Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2013, Vol. 49, No. 4, 1080–1089
DOI: 10.1214/12-AIHP502

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

On random fractals with infinite branching: Definition, measurability, dimensions[1]

Artemi Berlinkov

Komendantski pr. 7-1-25, St. Petersburg, 197227, Russian Federation. E-mail: artemiberlinkov@hotmail.com

Received 17 April 2011; revised 13 June 2012; accepted 15 June 2012

Abstract. We investigate the definition and measurability questions of random fractals with infinite branching, and find, under certain conditions, a formula for the upper and lower Minkowski dimensions. For the case of a random self-similar set we obtain the packing dimension.

Résumé. Nous étudions les questions de la définition et de la mesurabilité des fractales aléatoires avec ramification infinie. Nous trouvons sous certaines conditions une formule pour les dimensions de Minkowski supérieure et inférieure. Pour un d'ensemble aléatoire auto-similaire nous obtenons la dimension.

MSC: Primary 28A80; secondary 28A78; 60D05; 37F40

Keywords: Packing dimension; Minkowski dimension; Random fractal

1. Introduction

In this paper we study the Minkowski and packing dimensions of random fractals with infinite branching.

The almost sure Hausdorff dimension of random fractals was independently found by Mauldin and Williams in [13], and Falconer in [5]. Packing dimension and measures in case of finite branching were investigated by Berlinkov and Mauldin in [4]. It was shown that if the number of offspring is uniformly bounded, the Hausdorff, packing, lower and upper Minkowski dimensions coincide a.s.

Barnsley et al. in [1] introduced the notion of V-variable fractals and in [2] find their Hausdorff dimension. Fraser in [7] discusses the Minkowski dimension, packing and Hausdorff measures from topological (in the Baire sense) point of view rather than probabilistic. Random fractals find interesting and important applications in other areas, e.g. harmonic analysis [3], stochastic processes and random fields [14].

However most authors focus on the situation when the fractals are finitely branching, or, in other words, the number of offspring is bounded. In this paper we investigate the case when the number of offspring may be infinite. If it is bounded but not uniformly, the results of this paper show that all of these dimensions still coincide. If the number of offspring is unbounded, these dimensions may differ from each other, as shown in Examples 1, 2 from Section 6. As we see in these examples, the Minkowski dimensions may be non-degenerate random variables, whereas in [4] for the case of finite branching they have been shown to coincide with the a.s. constant Hausdorff dimension.

In Section 2 we give a precise definition of a random recursive construction and show that another definition used in [1] for random fractals coincides with it. Since in the case of infinite branching the Minkowski dimensions no longer

[1]Research was partially supported by NSF Grant DMS-0100078 and DFG-Graduiertenkolleg "Approximation und algorithmische Verfahren" in Jena, Germany.

have to be constant, their measurability is proven in Section 4. In Section 5 we derive the Minkowski dimensions of random recursive constructions under some additional conditions and a formula for the packing dimensions of random self-similar sets with infinite branching.

2. On the definition of random fractals

Let $n \in \mathbb{N} \cup \{\infty\}$, $\Delta = \{1, \ldots, n\}$ if $n < \infty$, and $\Delta = \mathbb{N}$ if $n = \infty$. Denote by $\Delta^* = \bigcup_{j=0}^{\infty} \Delta^j$ the set of all finite sequences of numbers in Δ, and by $\Delta^{\mathbb{N}}$ the set of all their infinite sequences. The result of concatenation of two finite sequences σ and τ from Δ^* is denoted by $\sigma * \tau$. For a finite sequence σ, its length will be denoted by $|\sigma|$. For a sequence σ of length at least k, $\sigma|_k$ is a sequence consisting of the first k numbers in σ. There is a natural partial order on the n-ary tree Δ^*: $\sigma \prec \tau$ if and only if the sequence τ starts with σ. A set $S \subset \Delta^*$ is called an antichain, if $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$ for all $\sigma, \tau \in \Delta^*$.

The following construction was proposed by Mauldin and Williams in [13]. We have to modify the original definition to fully take into account the case of offspring degeneration (see condition (vi)) below.

Suppose that J is a compact subset of $\mathbb{R}^d$ such that $J = \mathrm{Cl}(\mathrm{Int}(J))$, without loss of generality its diameter equals one. The construction is a probability space (Ω, Σ, P) with a collection of random subsets of $\mathbb{R}^d$ – $\{J_\sigma(\omega) | \omega \in \Omega, \sigma \in \Delta^*\}$, so that the following conditions hold.

(i) $J_\varnothing(\omega) = J$ for almost all $\omega \in \Omega$.
(ii) For all $\sigma \in \Delta^*$ the maps $\omega \to J_\sigma(\omega)$ are measurable with respect to Σ and the topology generated by the Hausdorff metric on the space of compact subsets.
(iii) For all $\sigma \in \Delta^*$ and $\omega \in \Omega$, the sets J_σ, if non-empty, are geometrically similar to J[2].
(iv) For almost every $\omega \in \Omega$ and all $\sigma \in \Delta^*$, $i \in \Delta$, $J_{\sigma * i}$ is a proper subset of J_σ provided $J_\sigma \neq \varnothing$.
(v) The construction satisfies the random *open set condition*: if σ and τ are two distinct sequences of the same length, then $\mathrm{Int}(J_\sigma) \cap \mathrm{Int}(J_\tau) = \varnothing$ a.s. and, finally.
(vi) The random vectors $\mathbf{T}_\sigma = (T_{\sigma*1}, T_{\sigma*2}, \ldots)$, $\sigma \in \Delta^*$, are conditionally i.i.d. given that $J_\sigma(\omega) \neq \varnothing$, where $T_{\sigma*i}(\omega)$ equals the ratio of the diameter of $J_{\sigma*i}(\omega)$ to the diameter of $J_\sigma(\omega)$.

The object of study is the random set

$$K(\omega) = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in \Delta^k} J_\sigma(\omega).$$

In general in condition (iii) other classes of functions instead of similarities may be used, e.g. conformal or affine mappings.

The meaning of condition (vi) is the following. Given that J_σ is non-empty, we ask that the random vectors of reduction ratios $\mathbf{T}_\sigma = (T_{\sigma*1}, T_{\sigma*2}, \ldots)$, have the same conditional distribution and be conditionally independent, i.e. for any finite antichain $S \subset \Delta^*$ and any collection of Borel sets $B_s \subset [0, 1]^\Delta$, $s \in S$,

$$P(\mathbf{T}_s \in B_s \, \forall s \in S | J_s \neq \varnothing \, \forall s \in S) = \prod_{s \in S} P(\mathbf{T}_s \in B_s | J_s \neq \varnothing)$$

and $\mathbf{T}_\sigma$ has the same distribution as $\mathbf{T}_\varnothing$, provided $J_\sigma \neq \varnothing$, i.e. for any $\sigma \in \Delta^*$ and any Borel set $B \subset \mathbb{R}^\Delta$,

$$P(\mathbf{T}_\sigma \in B | J_\sigma \neq \varnothing) = P(\mathbf{T}_\varnothing \in B).$$

Following [13] the above is called a random recursive construction.

The second term commonly used is “random fractals” (see, e.g. [1]), where condition (vi) is replaced by existence of an i.i.d. sequence of random vectors of reduction ratios. We note that the following holds:

[2]The sets $A, B \subset \mathbb{R}^d$ are geometrically similar, if there exist $S : \mathbb{R}^d \to \mathbb{R}^d$ and $r > 0$ such that for all $x, y \in \mathbb{R}^d$ $\mathrm{dist}(S(x), S(y)) = r\,\mathrm{dist}(x, y)$ and $S(A) = B$, such S is called a similarity map.

Proposition 1. *Random fractals and random recursive constructions are the same class of sets.*

Proof. That every random recursive construction is a random fractal is obvious because we can set the distributions of $\mathbf{T}_\sigma$ given $J = \varnothing$ the same as $\mathbf{T}_\varnothing$.

Suppose that we have a random fractal. Then the random vector $\mathbf{T}_\sigma$ is independent of vectors $\mathbf{T}_\tau$ with $\tau \prec \sigma$ and, in particular, of the event $J_\sigma \neq \varnothing$, therefore the second equality for the random vectors being conditionally i.i.d. holds. In the first equality the right-hand side equals

$$\prod_{s \in S} P(\mathbf{T}_s \in B_s)$$

because S is an antichain and $\mathbf{T}_s$ do not depend on events $\{J_s \neq \varnothing\}$, $s \in S$, while the left hand side equals the same expression for the same reason. □

Another definition in [13] for random stochastically geometrically self-similar sets made no reference to independence in the construction but a similar kind of conditional independence condition is needed to find the dimension of the limit set. We call such sets *random self-similar sets*, and for them not only the reduction ratios but also the maps (see Section 5) that map parent to its offspring are conditionally i.i.d.

3. Preliminaries

If the average number of offspring does not exceed one, then $K(\omega)$ is almost surely an empty set or a point, and we exclude that case from further consideration. Mauldin and Williams in [13] have found the Hausdorff dimension of almost every non-empty set $K(\omega)$,

$$\alpha = \inf\left\{\beta \,\middle|\, E\left[\sum_{i=1}^{n} T_i^\beta\right] \le 1\right\}.$$

In case $n < \infty$, α is the solution of equation

$$E\left[\sum_{i=1}^{n} T_i^\alpha\right] = 1.$$

The definitions and properties of Hausdorff and packing measures and dimensions, as well as definitions of upper and lower Minkowski dimensions, can be found in the book of Mattila [9]. We denote the Hausdorff, packing, lower and upper Minkowski dimension by $\dim_H$, $\dim_P$, $\underline{\dim}_B$ and $\overline{\dim}_B$ respectively.

For any $K \subset \mathbb{R}^d$ denote by $N_r(K)$ the smallest number of closed balls with radii r, needed to cover K. Then the upper Minkowski dimension,

$$\overline{\dim}_B K = \varlimsup_{r \to 0} -N_r(K)/\log r$$

and the lower Minkowski dimension,

$$\underline{\dim}_B K = \varliminf_{r \to 0} -N_r(K)/\log r.$$

Denote by $\overline{M}$ the closure of a set M. Obviously, if M is bounded, then $\overline{\dim}_B M = \overline{\dim}_B \overline{M}$ and $\underline{\dim}_B M = \underline{\dim}_B \overline{M}$ (see, e.g., [6], Proposition 3.4). One can use the maximal number of disjoint balls of radii r with centers in K (which will be denoted by $P_r(K)$) instead of the minimal number of balls needed to cover set K in the definition of Minkowski dimensions because of the following relation ([6], (3.9) and (3.10)):

$$N_{2r}(K) \le P_r(K) \le N_{r/2}(K).$$

The packing dimension can be defined using upper Minkowski dimension:

$$\dim_P K = \inf\left\{\sup \overline{\dim}_B F_i \,|\, K \subset \bigcup_i F_i\right\}.$$

4. Measurability of Minkowski dimensions

The measurability questions of dimension functions in deterministic case have been studied by Mattila and Mauldin in [10]. We start by exploring these questions for random fractals. In case of finite branching there is an obvious topology with respect to which the functions $\omega \mapsto \overline{\dim}_B K(\omega)$ and $\omega \mapsto \underline{\dim}_B K(\omega)$ are measurable – the topology generated on the space of compact subsets of J by the Hausdorff metric. However, it is unknown to the author, with repect to which topology these maps would be measurable in the case of infinite branching. Therefore we circumvent this problem as follows.

Denote by $\mathcal{K}(J)$ the space of compact subsets of J equipped with the Hausdorff metric

$$d_H(L_1, L_2) = \max\left\{\sup_{x\in L_1} \operatorname{dist}(x, L_2),\ \sup_{y\in L_2} \operatorname{dist}(L_1, y)\right\}.$$

Lemma 2. *Suppose that $L_i \in \mathcal{K}(J)$, $i \in \mathbb{N}$. Then*

$$\lim_{k\to+\infty} \bigcup_{i=1}^{k} L_i = \overline{\bigcup_{i=1}^{+\infty} L_i} \quad \textit{in the Hausdorff metric}.$$

Proof. Suppose that

$$\lim_{n\to+\infty} d_H\left(\bigcup_{i=1}^{n} L_i, \overline{\bigcup_{i=1}^{+\infty} L_i}\right) > 0.$$

Since $\bigcup_{i=1}^{n} L_i \subset \overline{\bigcup_{i=1}^{+\infty} L_i}$, there exists an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $p_n \in \overline{\bigcup_{i=1}^{+\infty} L_i}$ with $\operatorname{dist}(p_n, \bigcup_{i=1}^{n} L_i) \geq \varepsilon$. Without loss of generality we can assume that p_n converges to some $p \in \overline{\bigcup_{i=1}^{+\infty} L_i}$. Then $\operatorname{dist}(p, \bigcup_{i=1}^{+\infty} L_i) \geq \varepsilon/2$ which is a contradiction. □

Corollary 3. *The map $\omega \mapsto \overline{\bigcup_{|\tau|=n, J_\tau\cap K\neq\varnothing} J_\tau(\omega)}$ is measurable.*

Corollary 4. *If τ_i, $i \in \mathbb{N}$, is an enumeration of $\{\tau \in \Delta^n \,|\, J_\tau \cap K \neq \varnothing\}$, then*

$$\lim_{k\to\infty} P_r\left(\bigcup_{i=1}^{k} J_{\tau_i}\right) = P_r\left(\overline{\bigcup_{\substack{|\tau|=n\\ J_\tau\cap K\neq\varnothing}} J_\tau}\right).$$

Proof. The statement follows from the fact that the function $P_r : \mathcal{K}(J) \to \mathbb{R}$ is lower semicontinuous (see, [10], remark after Lemma 3.1). □

Lemma 5. *In the Hausdorff metric, $\lim_{n\to\infty} \overline{\bigcup_{|\tau|=n, J_\tau\cap K\neq\varnothing} J_\tau(\omega)} = \overline{K(\omega)}$ for a.e. $\omega \in \Omega$.*

Proof. According to [13], (1.14), $\lim_{n\to\infty} \sup_{\tau\in\Delta^n} l_\tau = 0$ for a.e. $\omega \in \Omega$. Consider such an ω. Suppose that

$$\lim_{n\to\infty} d_H\left(\overline{\bigcup_{\substack{|\tau|=n\\ J_\tau\cap K\neq\varnothing}} J_\tau(\omega)}, \overline{K(\omega)}\right) > 0,$$

then there exists an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $p_n \in \overline{\bigcup_{|\tau|=n, J_\tau \cap K \neq \varnothing} J_\tau(\omega)}$ with $\operatorname{dist}(p_n, \overline{K(w)}) \geq \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that for all $\tau \in \Delta^*$ of length at least n_0 the following holds:

$$l_\tau(\omega) < \varepsilon/4.$$

Without loss of generality p_n converges to some $p \in J$. Thus $\operatorname{dist}(p, K(w)) \geq \varepsilon$. Next choose $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that for all $n \geq n_1$

$$\operatorname{dist}(p_n, p) < \varepsilon/4.$$

Since a $3\varepsilon/4$ neighborhood of p_n contains a point of $K(\omega)$, we get a contradiction. □

Corollary 6. $\lim_{n\to+\infty} N_r(\bigcup_{|\tau|=n, J_\tau \cap K \neq \varnothing} J_\tau(\omega)) = N_r(K(\omega))$ *for a.e.* ω. *The equality holds if either set is replaced with its closure.*

Proof. This follows from the facts that the function $N_r : \mathcal{K}(J) \to \mathbb{R}$ is upper semicontinuous (see, e.g., [10], proof of Lemma 3.1) and $N_r(A) = N_r(\overline{A})$. □

From the statements above follows:

Theorem 7. *The maps* $\omega \to \overline{\dim}_B K(\omega)$ *and* $\omega \to \underline{\dim}_B K(\omega)$ *are measurable.*

Proof. Since the maps

$$\omega \to \overline{K(\omega)}, \qquad \omega \to N_r\big(\overline{K(\omega)}\big) \quad \text{and} \quad \omega \to N_r\big(K(\omega)\big)$$

are measurable, the measurability of the lower and upper Minkowski dimensions of $K(\omega)$ follows from their definition. □

5. Dimensions of random fractals

In this section we derive several expressions for Minkowski and packing dimensions of random self-similar fractals with infinite branching.

Lemma 8. *Suppose that* $t > \dim_H K$ *a.s.,* $0 < p = E[\sum_{i\in\Delta} T_i^t] < 1$ *and* $q \in \mathbb{N}$. *If* Γ *is an arbitrary* (*random*) *antichain such that* $|\tau| \geq q$ *for all* $\tau \in \Gamma$ *a.s., then* $E[\sum_{\tau\in\Gamma} l_\tau^t] \leq \frac{p^q}{1-p}$.

Proof. Indeed, $E[\sum_{\tau\in\Gamma} l_\tau^t] \leq \sum_{k=q}^{+\infty} E[\sum_{|\tau|=k} l_\tau^t] \leq \sum_{k=q}^{+\infty} p^k = \frac{p^q}{1-p}$. □

We will also need the following 2 conditions:

(vii) the construction is pointwise finite, i.e. each element of J belongs a.s. to at most finitely many sets J_i, $i \in \Delta$ (see [11]) and

(viii) J possesses the *neighborhood boundedness property* (see [8]): there exists an $n_0 \in \mathbb{N}$ such that for every $\varepsilon > \operatorname{diam}(J)$, if $J_1, \ldots, J_k$ are non-overlapping sets which are all similar to J with $\operatorname{diam}(J_i) \geq \varepsilon > \operatorname{dist}(J, J_i)$; $i = 1, \ldots, k$, then $k \leq n_0$.

As we will see, knowledge of similarity maps is essential to find the Minkowski dimension. For $\tau \in \Delta^*$, let $K_\tau(\omega) = \bigcup_{\eta\in\Delta^{\mathbb{N}}, \eta|_{|\tau|}=\tau} \bigcap_{i=1}^{\infty} J_{\eta|_i}(\omega) \subset J_\tau(\omega) \cap K(\omega)$. Fix a point $a \in \mathbb{R}^d$ with $\operatorname{dist}(a, J) \geq 1$. Denote by $S_\sigma^\tau : \mathbb{R}^d \to \mathbb{R}^d$ a random similarity map such that $S_\sigma^\tau(J_\tau) = J_{\tau*\sigma}$. If $J_\tau = \varnothing$ or $J_{\tau*\sigma} = \varnothing$, then we let $S_\sigma^\tau(\mathbb{R}^d) = a$. For a finite word $\sigma \in \Delta^*$, let $l_\sigma = \operatorname{diam}(J_\sigma)$. From [13] we know that $\lim_{k\to\infty} \sup_{|\tau|=k} l_\tau = 0$ a.s. For $x \in J_\tau$ and $n \in \mathbb{N}$, consider the random n-orbit of x within J_τ, $\mathrm{O}_\tau(x, n) = \bigcup_{|\sigma|=n, J_{\tau*\sigma}\cap K\neq\varnothing} S_\sigma^\tau(x)$. For $I \subset \Delta^*$, let $\mathrm{O}_\tau(x, I) = \bigcup_{\sigma\in I, J_{\tau*\sigma}\cap K\neq\varnothing} S_\sigma^\tau(x)$. In case $\tau = \varnothing$, $\mathrm{O}_\tau(x, I)$ is denoted by $\mathrm{O}(x, I)$, $\mathrm{O}_\tau(x, n)$ by $\mathrm{O}(x, n)$, and S_σ^τ by S_σ.

Lemma 9. *For all $\omega \in \Omega$, $n \in \mathbb{N}$, and any two collections of points $X = \{x_k\}_{k=1}^{\infty}$, $Y = \{y_k\}_{k=1}^{\infty} \subset \bigcup_{|\sigma|=n} J_\sigma$ such that for all $\sigma \in \Delta^n$ $\operatorname{card}(Y \cap J_\sigma) = \operatorname{card}(X \cap J_\sigma) = 1$ or 0, $\overline{\dim}_B X = \overline{\dim}_B Y$ and $\underline{\dim}_B X = \underline{\dim}_B Y$.*

Proof. Without loss of generality we assume that $n = 1$ since for every $n > 1$ the collection of sets $\{J_\tau\}$ such that $|\tau|$ is divisible by n forms a random recursive construction. First we note that there exists an $M > 0$ such that

$$\forall r > 0, \forall z \in \mathbb{R}^d \quad \operatorname{card}\big\{i \in \Delta | B(z,r) \cap J_i(\omega) \neq \varnothing \text{ and } l_i(\omega) \geq r/2\big\} \leq M.$$

Fix $\omega \in \Omega$, $z \in \mathbb{R}^d$, $r > 0$. Obviously $B(z,r)$ can be covered by 12^d balls of radius $r/6$. Let B_1 be one of them and place inside B_1 a set similar to J. By the neighborhood boundedness property with $\varepsilon = r/2$, we obtain $\operatorname{card}\{i \in \Delta | B_1 \cap J_i \neq \varnothing$ and $l_i \geq r/2\} \leq n_0$. Therefore it suffices to take $M = 12^d n_0$.

Finally take $0 < r \leq 2$, let $I_r(\omega) = \bigcup_{l_i(\omega)<r/2} J_i(\omega)$ and $I_r'(\omega) = \bigcup_{l_i(\omega)\geq r/2} J_i(\omega)$. Then $N_r(Y \cap I_r) \leq N_{r/2}(X \cap I_r)$. Clearly, for any collection of points $Z = \{z_k\}_{k=1}^{\infty}$, such that $\operatorname{card}(Z \cap J_i) = 0$ or 1 for all i, we have $N_r(Z \cap I_r') \leq \operatorname{card}(I_r')$. On the other hand $N_r(Z \cap I_r') \geq \operatorname{card}(I_r')/M$. Hence,

$$N_r(Y) \leq N_{r/2}(X \cap I_r) + N_r\big(Y \cap I_r'\big) \leq N_{r/2}(X) + M N_r\big(X \cap I_r'\big) \leq (1+M) N_{r/2}(X).$$

The result follows. □

Remark. *From the proof of Lemma 9, we see that if for some $x \in J$, $D > 0$ and $0 \leq u \leq d$ for all $0 < r \leq 2$, $N_r(\mathrm{O}(x,1)) \leq D r^{-u}$, then for all $y \in J$, $N_r(\mathrm{O}(y,1)) \leq 2^d(12^d n_0 + 1) D r^{-u}$.*

For $\tau \in \Delta^*$, let $\overline{\gamma}_\tau = \overline{\dim}_B \mathrm{O}_\tau(x,1)$ for some $x \in J_\tau$ and let $\overline{\gamma} = \sup_{\tau\in\Delta^*} \overline{\gamma}_\tau$. By Lemma 9, $\overline{\gamma}_\tau$ does not depend on the choice of x. Similarly we define $\underline{\gamma}_\tau = \underline{\dim}_B \mathrm{O}_\tau(x,1)$ and $\underline{\gamma} = \sup_{\tau\in\Delta^*} \underline{\gamma}_\tau$. For the rest of the paper, suppose additionally that

(ix) there exists $A > 0$ such that for all $\tau \in \Delta^*$, $x \in J_\tau$, $t > 0$ and $0 < r \leq 2$ we have $N_r(\mathrm{O}_\tau(x,1)) \mathbf{1}_{\{\overline{\gamma}_\tau < t\}} \leq A r^{-t} l_\tau^t$.

Lemma 10. *For any $x \in J$,*

$$\max\Big\{\dim_H K, \sup_n \overline{\dim}_B \mathrm{O}(x,n)\Big\} = \max\{\dim_H K, \overline{\gamma}\} \quad \textit{and}$$

$$\max\Big\{\dim_H K, \sup_n \underline{\dim}_B \mathrm{O}(x,n)\Big\} = \max\{\dim_H K, \underline{\gamma}\} \quad \textit{a.s.}$$

Proof. Fix $\omega \in \Omega$. Since for any $\tau \in \Delta^*$, $\mathrm{O}_\tau(S_\tau(x),1) \subset \mathrm{O}(x, |\tau|+1)$, we have $\overline{\gamma}_\tau = \overline{\dim}_B \mathrm{O}_\tau(S_\tau(x),1) \leq \overline{\dim}_B \mathrm{O}(x,|\tau|+1) \leq \sup_n \overline{\dim}_B \mathrm{O}(x,n)$, and $\overline{\gamma} \leq \sup_n \overline{\dim}_B \mathrm{O}(x,n)$.

In the opposite direction we prove by induction on n that if $P(\max\{\dim_H K, \overline{\gamma}\} < t) > 0$ for some $t > 0$, then there exists a random variable $B_n > 0$ such that $E[B_n] < +\infty$ and $N_r(\mathrm{O}(x,n)) \mathbf{1}_{\{\overline{\gamma}<t\}} \leq B_n r^{-t}$ a.s. for all $0 < r \leq 1$. When $n = 1$, we let $B_1 = A$. Suppose that for all $n \leq k$ and for all $0 < r \leq 1$, there exists $B_n > 0$ with $E[B_n] < +\infty$ such that $N_r(\mathrm{O}(x,n)) \mathbf{1}_{\{\overline{\gamma}<t\}} \leq B_n r^{-t}$ a.s. To prove the statement for $n = k+1$, fix $r > 0$ and set $I_r(\omega) = \{\tau \in \Delta^k | l_\tau(\omega) < r/2\}$. Then

$$N_r\big(\mathrm{O}(x, I_r \times \Delta)\big) \leq N_{r/2}\big(\mathrm{O}(x, I_r)\big) \leq N_{r/2}\big(\mathrm{O}(x,k)\big).$$

For a fixed $\tau \in \Delta^k$,

$$N_r\big(\mathrm{O}_\tau\big(S_\tau(x),1\big)\big) \mathbf{1}_{\tau\notin I_r} \mathbf{1}_{\{\overline{\gamma}<t\}} \leq A l_\tau^t r^{-t}.$$

Therefore

$$N_r\big(\mathrm{O}(x,k+1)\big)\mathbf{1}_{\{\overline{\gamma}<t\}} \leq N_{r/2}\big(\mathrm{O}(x,k)\big)\mathbf{1}_{\{\overline{\gamma}<t\}} + \sum_{|\tau|=k} N_r\big(\mathrm{O}_\tau\big(S_\tau(x),1\big)\big)\mathbf{1}_{\tau\notin I_r}\mathbf{1}_{\{\overline{\gamma}<t\}} \leq 2^t B_k r^{-t} + A r^{-t} \sum_{|\tau|=k} l_\tau^t.$$

Set $B_{k+1} = 2^t B_k + A \sum_{|\tau|=k} l_\tau^t$. If we fix n, then by Markov's inequality for every $\varepsilon > 0$

$$\sum_{i=0}^{\infty} P\left(B_n 2^{it} > 2^{i(t+\varepsilon)}\right) \le \sum_{i=0}^{\infty} E[B_n] 2^{-i\varepsilon} < \infty$$

and therefore by Borel–Cantelli lemma for a.e. $\omega \in \Omega$ $B_n 2^{it} > 2^{i(t+\varepsilon)}$ only finitely many times, hence for a.e. $\omega \in \Omega$ $N_{2^{-i}}(\mathrm{O}(x,n))\mathbf{1}_{\{\overline{\gamma}<t\}} > 2^{i(t+\varepsilon)}$ only finitely many times. Therefore

$$\overline{\lim_{i\to\infty}} \frac{\log N_{2^{-i}}(\mathrm{O}(x,n))}{i \log 2} < t + \varepsilon$$

for almost every ω such that $\max\{\dim_H K(\omega), \overline{\gamma}(\omega)\} < t$ for every $\varepsilon > 0$. Thus for almost every such ω we have $\overline{\dim}_B \mathrm{O}(x,n) \le t$. The same argument holds for the lower Minkowski dimension. □

From the proof of the last lemma and the fact that there cannot be more than 10^d offspring in the construction of diameter at least $1/5$ follows

Corollary 11. *Suppose that $q \in \mathbb{N}$, construction satisfies property (ix), for some $t > 0$ $P(\max\{\dim_H K, \overline{\gamma}\} < t) > 0$ and let*

$$\Gamma_{\tau,q} = \left\{\eta \in \Delta^{|\tau|+q} \colon l_\eta < l_\tau/5\right\} \cup \left\{\eta \in \Delta^* \colon |\eta| > |\tau| + q, l_\eta < l_\tau/5, l_{\eta|_{|\eta|-1}} \ge l_\tau/5\right\}.$$

Then there exists a random variable B_q' with $E[B_q'] < +\infty$ such that

$$N_r\left(\mathrm{O}(x, \Gamma_{\tau,q})\right)\mathbf{1}_{\{\overline{\gamma}<t\}} \le B_q' l_\tau^t r^{-t}.$$

Proof. Let

$$\Gamma_{0,\tau,q} = \left\{\sigma \in \Delta^* \colon |\sigma| \ge |\tau| + q, l_\sigma \ge l_\tau/5, \exists \tau \in \Gamma_{\tau,q} \colon \tau|_{|\tau|-1} = \sigma\right\}.$$

Then

$$\begin{aligned} N_r\left(\mathrm{O}(x, \Gamma_{\tau,q})\right)\mathbf{1}_{\{\overline{\gamma}<t\}} &\le N_r\left(\mathrm{O}_\tau\left(S_\tau(x), q\right)\right)\mathbf{1}_{\{\overline{\gamma}<t\}} + \sum_{\sigma \in \Gamma_{0,\tau,q}} N_r\left(\mathrm{O}_\sigma\left(S_\sigma(x), 1\right)\right)\mathbf{1}_{\{\overline{\gamma}<t\}} \\ &\le B_q l_\tau^t r^{-t} + A l_\tau^t r^{-t} \operatorname{card}\left\{\sigma \in \Delta^* | l_\sigma \ge 1/5\right\}, \end{aligned}$$

where B_q and the estimate on the first term come from the proof of Lemma 10, and the second term is bounded according to condition (ix).

Note that if $0 < p = E[\sum_{i\in\Delta} T_i^t] < 1$, then

$$E\left[\sum_{|\tau|=q} \frac{l_\tau^t}{(1/5)^t}\right] = 5^t E\left[\sum_{|\tau|=q} l_\tau^t\right] = 5^t p^q \ge E\left[\operatorname{card}\left\{\tau | \tau \in \Delta^q, l_\tau \ge 1/5\right\}\right].$$

Hence

$$E\left[\operatorname{card}\left\{\sigma \in \Delta^* | l_\sigma \ge 1/5\right\}\right] = \sum_{k=1}^{+\infty} E\left[\operatorname{card}\left\{\sigma \in \Delta^k | l_\sigma \ge 1/5\right\}\right] \le \frac{5^t}{1-p}$$

and we can put $B_q' = B_q + A \operatorname{card}\{\sigma \in \Delta^* | l_\sigma \ge 1/5\}$. □

Lemma 12. *For every $t \in \mathbb{R}$ such that $P(\max\{\dim_H K, \overline{\gamma}\} < t) > 0$, $\overline{\dim}_B K \le t$ for a.e. ω such that $\overline{\gamma}(\omega) < t$.*

Proof. Suppose that $P(\max\{\dim_H K, \overline{\gamma}\} < t) > 0$. Let $p \in (0,1)$ be defined by equality $p = E[\sum_{i\in\Delta} l_i^t]$. We will prove by induction on n that there exists $B > 0$ such that for each n, for every $\tau \in \Delta^*$ there exists a random variable $B_{\tau,n}$, independent of the σ-algebra generated by the maps $\omega \mapsto l_{\tau|_i}(\omega)$, $1 \le i \le |\tau|$, with $E[B_{\tau,n}] \le B$ such that

$$N_r(K_\tau)\mathbf{1}_{\{\overline{\gamma}<t\}} \le B_{\tau,n} r^{-t} l_\tau^t$$

for a.e. ω such that $1/n \le r/l_\tau(w) \le 1$.

Choose $q \in \mathbb{N}$ such that $p^q < 1/2$. Then put $B = \max\{2^d, 4^{t+1}E[B'_q]\}$, where B'_q is the random variable from Corollary 11. The induction base obviously holds for $n = 1, 2$.

Suppose the statement is true for $n_0 \in \mathbb{N}$, and $1/(n_0+1) \le r < 1/n_0$. We can assume that $K_\tau \ne \varnothing$. Let

$$C_{\tau,1}(\omega) = \left\{\sigma \in \Gamma_{\tau,q} \,\middle|\, l_\sigma \le \frac{l_\tau}{2n_0+2}\right\}, \qquad C_{\tau,2}(\omega) = \left\{\sigma \in \Gamma_{\tau,q} \,\middle|\, l_\sigma > \frac{l_\tau}{2n_0+2}\right\},$$

where

$$\Gamma_{\tau,q} = \left\{\sigma \in \Delta^{q+|\tau|}\colon\, l_\sigma < l_\tau/5\right\} \cup \left\{\sigma \in \Delta^*\colon\, |\sigma| > q+|\tau|, l_\sigma < l_\tau/5, l_{\sigma|_{|\sigma|-1}} \ge l_\tau/5\right\}.$$

Since

$$K_\tau = \left(\bigcup_{\sigma\in C_{\tau,1}} K_\sigma\right) \cup \left(\bigcup_{\sigma\in C_{\tau,2}} K_\sigma\right),$$

we have

$$N_r(K_\tau) \le N_{1/(n_0+1)}\left(\bigcup_{\sigma\in C_{\tau,1}} K_\sigma\right) + \sum_{\sigma\in C_{\tau,2}} N_r(K_\sigma).$$

We note that $N_{1/(n_0+1)}(\bigcup_{\sigma\in C_{\tau,1}} K_\sigma) \le N_{1/(2n_0+2)}(\mathrm{O}(x, \Gamma_{\tau,q}))$ because if $B(y_j, \frac{1}{2n_0+2})$ is a collection of balls of radius $\frac{1}{2n_0+2}$ covering $\mathrm{O}(x, \Gamma_{\tau,q})$, then the balls $B(y_j, \frac{1}{n_0+1})$ cover $\bigcup_{\sigma\in C_{\tau,1}} K_\sigma$, since $\mathrm{diam}(J_\sigma) < \frac{1}{2n_0+2}$ for all $\sigma \in C_{\tau,1}$. Therefore by Corollary 11

$$N_r(K_\tau)\mathbf{1}_{\{\overline{\gamma}<t\}} \le B'_q l_\tau^t 2^t (n_0+1)^t + \sum_{\sigma\in\Gamma_{\tau,q}} N_r(K_\sigma)\mathbf{1}_{\{l_\sigma\in C_{\tau,2}\}}\mathbf{1}_{\{\overline{\gamma}<t\}} \quad \text{a.s.}$$

The following chain of inequalities ensures applicability of the induction hypothesis to estimate the terms in the last sum:

$$\frac{r}{l_\sigma} > 5r \ge \frac{5}{n_0+1} > \frac{1}{n_0},$$

therefore

$$N_r(K_\sigma)\mathbf{1}_{\{l_\sigma\in C_{\tau,2}\}}\mathbf{1}_{\{\overline{\gamma}<t\}} \le B_{\sigma,n} r^{-t} l_\sigma(\omega)^t \quad \text{a.s.}$$

Since $r \le 2/(n_0+1)$,

$$N_r(K_\tau)\mathbf{1}_{\{\overline{\gamma}<t\}} \le r^{-t}\left(4^t l_\tau^t B'_q + \sum_{\sigma\in\Gamma_{\tau,q}} B_{\sigma,n_0} l_\sigma^t\right) = r^{-t} l_\tau^t \left(4^t B'_q + \sum_{\sigma\in\Gamma_{\tau,q}} B_{\sigma,n_0} l_\sigma^t / l_\tau^t\right) \quad \text{a.s.}$$

Note that

$$E\left[\left(4^t B'_q + \sum_{\sigma\in\Gamma_{\tau,q}} B_{\sigma,n} l_\sigma^t / l_\tau^t\right)\right] \le 4^t E\left[B'_q\right] + Bp^q < B/4 + B/2 < B.$$

Applying the same argument as in Lemma 10 we come to the desired conclusion. □

Theorem 13. *If there exists $A > 0$ such that for all $x \in J_\tau$, $t > 0$ and $0 < r \le 2$ we have*

$$N_r\big(\mathrm{O}_\tau(x,1)\big)\mathbf{1}_{\{\overline{\gamma}_\tau < t\}} < Ar^{-t} l_\tau^t,$$

then $\overline{\dim}_B K = \max\{\dim_H K, \overline{\gamma}\}$ a.s. provided $K \neq \varnothing$. Similarly, if

$$N_r\big(\mathrm{O}_\tau(x,1)\big)\mathbf{1}_{\{\underline{\gamma}_\tau < t\}} < Ar^{-t} l_\tau^t,$$

then $\overline{\dim}_B K = \max\{\dim_H K, \underline{\gamma}\}$ a.s. on $\{K \neq \varnothing\}$.

Proof. Fix $n \in \mathbb{N}$ and consider a collection of points $X = \{x_i\}_{i=1}^\infty \subset K$ such that for all $\sigma \in \Delta^n$, $J_\sigma \cap K \neq \varnothing \Rightarrow \mathrm{card}(X \cap J_\sigma) = 1$ and $J_\sigma \cap K = \varnothing \Rightarrow \mathrm{card}(X \cap J_\sigma) = 0$. By Lemma 9, $\overline{\dim}_B X = \overline{\dim}_B \mathrm{O}(x,n)$, and therefore $\overline{\dim}_B K \ge \max\{\dim_H K, \sup_{n\in\mathbb{N}} \overline{\dim}_B \mathrm{O}(x,n)\}$. By Lemma 12, $P(\overline{\dim}_B K > \max\{\dim_H K, \overline{\gamma}\}) = 0$. □

Corollary 14. *If the number of offspring is finite almost surely, then $\dim_H K = \dim_P K = \underline{\dim}_B K = \overline{\dim}_B K$ a.s.*

Theorem 15. *Suppose that we have a random self-similar set and there exists $A > 0$ such that*

$$N_r\big(\mathrm{O}(x,1)\big) < Ar^{-\overline{\gamma}}$$

a.s. for all $0 < r \le 2$. Then $\dim_P K = \overline{\dim}_B K = \max\{\dim_H K, \operatorname{ess\,sup} \overline{\dim}_B \mathrm{O}(x,1)\}$ and $\underline{\dim}_B K = \max\{\dim_H K, \operatorname{ess\,sup} \underline{\dim}_B \mathrm{O}(x,1)\}$ a.s. on $\{K \neq \varnothing\}$.

Proof. Since for a random self-similar set $\overline{\gamma_\tau}$, $\tau \in \Delta^*$ are conditionally i.i.d., we obtain that if $K(\omega) \neq \varnothing$, then $\overline{\gamma} = \operatorname{ess\,sup} \overline{\dim}_B \mathrm{O}(x,1)$ a.s. To see this, let $z = \operatorname{ess\,sup} \overline{\dim}_B \mathrm{O}(x,1)$, then $\operatorname{ess\,sup} \overline{\gamma_\tau} \le z$ for all $\tau \in \Delta^*$ and $\overline{\gamma} = \sup_\tau \overline{\gamma_\tau} \le z$ a.s. If $z = 0$ or $\overline{\gamma_\varnothing} = z$ a.s., we are done. Otherwise consider $0 < y < z$ such that

$$0 < P\big(\overline{\dim}_B \mathrm{O}(x,1) \le y\big) = b < 1.$$

For all $\tau \in \Delta^*$, $b = P(\overline{\gamma_\tau} \le y \mid J_\tau \neq \varnothing)$. Now we prove that for every $\varepsilon \in (0,1)$

$$P\big(\{\forall \tau\, \overline{\gamma_\tau} \le y\} \cap \{K \neq \varnothing\}\big) \le \varepsilon P(K \neq \varnothing).$$

Find $m \in \mathbb{N}$ such that $b^m < \varepsilon P(K \neq \varnothing)/2$. From [13] it is known that if S_k denotes the number of non-empty offspring on level k, then for almost every $\omega \in \{K \neq \varnothing\}$, $\lim_{k\to\infty} S_k = \infty$, and for almost every $\omega \in \{K = \varnothing\}$, $\lim_{k\to\infty} S_k = 0$. Therefore we can find $\Omega_0 \subset \{K \neq \varnothing\}$, $k_0 \in \mathbb{N}$, and perhaps a bigger m such that

$$P\big(\{K \neq \varnothing\} \setminus \Omega_0\big) < \varepsilon P(K \neq \varnothing)/2 \quad \text{and} \quad \forall \omega \in \Omega_0, \quad S_{k_0}(w) \ge m.$$

Next we enumerate somehow all indices of Δ^{k_0} and fix this enumeration, then denote all m-element subsets of Δ^{k_0} by F_i, $i \in \mathbb{N}$. For $\omega \in \Omega_0$ denote the event, that the first m non-empty sets $J_\sigma(\omega), \sigma \in \Delta^{k_0}$, concide with F_i, by Ω_i. Then Ω_i form a partition of Ω_0 and

$$\begin{aligned} P\big(\{\overline{\gamma} \le y\} \cap \{K \neq \varnothing\}\big) &= P\big(\{\overline{\gamma} \le y\} \cap \Omega_0\big) + P\big(\{K \neq \varnothing\} \setminus \Omega_0\big) \\ &\le \sum_i P\big(\{\overline{\gamma} \le y\} \cap \Omega_i\big) + \varepsilon P(K \neq \varnothing)/2 = \sum_i P(\overline{\gamma} \le y \mid \Omega_i) P(\Omega_i) + \varepsilon P(K \neq \varnothing)/2 \\ &\le \sum_i b^m P(\Omega_i) + \varepsilon P(K \neq \varnothing)/2 \le \varepsilon P(K \neq \varnothing). \end{aligned}$$

Examination of the proofs of Lemmas 10, 12 and Theorem 13 shows that for every $\tau \in \Delta^*$, $\overline{\dim}_B K_\tau = \max\{\dim_H K, \operatorname{ess\,sup} \underline{\mathrm{O}(x}, 1)\}$ provided $K_\tau \neq \varnothing$. Now using Baire's category theorem we see that for $t < \max\{\dim_H K, \operatorname{ess\,sup} \overline{\dim}_B \mathrm{O}(x,1)\}$, $\mathcal{P}^t(K) = \infty$. The result follows. □

What is the packing dimension of infinitely branching random fractals in general is unknown.

6. Examples

As we see for a random self-similar set the packing dimension is almost surely constant even with infinite branching. In the following example we see that if we drop the condition that the similarity maps are conditionally independent, packing dimension is no longer a constant.

Example 1 (Random fractal for which the zero–one law does not hold). *Let $J = [0, 1]$ and take $p(\omega), \omega \in \Omega$ with respect to the uniform distribution on $[1, 2]$. We build a random recursive construction so that on level 1, the right endpoints of the offspring are the points $1/n^p$, $n \in \mathbb{N}$, and the length of the nth subinterval is $V_n = (1/16^n)\inf_{1\le p\le 2}\{1/n^p - 1/(n+1)^p\}$. On all other levels, the offspring are formed from a scaled copy of $[0, 1]$ and its disjoint subintervals of length V_n with right endpoints at $1/n^p$, $n \in \mathbb{N}$. Obviously, $\sum_{n=1}^{\infty} V_n^{1/4} < \infty$, and hence for each $\omega \in \Omega$, we have $\dim_H K \le 1/4$. On the other hand we can use the results from [12] to determine that for each $\omega \in \Omega$, $\dim_P K(\omega) = \overline{\dim}_B K(\omega) = \frac{1}{p(\omega)+1}$. So, the reduction ratios are constant, but random placement of subintervals gives non-trivial variation of the packing dimension.*

Example 2 (Random recursive construction for which $\overline{\dim}_B K$ is a non-degenerate random variable and $\dim_H K < \dim_P K < \operatorname{ess\,inf} \overline{\dim}_B K$ a.s.). *Note that for $p > 0$, $\overline{\dim}_B\{1/n^p, n \in \mathbb{N}\} = 1/(p+1)$. Let $J = [0, 1]$ and take p with respect to the uniform distribution on $[1, 2]$. We build a random recursive construction so that on level 1, the right endpoints of offspring are the points $1/n^p$, $n \in \mathbb{N}$. On all other levels, the offspring are formed from a scaled copy of $[0, 1]$ and its disjoint subintervals with right endpoints at $1/n^4$, $n \in \mathbb{N}$. Let $(V_1, V_2, \ldots)$ be a fixed vector of reduction ratios so that $V_n = (1/1024)^n \inf_{1\le p\le 4}\{1/i^p - 1/(i+1)^p\}$. Then $\sum_{n=1}^{\infty} V_n^{1/8} < 1$, $K(\omega) \ne \varnothing$, $\dim_H K \le 1/8$ and $\overline{\dim}_B K = \max\{\dim_H K, 1/(p+1)\} = 1/(p+1)$, where p is chosen according to the uniform distribution on $[1, 2]$. Hence, $\operatorname{ess\,inf} \overline{\dim}_B K = 1/3$. By Theorem 15, $\dim_P K = 1/5$.*

Acknowledgement

Sufficiency of neighborhood boundedness property, condition (viii), for the proof of Lemma 9, instead of the "cone condition" from reference [11] as in [12], Proposition 2.9, became known to the author during conversation with R. D. Mauldin.

References

[1] M. F. Barnsley, J. E. Hutchinson and Ö. Stenflo. V-variable fractals: Fractals with partial self similarity. *Adv. Math.* **218** (2008) 2051–2088. MR2431670

[2] M. F. Barnsley, J. E. Hutchinson and Ö. Stenflo. V-variable fractals: dimension results. *Forum Math.* **24** (2012) 445–470.

[3] D. Beliaev and S. Smirnov. Random conformal snowflakes. *Ann. of Math. (2)* **172** (2010) 597–615. MR2680427

[4] A. Berlinkov and R. D. Mauldin. Packing measure and dimension of random fractals. *J. Theoret. Probab.* **15** (2002) 695–713. MR1922443

[5] K. J. Falconer. Random Fractals. *Math. Proc. Cambrige Philos. Soc.* **100** (1986) 559–582. MR0857731

[6] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edition. Wiley, Chichester, UK, 2003. MR2118797

[7] J. M. Fraser. Dimensions and measure for typical random fractals. Preprint, 2011. Available at arXiv:1112.4541.

[8] S. Graf, R. D. Mauldin and S. C. Williams. The exact Hausdorff dimension in random recursive constructions. *Mem. Amer. Math. Soc.* **381** (1988). MR0920961

[9] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge Univ. Press, Cambridge, UK, 1995. MR1333890

[10] P. Mattila and R. D. Mauldin. Measure and dimension functions: measurability and densities. *Math. Proc. Cambridge Philos. Soc.* **121** (1997) 81–100. MR1418362

[11] R. D. Mauldin and M. Urbanski. Dimensions and measures in iterated function systems. *Proc. London Math. Soc. (3)* **73** (1996) 105–154. MR1387085

[12] R. D. Mauldin and M. Urbanski. Conformal iterated function systems with applications to the geometry of continued fractions. *Trans. Amer. Math. Soc.* **351** (1999) 4995–5025. MR1487636

[13] R. D. Mauldin and S. C. Williams. Random recursive constructions: asymtotic geometric and topological properties. *Trans. Amer. Math. Soc.* **295** (1986) 325–346. MR0831202

[14] N.-R. Shieh and Y. Xiao. Hausdorff and packing dimensions of the images of random fields. *Bernoulli* **16** (2010) 926–952. MR2759163