality. Set $(\mathcal{Q}, \mathcal{D}) = (\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ as above. Let $v \in L^1(\Sigma^+, \mu) \cap \mathcal{D}_\Gamma$ and let $u_n = \chi_{\Sigma^+_{\phi_n^-}}$. Since $u_n \in \mathcal{C}$, Lemma 3.16 implies that $u_n \in \mathcal{D}$. By (3.6),

$$\mathcal{Q}(u_n, v) = \big(\lambda\big(\phi_{n+1}^-\big) - 2J\big(\phi_{n+1}^-\big)\mu\big(\phi_n^-\big)\big)\int_{\Sigma^+_{\phi_n^-}} v\,\mathrm{d}\mu - 2\mu\big(\phi_n^-\big)\sum_{m=n+1}^{\infty} J\big(\phi_m^-\big)\int_{\Sigma^+_{\phi_m^-}\setminus\Sigma^+_{\phi_{m-1}^-}} v\,\mathrm{d}\mu.$$

Using (3.5), (λ1) and the fact that $v \in L^1(\Sigma^+, \mu)$, we see that

$$\mu(\phi_n^-) \sum_{m=n+1}^{\infty} J(\phi_m^-) \int_{\Sigma^+_{\phi_m^-} \setminus \Sigma^+_{\phi_{m-1}^-}} |v| \,\mathrm{d}\mu \le \|v\|_1 \sup_{m \ge n+1} J(\phi_m^-)\mu(\phi_m^-)$$
$$\le \|v\|_1 \sum_{k=n}^{\infty} \left|\lambda(\phi_k^-) - \lambda(\phi_{k+1}^-)\right| \to 0 \tag{3.7}$$

as $n \to \infty$. Moreover, since $\lambda(\phi_{n+1}^-) - 2J(\phi_{n+1}^-)\mu(\phi_n^-) \ge 0$ by Lemma 3.15,

$$\left|\left(\lambda(\phi_{n+1}^-) - 2J(\phi_{n+1}^-)\mu(\phi_n^-)\right) \int_{\Sigma^+_{\phi_n^-}} v \,\mathrm{d}\mu\right| \le \lambda(\phi_{n+1}^-)\|v\|_1 \to 0 \tag{3.8}$$

as $n \to \infty$. Thus it follows that $\mathcal{Q}(u_n, v) \to 0$ as $n \to \infty$. Therefore, we have verified the condition (ii) of Theorem 1.6.6 of [14], and hence $(\mathcal{Q}, \mathcal{D})$ is conservative. □

4. Transition density

In this section, we establish the existence of the transition density $p_\Gamma(t, \omega, \tau)$ associated with the regular Dirichlet form $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$. See (4.8) for the exact definition of "transition density." As in the last section, $(T, \mathcal{A})$ is a locally finite tree satisfying Assumption 2.2, $\phi_* \in \Sigma$ and $\phi \in T$. We let $\mathcal{T} = (T, \mathcal{A}, \phi_*)$ and use π to denote π_{ϕ_*}.

Notation. *Let $(\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$. We use $\lambda_m(\omega) = \lambda([\omega]_m)$ and $\mu_m(\omega) = \mu(\Sigma^+_{[\omega]_m})$ for $\omega \in \Sigma^+$ and $m \in \mathbb{Z}$.*

By Theorem 3.4, we have all the eigenfunctions and eigenvalues of the self-adjoint operator L_Γ associated with the Dirichlet form $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$. Formally we may obtain a fundamental solution of the equation $\partial u/\partial t = -L_\Gamma u$ from those data. In this case, the formal expression of transition density turns out to be (4.1), which is shown to be convergent by the next proposition.

Proposition 4.1. *Let $\Gamma = (\lambda, \mu) \in \ell^+(T) \times \mathcal{M}(\Sigma^+)$. Assume that ($\lambda$1) holds. Define*

$$p_\Gamma(z, \omega, \tau) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\lambda_{n-1}(\omega)z} - \mathrm{e}^{-\lambda_n(\omega)z}}{\mu_n(\omega)} \chi_{\Sigma^+_{[\omega]_n}}(\tau) \tag{4.1}$$

for $\omega \ne \tau \in \Sigma^+$ and $z \in \mathbb{C}$. Then for any $z \in \mathbb{C}$ and any $\omega \ne \tau \in \Sigma^+$, the infinite sum in (4.1) is convergent, $p_\Gamma(z, \omega, \tau) = p_\Gamma(z, \tau, \omega)$, $p_\Gamma(z, \omega, \tau)$ is continuous on $\{(z, \omega, \tau) | z \in \mathbb{C}, \omega \ne \tau \in \Sigma^+\}$ and is an entire function of z for each $\omega \ne \tau$. Moreover, if $\Gamma \in \Theta^+(\mathcal{T})$, then $p_\Gamma(t, \omega, \tau) \ge 0$ for any $t \in [0, \infty)$ and any $\omega \ne \tau \in \Sigma^+$.

For AK class on p-adic numbers, the expansion (4.1) has been given in [3]. Note that $\mu_n(\omega) = p^{-n}$ and $\lambda_n(\omega)$ does not depend on ω in such a case.

If no confusion may occur, we write $p(t, \omega, \tau)$ in place of $p_\Gamma(t, \omega, \tau)$. To prove the above proposition, we need the next lemma.

Lemma 4.2. *Let $\{\alpha_n\}_{n \ge 1}$ and $\{\beta_n\}_{n \ge 1}$ be sequences of real numbers which satisfy that $0 < \alpha_n \le \alpha_{n+1}$ for any $n \ge 1$, that $\sup_{n \ge 1} |\beta_n| < +\infty$, and that $\sum_{n \ge 1} |\beta_n - \beta_{n+1}|/\alpha_n < +\infty$. Define $F(z) = \sum_{n \ge 1} (\mathrm{e}^{-\beta_{n+1} z} - \mathrm{e}^{-\beta_n z})/\alpha_n$ for any $z \in \mathbb{Z}$. Then $F(z)$ is an entire function. Moreover, let $\kappa = \sum_{n \ge 1} (\beta_n - \beta_{n+1})/\alpha_n$. If $\kappa \ge 0$, then, for any $t > 0$,*

$$\kappa t \mathrm{e}^{-\beta_1 t} \le F(t) \le \beta_1 t F(t) + t F'(t). \tag{4.2}$$

Proof. Let $\gamma_n = \beta_n - \beta_1$ and define $G(z) = \sum_{n\geq 1}(\mathrm{e}^{-\gamma_{n+1}z} - \mathrm{e}^{-\gamma_n z})/\alpha_n$. Set $M = \sup_{n\geq 1}|\gamma_n|$. If $\mathrm{Re}z \leq T$, then

$$\frac{|\mathrm{e}^{-\gamma_{n+1}z} - \mathrm{e}^{-\gamma_n z}|}{\alpha_n} \leq \frac{|\gamma_n - \gamma_{n+1}|}{\alpha_n}\mathrm{e}^{MT}.$$

Therefore, the infinite sum in the definition of $G(z)$ is uniformly convergent on any compact subset of $\mathbb{C}$. Therefore $F(z) = \mathrm{e}^{\beta_1 z}G(z)$ is an entire function. Assume $\kappa \geq 0$. It is easy to see that $G(0) = 0$ and $G'(0) = \kappa \geq 0$. Now let $G_m(z) = \sum_{n=1}^{m}(\mathrm{e}^{-\gamma_{n+1}z} - \mathrm{e}^{-\gamma_n z})/\alpha_n$. Then for any $t \in [0, \infty)$,

$$(G_m)''(t) = \sum_{n=2}^{m}\left(\frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n}\right)(\gamma_n)^2\mathrm{e}^{-\gamma_n t} + \frac{(\gamma_{m+1})^2}{\alpha_m}\mathrm{e}^{-\gamma_{m+1}t} \geq 0.$$

As $m \to \infty$, it follows that $G''(t) \geq 0$ for any $t \in [0, \infty)$. Hence $G(t) \geq 0$ and $G'(t) \geq 0$ for any $t \in [0, \infty)$. By the mean value theorem, we have $G'(0)t \leq G(t) \leq G'(t)t$. This suffices for (4.2). □

Proof of Proposition 4.1. Let $x \neq y \in T$ with $|x| = |y| = M$. It is enough to show the claims of the proposition in the case when $(\omega, \tau) \in \Sigma_x^+ \times \Sigma_y^+$. Then

$$p(z, \omega, \tau) = \sum_{n=-\infty}^{|\omega\wedge\tau|} \frac{\mathrm{e}^{-\lambda_{n-1}(\omega)z} - \mathrm{e}^{-\lambda_n(\omega)z}}{\mu_n(\omega)}. \tag{4.3}$$

Now applying Lemma 4.2, we have the desired statements. □

To verify that $p_\Gamma(t, \omega, \tau)$ is really the transition density, we need to assume that $\lambda_m(\omega) \to +\infty$ as $m \to \infty$ for any $\omega \in \Sigma^+$.

Definition 4.3.

(1) *Define*

$$\ell_\infty^+(T) = \left\{\lambda|\lambda : T \to [0, \infty),\ \lim_{m\to\infty}\lambda_m(\omega) = +\infty \textit{ for any } \omega \in \Sigma^+\right\}$$

and $\Theta_\infty^+(\mathcal{T}) = \{(\lambda, \mu)|(\lambda, \mu) \in \Theta^+(\mathcal{T}), \lambda \in \ell_\infty^+(T)\}$.

(2) *For* $\lambda \in \ell^+(T)$, *define*

$$\Phi_\lambda(t, \omega) = \sum_{m\in\mathbb{Z}}\left|\mathrm{e}^{-\lambda_{m-1}(\omega)t} - \mathrm{e}^{-\lambda_m(\omega)t}\right|.$$

Proposition 4.4. *Let* $\Gamma = (\lambda, \mu) \in \ell_\infty^+(T) \times \mathcal{M}(\Sigma^+)$. *Define*

$$(p_t u)(\omega) = \int_{\Sigma^+} p(t, \omega, \tau)u(\tau)\mu(\mathrm{d}\tau)$$

for a measurable function $u : \Sigma^+ \to \mathbb{R}$ *whenever the integral makes sense. Assume* (λ1) *and the following condition* (λ2)

(λ2) *For any* $t > 0$, $\Phi_\lambda(t, \omega)$ *is continuos on* Σ^+ *and* $\sup_{\omega\in\Sigma^+}\Phi_\lambda(t, \omega) < +\infty$.

Then

(1) *Define* $p^{t,\omega}(\tau) = p(t, \omega, \tau)$. *For any* $\omega \in \Sigma^+$ *and any* $t \in (0, \infty)$, $p^{t,\omega}$ *is* μ*-integrable on* Σ^+ *and*

$$(p_t 1)(\omega) = \mathrm{e}^{-\lambda_I t}. \tag{4.4}$$

(2) *For any $x \in T$, any $\omega \in \Sigma^+$ and any $t \in (0, \infty)$*

$$(p_t \chi_{\Sigma_x^+})(\omega) = \mu(x) p(t, \omega, \tau) + \mathrm{e}^{-\lambda(x)t} \chi_{\Sigma_x^+}(\omega), \tag{4.5}$$

where τ is chosen so that $\tau \in \Sigma_x^+$ and $|\omega \wedge \tau| \le |x|$.

(3) *For any $x \in T$ and for any $\varphi \in E_{x,\mu}$,*

$$(p_t \varphi)(\omega) = \mathrm{e}^{-\lambda(x)t} \varphi(\omega). \tag{4.6}$$

(4) *For any $\omega \neq \xi \in \Sigma^+$ and for any $s, t \in (0, \infty)$,*

$$\int_{\Sigma^+} p(t, \omega, \tau) p(s, \tau, \xi) \mu(\mathrm{d}\tau) = p(t + s, \omega, \xi). \tag{4.7}$$

For the monotone class, the condition (λ2) always holds if $\lambda \in \ell_\infty^+(T)$ by Proposition 4.6.

Theorem 4.5. *Let $\Gamma = (\lambda, \mu) \in \Theta_\infty^+(\mathcal{T})$. Assume that ($\lambda$2) holds. Then there exists a Hunt process $(\{X_t\}_{t>0}, \{P_\omega\}_{\omega \in \Sigma^+})$ on Σ^+ which is associated with the Dirichlet form $(\mathcal{Q}, \mathcal{D})$ on $L^2(\Sigma^+, \mu)$ and whose transition function is $p_\Gamma(t, \omega, \tau)$, i.e.*

$$E_\omega\big(f(X_t)\big) = (p_t f)(\omega) \tag{4.8}$$

for any $\omega \in \Sigma^+$ and any Borel measurable bounded function $f : \Sigma^+ \to \mathbb{R}$.

Remark. *Assume that $\Gamma = (\lambda, \mu) \in \Theta_\infty^+(\mathcal{T})$ and λ satisfies (λ2). Then it is easy to see that $\Gamma^c \in \Theta_\infty^+(\mathcal{T})$ and λ^c satisfies (λ2). Hence by the above theorem, there exists a Hunt process associated with the Dirichlet form $(\mathcal{Q}_{\Gamma^c}, \mathcal{D}_\Gamma)$ on $L^2(\Sigma^+, \mu)$ whose transition density is $p_{\Gamma^c}(t, \omega, \tau)$. Note that $p_\Gamma(t, \omega, \tau) = \mathrm{e}^{-\lambda_I t} p_{\Gamma^c}(t, \omega, \tau)$.*

We will give a proof of the above theorem at the end of this section.

For the monotone class, $\lambda \in \ell_\infty^+(T)$ is enough for the condition (λ2). More precisely we have the following proposition.

Proposition 4.6. *Define $\ell_{M,\infty}^+(T) = \ell_M^+(T) \cap \ell_\infty^+(T)$ and*

$$\Theta_{M,\infty}^+(\mathcal{T}) = \big\{(\lambda, \mu) | (\lambda, \mu) \in \Theta_M^+(\mathcal{T}), \lambda \in \ell_{M,\infty}^+(T)\big\}.$$

Then (λ2) holds for any $(\lambda, \mu) \in \Theta_{M,\infty}^+(\mathcal{T})$.

Proof. If $(\lambda, \mu) \in \Theta_{M,\infty}^+(\mathcal{T})$, then $\Phi(t, \omega) = \mathrm{e}^{-\lambda_I t}$ for any $t \ge 0$ and any $\omega \in \Sigma^+$. Thus we have (λ2). □

For the Albeverio–Karwowski class, we have the next results.

Proposition 4.7.

(1) *For $\lambda_\eta \in \ell_{AK}^+(T)$, $\lambda_\eta \in \ell_\infty^+(T)$ if and only if $\eta(m) \to +\infty$ as $m \to \infty$.*
(2) *Let $\lambda_\eta \in \ell_{AK}^+(T) \cap \ell_\infty^+(T)$. If $\sum_{m \ge 0} \mathrm{e}^{-\eta(m)t} < +\infty$ for any $t > 0$, then the condition (λ2) is satisfied.*

Proof. (1) is immediate by the definition. To verify (2), fix $t > 0$ and let

$$\Phi_-(\omega, t) = \sum_{m \le 0} \big|\mathrm{e}^{-\lambda_{m-1}(\omega)t} - \mathrm{e}^{-\lambda_m(\omega)t}\big|.$$

Then $\Phi_-(\omega, t) \le t \sum_{m \le 0} |\lambda_{m-1}(\omega) - \lambda_m(\omega)|$. By the similar discussion as the proof of Proposition 3.13, we see that $\Phi_-(\omega, t)$ is bounded and continuous on Σ^+. Since $\lambda_m(\omega) = (N([\omega]_m)\eta(m) - \eta(m-1))/(N([\omega]_m) - 1) \ge$

$\eta(m-1)$, it follows that $\mathrm{e}^{-\lambda_m(\omega)} \leq \mathrm{e}^{-\eta(m-1)t}$. Hence $|\mathrm{e}^{-\lambda_{m-1}(\omega)t} - \mathrm{e}^{-\lambda_m(\omega)t}| \leq \mathrm{e}^{-\eta(m-2)t} + \mathrm{e}^{-\eta(m-1)t}$. Therefore if $\sum_{m\geq 0} \mathrm{e}^{-\eta(m)t} < +\infty$ for any $t > 0$, the infinite sum

$$\Phi_+(\omega, t) = \sum_{m\geq 1} \left|\mathrm{e}^{-\lambda_{m-1}(\omega)t} - \mathrm{e}^{-\lambda_m(\omega)t}\right|,$$

is uniformly convergent on Σ^+. This immediately implies our claim. □

The rest of this section is devoted to proving Theorem 4.5. Note that $\{p_t\}_{t>0}$ is a Markovian transition function in the sense of [14] by Propositions 4.1 and 4.4. By Theorem A.2.2 of [14], the above theorem follows if $\{p_t\}_{t>0}$ is a Feller transition function. Namely it is enough to show that $p_t(C_\infty(\Sigma^+)) \subseteq C_\infty(\Sigma^+)$ and that $(p_t f)(\omega) \to f(\omega)$ as $t \downarrow 0$ for any $\omega \in \Sigma^+$ and any $f \in C_\infty(\Sigma^+)$, where $C_\infty(\Sigma^+)$ is the continuous function which vanishes at the infinity. Since $\Sigma^+ = \Sigma\backslash\{\phi_*\}$, $C_\infty(\Sigma^+)$ is identified with $\{u|u \in C(\Sigma), u(\phi_*) = 0\}$.

We assume that $\Gamma = (\lambda, \mu) \in \Theta^+_\infty(\mathcal{T})$ and λ satisfies (λ2) in the following lemmas. Moreover, we write

$$\Psi_m(t, \omega) = \sum_{n\geq m+1} \left|\mathrm{e}^{-\lambda_{n-1}(\omega)t} - \mathrm{e}^{-\lambda_n(\omega)t}\right|.$$

Lemma 4.8. *For any $x \in T$ and any $t > 0$,*

$$\lim_{m\to\infty} \sup_{\omega\in\Sigma^+_x} \Psi_m(t, \omega) = 0.$$

Proof. Fix $x \in T$ and $t > 0$. Then there exists $N \in \mathbb{Z}$ such that $[\omega]_m = \phi^-_{|m|}$ for any $m \leq N$ and any $\omega \in \Sigma^+_x$. Since $|\mathrm{e}^{-\lambda_{n-1}(\omega)t} - \mathrm{e}^{-\lambda_n(\omega)t}| \leq t|\lambda_{n-1}(\omega) - \lambda_n(\omega)|$, ($\lambda$1) shows that

$$\sum_{n\leq m} \left|\mathrm{e}^{-\lambda_{n-1}(\omega)t} - \mathrm{e}^{-\lambda_n(\omega)t}\right|$$

is continuous on Σ^+_x. By (λ2), $\Psi_m(t, \omega)$ is continuous on Σ^+_x. Moreover, $\Psi_m(t, \omega) \geq \Psi_{m+1}(t, \omega) \geq 0$ and $\Psi_m(t, \omega) \to 0$ as $m \to \infty$ for any $\omega \in \Sigma^+_x$. Hence by Dini's theorem, $\Psi_m(t, \omega)$ converges to 0 as $m \to \infty$ uniformly on Σ^+_x. □

Lemma 4.9. *$p_t f \in C(\Sigma^+)$ for any $f \in L^\infty(\Sigma^+, \mu)$.*

Proof. For $\omega \neq \tau$,

$$\left|(p_t f)(\omega) - (p_t f)(\tau)\right| \leq \int_{\Sigma^+} \left|p(t, \omega, \xi) - p(t, \tau, \xi)\right|\left|f\right|\mu(\mathrm{d}\xi) \leq \left(\Psi_{|\omega\wedge\tau|}(t, \omega) + \Psi_{|\omega\wedge\tau|}(t, \tau)\right)\|f\|_\infty. \tag{4.9}$$

Fix $\omega \in \Sigma^+$. If $\tau_n \to \omega$ as $n \to \infty$, then there exists $x \in T$ such that $\omega \in \Sigma^+_x$ and $\tau_n \in \Sigma^+_x$ for any $n \geq 1$. Since $|\omega \wedge \tau_n| \to +\infty$ as $n \to \infty$, Lemma 4.8 and (4.9) yield $\lim_{n\to\infty} |(p_t f)(\omega) - (p_t f)(\tau_n)| \to 0$ as $n \to \infty$. □

Lemma 4.10. *$p_t f \in C_\infty(\Sigma^+)$ for any $f \in C_\infty(\Sigma^+)$.*

Proof. Define $I(\omega) = \max\{m|m \leq 0, [\omega]_m = \phi^-_{|m|}\}$ for any $\omega \in \Sigma^+$. Then $f \in C_\infty(\Sigma^+)$ if and only if $f \in C(\Sigma^+)$ and, for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $I(\omega) \leq -N_\varepsilon$ implies $|f(\omega)| < \varepsilon$. Now if $I(\omega) \leq -N_\varepsilon$, then

$$\begin{aligned}
\left|(p_t f)(\omega)\right| &\leq \sum_{n=-\infty}^{\infty} \left|\mathrm{e}^{-\lambda_{n-1}(\omega)t} - \mathrm{e}^{-\lambda_n(\omega)t}\right|\left|(f)_{[\omega]_n,\mu}\right| \\
&= \sum_{n\leq I(\omega)} \left|\mathrm{e}^{-\lambda_{n-1}(\omega)t} - \mathrm{e}^{-\lambda_n(\omega)t}\right|\left|(f)_{[\omega]_n,\mu}\right| + \sum_{n>I(\omega)} \left|\mathrm{e}^{-\lambda_{n-1}(\omega)t} - \mathrm{e}^{-\lambda_n(\omega)t}\right|\left|(f)_{[\omega]_n,\mu}\right| \\
&\leq \sum_{n\leq I(\omega)} \left|\lambda\left(\phi^-_{|n-1|}\right) - \lambda\left(\phi^-_{|n|}\right)\right|\|f\|_\infty + \varepsilon\Phi_\lambda(t, \omega).
\end{aligned}$$

Using (λ1) and (λ2), we see that $|(p_t f)(\omega)| \le 2\varepsilon \sup_{\tau\in\Sigma^+} \Phi_\lambda(t,\tau)$ for sufficiently small $I(\omega)$. Hence $p_t f \in C_\infty(\Sigma^+)$. □

Lemma 4.11. *$(p_t f)(\omega) \to f(\omega)$ as $t \downarrow 0$ for any $f \in C_\infty(\Sigma^+)$.*

Proof. Note that $p(0,\omega,\tau) = 0$ for any $\omega \ne \tau \in \Sigma^+$. By (4.5), it follows that $(p_t \chi_{\Sigma_x^+})(\omega) \to \chi_{\Sigma_x^+}(\omega)$ as $t \downarrow 0$ for any $x \in T$. This implies that $(p_t u)(\omega) \to u(\omega)$ as $t \downarrow 0$ for any $u \in \mathcal{C}$, where $\mathcal{C}$ is defined in Definition 3.14. Let $f \in C_\infty(\Sigma^+)$. Then, for any $\varepsilon > 0$, there exists $u \in \mathcal{C}$ such that $\|f - u\|_\infty < \varepsilon$. Note that $\|p^{t,\omega}\|_1 = 1$ by (4.4) and Proposition 4.1. Hence we have $\|p_t f - p_t u\|_\infty \le \|f - u\|_\infty$. Therefore,

$$\left|(p_t f)(\omega) - f(\omega)\right| \le 2\|f-u\|_\infty + \left|(p_t u)(\omega) - u(\omega)\right|.$$

This yields $|(p_t f)(\omega) - f(\omega)| < 3\varepsilon$ for sufficiently small $t > 0$. □

Proof of Theorem 4.5. The above lemmas shows that $\{p_t u\}_{t>0}$ is a Feller transition function. Then by Theorem A.2.2 of [14] (see also Theorem I.9.4 of [8]), there exists a Hunt process $((X_t)_{t>0}, (P_\omega)_{\omega\in\Sigma^+})$ on Σ^+ such that (4.8) holds for any ω, any $t > 0$ and any Borel measurable bounded function $f : \Sigma^+ \to \mathbb{R}$. Using (4.6), we verify that $(\mathcal{Q}, \mathcal{D})$ is the regular Dirichlet form associated with the Hunt process $((X_t)_{t>0}, (P_\omega)_{\omega\in\Sigma^+})$. □

5. Intrinsic metric and volume doubling property

In this section, we are going to introduce a metric on Σ^+ which is suitable for describing asymptotic behaviors of the transition density $p_\Gamma(t,\omega,\tau)$. We continue to assume that $(T, \mathcal{A}, \phi_*)$ is a locally finite ordered tree and $\phi \in T$. Recall that $N(x) = \#(S^+(x)) \ge 2$ for any $x \in T$ by Assumption 2.2 and that $\pi = \pi_{\phi_*}$.

Definition 5.1. *Let $\lambda \in \ell^+(T)$. Define $D_\lambda(x) = \inf_{m\ge 0} \lambda(x_m^-)^{-1}$. For any $\omega, \tau \in \Sigma^+$, define*

$$d_\lambda(\omega,\tau) = \begin{cases} D_\lambda(\omega\wedge\tau) & \text{if } \omega \ne \tau, \\ 0 & \text{if } \omega = \tau. \end{cases}$$

Note that $D_\lambda(\pi(x)) \ge D_\lambda(x)$ for any $x \in T$ by definition.

Proposition 5.2. *Let $\lambda \in \ell^+(T)$. Then $d_\lambda(\omega,\tau)$ is an (ultra-)metric on Σ^+, i.e.*

$$d_\lambda(\omega,\tau) \le \max\left\{d_\lambda(\omega,\xi), d_\lambda(\xi,\tau)\right\}$$

for any $\omega, \tau, \xi \in \Sigma^+$. Moreover, if $\lambda \in \ell^+_\infty(T)$, then the topology induced by $d_\lambda(\cdot,\cdot)$ coincides with the original topology of Σ^+.

Recall that the fundamental system of neighborhoods of $\omega \in \Sigma^+$ is given by $\{\Sigma^+_{[\omega]_m}\}_{m\ge 0}$ in the original topology of $\Sigma^+ = \Sigma\backslash\{\phi_*\}$.

Proof. Set $D(x) = D_\lambda(x)$ for ease of notation. Let $\omega, \tau, \xi \in \Sigma^+$. If $|\omega\wedge\tau| \ge |\omega\wedge\xi|$, then the definition of D_λ implies that $d_\lambda(\omega,\tau) = D([\omega]_{|\omega\wedge\tau|}) \le D([\omega]_{|\omega\wedge\xi|}) = d_\lambda(\omega,\xi)$. Otherwise, we have $\omega\wedge\tau = \xi\wedge\tau$. Hence $d_\lambda(\omega,\tau) = d_\lambda(\xi,\tau)$. Thus, d_λ is an ultra-metric.

Note that $\tau \in \Sigma^+_{[\omega]_n}$ whenever $d_\lambda(\omega,\tau) < D([\omega]_n)$. Hence if $d_\lambda(\omega_m,\omega) \to 0$ as $m \to \infty$, then $\omega_m \to \omega$ as $m \to \infty$ in the original topology. Conversely assume that $\omega_m \to \omega$ as $m \to \infty$ in the original topology. For any n, we have $d_\lambda(\omega_m,\omega) \le D([\omega]_n)$ for sufficiently large m. Since $\lambda \in \ell^+_\infty(T)$, we have $D([\omega]_n) \to 0$ as $n \to \infty$. Therefore $d_\lambda(\omega_m,\omega) \to 0$ as $m \to \infty$. □

Next we introduce the volume doubling property of a measure. It is known that the volume doubling property is an indispensable part of the set of conditions which implies “good behaviors” of transition densities on the Riemannian manifolds, fractals and metric measure spaces. See [6,7,15,16,20] for example.

Definition 5.3. *Let (X,d) be a metric space. A Borel regular measure on ν on X is said to have the volume doubling property with respect to the metric d if and only if there exists $c>0$ such that $\nu(B_d(x,2r)) \le c\nu(B_d(x,r)) < +\infty$ for any $r>0$ and any $x \in X$.*

Now we have an equivalent condition for μ being volume doubling with respect to d_λ. Recall the notation that $\mu(x) = \mu(\Sigma_x^+)$.

Theorem 5.4. *Let $(\lambda,\mu) \in \ell_\infty^+(T) \times \mathcal{M}(\Sigma^+)$. Then μ has the volume doubling property with respect to d_λ if and only if the following two conditions are satisfied*:

(EL) *There exists $\gamma \in (0,1)$ such that $\gamma\mu(\pi(x)) \le \mu(x)$ for any $x \in T$.*
(λ3) *There exist $m \ge 1$ and $\alpha \in (0,1)$ such that $D_\lambda([\omega]_{n+m}) \le \alpha D_\lambda([\omega]_n)$ for any $n \in \mathbb{Z}$ and any $\omega \in \Sigma^+$.*

The condition (EL) implies several good property of μ as follows, which are used to show the above theorem.

Proposition 5.5. *Let $\mu \in \mathcal{M}(\Sigma^+)$. Assume* (EL). *Then $\sup_{x\in T} N(x) < +\infty$. Furthermore, $\mu(x) \le (1-\gamma)\mu(x_1^-)$ for any $x \in T$. In particular, $\mu(\Sigma^+) = +\infty$ and $\mu(\{\omega\}) = 0$ for any $\omega \in T$.*

Proof. $\gamma\mu(x) \le \mu(y)$ for any $x \in T$ and any $y \in S^+(x)$. Summing this for all $y \in S^+(x)$, we have $\gamma\#(S^+(x))\mu(x) \le \mu(x)$. Hence $N(x) = \#(S^+(x)) \le \gamma^{-1}$ for any $x \in T$. Let $y \in S^+(x)$. Since $N(x) \ge 3$, there exists $z \in S^+(x)$ such that $z \ne y$. $\mu(y) \le \mu(x) - \mu(z) \le (1-\gamma)\mu(x)$. The rest is immediate. □

Proof of Theorem 5.4. Set $D(x) = D_\lambda(x)$ for ease of notation. Write $B(\omega,r) = B_{d_\lambda}(\omega,r)$ for $\omega \in \Sigma^+$ and $r>0$. First assume that μ has the volume doubling property with respect to d_λ, i.e. $\beta\mu(B(\omega,r)) \ge \mu(B(\omega,2r))$ for any $\omega \in \Sigma^+$ and $r>0$, where $\beta>1$ is independent of ω and r. Let $R = D([\omega]_n)$ and let $k = \min\{i \mid D([\omega]_{n+i}) < D([\omega]_n)\}$. Then $B(\omega,R) = \Sigma^+_{[\omega]_{n+k}}$ and $B(\omega,R+\varepsilon) = \Sigma^+_{[\omega]_n}$ for sufficiently small $\varepsilon>0$. Hence

$$\mu\big([\omega]_n\big) = \mu\big(B(\omega,R+\varepsilon)\big) \le \beta\mu\left(B\left(\omega,\frac{R+\varepsilon}{2}\right)\right) \le \beta\mu\big(B(\omega,R)\big) \le \beta\mu\big([\omega]_{n+1}\big).$$

Hence (EL) follows with $\gamma = 1/\beta$. Using Proposition 5.5, we have

$$\begin{aligned}\beta\mu\left(B\left(\omega,\frac{2D([\omega]_n)}{3}\right)\right) &\ge \mu\left(B\left(\omega,\frac{4D([\omega]_n)}{3}\right)\right) \ge \mu\big([\omega]_n\big)\\ &\ge (1-\gamma)^{-m}\mu\big([\omega]_{n+m}\big) \ge (1-\gamma)^{-m}\mu\big(B\big(\omega,D\big([\omega]_{n+m}\big)\big)\big)\end{aligned}$$

for any $m \ge 0$. Choose m so that $\beta(1-\gamma)^m < 1$. Then the above inequality yields $2D([\omega]_n)/3 > D([\omega]_{n+m})$. Thus we have shown ($\lambda$3).

Conversely, assume (EL) and (λ3). For any $r>0$, we may choose n which satisfies $D([\omega]_n) < r \le D([\omega]_{n-1})$. By ($\lambda$3), $D([\omega]_{n+m}) < \alpha r$ and hence $B(\omega,\alpha r) \supseteq \Sigma^+_{[\omega]_{n+m}}$. Hence by (EL),

$$\mu\big(B(\omega,\alpha r)\big) \ge \mu\big([\omega]_{n+m}\big) \ge \gamma^m\mu\big([\omega]_n\big) = \gamma^m\mu\big(B(\omega,r)\big).$$

This implies the volume doubling property. □

Even if μ has the volume doubling property with respect to d_λ, (Σ^+,d_λ) may not be uniformly perfect. See Example 8.5.

Definition 5.6. *Let (X,d) be a metric space. Define an annulus $A_d(x,r_1,r_2)$ for $x \in X$, $r_1,r_2>0$ by $A_d(x,r_1,r_2) = B_d(x,r_2)\backslash B_d(x,r_1)$. (X,d) is called uniformly perfect if and only if there exists $\alpha \in (0,1)$ such that $A(x,\alpha r,r) \ne \emptyset$ whenever $X \ne B_d(x,r)$.*

In the rest of this section, we consider equivalent conditions for the uniformly perfectness of (Σ^+, d_λ) in terms of $\Gamma = (\lambda, \mu)$.

Lemma 5.7. *Assume that $\lambda \in \ell^+_\infty(T)$. Let $\overline{D}_\lambda = \lim_{m\to\infty} D_\lambda(\phi^-_m)$. Define*

$$F_\lambda(s, \omega) = \min\bigl\{D_\lambda\bigl([\omega]_m\bigr) | m \in \mathbb{Z}, s \le D_\lambda\bigl([\omega]_m\bigr)\bigr\}$$

for any $s \in (0, \overline{D}_\lambda)$ and any $\omega \in \Sigma^+$. Then $s \le F_\lambda(s, \omega)$ for any $s \in (0, \overline{D}_\lambda)$. Moreover, $F_\lambda(D_\lambda([\omega]_m), \omega) = D_\lambda([\omega]_m)$ for any $m \in \mathbb{Z}$ and any $\omega \in \Sigma^+$.

Theorem 5.8. *Let $\lambda \in \ell^+_\infty(T)$. Then the following three conditions* (UP1), (UP2) *and* (UP3) *are equivalent*:

(UP1) *(Σ^+, d_λ) is uniformly perfect.*
(UP2) *There exists $a_1 > 1$ such that $F_\lambda(s, \omega) \le a_1 s$ for any $s \in (0, \overline{D}_\lambda)$ and any $\omega \in \Sigma^+$.*
(UP3) *There exists $a_2 > 1$ such that $a_2 D_\lambda(x) \ge D_\lambda(\pi(x))$ for any $x \in T$.*

We use $A(\omega, r_1, r_2)$ to denote $A_{d_\lambda}(\omega, r_1, r_2)$ in the following proof. Also we set $D(x) = D_\lambda(x)$ for ease of notation.

Proof.
(UP3) $\Rightarrow$ (UP2): By (UP3), it follows that $a_2 s \le D([\omega]_m) \le s$ for some $m \in \mathbb{Z}$ if $s \in (0, \overline{D}_\lambda)$. Therefore $F_\lambda(s, \omega) \le a_2 s$ for any $s \in (0, \overline{D}_\lambda)$.
(UP2) $\Rightarrow$ (UP1): If $B(\omega, r) \ne X$, then $r \le \overline{D}_\lambda$. By (UP2), there exists $D([\omega]_m)$ such that $r/(2a_1) \le D([\omega]_m) \le r/2$. Therefore, $A(\omega, r/(2a_1), r) \ne \emptyset$.
(UP1) $\Rightarrow$ (UP3): Choose ω and m so that $x = [\omega]_{m+1}$ and $\pi(x) = [\omega]_m$. Since $A(\omega, \alpha D([\omega]_m), D([\omega]_m)) \ne \emptyset$, we have $\alpha D([\omega]_m) \le D([\omega]_{m+1})$. Hence $a_2 D(x) \ge D(\pi(x))$, where $a_2 = 1/\alpha$. $\square$

6. Asymptotic behaviors of transition density

In this section, we study asymptotic behaviors of $p_\Gamma(t, \omega, \tau)$. Let $(T, \mathcal{A}, \phi_*)$ be a locally finite ordered tree satisfying Assumption 2.2 and let $\phi \in T$. We keep using $\pi = \pi_{\phi_*}$.

First we present estimates of $p_\Gamma(t, \omega, \tau)$ and $J(\omega, \tau)$ which hold whenever $p_\Gamma(t, \omega, \tau)$ makes sense.

Proposition 6.1. *Let $\Gamma \in (\lambda, \mu) \in \Theta^+_\infty(\mathcal{T})$ and assume that* (λ2) *holds. Then, for $\omega \ne \tau \in \Sigma^+$ and any $t > 0$,*

$$J(\omega, \tau) t \mathrm{e}^{-\lambda(\omega\wedge\tau)t} \le p_\Gamma(t, \omega, \tau) \le \frac{\lambda(\omega \wedge \tau)t}{\mu(\omega \wedge \tau)} \tag{6.1}$$

and

$$J(\omega, \tau) \le \frac{\lambda(\omega \wedge \tau)}{\mu(\omega \wedge \tau)}. \tag{6.2}$$

Proof. Let

$$F(z) = \sum_{n\ge 0} \frac{\mathrm{e}^{-\lambda_{M-n-1}(\omega)z} - \mathrm{e}^{-\lambda_{M-n}(\omega)z}}{\mu_{M-n}(\omega)},$$

where $M = |\omega \wedge \tau|$. Note that $p_\Gamma(t, \omega, \tau) = F(t)$ and $F'(0) = J(\omega, \tau)$. Applying (4.2), we have

$$J(\omega, \tau) t \mathrm{e}^{-\lambda_M(\omega)t} \le F(t), \tag{6.3}$$

which is the lower part of (6.1). For the upper estimate, we have

$$\frac{\partial}{\partial t}p_\Gamma(t,\omega,\tau)=F'(t)=\frac{\lambda_M(\omega)e^{-\lambda_M(\omega)t}}{\mu_M(\omega)}-\sum_{m\leq M-1}\left(\frac{1}{\mu_{m+1}(\omega)}-\frac{1}{\mu_m(\omega)}\right)e^{-\lambda_m(\omega)t}-\frac{\lambda_I e^{-\lambda_I t}}{\mu(\Sigma^+)}. \tag{6.4}$$

This shows $F'(t)\leq\lambda_M(\omega)/\mu_M(\omega)$ and hence we obtain (6.1). Thus it follows that $J(\omega,\tau)\leq\lambda_M(\omega)e^{\lambda_M(\omega)t}/\mu_M(\omega)$. Letting $t\downarrow 0$, we have (6.2). □

With further assumptions, we have more detailed asymptotic estimate of $p_\Gamma(t,\omega,\tau)$. In the case of the monotone class, the assumptions of the following theorem essentially mean that μ has the volume doubling property with respect to d_λ. See Remark(2) after the theorem. For the rest of this paper, we use $B(\omega,r)$ to denote $B_{d_\lambda}(\omega,r)$ for $\Gamma=(\lambda,\mu)\in\Theta^+_\infty(\mathcal{T})$ if no coufusion may occur.

Theorem 6.2. *Let $\Gamma=(\lambda,\mu)\in\Theta^+(\mathcal{T})$. Assume that μ satisfies* (EL) *and that λ satisfies* (λ2) *and the following condition* (λ4):

(λ4) *There exist $\alpha\in(0,1)$ and $m\geq 1$ such that $\lambda([\omega]_{n-m})\leq\alpha\lambda([\omega]_n)$ for any $\omega\in\Sigma^+$ and any $n\in\mathbb{Z}$.*

(1) *$\Gamma\in\Theta^+_\infty(\mathcal{T})$, μ has the volume doubling property with respect to d_λ and $p_\Gamma(t,\omega,\tau)$ is continuous on $(0,\infty)\times\Sigma^+\times\Sigma^+$.*

(2) Upper estimate: *There exists $c_1>0$ such that*

$$p_\Gamma(t,\omega,\tau)\leq\min\left\{\frac{t}{\mu(\omega\wedge\tau)d_\lambda(\omega,\tau)},\frac{c_1}{\mu(B(\omega,t))}\right\} \tag{6.5}$$

for any $t>0$ and any $\omega,\tau\in\Sigma^+$.

(3) Near diagonal lower estimate: *There exist $\varepsilon>0$ and $c_2>0$ such that*

$$\frac{c_2}{\mu((B(\omega,t))}\leq p_\Gamma(t,\omega,\tau) \tag{6.6}$$

whenever $d_\lambda(\omega,\tau)\leq\varepsilon t$.

(4) Off diagonal lower estimate: *Define*

$$U_\lambda=\big\{(\omega,\tau)|\omega,\tau\in\Sigma^+,\omega\neq\tau,\lambda(\omega\wedge\tau)\geq\lambda\big([\omega]_m\big)\text{ for any }m\leq|\omega\wedge\tau|\big\}.$$

Then

$$\mu\big(U_{\lambda,\omega}\cap A(\omega,r_1,r_2)\big)\geq\gamma\mu\big(A(\omega,r_1,r_2)\big) \tag{6.7}$$

for any $r_1,r_2>0$ and any $\omega\in\Sigma^+$, where $\gamma\in(0,1)$ is the constant in the definition of (EL) *and $U_{\lambda,\omega}=\{\tau|(\omega,\tau)\in U_\lambda\}$ is the ω-section of U_λ, and there exists $c_3>0$ such that*

$$c_3\frac{1}{\mu(\omega\wedge\tau)d_\lambda(\omega,\tau)}\leq J(\omega,\tau) \tag{6.8}$$

for any $(\omega,\tau)\in U_\lambda$. Furthermore, for any $\varepsilon>0$, there exists $c_4>0$ such that

$$c_4\frac{t}{\mu(\omega\wedge\tau)d_\lambda(\omega,\tau)}\leq p_\Gamma(t,\omega,\tau) \tag{6.9}$$

if $(\omega,\tau)\in U_\lambda$ and $d_\lambda(\omega,\tau)\geq\varepsilon t$.

(5) Moments of displacement: *For any* $\theta \geq 1$, $E_\omega(d_\lambda(\omega, X_t)^\theta) = +\infty$ *for any* $\omega \in \Sigma^+$ *and any* $t > 0$. *For any* $\theta \in (0, 1)$, *there exist* $c_5, c_6 > 0$ *such that*

$$c_5 t F_\lambda(\varepsilon t, \omega)^{\theta-1} \leq E_\omega\big(d_\lambda(\omega, X_t)^\theta\big) \leq c_6 F_\lambda(\varepsilon t, \omega)^\theta$$

for any $t > 0$ *and any* $\omega \in \Sigma^+$, *where* $\varepsilon > 0$ *is the constant given in the above statement* (3). *In particular, if* (Σ^+, d_λ) *is uniformly perfect, then, for any* $\theta \in (0, 1)$,

$$c_7 t^\theta \leq E_\omega\big(d_\lambda(\omega, X_t)^\theta\big) \leq c_8 t^\theta$$

for any $t > 0$ *and any* $\omega \in \Sigma^+$, *where* c_7 *and* c_8 *is a positive real number which are independent of* t *and* ω.

The asymptotic behaviors (6.5), (6.6) and (6.9) of the transition density are similar to those obtained for certain class of jump processes on a metric measure space with the volume doubling property. See [11] for details. Also, in [18], we have the same estimate for the transition density of the Dirichlet form on the Cantor set derived from a random walk on the associated tree.

We will prove this theorem in the next section.

Remark 1. *If* $(\lambda 1)$ *and* $(\lambda 4)$ *hold, then* $\lambda_I = 0$ *and* $\lambda_m(\omega) \to +\infty$ *as* $m \to +\infty$ *for any* $\omega \in \Sigma^+$.

Remark 2. *If* $\lambda \in \ell_M^+(T)$, *then* $D_\lambda(x) = \lambda(x)^{-1}$. *Hence* $(\lambda 3)$ *holds if and only if* $(\lambda 4)$ *holds. In this case, the assumption of the above theorem, which is that* μ *satisfies* (EL) *and that* λ *satisfies* $(\lambda 2)$ *and* $(\lambda 4)$, *is equivalent to the fact that* μ *has the volume doubling property with respect to* d_λ *by Theorem* 5.4.

Remark 3. (6.7) *shows that* U_λ *is a relatively large subset of* $\Sigma^+ \times \Sigma^+$. *In particular, every annulus* $A(\omega, r_1, r_2)$ *contains* $\tau \in U_{\lambda,\omega}$ *unless it is empty. Moreover,* (6.7) *yields*

$$\mu\big(U_{\lambda,\omega} \cap B(\omega, r)\big) \geq \gamma\mu\big(B(\omega, r)\big)$$

for any $\omega \in \Sigma^+$ *and any* $r > 0$.

Remark 4. *If* $\lambda \in \ell_M^+(T)$, *then* $U_\lambda = \{(\omega, \tau) | \omega \neq \tau \in \Sigma^+\}$. *Hence the lower estimates* (6.8) *and* (6.9) *of the jump kernel* J *and the transition density* $p_\Gamma(t, \omega, \tau)$ *hold for all* $\omega \neq \tau \in \Sigma^+$. *On the contrary, if* λ *is not monotone, then* (6.8) *and* (6.9) *may fail in general on the complement of* U_λ. *In Example* 8.4, *we will present an example of* $\Gamma = (\lambda, \mu) \in \Theta_{AK}^+(\mathcal{T}) \cap \Theta_\infty^+(\mathcal{T})$ *where* $J(\omega, \tau) = \frac{\partial}{\partial t} p_\Gamma(0, \omega, \tau) = 0$ *for infinitely many pairs* $(\omega, \tau) \in (\Sigma^+)^2$. *Note that if there exists* $\delta > 0$ *such that* (6.9) *holds on* $t \in (0, \delta)$ *for given pair* (ω, τ) *with* $\omega \neq \tau$, *then* $\frac{\partial}{\partial t} p_\Gamma(0, \omega, \tau) > 0$.

Remark 5. *Assume that* $\Gamma = (\lambda, \mu) \in \Theta_\infty^+(\mathcal{T})$ *and that* $(\lambda 2)$ *and* (EL) *are satisfied. If* $\lambda_I > 0$, *then* $(\lambda 4)$ *does not hold as we mentioned in the above remark. Even in such a case, however, we may have an asymptotic estimate of* p_{Γ^c} *in place of* p_Γ *if the conservative part* Γ^c *satisfies* $(\lambda 4)$. *Recall that* $p_\Gamma(t, \omega, \tau) = \mathrm{e}^{-\lambda_I} p_{\Gamma^c}(t, \omega, \tau)$ *as in the remark after Theorem* 4.5.

If λ belongs to the monotone class or the AK class with additional regularity, then $J(\omega, \tau)$ and $p_\Gamma(t, \omega, \tau)$ have simpler asymptotic behaviors.

Corollary 6.3. *Let* $\Gamma = (\lambda, \mu) \in \Theta^+(\mathcal{T})$. *Assume that* μ *satisfies* (EL) *and that* λ *satisfies* $(\lambda 2)$ *and* $(\lambda 4)$. *Furthermore, assume either of the following two conditions* (M) *or* (AK+) *is satisfied*:

(M) λ *belongs to the monotone class, i.e.* $\lambda \in \ell_M^+(T)$,

(AK+) λ *belongs to the AK class, i.e.* $\lambda \in \ell_{AK}^+(T)$ *and there exists* $\xi \in (0, 1)$ *such that* $\eta(m+1) \leq \xi\eta(m)$ *for any* $m \in \mathbb{Z}$.

Then

$$J(\omega,\tau) \asymp \frac{1}{\mu(\omega\wedge\tau)d_\lambda(\omega,\tau)} \tag{6.10}$$

on $(\Sigma^+\times\Sigma^+)\backslash\{(\omega,\omega)|\omega\in\Sigma^+\}$ *and*

$$p_\Gamma(t,\omega,\tau) \asymp q_\Gamma(t,\omega,\tau) \tag{6.11}$$

on $(0,\infty)\times\Sigma^+\times\Sigma^+$, *where*

$$q_\Gamma(t,\omega,\tau) = \begin{cases} \frac{t}{\mu(\omega\wedge\tau)d_\lambda(\omega,\tau)} & \text{if } t < d_\lambda(\omega,\tau), \\ \frac{1}{\mu(B(\omega,t))} & \text{if } t \geq d_\lambda(\omega,\tau). \end{cases}$$

One can find a proof of this corollary in the next section.

7. Proof of the results in Section 6

This section is devoted to proving the results in Section 6. As in the previous sections, $(T,\mathcal{A},\phi_*)$ is a locally finite ordered tree and $\phi\in T$. Throughout this section, we assume that $\Gamma=(\lambda,\mu)\in\Theta^+(\mathcal{T})$, that μ satisfies (EL) and that λ satisfies (λ2) and (λ4). By (EL), $\gamma\mu_m(\omega)\leq\mu_{m+1}(\omega)$ for any $\omega\in\Sigma^+$ and any $m\in\mathbb{Z}$, where $\gamma\in(0,1)$ is independent of ω and m. Hereafter in this section, we omit ω from the notation and write μ_m and λ_m in place of $\mu_m(\omega)$ and $\lambda_m(\omega)$ respectively if no confusion may occur. Also we write D_m in place of $D_\lambda([\omega]_m)$ for ease of the notation. By Proposition 5.5, we have

$$\gamma^n\mu_m \leq \mu_{m+n} \leq (1-\gamma)^n\mu_m \tag{7.1}$$

for any $m\in\mathbb{Z}$ and any $n\geq 0$. Other useful estimates are

$$\begin{aligned} \frac{\gamma}{1-\gamma} &\leq \frac{1}{\mu_{m+1}} - \frac{1}{\mu_m} \leq \frac{1-\gamma}{\gamma}\frac{1}{\mu_m}, \\ \frac{\gamma}{\mu_{m+1}} &\leq \frac{1}{m+1} - \frac{1}{\mu_m} \leq \frac{1-\gamma}{\mu_{m+1}}. \end{aligned} \tag{7.2}$$

Lemma 7.1. *There exist* $\delta\geq 1$ *and* $\beta\in(0,1)$ *such that*

$$\lambda_{n-k} \leq \delta\beta^{k-1}\lambda_n \tag{7.3}$$

for any $\omega\in\Sigma^+$, *any* $n\in\mathbb{Z}$ *and any* $k\in\mathbb{N}$. *Moreover,* (λ3) *holds and* μ *has the volume doubling property with respect to* d_λ.

Proof. Obviously, $\lambda\in\ell^+_\infty(T)$ under (λ4). Since $(\lambda,\mu)\in\Theta^+(\mathcal{T})$, (6.4) implies that

$$\sum_{m\leq M}\frac{\lambda_m-\lambda_{m-1}}{\mu_m} = \frac{\lambda_M}{\mu_M} - \sum_{m\leq M-1}\left(\frac{1}{\mu_{m+1}}-\frac{1}{\mu_m}\right)\lambda_m \geq 0$$

for any $M\in\mathbb{Z}$. By (7.2),

$$\frac{\lambda_M}{\mu_M} \geq \left(\frac{1}{\mu_M}-\frac{1}{\mu_{M-1}}\right)\lambda_{M-1} \geq \gamma\frac{\lambda_{M-1}}{\mu_M}.$$

Hence $\lambda_M\geq\gamma\lambda_{M-1}$ for any $M\in\mathbb{Z}$ and any $\omega\in\Sigma^+$. Combining this with (λ4), we see that there exists $\delta\geq 1$ and $\beta\in(0,1)$ such that (7.3) holds for any $\omega\in\Sigma^+$, any $n\in\mathbb{Z}$ and any $k\in\mathbb{N}$. The rest of the statement is straightforward. $\square$

Lemma 7.2.

$$\delta^{-1}\lambda_m^{-1} \leq D_m \leq \lambda_m^{-1}$$

for any $\omega \in \Sigma^+$ and any $m \in \mathbb{Z}$. In particular,

$$\delta^{-1}\lambda(\omega \wedge \tau)^{-1} \leq d_\lambda(\omega, \tau) \leq \lambda(\omega \wedge \tau)^{-1}$$

for any $(\omega, \tau) \in \Sigma^+$, where we let $\lambda(\omega \wedge \omega) = +\infty$.

Proof. Obviously $D_m \leq \lambda_m^{-1}$ by definition. Lemma 7.1 implies that $\lambda_m \leq \delta\lambda_n$ for any $m \leq n$ and any $\omega \in \Sigma^+$. Hence $\delta^{-1}\lambda_m^{-1} \leq D_m$. □

Lemma 7.3. *$p_\Gamma(t, \omega, \tau)$ is continuous on $(0, +\infty) \times \Sigma^+ \times \Sigma^+$. Moreover,*

$$p_\Gamma(t, \omega, \tau) = -\frac{\mathrm{e}^{-\lambda_M t}}{\mu_M} + \sum_{m \leq M-1} \left(\frac{1}{\mu_{m+1}} - \frac{1}{\mu_m}\right)\mathrm{e}^{-\lambda_m t}, \tag{7.4}$$

where $M = |\omega \wedge \tau|$ and

$$p_\Gamma(t, \omega, \omega) = \sum_{m \in \mathbb{Z}} \left(\frac{1}{\mu_{m+1}} - \frac{1}{\mu_m}\right)\mathrm{e}^{-\lambda_m t}. \tag{7.5}$$

In particular, $p_\Gamma(t, \omega, \tau) \leq p_\Gamma(t, \omega, \omega)$.

Combining Lemmas 7.1 and 7.3, we immediately obtain Theorem 6.2(1).

Proof. We show that the infinite sum in the right-hand side of (4.1) is uniformly and absolutely convergent on $[T, \infty) \times \Sigma_x^+ \times \Sigma_x^+$ for any $x \in T$ and any $T > 0$. Fix $x \in T$. Set $|x| = M$ and let $\omega \in \Sigma_x^+$. Since $|\mathrm{e}^{-\lambda_{m-1} t} - \mathrm{e}^{-\lambda_m t}|/\mu_m \leq |\lambda_m - \lambda_{m-1}|/\mu_M$, ($\lambda$1) shows that

$$\sum_{n \leq M} \frac{\mathrm{e}^{-\lambda_{m-1} t} - \mathrm{e}^{-\lambda_m t}}{\mu_m}$$

is uniformly and absolutely convergent.

By (EL), we have $\mu_{M+m}^{-1} \leq \gamma^{-m}(\mu_M)^{-1}$. Also Lemma 7.1 implies $\lambda_{M+m} \geq \delta^{-1}\beta^{-m+1}\lambda_M$. Therefore

$$\left|\frac{\mathrm{e}^{-\lambda_{m+M-1} t} - \mathrm{e}^{-\lambda_{m+M} t}}{\mu_{m+M}}\right| \leq \frac{2\gamma^{-m}\mathrm{e}^{-c^{-1}\beta^{-m+2}\lambda_M t}}{\mu_M}.$$

Since $\beta > 1$, the infinite sum

$$\sum_{m > M} \frac{\mathrm{e}^{-\lambda_{m-1} t} - \mathrm{e}^{-\lambda_m t}}{\mu_m}$$

is uniformly and absolutely convergent on $[T, \infty) \times \Sigma_x^+ \times \Sigma_x^+$. Hence $p_\Gamma(t, \omega, \tau)$ is continuous on $(0, \infty) \times \Sigma^+ \times \Sigma^+$. The rest is straightforward by the definition of $p_\Gamma(t, \omega, \tau)$. □

Using (7.1), (7.2) and (7.3), we have the following estimate.

Lemma 7.4. *Define*

$$w(s) = \frac{1-\gamma}{\gamma}\left(\mathrm{e}^{-s} + \sum_{n \geq 1} \frac{1}{\gamma^n}\mathrm{e}^{-s/(\delta\beta^{n-1})}\right).$$

Then

$$\sum_{m\geq M}\left(\frac{1}{\mu_{m+1}}-\frac{1}{\mu_m}\right)\mathrm{e}^{-\lambda_m t}\leq\frac{w(\lambda_M t)}{\mu_M}. \tag{7.6}$$

Note that $w(s)$ is monotonically decreasing and $sx^{\alpha}w(s)\to 0$ as $s\to+\infty$ for any $\alpha\geq 0$.

Proof of Theorem 6.2(2). By Lemma 7.2, $\lambda(\omega\wedge\tau)\leq d_\lambda(\omega,\tau)^{-1}$. Proposition 6.1 implies that $p(t,\omega,\tau)\leq t/(\mu(\omega\wedge\tau)d_\lambda(\omega,\tau))$ for any $t>0$, $\omega\neq\tau\in\Sigma^+$.

Next assume $d_\lambda(\omega,\tau)\leq t$. Define $k=\max\{i|t\lambda_i<\delta^{-1}\}$. Then $t\lambda_k<\delta^{-1}\leq t\lambda_{k+1}$. Using Lemma 7.2, we have $t<(\delta\lambda_k)^{-1}\leq D([\omega]_k)$. This shows that $B(\omega,t)\subseteq\Sigma^+_{[\omega]_k}$. Hence

$$\mu\bigl(B(\omega,t)\bigr)\leq\mu_k. \tag{7.7}$$

Now

$$\sum_{m\leq k-1}\left(\frac{1}{\mu_{m+1}}-\frac{1}{\mu_m}\right)\mathrm{e}^{-\lambda_m t}\leq\sum_{m\leq k-1}\left(\frac{1}{\mu_{m+1}}-\frac{1}{\mu_m}\right)\leq\frac{1}{\mu_k}. \tag{7.8}$$

By Lemma 7.4,

$$\sum_{m\geq k}\left(\frac{1}{\mu_{m+1}}-\frac{1}{\mu_m}\right)\mathrm{e}^{-\lambda_m t}\leq\frac{w(\lambda_k t)}{\mu_k}\leq\frac{w(1/\delta)}{\mu_k}. \tag{7.9}$$

Combining (7.7), (7.8) and (7.9) with Lemma 7.3, we obtain

$$p(t,\omega,\tau)\leq p(t,\omega,\omega)\leq\frac{c}{\mu_k}\leq\frac{c}{\mu(B(\omega,t))},$$

where $c=w(1/\delta)+1$. □

Lemma 7.5. *If $\lambda_{m-1}\leq r^{-1}<\lambda_m$, then*

$$\left(\frac{1}{\mu_m}-\frac{1}{\mu_{m-1}}\right)\mathrm{e}^{-\lambda_{m-1}t}\geq\frac{\gamma}{\mu_m}\mathrm{e}^{-t/r}\geq\frac{\gamma}{\mu(B(\omega,r))}\mathrm{e}^{-t/r}.$$

Proof. The first part of the inequality is immediate from (7.2) and the fact that $\lambda_{m-1}t\leq t/r$. If $\tau\in\Sigma^+_{[\omega]_m}$, then $d_\lambda(\omega,\tau)\leq D_m\leq 1/\lambda_m<r$. Hence $\Sigma^+_{[\omega]_m}\subseteq B(\omega,r)$. This implies the second part of the inequality. □

Proof of Theorem 6.2(3). Let $M=|\omega\wedge\tau|$ and let $s=\lambda_M t$. Assume that $s>\delta$. Define $m_0=\min\{n|n\geq 0,\lambda_{M-n-1}\leq 1/t\}$. Since $\lambda_{M+k}\geq\lambda_M/\delta>1/t$ for any $k\geq 0$, it follows that $\lambda_{M-m_0-1}\leq\frac{1}{t}<\lambda_{M-m_0}$. By Lemma 7.5 and (7.5),

$$p_\Gamma(t,\omega,\omega)\geq\frac{\gamma}{\mu_{M-m_0}}\mathrm{e}^{-1}\geq\frac{\gamma}{\mu(B(\omega,t))}\mathrm{e}^{-1}. \tag{7.10}$$

Since $\lambda_{M-n-1}\leq\delta\beta^n\lambda_M$ for any $n\geq 1$, we have $m_0\leq-\frac{\log s\delta}{\log\beta}+1$. This implies

$$\frac{\gamma}{\mu_{M-m_0}}\mathrm{e}^{-1}\geq\gamma\frac{\gamma^{m_0}}{\mu_M}\mathrm{e}^{-1}\geq c\frac{s^{-\alpha}}{\mu_M},$$

where $c=\gamma^{2-\log\delta/\log\beta}/e$ and $\alpha=\log\gamma/\log\beta$. Note that there exists $s_0>0$ such that $s^{\alpha}(w(s)+\mathrm{e}^{-s})\leq c/2$ for any $s\geq s_0$. Combining the above inequalities with (7.4), (7.5) and (7.6), we see that

$$p_\Gamma(t,\omega,\omega)-p_\Gamma(t,\omega,\tau)\leq\frac{w(s)+\mathrm{e}^{-s}}{\mu_M}\leq\frac{c}{2}\frac{s^{-\alpha}}{\mu_M}\leq\frac{\gamma}{2\mu_{M-m_0}}\mathrm{e}^{-1}\leq\frac{1}{2}p_\Gamma(t,\omega,\omega)$$

for any $s \geq s_0$. Hence by (7.10), if $\lambda_M t \geq s_0$, then

$$p_\Gamma(t,\omega,\tau) \geq \frac{\gamma}{2\mathrm{e}} \frac{1}{\mu(B(\omega,t))}.$$

Let $\varepsilon = 1/(\delta s_0)$. Then Lemma 7.2 shows that $\lambda_M t \geq s_0$ if $d_\lambda(\omega,\tau) \leq \varepsilon t$. □

The following lemma can be shown by an inductive argument.

Lemma 7.6. *Let $\alpha_1,\ldots,\alpha_m > 0$ and let $\beta_1,\beta_2,\ldots,\beta_{m+1} \in \mathbb{R}$. Assume that $\alpha_n \leq \alpha_{n+1}$ for any $n = 1,\ldots,m-1$. If $\beta_1 \geq \beta_n$ for any $n = 1,\ldots,m$, then*

$$\sum_{n=1}^{m} \frac{\beta_n - \beta_{n+1}}{\alpha_n} \geq \frac{\beta_1 - \beta_{m+1}}{\alpha_m}.$$

Lemma 7.7. *If $D_{m-1} < D_m$, then $\Sigma^+_{[\omega]_m} \backslash \Sigma^+_{[\omega]_{m+1}} \subseteq U_{\lambda,\omega}$.*

Proof. If $D_{m-1} \leq D_m$, then $\lambda_m > \lambda_n$ for any $n < m$. Let $\tau \in \Sigma^+_{[\omega]_m} \backslash \Sigma^+_{[\omega]_{m+1}}$. Then $\omega \wedge \tau = [\omega]_m$. Hence $(\omega,\tau) \in U_\lambda$. □

Proof of Theorem 6.2(4). First we prove (6.7). If $A(\omega,r_1,r_2) = \varnothing$, then (6.7) is trivial. Assume that $A(\omega,r_1,r_2) \neq \varnothing$. In particular we have $r_2 > r_1$. Let $M_1 = \min\{m | r_1 \leq D_m < r_2\}$. Then $A(\omega,r_1,r_2) \subseteq \Sigma^+_{[\omega]_{M_1}}$. Hence $\mu(A(\omega,r_1,r_2)) \leq \mu_{M_1}$. On the other hand, since $D_{M_1} < D_{M_1-1}$, Lemma 7.7 implies $\Sigma^+_{[\omega]_{M_1}} \backslash \Sigma^+_{[\omega]_{M_1+1}} \subseteq U_{\lambda,\omega} \cap A(\omega,r_1,r_2)$. By (7.1),

$$\mu\bigl(U_{\lambda,\omega} \cap A(\omega,r_1,r_2)\bigr) \geq \gamma\mu_{M_1} \geq \gamma\mu\bigl(A(\omega,r_1,r_2)\bigr).$$

Next we show (6.8). Let $(\omega,\tau) \in U_\lambda$. Choose $N \geq 1$ so that $\delta\beta^{N-1} < 1$. Then by (λ2), Lemmas 7.6 and 7.2,

$$\begin{aligned} J(\omega,\tau) &= \sum_{m \leq M} \frac{\lambda_m - \lambda_{m-1}}{\mu_m} = \sum_{n=1}^{N-1} \frac{\lambda_{M-n} - \lambda_{M-n-1}}{\mu_{M-n}} + J\bigl([\omega]_{M-n}\bigr) \\ &\geq \frac{\lambda_M - \lambda_{M-n}}{\mu_{M-n+1}} \geq \gamma^{N-1}\bigl(1 - \delta\beta^{N-1}\bigr)\frac{\lambda_M}{\mu_M} \geq \frac{\gamma^{N-1}(1-\delta\beta^{N-1})}{\delta\mu_M d_\lambda(\omega,\tau)}. \end{aligned}$$

Finally (6.8) along with (6.1) implies (6.9). □

Proof of Theorem 6.2(5). Note that

$$E_\omega\bigl(d_\lambda(\omega,X_t)^\theta\bigr) = \int_{\Sigma^+} p_\Gamma(t,\omega,\tau) d_\lambda(\omega,\tau)^\theta \mu(\mathrm{d}\tau).$$

First we consider the lower estimate. Define $m_0 = \min\{m | F_\lambda(\varepsilon t,\omega) = D_m\}$. Since $D_{m_0-1} < D_{m_0}$, Lemma 7.7 shows that $\Sigma^+_{[\omega]_{m_0}} \backslash \Sigma^+_{[\omega]_{m_0+1}} \subseteq U_{\lambda,\omega}$. By (6.7), there exists $\{m_i\}_{i\geq 1}$ such that $m_{i-1} > m_i$ and $\Sigma^+_{[\omega]_{m_i}} \backslash \Sigma^+_{[\omega]_{m_i+1}} \subseteq U_{\lambda,\omega}$ for any $i \geq 1$. Hence by (6.9),

$$\int_{\Sigma^+_{[\omega]_{m_i}} \backslash \Sigma^+_{[\omega]_{m_i+1}}} p_\Gamma(t,\omega,\tau) d_\lambda(\omega,\tau)^\theta \mu(\mathrm{d}\tau) \geq c_4 \frac{(\mu_{m_i} - \mu_{m_i+1}) t (D_{m_i})^{\theta-1}}{\mu_{m_i}} \geq c_4\gamma t (D_{m_i})^{\theta-1}.$$

If $\theta > 1$, then $D_{m_i} \geq D_{m_0}$ and hence $\sum_{i\geq 0}(D_{m_i})^{\theta-1} = +\infty$. Hence we have $E_\omega(d_\lambda(\omega,X_t)^\theta) = +\infty$. If $\theta \in (0,1)$, then we obtain

$$E_\omega\bigl(d_\lambda(\omega,X_t)^\theta\bigr) \geq c_4\gamma t (D_{m_0})^{\theta-1} = c_4\gamma t F_\lambda(\varepsilon t,\omega)^{\theta-1}.$$

For the upper estimate, we divide Σ^+ into two parts $\Sigma_1^+ = \{\tau | d_\lambda(\omega,\tau) \geq \varepsilon t\}$ and $\Sigma_2^+ = \{\tau | d_\lambda(\omega,\tau) < \varepsilon t\}$. Define $m_* = \max\{m | D_m = F_\lambda(\varepsilon t, \omega)\}$. Using (7.3), we see that $\delta\beta^{n-1} D_{m_*-n} \geq D_{m_*}$ for any $n \geq 0$. Hence by (6.5),

$$\begin{aligned}\int_{\Sigma_1^+} p_\Gamma(t,\omega,\tau) d_\lambda(\omega,\tau)^\theta \mu(\mathrm{d}\tau) &\leq t \sum_{n\geq 0} \frac{(\mu_{m_*-n} - \mu_{m_*-n+1})(D_{m_*-n})^{\theta-1}}{\mu_{m_*}} \\ &\leq t \sum_{n\geq 0} \delta^{1-\theta} \beta^{(1-\theta)(n-1)} (D_{m_*})^{\theta-1} \leq ct F_\lambda(\varepsilon t, \omega)^{\theta-1},\end{aligned}$$

where c is independent of ω and t. On the other hand by (6.5),

$$\begin{aligned}\int_{\Sigma_2^+} p_\Gamma(t,\omega,\tau) d_\lambda(\omega,\tau)^\theta \mu(\mathrm{d}\tau) &\leq \frac{c_1}{\mu(B(\omega,t))} \sum_{n\geq 1} (D_{m_*+n})^\theta (\mu_{m_*+n} - \mu_{m_*+n+1}) \\ &\leq \frac{c_1 \mu_{m_*} (D_{m_*})^\theta}{\mu(B(\omega,t))} \sum_{n\geq 0} \delta^\theta \beta^{\theta(n-1)} (1-\gamma)^n = c F_\lambda(\varepsilon t, \omega)^\theta \frac{\mu_{m_*}}{\mu(B(\omega,t))}.\end{aligned}$$

Now it follows that $\mu(\omega, \varepsilon t) = \mu(\omega, D_{m_*}(1-h))$ for sufficiently small h. Choose h so that $2(1-h) > 1$. Then by the volume doubling property, $c'\mu(B(\omega,t)) \geq c''\mu(B(\omega,\varepsilon t)) = c''\mu(B(\omega, D_{m_*}(1-h))) \geq \mu(B(\omega, 2D_{m_*}(1-h))) \geq \mu_{m_*}$. This and the above inequality yields

$$\int_{\Sigma_2^+} p_\Gamma(t,\omega,\tau) d_\lambda(\omega,\tau)^\theta \mu(\mathrm{d}\tau) \leq c''' F_\lambda(\varepsilon t, \omega)^\theta.$$

Thus we have the upper estimate. If (Σ^+, d_λ) is uniformly perfect, then Theorem 5.8 shows that $F_\lambda(\varepsilon t, \omega) \leq a_1 \varepsilon t$, where $a_1 > 1$ is independent of ω and t. This completes the proof. □

Proof of Corollary 6.3. First assume that (M) holds, i.e. $\lambda \in \ell_M^+(T)$. Then $U_\lambda(\omega,\tau) = (\Sigma^+ \times \Sigma^+) \backslash \{(\omega,\omega) | \omega \in \Sigma^+\}$. (6.10) is immediate from (6.2) and (6.8). We easily have (6.11) from Theorem 6.2 except the near diagonal estimate for $\varepsilon t \leq d_\lambda(\omega,\tau) \leq t$ if ε given in Theorem 6.2(3) is less than 1. Even so, we still have (6.9) for $\varepsilon t \leq d_\lambda(\omega,\tau) \leq t$. Since $B(\omega, d_\lambda(\omega,\tau)) \subseteq \mu(\omega \wedge \tau) \subseteq B(\omega, 2d_\lambda(\omega,\tau))$, the volume doubling property shows that there exists $c' > 0$ such that

$$\frac{t}{\mu(\omega\wedge\tau) d_\lambda(\omega,\tau)} \geq \frac{c'}{\mu(B(\omega,t))}$$

for any (t,ω,τ) with $\varepsilon t \leq d_\lambda(\omega,\tau) \leq t$. Thus we have the near diagonal lower estimate.

Next we assume that (AK+) holds. Upper estimates in (6.10) and (6.11) are obvious by (6.2) and (6.5). Since $\eta(|x|) \leq \lambda_\eta(x) \leq 2\eta(|x|)$, we have

$$J(\omega\wedge\tau) = \frac{B(\omega\wedge\tau)(\eta(m) - \eta(m-1))}{(B(\omega\wedge\tau) - 1)\mu(\omega\wedge\tau)} \geq \frac{(1-\xi)\lambda_\eta(\omega\wedge\tau)}{2\mu(\omega\wedge\tau)} \geq \frac{1-\xi}{2\delta\mu(\omega\wedge\tau) d_\lambda(\omega,\tau)}$$

for any $\omega \neq \tau \in \Sigma^+$, where $m = |\omega\wedge\tau|$. Hence we have the lower estimate in (6.10) and the off diagonal lower estimate in (6.11). The near diagonal lower estimate is verified by the same argument as in the monotone case. □

8. Examples

In this section, we present examples. For simplicity, we set $T = T^{(2)}$ and adopt the settings of Example 2.9. Consequently, Σ^+ is identified with the 2-adic numbers $\mathbb{Q}_2$. In this case, $N(x) = 2$ for any $x \in T$. Hence $\mu_{\mathcal{T}}(\Sigma_x^+) = \mu_{\mathcal{T}}(\Sigma_{\pi(x)}^+)/2$ for any $x \in T$ and $\mu_{\mathcal{T}}(\Sigma_\phi^+) = 1$. We use μ_* to denote $\mu_{\mathcal{T}}$. Next we define a class of self-similar measures on Σ^+ including μ_* as a special case.

Proposition 8.1. *Let $\nu_0, \nu_1 \in (0, 1)$ with $\nu_0 + \nu_1 = 1$. Then there exists a unique Borel regular measure on Σ^+ which satisfies $\nu(\Sigma^+_\phi) = 1$ and $\nu(\Sigma^+_x) = \nu_{\alpha_m}\nu(\Sigma^+_{\pi(x)})$ for any $x = (\alpha_i)_{i\le m} \in T$. This measure ν belongs to $\mathcal{M}(\Sigma^+)$ and satisfies* (EL).

Note that if $x = (\alpha_i)_{i\le m}$, then $\pi(x) = (\alpha_i)_{i\le m-1}$.

Definition 8.2. *For $\nu_0, \nu_1 \in (0, 1)$ with $\nu_0 + \nu_1 = 1$, the unique Borel regular measure ν given in Proposition 8.1 is called the self-similar measure on Σ^+ with weight (ν_0, ν_1).*

μ_* is the self-similar measure on Σ^+ with weight $(1/2, 1/2)$.

Example 8.3. *Let $\mathbf{s} = (s_0, s_1) \in (1,\infty)^2$. Define $\lambda^{\mathbf{s}} \in \ell^+(T)$ inductively by $\lambda^{\mathbf{s}}(\phi) = 1$ and $\lambda^{\mathbf{s}}(x) = s_{\alpha_m}\lambda^{\mathbf{s}}(\pi(x))$ for any $x = (\alpha_i)_{i\le m} \in \Sigma^+$. More directly,*

$$\lambda^{\mathbf{s}}(x) = (s_0)^K s_{\alpha_{K+1}} \cdots s_{\alpha_m}$$

for any $(\alpha_i)_{i\le m} \in T$, where $K = \max\{j \mid j \le m, \alpha_i = 0 \text{ for any } i \le j\}$. Note that $\lambda^{\mathbf{s}} \notin \ell^+_{AK}(T)$ if $s_0 \ne s_1$. The basic properties of $\lambda^{\mathbf{s}}$ are:

(1) $\lambda^{\mathbf{s}} \in \ell^+_{M,\infty}(T)$.
(2) $\lim_{m\to\infty} \lambda^{\mathbf{s}}(\phi^-_m) = 0$.
(3) $\lambda^{\mathbf{s}}$ *satisfies* (λ4).
(4) $(\Sigma^+, d_{\lambda^{\mathbf{s}}})$ *is uniformly perfect.*

The last property (4) *can be verified by Theorem 5.8. Thus if ν is a self-similar measure with weight (ν_0, ν_1), then $(\lambda^{\mathbf{s}}, \nu)$ satisfies all the assumptions of Theorems 4.5, 6.2 and* (M) *of Corollary 6.3. Hence we have the asymptotic properties of the transition density given in those theorems and corollary.*

Let h be the unique positive number which satisfies

$$(s_0)^{-h} + (s_1)^{-h} = 1.$$

Then, h is the Hausdorff dimension of $(\Sigma^+, d_{\lambda^{\mathbf{s}}})$. Let $\mu^{\mathbf{s}}$ be the self-similar measure with weight $((s_0)^{-h}, (s_1)^{-h})$. One can directly show that $\mu^{\mathbf{s}}$ coincides with the h-dimensional Hausdorff measure on $(\Sigma^+, d_{\lambda^{\mathbf{s}}})$. Moreover, it follows that

$$\mu^{\mathbf{s}}\big(B_{d_{\lambda^{\mathbf{s}}}}(\omega, r)\big) \asymp r^h$$

on $\Sigma^+ \times (0,\infty)$. Hence by Corollary 6.3, if $\Gamma_{\mathbf{s}} = (\lambda^{\mathbf{s}}, \mu^{\mathbf{s}})$, then

$$p_{\Gamma_{\mathbf{s}}}(t, \omega, \tau) \asymp \begin{cases} \frac{t}{d_{\lambda^{\mathbf{s}}}(\omega,\tau)^{h+1}} & \text{if } d_{\lambda^{\mathbf{s}}}(\omega, \tau) \ge t, \\ t^{-h} & \text{if } d_{\lambda^{\mathbf{s}}}(\omega, \tau) < t \end{cases}$$

on $(0,\infty) \times \Sigma^+ \times \Sigma^+$. In particular, if $s = s_0 = s_1$ then $h = \log 2/\log s$ and $d_{\lambda^{\mathbf{s}}}(\omega, \tau) = n_2(\omega - \tau)^{\log s/\log 2}$, where $n_2(\cdot)$ is the 2-adic norm. In this case, $\mu^{\mathbf{s}} = \mu_*$ and $(\lambda^{\mathbf{s}}, \mu^{\mathbf{s}}) \in \Theta^+_{AK}(\mathcal{T})$ with $\lambda^{\mathbf{s}} = \lambda_{\eta^{\mathbf{s}}}$, where $\eta^{\mathbf{s}}(m) = s^{m+1}/(2s-1)$.

Example 8.4. *For $\varepsilon \in [0, 1]$, define $\eta_\varepsilon : T \to (0,\infty)$ by*

$$\eta_\varepsilon(2n) = 2^n \quad \text{and} \quad \eta_\varepsilon(2n+1) = (1+\varepsilon)2^n$$

for any $n \in \mathbb{Z}$. Then $\lambda_{\eta_\varepsilon} \in \ell^+_{AK}(T)$ and $(\lambda_{\eta_\varepsilon}, \mu_*) \in \Theta^+_{AK}(\mathcal{T}) \cap \Theta^+_\infty(\mathcal{T})$. It follows that

$$\lambda_{\eta_\varepsilon}(x) = \begin{cases} (3-\varepsilon)2^{n-1} & \text{if } |x| = 2n, \\ (1+2\varepsilon)2^n & \text{if } |x| = 2n+1. \end{cases}$$

Thus we may easily verify (λ2) *and* (λ4) *and apply Theorem* 6.2. *Theorem* 5.8 *shows that* $(\Sigma^+, d_{\lambda_\varepsilon})$ *is uniformly perfect. Note that* $\lambda_{\eta_\varepsilon} \in \ell^+_M(T)$ *if and only if* $\varepsilon \in [1/5, 2/3]$. *Moreover, we have*

$$U_{\lambda_{\eta_\varepsilon}} = \begin{cases} \{(\omega,\tau) \mid |\omega \wedge \tau| \text{ is even}\} & \text{if } \varepsilon \in [0, 1/5), \\ \{(\omega,\tau) \mid |\omega \wedge \tau| \text{ is odd}\} & \text{if } \varepsilon \in (2/3, 1]. \end{cases}$$

However, if $\varepsilon \in (0, 1)$, *then the condition* (AK+) *in Corollary* 6.3 *holds and hence we have asymptotic estimates* (6.10) *and* (6.11) *on* $(0,\infty) \times \Sigma^+ \times \Sigma^+$. *On the contrary, if* $\varepsilon \in \{0, 1\}$, *then lower parts of these estimates fail on the complement of* U_{λ_ε}. *As a matter of fact,* (3.3) *implies that* $J(\omega,\tau) = 0$ *for any* $(\omega,\tau) \notin U_{\lambda_{\eta_\varepsilon}}$ *in case* $\varepsilon \in \{0, 1\}$.

Example 8.5. *Define*

$$\lambda(x) = \begin{cases} (|x|+1)! & \text{if } |x| \geq 0, \\ \frac{1}{(-|x|+1)!} & \text{if } |x| < 0. \end{cases}$$

Then $(\lambda, \mu_*) \in \Theta^+_{M,\infty}(\mathcal{T})$ *and* (λ4) *is satisfied. Also* μ_* *satisfies* (EL). *Hence we have* (6.10) *and* (6.11). *In this case, however,* (Σ^+, d_λ) *is not uniformly perfect. In fact, we have*

$$\limsup_{s \downarrow 0} \frac{F_\lambda(s,\omega)}{s} = \limsup_{s \to \infty} \frac{F_\lambda(s,\omega)}{s} = +\infty.$$

Therefore, for any $\theta \in (0, 1)$ *and any* $\omega \in \Sigma^+$,

$$\limsup_{s \downarrow 0} \frac{E_\omega(d_\lambda(\omega, X_t)^\theta)}{t^\theta} = \limsup_{s \to \infty} \frac{E_\omega(d_\lambda(\omega, X_t)^\theta)}{t^\theta} = +\infty.$$

9. Random walks on trees

In the rest of this paper, we study Dirichlet forms on noncompact Cantor sets induced by transient random walks on trees. We will review fundamental notions and results on random walks on trees and their Martin boundaries in this section. As a reference, one can see [22] for the details on random walks.

In this and the following sections, $\mathcal{T} = (T, \mathcal{A}, \phi_*)$ is a locally finite ordered tree where Assumption 2.2 holds and $\phi \in T$ is a fixed reference point. We use π to denote π_{ϕ_*}.

First we define random walks on trees. In the following definition, $(T, \mathcal{A})$ can be general non-directed graph.

Definition 9.1. $C : T \times T \to [0, \infty)$ *is called a weight on the tree* $(T, \mathcal{A})$ *if* $C(x, y) = C(y, x)$ *and* $C(x, y) > 0$ *if and only if* $\mathcal{A}(x, y) > 0$. *For a weight* C, *define* $C(x) = \sum_{y \in T} C(x, y)$, $q(x, y) = C(x, y)/C(x)$. *Moreover, we define* $q^{(n)}(x, y)$ *inductively by* $q^{(0)}(x, y) = \delta_{xy}$, $q^{(n+1)}(x, y) = \sum_{z \in T} q(x, y)q^{(n)}(x, y)$. *Also* $G(x, y) = \sum_{n \geq 0} q^{(n)}(x, y)$. $G(x, y)$ *is called the Green function of* (T, C).

The quantity $q^{(n)}(x, y)$ is the transition probability from x to y at time n. There exists a reversible Markov chain $(\{Z_n\}_{n \geq 0}, \{Q_x\}_{x \in T})$ on T such that $q^{(n)}(x, y) = Q_x(Z_n = y)$ for any $x, y \in T$. We call (T, C) a random walk on T.

Definition 9.2. *A random walk* (T, C) *is called transient if and only if* $G(x, y) < +\infty$ *for any* $x, y \in T$.

It is well-known that (T, C) is transient if $G(x, y) < +\infty$ for some $x, y \in T$.

Next we introduce a quadratic form associated with a random walk (T, C).

Definition 9.3. *Define*

$$\mathcal{F}_{(T,C)} = \left\{ u \,\middle|\, u : T \to \mathbb{R}, \sum_{x,y \in T} C(x, y)\big(u(x) - u(y)\big)^2 < +\infty \right\}$$

and, for any $u, v \in \mathcal{F}_{(T,C)}$,

$$\mathcal{E}_{(T,C)}(u,v) = \frac{1}{2}\sum_{x,y\in T} C(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big).$$

For $x \in T$, *define* $\mathcal{E}_{(T,C),x}(u,v) = \mathcal{E}_{(T,C)}(u,v) + u(x)v(x)$ *for any* $u, v \in \mathcal{F}_{(T,C)}$.

It is easy to see that $(\mathcal{F}_{(T,C)}, \mathcal{E}_{(T,C),x})$ is a Hilbert space.

Definition 9.4. *Let* $\mathcal{F}_0$ *be the closure of* $C_0(T)$ *with respect to* $\mathcal{E}_{(T,C),x}$, *where* $C_0(T) = \{u \mid \mathrm{supp}(u)$ *is finite*$\}$.

Note that $\mathcal{F}_0$ is independent of the choice of x. If no confusion may occur, we use $\mathcal{F}, \mathcal{E}$ and $\mathcal{E}_x$ in place of $\mathcal{F}_{(T,C)}, \mathcal{E}_{(T,C)}$ and $\mathcal{E}_{(T,C),x}$ respectively. In terms of $(\mathcal{E}, \mathcal{F})$, we have the following equivalent condition for transience by [23,24]. See 4.51 of [22] for details.

Proposition 9.5. (T, C) *is transient if and only if* $\sup_{u\in C_0(T), u\neq 0} \frac{u(x)^2}{\mathcal{E}(u,u)} < +\infty$ *for any* $x \in T$.

Definition 9.6. *For* $x \in T$, *define*

$$R_x(T,C) = \sup\left\{\frac{u(x)^2}{\mathcal{E}_{(T,C)}(u,u)}\,\middle|\, u \in C_0(T), \mathcal{E}_{(T,C)}(u,u) \neq 0\right\}.$$

By Proposition 9.5, (T, C) is transient if and only if $R_x(T,C) < +\infty$ for any $x \in T$. $R_x(T,C)$ is regarded as the resistance between x and the infinity.

For $x, y \in T$, if $C^x_y = C|_{T^x_y \times T^x_y}$, then (T^x_y, C^x_y) is a random walk on T^x_y. Hereafter we assume the followings.

Assumption 9.7. (T^x_y, C^x_y) *is transient for any* $x, y \in T$.

By the celebrated work of Cartier [10], we may identify the Martin boundary of the random walk (T, C) with Σ, which is the collection of ends. Consequently we obtain Proposition 9.8 and Theorem 9.11 below. See [22], Chapter 7, for details.

Proposition 9.8. *There exists a* Σ*-valued random variable* Z_∞ *such that*

$$Q_x\Big(\lim_{n\to\infty} Z_n = Z_\infty\Big) = 1$$

for any $x \in T$.

Definition 9.9. *Let* $x \in T$. *For any Borel set* $A \subseteq \Sigma$, *define* $\nu_x(A) = Q_x(Z_\infty \in A)$. ν_x *is called the hitting distribution of* Σ *starting from* x.

ν_x is naturally extended to a Borel regular complete probability measure on A. We use ν_x to denote the extended measure as well. Assumption 9.7 is equivalent to the condition that the support of ν_x is Σ.

Since $\nu_x = \sum_{y\in V(x)} q(x,y)\nu_y$, all ν_x's for $x \in T$ are mutually absolutely continuous and the Radon–Nikodym derivative $d\nu_y/d\nu_x$ is bounded. Hence $L^p(\Sigma,\nu_x) = L^p(\Sigma,\nu_y)$ for any $x, y \in T$ and $p \geq 1$. We use $L^p(\Sigma)$ to denote $L^p(\Sigma,\nu_x)$.

Definition 9.10. $f : T \to \mathbb{R}$ *is said to be harmonic on* T *if and only if* $\sum_{y\in T} q(x,y)f(y) = f(x)$ *for any* $x \in T$. *The collection of all harmonic functions is denoted by* $\mathcal{H}$.

Theorem 9.11. *For $z \in T$, let $K_z(x, y) = G(x, y)/G(z, y)$ for any $x, y \in T$. Then $K_z(x, y)$ is extended to a continuous function on $T \times \widehat{T}$. Moreover, define $H : L^1(\Sigma) \to \ell(T)$ by*

$$(Hf)(x) = \int_{\Sigma} K_z(x, y) f(y) \nu_z(\mathrm{d}y)$$

for any $x \in T$, where $\ell(T) = \{u | u : T \to \mathbb{R}\}$. Then Hf is harmonic on T and independent of z.

The kernel $K_z(x, y)$ is called the Martin kernel.

10. Resistances and hitting distributions

A random walk (T, C) can be regarded as a electrical network consisting of resistors with resistances $r(x, y) = C(x, y)^{-1}$ between x and y belonging to T. In this respect we consider $R_y(T_y^x, C_y^x)$ as the effective resistance between y and the infinity with respect to the electrical network (T_y^x, C_y^x). In this section, we study relations between such resistances and hitting distributions.

As in the last section, $\mathcal{T} = (T, \mathcal{A}, \phi_*)$ is a locally finite ordered tree which satisfies Assumption 2.2, $\pi = \pi_{\phi_*}$, $\phi \in T$ is a fixed referee point and (T, C) is a weight on $(T, \mathcal{A})$ which satisfies Assumption 9.7 throughout this section.

Definition 10.1. *Define $R_y^x = R_y(T_y^x, C_y^x)$. We write $R_x^+ = R_x^{\pi(x)}$ for any $x \in T$, where $\pi = \pi_{\phi_*}$.*

By Assumption 9.7 along with Proposition 9.5, it follows that $R_y^x < +\infty$ for any $x, y \in T$.

Recall that $\Sigma_z^x = \bigcup_{y \in S^x(z)} \Sigma_y^x$, where the union is the disjoint union and $S^x(z) = V(x) \backslash \pi^x(z)$. Furthermore note that

$$\sum_{y \in S^x(z)} \frac{R_z^x}{r_y^x + R_y^x} = 1 \tag{10.1}$$

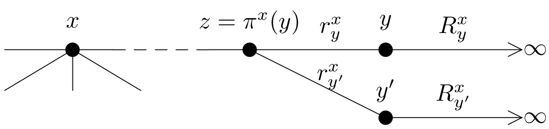
since the effective resistance R_z^x between z and the infinity is a parallel combination of $r_y^x + R_y^x$ for $y \in S^x(z)$. See Fig. 3 for the case where $S^x(z) = \{y, y'\}$. In conjunction with these facts, the following relation on the hitting distribution has been shown in [18], Theorem 3.8.

Theorem 10.2. *For any x, y and $z \in T$ with $\pi^x(y) = z$,*

$$\nu_x(\Sigma_y^x) = \frac{R_z^x}{r_y^x + R_y^x} \nu_x(\Sigma_z^x), \tag{10.2}$$

where $r_y^x = C(y, z)^{-1}$.

Next we introduce a Borel regular measure ν_* on Σ^+ associated with the random walk (T, C). Intuitively, ν_* is a kind of "hitting distribution from ϕ_*" although $\nu_*(\Sigma^+)$ can be $+\infty$.



R_z^x = the parallel combination of R_y^x and $R_{y'}^x$

Fig. 3. Calculation of R_z^x if $S^x(z) = \{y, y'\}$.

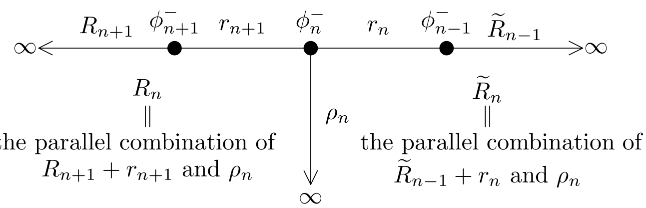


Fig. 4. R_n, $\widetilde{R}_n$, r_n and ρ_n.

Proposition 10.3. *There exists a unique Borel regular complete measure ν_* on Σ^+ which satisfies $\nu_*(\Sigma^+_\phi)=1$ and*

$$\nu_*\big(\Sigma^+_y\big)=\frac{R^+_z}{r^z_y+R^+_y}\nu_*\big(\Sigma^+_z\big) \tag{10.3}$$

for any $z\in T$ and any $y\in S^+(z)$.

Let $x=\pi(z)$. Then $S^+(z)=S^x(z)$, $\Sigma^+_y=\Sigma^x_y$, $\Sigma^+_z=\Sigma^x_z$, $R^+_z=R^x_z$, $r^z_y=r^x_y$ and $R^+_y=R^x_y$. These relations shows that (10.3) is obtained by replacing ν_x by ν_* in (10.2).

Using (10.3) inductively, we obtain $\nu_*(\Sigma^+_x)$ for every $x\in T$. It is straight forward to show the above proposition. For example, (10.1) yields the consistency of ν_*.

Comparing (10.2) with (10.3), we immediately obtain the next lemma. In the followings, we regard ν_* as a Borel regular measure on Σ by letting $\nu_*(\{\phi_*\})=0$.

Lemma 10.4. *Let $x\in T$. If $y\notin\{x^-_n|n\geq 0\}$, then $T^x_y=T^+_y$, $R^x_y=R^+_y$, $\Sigma^x_y=\Sigma^+_y$ and $\nu_x(A\cap\Sigma^+_y)/\nu_x(\Sigma^+_y)=\nu_*(A\cap\Sigma^+_y)/\nu_*(\Sigma^+_y)$ for any Borel set $A\subseteq\Sigma$.*

Using the above lemma, we see that ν_x is absolutely continuous with respect to ν_*.

Lemma 10.5. *For any $x\in T$, there exists $c>0$ such that $\nu_*(A)\geq c\nu_x(A)$ for any Borel set $A\subseteq\Sigma^+$.*

Proof. Without loss of generality, we may assume that $x=\phi$. Write $r_m=C(\phi^-_m,\phi^-_{m-1})^{-1}$ for $m\geq 1$ and $R_m=R^\phi_{\phi^-_m}$ and $\widetilde{R}_m=R^+_{\phi^-_m}$ for $m\geq 0$. (See Fig. 4.) Let $y\in S^+(\phi^-_n)\cap S^\phi(\phi^-_n)$. Set $r_y=C(\phi^-_n,y)^{-1}$. Then by Theorem 10.2,

$$\nu_\phi\big(\Sigma^\phi_y\big)=\frac{R_n}{r_y+R^\phi_y}\frac{R_{n-1}}{r_n+R_n}\cdots\frac{R_0}{r_1+R_1}\nu_\phi(\Sigma). \tag{10.4}$$

By the definition of ν_*, we also have

$$\nu_*\big(\Sigma^+_\phi\big)=\frac{\widetilde{R}_n}{r_n+\widetilde{R}_{n-1}}\frac{\widetilde{R}_{n-1}}{r_{n-1}+\widetilde{R}_{n-2}}\cdots\frac{\widetilde{R}_1}{r_1+\widetilde{R}_0}\nu_*\big(\Sigma^+_{\phi^-_n}\big) \tag{10.5}$$

and

$$\nu_*\big(\Sigma^+_y\big)=\frac{\widetilde{R}_n}{r_y+R^+_y}\nu_*\big(\Sigma^+_{\phi^-_n}\big). \tag{10.6}$$

By Lemma 10.4, (10.4), (10.5) and (10.6) implies

$$\frac{\nu_*(\Sigma^+_y)}{\nu_\phi(\Sigma^+_y)}=\left(\frac{r_n+\widetilde{R}_{n-1}}{\widetilde{R}_{n-1}}\cdots\frac{r_2+\widetilde{R}_1}{\widetilde{R}_1}\right)\left(\frac{r_n+R_n}{R_n}\cdots\frac{r_1+R_1}{R_1}\right)\frac{r_1+\widetilde{R}_0}{R_0}. \tag{10.7}$$

Using Lemma 10.4 again, we obtain

$$\nu_*(A) \geq \frac{r_1 + \widetilde{R}_0}{R_0} \nu_\phi(A)$$

for any Borel set $A \subseteq \Sigma^+$. □

Definition 10.6. *For any $x \in T$, we define $D_x^+ = \nu_*(\Sigma_x^+) R_x^+$ and $\lambda_x^+ = 1/D_x^+$.*

In the next section, $\{\lambda_x^+\}_{x \in T}$ is shown to be the collection eigenvalues of the self-adjoint operator associate with the Dirichlet form on Σ^+ induced by the random walk (T, C).

By (10.3), we have the next proposition.

Proposition 10.7. *For any $x \in T$ and any $y \in S^+(x)$,*

$$D_x^+ - D_y^+ = r_y^x \nu_*(\Sigma_y^+) \quad \text{and} \quad \frac{D_y^+}{D_x^+} = \frac{R_y^+}{r_y^x + R_y^+}.$$

In particular, $D_x^+ > D_y^+$ and $\lambda_x^+ < \lambda_y^+$.

By the above proposition, $D_{[\omega]_n}^+$ and $\lambda_{[\omega]_n}^+$ converges as $n \to \infty$ and as $n \to -\infty$ for any $\omega \in \Sigma^+$, if we allow ∞ as a value of a limit. Note that $[\omega]_{-m} = \phi_m^-$ for sufficiently large m. So the limits as $n \to -\infty$ do not depend on ω.

Theorem 10.8. *Define $\lambda^+ = \lim_{m\to\infty} \lambda_{\phi_m^-}^+$. Then $\lambda^+ > 0$ if and only if $\nu_\phi(\{\phi_*\}) > 0$.*

By letting $\lambda(x) = \lambda_x^+$, the constant λ^+ turns out to correspond to λ_I in Section 11. The above theorem says that the Dirichlet form on Σ^+ associated with (T, C) is conservative if and only if the hitting probability of the single point ϕ_* by the original random walk (T, C) is 0.

To prove Theorem 10.8, we need series of lemmas.

Lemma 10.9. *Define $\widehat{T}_n = T_{\phi_n^-}^+ \cap T_{\phi_n^-}^\phi$ and $\rho_n = R_{\phi_n^-}(\widehat{T}_n, C|_{\widehat{T}_n \times \widehat{T}_n})$. Let R_n, r_n and $\widetilde{R}_n$ be the same as in the proof of Lemma 10.5. Then*

(1) $\nu_\phi(\{\phi_*\}) > 0 \Leftrightarrow \sum_{n \geq 0} \frac{R_n}{\rho_n} < +\infty$,
(2) $\lambda^+ > 0 \Leftrightarrow \sum_{n \geq 1} \frac{r_n}{\widetilde{R}_{n-1}} < +\infty$.

Proof. As is indicated in Fig. 4, the definitions of ρ_n, R_n and $\widetilde{R}_n$ imply

$$\frac{1}{R_n} = \frac{1}{r_{n+1} + R_{n+1}} + \frac{1}{\rho_n} \tag{10.8}$$

and

$$\frac{1}{\widetilde{R}_n} = \frac{1}{r_n + \widetilde{R}_{n-1}} + \frac{1}{\rho_n}. \tag{10.9}$$

(1) By (10.8), it follows that

$$\nu_\phi(\Sigma_{\phi_{n+1}^-}^\phi) = \left(1 - \frac{R_n}{\rho_n}\right) \nu_\phi(\Sigma_{\phi_n^-}^\phi). \tag{10.10}$$

Since $\nu_\phi(\{\phi_*\}) = \lim_{n\to\infty} \nu_\phi(\Sigma_{\phi_n^-}^\phi)$, we obtain the desired statement.

(2) By (10.5), we have

$$\lambda^+_{\phi_n^-} = \frac{\widetilde{R}_{n-1}}{r_n + \widetilde{R}_{n-1}} \frac{\widetilde{R}_{n-2}}{r_{n-1} + \widetilde{R}_{n-2}} \cdots \frac{\widetilde{R}_1}{r_2 + \widetilde{R}_1} \times \frac{1}{r_1 + \widetilde{R}_0}.$$

This implies the desired equivalence. □

Let $a_n = \widetilde{R}_n/\rho_n$ and let $b_n = r_n/\widetilde{R}_{n-1}$. Then by (10.9), we have $\widetilde{R}_n = (1-a_n)(1+b_n)\widetilde{R}_{n-1}$. Hence,

$$\widetilde{R}_n = A_n B_n, \tag{10.11}$$

where $A_0 = 1$, $A_n = (1-a_1)(1-a_2)\cdots(1-a_n)$ for $n \ge 1$ and $B_n = (1+b_1)(1+b_2)\cdots(1+b_n)\widetilde{R}_0$ for $n \ge 1$. Moreover,

$$\frac{1}{\rho_n} = a_n/(A_n B_n) = \left(\frac{1}{A_n} - \frac{1}{A_{n-1}}\right)\frac{1}{B_n}, \tag{10.12}$$

$$r_n = b_n A_{n-1} B_{n-1} = A_{n-1}(B_n - B_{n-1}). \tag{10.13}$$

Note that $A_n < A_{n-1}$ and $B_n > B_{n-1}$. From Lemma 10.9, we have the following statements.

Lemma 10.10. $\lambda^+ > 0 \Leftrightarrow \sum_{n\ge 1} b_n < +\infty \Leftrightarrow \sup_{n\ge 1} B_n < +\infty$.

The following lemma is the essential part of our proof of Theorem 10.8.

Lemma 10.11. $\lambda^+ > 0 \Leftrightarrow$

$$\sum_{n\ge 2}\left(\sum_{i=1}^{n-1}\frac{1}{\rho_i}\right) r_n < +\infty. \tag{10.14}$$

If (10.14) holds then $\sum_{n\ge 1} r_n < +\infty$. Note that

$$\sum_{n\ge 2}\left(\sum_{i=1}^{n-1}\frac{1}{\rho_i}\right) r_n = \sum_{n\ge 1}\left(\sum_{j=n+1}^{\infty} r_j\right)\frac{1}{\rho_n}. \tag{10.15}$$

Proof. $\Rightarrow$: (10.12) shows that

$$\sum_{i=1}^{n-1}\frac{1}{\rho_i} \le \sum_{i=1}^{n-1}\left(\frac{1}{A_i} - \frac{1}{A_{i-1}}\right)\frac{1}{B_1} \le \frac{1}{A_{n-1}}\frac{1}{B_1}.$$

Hence by (10.13) and Lemma 10.10(2),

$$\sum_{n\ge 2}\left(\sum_{i=1}^{n-1}\frac{1}{\rho_i}\right) r_n \le \sum_{n\ge 2}\frac{B_n - B_{n-1}}{B_1} = \frac{1}{B_1}\left(\lim_{n\to\infty} B_n - B_1\right) < +\infty.$$

$\Leftarrow$: Using (10.12), we have

$$\sum_{i=1}^{n-1}\frac{1}{\rho_i} = \frac{1}{A_{n-1}B_{n-1}} - \frac{1}{B_1} + \sum_{j=1}^{n-2}\frac{1}{A_j}\left(\frac{1}{B_j} - \frac{1}{B_{j+1}}\right) \ge \frac{1}{A_{n-1}B_{n-1}} - \frac{1}{B_1}.$$

Therefore by (10.13),

$$\sum_{n\geq 2}\left(\sum_{i=1}^{n-1}\frac{1}{\rho_i}\right)r_n+\frac{1}{B_1}\sum_{n\geq 2}r_n\geq\sum_{n\geq 2}b_n.$$

Since (10.14) implies $\sum_{n\geq 1} r_n < +\infty$, $\lambda^+ > 0$ follows from the above inequality. □

Next, let $x_n = R_n/\rho_n$, where $\rho_0 = \widetilde{R}_0$, and let $y_n = r_n/R_n$. Then by (10.8), it follows that $R_{n+1} = R_n/(1-x_n)(1+y_{n+1})$. Hence

$$R_n = \frac{1}{X_n Y_n}, \tag{10.16}$$

where $X_n = (1-x_{n-1})(1-x_{n-2})\cdots(1-x_0)$ and $Y_n = (1+y_n)(1+y_{n-1})\cdots(1+y_1)/R_0$. Furthermore,

$$\frac{1}{\rho_n} = x_n X_n Y_n = (X_n - X_{n+1})Y_n, \tag{10.17}$$

$$r_n = y_n R_n = \frac{1}{X_n}\left(\frac{1}{Y_{n-1}}-\frac{1}{Y_n}\right). \tag{10.18}$$

By Lemma 10.9, we have the following fact.

Lemma 10.12. $\nu_\phi(\{\phi_*\}) > 0 \Leftrightarrow \sum_{n\geq 1} x_n < +\infty \Leftrightarrow \inf_{n\geq 1} X_n > 0$.

Finally we conclude our proof of Theorem 10.8.

Proof of Theorem 10.8. By Lemma 10.11, it is enough to show that $\nu_\phi(\{\phi_*\}) > 0 \Leftrightarrow$ (10.14).

$\Leftarrow$: Since $R_n \leq \sum_{i\geq n+1} r_n$, (10.14) suffices to show $\nu_\phi(\{\phi_*\}) > 0$.

$\Rightarrow$: By Lemma 10.12, there exists $C > 0$ such that $1/X_n \leq C$ for any $n \geq 1$. Hence (10.18) shows that $\sum_{i\geq n+1} r_i \leq \frac{C}{Y_n}$. Combining this with (10.17)

$$\sum_{n\geq 1}\left(\sum_{i\geq n+1} r_i\right)\frac{1}{\rho_n} \leq C\sum_{n\geq 1}(X_n - X_{n+1}) \leq CX_1.$$

By (10.15), we have (10.14). □

At the end of this section, we present another fundamental relation between resistances, which will play an essential role in the proof of Theorem 11.3.

Theorem 10.13. *For any $m \geq 0$,*

$$\sum_{k=0}^{m}\frac{(X_{m+1})^2}{\rho_k X_k X_{k+1}} = \frac{1}{R_{m+1}+r_{m+1}+\widetilde{R}_m}, \tag{10.19}$$

where we let $X_0 = 1$ and $\widetilde{R}_0 = \rho_0$.

Proof. We use an induction on m. If $m = 0$, then (10.19) is

$$\frac{1}{\rho_0}\left(1-\frac{R_0}{\rho_0}\right) = \frac{1}{R_1+r_1+\rho_0}. \tag{10.20}$$

Since $1/R_0 = 1/\rho_0 + 1/(R_1+r_1)$, a routine calculation implies (10.20).

Assume that (10.19) holds for $m = M$. To establish (10.19) for $m = M + 1$, it is enough to show

$$\left(\frac{X_{M+2}}{X_{M+1}}\right)^2 \frac{1}{R_{M+1} + r_{M+1} + \widetilde{R}_M} + \frac{(X_{M+2})^2}{\rho_{M+1} X_{M+1} X_{M+2}} = \frac{1}{R_{M+2} + r_{M+2} + \widetilde{R}_{M+1}}. \tag{10.21}$$

Since $X_{M+2}/X_{M+1} = 1 - x_{M+1} = 1 - R_{m+1}/\rho_{m+1}$, (10.21) becomes

$$\frac{(\rho_{M+1} - R_{M+1})^2}{(\rho_{M+1})^2 (R_{M+1} + r_{M+1} + \widetilde{R}_M)} + \frac{\rho_{M+1} - R_{M+1}}{(\rho_{M+1})^2} = \frac{1}{R + \widetilde{R}_{M+1}}, \tag{10.22}$$

where $R = R_{M+2} + r_{M+2}$. Using the relations $1/\widetilde{R}_{M+1} = 1/\rho_{M+1} + 1/(r_{M+1} + \widetilde{R}_M)$ and $1/R_{M+1} = 1/\rho_{M+1} + 1/R$, we may eliminate $\widetilde{R}_{M+1}$ and R_{M+1} and show (10.22). Thus (10.19) for $m = M + 1$ holds. □

11. Dirichlet forms induced by random walks

In this section, we will show that the Dirichlet form on Σ^+ induced by a random walk (T, C) belongs to the class of Dirichlet forms studied in the earlier sections of this paper.

As in the last section, $\mathcal{T} = (T, \mathcal{A}, \phi_*)$ is an ordered tree which satisfies Assumption 2.2, $\pi = \pi_{\phi_*}$, $\phi \in T$ is a fixed referee point and (T, C) is a weight on $(T, \mathcal{A})$ which satisfies Assumption 9.7 throughout this section.

By Lemma 10.5, we may immediately verify the following statement.

Lemma 11.1. *For any ν_*-measurable function $f : \Sigma^+ \to \mathbb{R}$, we define $\tilde{f} : \Sigma \to \mathbb{R}$ by $\tilde{f}(\phi_*) = 0$ and $\tilde{f}(\omega) = f(\omega)$ for any $\omega \in \Sigma^+$. Then $\tilde{f}$ is a ν_ϕ-measurable function. In particular, if $f \in L^2(\Sigma^+, \nu_*)$, then $\tilde{f} \in L^2(\Sigma, \nu_\phi)$ and*

$$c \|\tilde{f}\|^2_{2,\nu_\phi} \le \|f\|^2_{2,\nu_*}, \tag{11.1}$$

where $c > 0$ is independent of f.

In light of the above lemma, we naturally regard $L^2(\Sigma^+, \nu_*)$ as a subset of $L^2(\Sigma, \nu_\phi)$. In this manner, we identify $\tilde{f}$ with f.

Now we present how we can induce a Dirichlet form on Σ^+ from a random walk on a tree. The basic idea is to extend a function on Σ^+ to a harmonic function on T by using the Martin kernel as in Theorem 9.11 and then consider the energy of the harmonic function.

Definition 11.2. *Define*

$$\mathcal{F}_{\Sigma^+} = \big\{ f \,|\, f \in L^2\big(\Sigma^+, \nu_*\big), Hf \in \mathcal{F} \big\} \quad \text{and} \quad \mathcal{E}_{\Sigma^+}(u, v) = \mathcal{E}(Hu, Hv)$$

for any $u, v \in \mathcal{F}_{\Sigma^+}$, where H is defined in Theorem 9.11.

Theorem 11.3. *Define $\lambda : T \to [0, \infty)$ by $\lambda(x) = \lambda^+_x$ for any $x \in T$. Then $(\lambda, \nu_*) \in \Theta^+_M(\mathcal{T})$, $\lambda_I = \lambda^+$ and $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+}) = (\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$, where $\Gamma = (\lambda, \nu_*)$. In particular, $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ is a regular Dirichlet form on $L^2(\Sigma^+, \nu_*)$.*

Since $\Gamma = (\lambda, \nu_*)$ is determined by the ordered tree $\mathcal{T} = (T, \mathcal{A}, \phi_*)$ and weight C on $(T, \mathcal{A})$, we write $\Gamma = \Gamma(\mathcal{T}, C)$.

We use the following result obtained in Section 5 of [18] to prove Theorem 11.3.

Theorem 11.4. *Let $\lambda^x_y = (\nu_x(\Sigma^x_y) R^x_y)^{-1}$. Define $\mathcal{F}_\Sigma = \{ f \,|\, f \in L^2(\Sigma), Hf \in \mathcal{F} \}$ and $\mathcal{E}_\Sigma(u, v) = \mathcal{E}(Hu, Hv)$ for any $u, v \in \mathcal{F}_\Sigma$. Then for any $x \in T$, $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a regular Dirichlet form on $L^2(\Sigma, \nu_x)$,*

$$\mathcal{F}_\Sigma = \left\{ u \,\middle|\, u \in L^2(\Sigma, \nu_x), \sum_{y \in T} \frac{\lambda^x_y}{2\nu_x(\Sigma^x_y)} \sum_{z, w \in S^x(y)} \nu_x\big(\Sigma^x_z\big) \nu_x\big(\Sigma^x_w\big) \big((u)_{z,\nu_x} - (u)_{w,\nu_x}\big)^2 < +\infty \right\}$$

and

$$\mathcal{E}_\Sigma(u,v)=\sum_{y\in T}\frac{\lambda_y^x}{2\nu_x(\Sigma_y^x)}\sum_{z,w\in S^x(y)}\nu_x\big(\Sigma_z^x\big)\nu_x\big(\Sigma_w^x\big)\big((u)_{z,\nu_x}-(u)_{w,\nu_x}\big)\big((v)_{z,\nu_x}-(v)_{w,\nu_x}\big)$$

for any $u,v\in\mathcal{F}_\Sigma$. *Moreover, if* $E_y^x=\{\varphi|\varphi=\sum_{z\in S^+(y)}a_z\chi_{\Sigma_z^x},\int_\Sigma\varphi\,d\nu_x=0\}$, *then* $\mathcal{E}(\varphi,u)=\lambda_y^x(\varphi,u)_{\nu_x}$ *for any* $y\in T$, *any* $\varphi\in E_y^x$, *and any* $u\in\mathcal{F}_\Sigma$.

Proof of Theorem 11.3. First we show that $(\mathcal{E}_{\Sigma^+},\mathcal{F}_{\Sigma^+})$ is closed. Note that $\mathcal{F}_{\Sigma^+}\subseteq\mathcal{F}_\Sigma$ and $\mathcal{E}_{\Sigma^+}=\mathcal{E}_\Sigma|_{\mathcal{F}_{\Sigma^+}\times\mathcal{F}_{\Sigma^+}}$ by Lemma 11.1. Let $\mathcal{E}^1_{\Sigma^+}(u,v)=\mathcal{E}_{\Sigma^+}(u,v)+(u,v)_{\nu_*}$ and $\mathcal{E}^1_\Sigma(u,v)=\mathcal{E}_\Sigma(u,v)+(u,v)_{\nu_\phi}$. Let $\{u_n\}_{n\ge1}$ be an $\mathcal{E}^1_{\Sigma^+}$-Cauchy sequence. By (11.1), $\{u_n\}_{n\ge1}$ is an $\mathcal{E}^1_\Sigma$-Cauchy sequence and at the same time an $L^2(\Sigma^+,\nu_*)$-Cauchy sequence. Hence there exists $u\in\mathcal{F}_{\Sigma^+}$ such that $\mathcal{E}^1_{\Sigma^+}(u-u_n,u-u_n)\to0$ as $n\to\infty$. Hence $(\mathcal{E}_{\Sigma^+},\mathcal{F}_{\Sigma^+})$ is closed.

It is immediate to see that $\Gamma\in\Theta_M^+(\mathcal{T})$. Theorem 3.7 and Proposition 3.10 imply that $(\mathcal{Q}_\Gamma,\mathcal{D}_\Gamma)$ is a regular Dirichlet form on $L^2(\Sigma^+,\nu_*)$. In particular, both $(\mathcal{E}_{\Sigma^+},\mathcal{F}_{\Sigma^+})$ and $(\mathcal{Q}_\Gamma,\mathcal{D}_\Gamma)$ are closed. Let L_1 and L_2 be the self-adjoint operators associated with closed forms $(\mathcal{E}_{\Sigma^+},\mathcal{F}_{\Sigma^+})$ and $(\mathcal{Q}_\Gamma,\mathcal{D}_\Gamma)$ respectively. To identify $(\mathcal{E}_{\Sigma^+},F_{\Sigma^+})$ with $(\mathcal{Q}_\Gamma,\mathcal{D}_\Gamma)$, it is enough to show that $L_1=L_2$.

Now let $y\in T$ and set $x=\pi(y)$. By Lemma 10.4, $S^+(y)=S^x(y)$, $R_y^x=R_y^+$, $\Sigma_y^+=\Sigma_y^x$ and there exists $c>0$ such that $\nu_x|_{\Sigma_y^+}=c\nu_*|_{\Sigma_y^+}$. This implies $E_{y,\nu_x}=E_{y,\nu_*}$. Theorem 11.4 shows that $\mathcal{E}_\Sigma(\varphi,u)=\lambda_y^x(\varphi,u)_{\nu_x}$ for any $\varphi\in E_{y,\nu_x}$ and any $u\in\mathcal{F}_\Sigma$. Since $\lambda_y^x=\lambda_y^+/c$ and $(\varphi,u)_{\nu_x}=c(\varphi,u)_{\nu_*}$, we have $\mathcal{E}_{\Sigma^+}(\varphi,u)=\lambda_y^+(\varphi,u)_{\nu_*}$. Hence $\varphi\in E_{y,\nu_*}$ is an eigenfunction with eigenvalue λ_y^+ of the self-adjoint operator on $L^2(\Sigma^+,\nu_*)$ associated with the closed form $(\mathcal{E}_{\Sigma^+},\mathcal{F}_{\Sigma^+})$.

Suppose $\nu_*(\Sigma^+)=+\infty$. Using Proposition 3.3, we see that $\bigoplus_{y\in T}E_{y,\nu_*}$ is dense in $L^2(\Sigma^+,\nu_*)$. Hence by the above discussion, we have all the eigenvalues and eigenfunctions of L_1. Theorem 3.4 shows that those eigenvalues and eigenfunctions are exactly the same as those of L_2. Thus we have $(\mathcal{E}_{\Sigma^+},\mathcal{F}_{\Sigma^+})=(\mathcal{Q}_\Gamma,\mathcal{D}_\Gamma)$.

Next suppose $\nu_*(\Sigma^+)<+\infty$. The orthogonal complement of $\bigoplus_{y\in T}E_{y,\nu_*}$ in $L^2(\Sigma^+,\nu_*)$ is $\{a\chi_{\Sigma^+}|a\in\mathbb{R}\}$. Assume $\nu_\phi(\{\phi_*\})=0$. Then $\chi_{\Sigma^+}(\omega)=\chi_\Sigma(\omega)$ for ν_ϕ-a.e. $\omega\in\Sigma$. Hence we have $\mathcal{E}_{\Sigma^+}(\chi_{\Sigma^+},\chi_{\Sigma^+})=\mathcal{E}_\Sigma(\chi_\Sigma,\chi_\Sigma)=0$. Therefore, χ_{Σ^+} is an eigenfunction with eigenvalue 0 of L_1. On the other hand, by Theorem 10.8-(1), it follows that $\lambda_I=0$ and hence χ_{Σ^+} is an eigenfunction with eigenvalue $\lambda_I=0$. Using Theorem 3.4, we see that χ_{Σ^+} is an eigenfunction with eigenvalue 0 of L_2. Thus all the eigenvalues and eigenfunctions of L_1 and L_2 coincide and hence $L_1=L_2$.

Finally suppose that $\nu_*(\Sigma^+)<+\infty$ and $\nu_\phi(\{\phi_*\})>0$. Since $\mathcal{E}_\Sigma(\chi_\Sigma,\chi_\Sigma)=\mathcal{E}_{(T,C)}(1,1)=0$, we see that $\mathcal{E}_{\Sigma^+}(\chi_{\Sigma^+},\chi_{\Sigma^+})=\mathcal{E}_\Sigma(\varphi_*,\varphi_*)$, where $\varphi_*=\chi_{\{\phi_*\}}=\chi_\Sigma-\chi_{\Sigma^+}$. Now write $\lambda_m=\lambda^\phi_{\phi_m^-}$, $S_m=S^\phi(\phi_m^-)$ and $\nu_m=\nu_\phi(\Sigma^\phi_{\phi_m^-})$ for $m\ge0$. Also let $\nu_y=\nu_\phi(\Sigma_y^\phi)$. Then

$$\begin{aligned}\mathcal{E}_\Sigma(\varphi_*,\varphi_*)&=\sum_{m\ge0}\frac{\lambda_m}{2\nu_m}\sum_{y,z\in S_m}\nu_y\nu_z\big((\varphi_*)_{y,\nu_\phi}-(\varphi_*)_{z,\nu_\phi}\big)^2\\&=\sum_{m\ge0}\frac{\lambda_m}{\nu_m}\sum_{z\in S_m\backslash\{\phi_{m+1}^-\}}\nu_{m+1}\nu_z\frac{\nu_\phi(\{\phi_*\})^2}{(\nu_{m+1})^2}=\nu_\phi\big(\{\phi_*\}\big)^2\sum_{m\ge0}\lambda_m\frac{\nu_m-\nu_{m+1}}{\nu_m\nu_{m+1}}.\end{aligned}$$

By (10.10), we have $X_m=\nu_m$. Also (10.16) shows that $\lambda_m=(\nu_mR_m)^{-1}=Y_m$. Therefore, using (10.17) and Theorem 10.13, we obtain

$$\mathcal{E}_{\Sigma^+}(\chi_{\Sigma^+},\chi_{\Sigma^+})=\Big(\lim_{m\to\infty}X_m\Big)^2\sum_{m\ge0}\frac{1}{\rho_mX_mX_{m+1}}=\lim_{m\to\infty}\frac{1}{R_{m+1}+r_{m+1}+\widetilde{R}_m}.$$

By Lemma 10.11, we have (10.14). In particular, $\sum_{m\geq 0} r_m < +\infty$. Since $R_m \leq \sum_{n\geq m+1} r_n$, it follows that $\lim_{m\to\infty} r_m = \lim_{m\to\infty R_m} = 0$. Therefore,

$$\mathcal{E}_{\Sigma^+}(\chi_{\Sigma^+}, \chi_{\Sigma^+}) = \lim_{m\to\infty} \frac{1}{\widetilde{R}_m} = \lim_{m\to\infty} \frac{\nu_*(\Sigma^+_{\phi_m^-})}{D^+_{\phi_m^-}} = \nu_*\big(\Sigma^+\big)\lambda^+.$$

This yields that χ_{Σ^+} is an eigenfunction of L_1 with eigenvalue $\lambda^+ = \lambda_I$. On the other hand, χ_{Σ^+} is an eigenfunction of L_2 with eigenvalue λ_I. Thus we have shown $L_1 = L_2$. □

Now that $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ is identified with $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ for $\Gamma = \Gamma(\mathcal{T}, C)$, we may apply the results from Section 3 to Section 6 to $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$. Note that Γ is in the monotone class, i.e. $\Gamma(\mathcal{T}, C) \in \Theta^+_M(\mathcal{T})$ and hence the conditions (λ1) and (λ2) are satisfied. For example, we have expressions of $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ by Theorems 3.4 and 3.7. Furthermore, if $\lambda^+_{[\omega]_m} \to +\infty$ as $m \to \infty$ for any $\omega \in \Sigma^+$, then there exists the transition density $p_\Gamma(t, \omega, \tau)$ by Theorem 4.5. Moreover, assume that ν_* satisfies (EL) and that λ satisfies (λ4), i.e. ν_* has the volume doubling property with respect to d_λ. Then we have the asymptotic behaviors of the transition density $p_\Gamma(t, \omega, \tau)$ and the jump kernel $J_*(\cdot, \cdot)$ by Corollary 6.3.

Example 11.5 (Self-similar weight on 2-adic numbers). *Let $T = T^{(2)}$ and let $(T, \mathcal{A}, \phi_*)$ be that given in Example 2.9. Naturally, Σ^+ is the set of 2-adic numbers $\mathbb{Q}_2$. We consider a self-similar weight C_γ on T. For $x = (\alpha_i)_{i\leq m} \in T^{(2)}_m$, define $K(x) = \max\{j | j \leq m, \alpha_i = 0$ for any $i \leq j\}$. Let $\gamma = (\gamma_0, \gamma_1) \in (0, \infty)^2$. We define C_γ by $C_\gamma(\pi(x), x)^{-1} = (\gamma_0)^{K(x)}\gamma_{\alpha_{K(x)+1}}\gamma_{\alpha_{K(x)+1}} \cdots \gamma_{\alpha_m}$ for $x = (\alpha_i)_{i\leq m} \in T^{(2)}_m$. Then, Assumption 9.7 holds if and only if $\gamma_0\gamma_1/(\gamma_0 + \gamma_1) < 1$. Under this condition, we see that $R^+_x = r_x R$, where $r_x = C_\gamma(\pi(x), x)^{-1}$ and $R = (\gamma_0 + \gamma_1)/(\gamma_0\gamma_1) - 1$. Letting $\nu_0 = \gamma_1/(\gamma_0 + \gamma_1)$ and $\nu_1 = \gamma_0/(\gamma_0 + \gamma_1)$ and using Proposition 10.3, we see that ν_* is the self-similar measure on Σ^+ with weight (ν_0, ν_1). These facts yield $D^+_x = \nu_*(\Sigma^+_x)R^+_x = (\gamma_0\gamma_1/(\gamma_0 + \gamma_1))^{|x|}R$. Hence $\lambda = \lambda(\mathcal{T}, C_\gamma)$ essentially belongs to the family discussed in Example 8.3. More precisely, $\lambda = \lambda_{\mathbf{s}}/R$ with $\mathbf{s} = (s_0, s_1)$, where $s_0 = s_1 = (\gamma_0)^{-1} + (\gamma_1)^{-1}$. Note that even if $\gamma_0 \neq \gamma_1$, λ is homogeneous, i.e. it only depends on $|x|$ and $d_\lambda(\omega, \tau) = n_2(\omega - \tau)^{\log s/\log 2}R$, where $s = s_0 = s_1$ and $n_2(\cdot)$ is the 2-adic norm. As is discussed in Example 8.3, $\Gamma = \Gamma(\mathcal{T}, C_\gamma)$ satisfies* (EL) *and* (λ4) *and hence we may apply Corollary 6.3 to obtain estimates of the transition density $p_\Gamma(t, \omega, \tau)$ and the jump kernel $J_*(\omega, \tau)$. Moreover, we have*

$$E_\omega\big(d_\lambda(\omega, X_t)^\theta\big) \asymp \begin{cases} +\infty & \text{if } \theta \geq 1, \\ t^\theta & \text{if } 0 < \theta < 1. \end{cases}$$

on $(t, \omega) \in (0, \infty) \times \Sigma^+$.

Example 11.6 (Homogeneous weight on 2-adic numbers). *Let $(T, \mathcal{A}, \phi_*)$ be the same as in Example 11.5. Let $r(n) > 0$ for any $n \in \mathbb{Z}$. Define $C(\pi(x), x) = r(|x|)^{-1}$. Then (T, C) is transient if and only if $\sum_{n\geq 0} r(n)/2^n < +\infty$. Assuming the transience, we have $R^+_x = 2^{|x|}\sum_{n\geq |x|+1} r(n)/2^n$ and $\nu_*(\Sigma^+_x) = 2^{-|x|}$ for any $x \in T$. In particular, $\nu_* = \mu_{\mathcal{T}}$. Hence $D^+_x = \sum_{n\geq |x|+1} r(n)/2^n$. Now let $\{\lambda(n)\}_{n\in\mathbb{Z}}$ satisfy $0 < \lambda(n) < \lambda(n+1)$ for any $n \in \mathbb{Z}$ and $\lim_{n\to\infty} \lambda(n) = +\infty$. Then there exists $\{r(n)\}_{n\in\mathbb{Z}}$ such that the associated (T, C) satisfies Assumption 9.7 and $\lambda^+_x = (D^+_x)^{-1} = \lambda(|x|)$ for any $x \in T$. Then by (4.1), we have the corresponding transition density $p(t, \omega, \tau)$ and the jump kernel $J_*(\omega, \tau)$ as follows*:

$$p(t, \omega, \tau) = \sum_{n=-\infty}^{|\omega\wedge\tau|} 2^n \big(e^{-\lambda(n-1)t} - e^{-\lambda(n)t}\big) \tag{11.2}$$

and

$$J_*(\omega, \tau) = \sum_{n=-\infty}^{|\omega\wedge\tau|} 2^n \big(\lambda(n) - \lambda(n-1)\big).$$

Hence choosing $\{\lambda(n)\}_{n\in\mathbb{Z}}$, one may build an example of (T, C) where $p(t, \omega, \tau)$ and $J_*(\omega, \tau)$ have desired asymptotic behaviors. Note that (11.2) *has been obtained in* [3] *as a formal expansion of a transition density in the case of* 2*-adic numbers.*

12. Inverse problem

In Section 11, we have observed that the regular Dirichlet form $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ on $L^2(\Sigma^+, \nu_*)$ associated with a transient random walk on T belongs to the class of Dirichlet forms defined in Section 3. To be precise, let $\lambda(x) = \lambda_x^+$ for any $x \in T$ and let $\Gamma = (\lambda, \nu_*)$. Then $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+}) = (\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$.

In this section, we consider the inverse problem. Namely, we first give $\lambda : T \to [0, \infty)$ and a Borel regular measure μ on Σ with $(\lambda, \mu) \in \Theta^+(\mathcal{T})$. Then we are going to ask whether or not there exists a transient random walk (T, C) such that the associated Dirichlet form $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ on $L^2(\Sigma^+, \nu_*)$ coincides with $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$ on $L^2(\Sigma^+, \mu)$. By Theorem 11.3, it is enough to show that $\Gamma(\mathcal{T}, C) = (\lambda, \mu)$ to identify $(\mathcal{E}_{\Sigma^+}, \mathcal{F}_{\Sigma^+})$ with $(\mathcal{Q}_\Gamma, \mathcal{D}_\Gamma)$. Proposition 10.7 implies that a necessary condition for the existence of a random walk is $\lambda(\pi(x)) < \lambda(x)$ for any $x \in T$. Under this condition, the assumption $(\lambda 1)$ holds as we have remarked.

As in the previous sections, $(T, \mathcal{A}, \phi_*)$ is a locally finite ordered tree which satisfies Assumption 2.2 and $\phi \in T$ in this section. We use π to denote π_{ϕ_*}.

Theorem 12.1. *Let $(\lambda, \mu) \in \Theta^+_{M,\infty}(T)$ with $\mu(\phi) = 1$. If $\lambda(\pi(x)) < \lambda(x)$ for any $x \in X$, then there exists a transient random walk (T, C) with Assumption 9.7 such that $\mu = \nu_*$ and $\lambda(x) = \lambda_x^+$ for any $x \in T$.*

The condition $\mu(\phi) = 1$ is merely a normalization. Recall that $\nu_*(\Sigma_\phi^+) = 1$ by definition.

Proof. First fix $x \in T$. Define a probability measure $\tilde{\mu}_x$ by $\tilde{\mu}_x(A) = \mu(A)/\mu(x)$ for any Borel set $A \subseteq \Sigma_x^+$. Then by Theorem 11.1 of [18], we have a transient random walk $(T_x^+, \widetilde{C}_x)$ on T_x^+ where $\tilde{\mu}_x$ is the hitting distribution starting from x and λ is the eigenvalue map. Moreover, let $\tilde{r}_y^x = \widetilde{C}_x(\pi(y), y)^{-1}$ for any $y \in T_x^+\backslash\{x\}$. Then using (11.3) of [18], we have

$$\tilde{r}_y^x = \frac{1}{\tilde{\mu}_x(\Sigma_y^+)}\left(\frac{1}{\lambda(\pi(y))} - \frac{1}{\lambda(y)}\right). \tag{12.1}$$

Also by Theorem 10.2,

$$\tilde{\mu}_x\bigl(\Sigma_y^+\bigr) = \frac{\widetilde{R}^x_{\pi(y)}}{\tilde{r}_y^x + \widetilde{R}_y^x}\tilde{\mu}_x\bigl(\Sigma^+_{\pi(y)}\bigr), \tag{12.2}$$

where $\widetilde{R}_y^x = R_y(T_y^+, \widetilde{C}_x|_{T_y^+\times T_y^+})$. Now define

$$r_z = \frac{1}{\mu(z)}\left(\frac{1}{\lambda(\pi(z))} - \frac{1}{\lambda(z)}\right)$$

for any $z \in T$. If $z \in T_x^+$, then

$$r_z = \tilde{r}_z^x/\mu(x). \tag{12.3}$$

Let $C(\pi(z), z) = (r_z)^{-1}$. We are going to show that (T, C) is the desired random walk. Let ν_* be the measure on Σ defined in Proposition 10.3 associated with (T, C). First, if $y \in T_x^+$, then (12.3) implies

$$R_y\bigl(T_y^+, C|_{T_y^+\times T_y^+}\bigr) = \widetilde{R}_y^x/\mu(x). \tag{12.4}$$

Let $R_y^+ = R_y(T_y^+, C|_{T_y^+\times T_y^+})$. Then (12.4) along with (12.3) yields

$$\mu(y) = \frac{R^+_{\pi(y)}}{r_y + R_y^+}\mu\bigl(\pi(y)\bigr).$$

Note that $\nu_*(\Sigma_\phi^+) = \mu(\phi) = 1$. Hence, by Proposition 10.3, the above equality shows that $\nu_* = \mu$. Since λ is the eigenvalue map of $(T_x^+, \widetilde{C}_x)$, if $y \in T_x^+$, we have

$$\frac{1}{\lambda(y)} = \tilde{\mu}_x(\Sigma_y^+)\widetilde{R}_y^x = \mu(y)\frac{\widetilde{R}_y^x}{\mu(x)} = \mu(y)R_y^+ = \frac{1}{\lambda_y^+}. \tag{12.5}$$

Thus we have shown that (T, C) is the desired random walk. □

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