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Supercritical super-Brownian motion with a general branching mechanism and travelling waves

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Abstract. We offer a probabilistic treatment of the classical problem of existence, uniqueness and asymptotics of monotone solutions to the travelling wave equation associated to the parabolic semi-group equation of a super-Brownian motion with a general branching mechanism. Whilst we are strongly guided by the reasoning in Kyprianou (*Ann. Inst. Henri Poincaré Probab. Stat.* **40** (2004) 53–72) for branching Brownian motion, the current paper offers a number of new insights. Our analysis incorporates the role of Seneta–Heyde norming which, in the current setting, draws on classical work of Grey (*J. Appl. Probab.* **11** (1974) 669–677). We give a *pathwise* explanation of Evans' immortal particle picture (the spine decomposition) which uses the Dynkin–Kuznetsov $\mathbb{N}$-measure as a key ingredient. Moreover, in the spirit of Neveu's stopping lines we make repeated use of Dynkin's exit measures. Additional complications arise from the general nature of the branching mechanism. As a consequence of the analysis we also offer an exact $X(\log X)^2$ moment dichotomy for the almost sure convergence of the so-called derivative martingale at its critical parameter to a non-trivial limit. This differs to the case of branching Brownian motion (*Ann. Inst. Henri Poincaré Probab. Stat.* **40** (2004) 53–72), and branching random walk (*Adv. in Appl. Probab.* **36** (2004) 544–581), where a moment 'gap' appears in the necessary and sufficient conditions. Our probabilistic treatment allows us to replicate known existence, uniqueness and asymptotic results for the travelling wave equation, which is related to a super-Brownian motion.

Résumé. Nous proposons une approche probabiliste au problème classique de l'existence, de l'unicité et du comportement asymptotique des solutions monotones de l'équation de propagation de front associée à l'équation parabolique du super-mouvement brownien de mécanisme de branchement général. Bien que largement inspiré par l'approche de Kyprianou (*Ann. Inst. Henri Poincaré Probab. Stat.* **40** (2004) 53–72) pour le mouvement brownien branchant, cet article ouvre plusieurs perspectives nouvelles. Notre analyse inclut le rôle de la normalisation de Seneta–Heyde qui, dans cette situation, s'inspire du travail classique de Grey (*J. Appl. Probab.* **11** (1974) 669–677). Nous donnons une explication *trajectorielle* de la décomposition en épine (la particule immortelle d'Evans), en utilisant la $\mathbb{N}$-mesure de Dynkin–Kuznetsov comme ingrédient clef. En outre, dans l'esprit des lignes d'arrêt de Neveu nous utilisons à plusieurs reprises les mesures de sortie de Dynkin. La nature générale du mécanisme de branchement rend l'analyse du problème plus délicate et nous proposons une dichotomie exacte basée sur un moment $X(\log X)^2$ pour la convergence presque-sûre de la *martingale dérivée* (pour la valeur critique de son paramètre) vers une limite non-triviale. Ceci diffère du cas du mouvement brownien branchant (*Ann. Inst. Henri Poincaré Probab. Stat.* **40** (2004) 53–72) et de la marche aléatoire branchante (*Adv. in Appl. Probab.* **36** (2004) 544–581), où un écart dans les hypothèses sur les moments apparaît entre les conditions nécessaires et suffisantes. Notre approche probabiliste permet de retrouver des résultats connus d'existence, d'unicité et de comportement asymptotique pour l'équation de propagation de front reliée au super-mouvement brownien.

MSC: 60J80; 60E10

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1. Introduction

Suppose that $X = \{X_t\colon t \geq 0\}$ is a (one-dimensional) ψ-super-Brownian motion with general branching mechanism ψ taking the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \big(\mathrm{e}^{-\lambda x} - 1 + \lambda x\big)\nu(\mathrm{d}x) \tag{1}$$

for $\lambda \geq 0$ where $\alpha = -\psi'(0^+) \in (0,\infty)$, $\beta \geq 0$ and ν is a measure concentrated on $(0,\infty)$ which satisfies $\int_{(0,\infty)}(x \wedge x^2)\nu(\mathrm{d}x) < \infty$. Let $\mathcal{M}_F(\mathbb{R})$ be the space of finite measures on $\mathbb{R}$ and note that X is a $\mathcal{M}_F(\mathbb{R})$-valued Markov process under $\mathbb{P}_\mu$ for each $\mu \in \mathcal{M}_F(\mathbb{R})$, where $\mathbb{P}_\mu$ is law of X with initial configuration μ. We shall use standard inner product notation, for $f \in C_b^+(\mathbb{R})$ and $\mu \in \mathcal{M}_F(\mathbb{R})$,

$$\langle f, \mu\rangle = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x).$$

Accordingly we shall write $\|\mu\| = \langle 1, \mu\rangle$.

The existence of our class of superprocesses is guaranteed by [8,9,11]. The following standard result from the theory of superprocesses describes the evolution of X as a Markov process. For all $f \in C_b^+(\mathbb{R})$, the space of positive, uniformly bounded, continuous functions on $\mathbb{R}$, and $\mu \in \mathcal{M}_F(\mathbb{R})$,

$$-\log \mathbb{E}_\mu\big(\mathrm{e}^{-\langle f, X_t\rangle}\big) = \int_{\mathbb{R}} u_f(x,t)\mu(\mathrm{d}x), \quad \mu \in \mathcal{M}_F(\mathbb{R}), t \geq 0, \tag{2}$$

where $u_f(x,t)$ is the unique positive solution to the evolution equation for $x \in \mathbb{R}$ and $t > 0$

$$\frac{\partial}{\partial t}u_f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_f(x,t) - \psi\big(u_f(x,t)\big), \tag{3}$$

with initial condition $u_f(x,0) = f(x)$. The reader is referred to Theorem 1.1 of Dynkin [7], Proposition 2.3 of Fitzsimmons [17] and Proposition 2.2 of Watanabe [43] for further details; see also Dynkin [9,11] for a general overview. The analogous object to (3) for branching Brownian motion is called the Fisher–Kolmogorov–Petrovski–Piscounov (FKPP) equation and hence in the current setting we name (3) the FKPP equation for ψ-super-Brownian motion.

Recall that the total mass of the process X is a continuous-state branching process with branching mechanism ψ. Since there is no interaction between spatial motion and branching we can characterise the ψ-super-Brownian motions into the categories of supercritical, critical and subcritical accordingly with the same categories for continuous-state branching processes. Respectively, these cases correspond to $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ and $\psi'(0^+) > 0$. The class of ψ-super-Brownian motions described above are necessarily supercritical. Such processes may exhibit explosive behaviour, however, under the conditions assumed above, X remains finite at all positive times. We insist moreover that $\psi(\infty) = \infty$ which means that with positive probability the event $\lim_{t\uparrow\infty}\|X_t\| = 0$ will occur. Equivalently this means that the total mass process does not have monotone increasing paths; see for example the summary in Chapter 10 of Kyprianou [27]. The probability of the event

$$\mathcal{E} := \Big\{\lim_{t\uparrow\infty}\|X_t\| = 0\Big\}$$

is described in terms of the largest root, say λ^*, of the equation $\psi(\lambda) = 0$. Note that it is known (cf. Chapter 8 of [27]) that ψ is strictly convex with $\psi(0) = 0$ and hence since $\psi(\infty) = \infty$ and $\psi'(0^+) < 0$ it follows that there are exactly two roots in $[0,\infty)$, one of which is always 0. For $\mu \in \mathcal{M}_F(\mathbb{R})$ we have

$$\mathbb{P}_\mu\Big(\lim_{t\uparrow\infty}\|X_t\| = 0\Big) = \mathrm{e}^{-\lambda^*\|\mu\|}. \tag{4}$$

In this article we shall also assume on occasion that

$$\int^{\infty} \frac{1}{\sqrt{\int_{\lambda^*}^{\xi} \psi(u)\,\mathrm{d}u}}\,\mathrm{d}\xi < \infty. \tag{5}$$

This condition is equivalent to requiring that $\int^{\infty}(\int_0^{\xi}(\psi(u)+\alpha u)\,\mathrm{d}u)^{-1/2}\,\mathrm{d}\xi < \infty$. In combination with additional assumptions on ψ given above, (5) has a number of implications for the underlying superprocess. Firstly, if we denote by $\mathcal{R}$ the smallest closed set in $\mathbb{R}$ such that $\operatorname{supp} X_t \subseteq \mathcal{R}$ for all $t \geq 0$, then Sheu [40] shows that for all $\mu \in \mathcal{M}_F(\mathbb{R})$ with compact support,

$$\mathbb{P}_\mu(\mathcal{R} \text{ is compact}) = \mathrm{e}^{-\lambda^*\|\mu\|}.$$

Secondly (5) implies that $\int^{\infty} 1/\psi(\xi)\,\mathrm{d}\xi < \infty$ (cf. [40]) which in turn guarantees that the event $\mathcal{E}$ agrees with the event of *extinction*, namely $\{\zeta < \infty\}$ where

$$\zeta = \inf\bigl\{t > 0\colon \|X_t\| = 0\bigr\}.$$

Note that (5) cannot be satisfied for branching mechanisms which belong to bounded variation spectrally positive Lévy processes.

Our primary concern in this paper will be to look at monotone travelling wave solutions to the FKPP equation (3) through entirely probabilistic means. Specifically, we are interested in non-increasing solutions to (3) of the form $\Phi_c(x-ct)$, where $\Phi_c \geq 0$ and $c \geq 0$ is the wave speed. That is to say Φ_c solves

$$\frac{1}{2}\Phi_c'' + c\Phi_c' - \psi(\Phi_c) = 0. \tag{6}$$

Moreover, for technical reasons which will become clear later, we shall be interested in the case that

$$\Phi_c(-\infty) = \lambda^* \quad \text{and} \quad \Phi_c(+\infty) = 0.$$

Henceforth we shall say that any solution to (6) which respects the aforementioned conditions of non-negativity, monotonicity and connecting the points λ^* at $-\infty$ to 0 at $+\infty$ is a *travelling wave with wave speed c*.

Starting with Kolmogorov et al. [24] and Fisher [16] there exists a variety of analytical treatments of travelling wave equations similar to (6). We name but a few, for example Aronson and Weinberger [1], Bramson [3], Fife and McLeod [15], Kametaka [23], Lau [29], Pinsky [37], Uchiyama [41], Volpert et al. [42]. Among these papers, the problem discussed in Uchiyama [41], and later in Bramson [3], is closely related to the problem we are going to investigate in this paper. The last two references consider travelling waves to the following semi-linear equation:

$$\frac{\partial}{\partial t}u_f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_f(x,t) + F\bigl(u_f(x,t)\bigr), \tag{7}$$

where the function F is assumed to satisfy

$$F \in C^1[0,1], \quad F(0) = F(1) = 0 \quad \text{and} \quad F(u) < 0, \quad 0 < u < 1. \tag{8}$$

It turns out that it is possible to transform (3) into the setting (7). Indeed, it is a straighforward exercise to show that, thanks to the smoothness and convexity properties of ψ our assumption that $\psi'(0+) < 0$ as well as the fact that $\psi(\lambda^*) = 0$,

$$F(u) := \psi\bigl(\lambda^*(1-u)\bigr)/\lambda^* \tag{9}$$

respects the requirements in (8). It thus follows that u_f is a solution to (7) with F given by (9) if and only if $\lambda^*(1-u_f)$ is a solution to (3). This transformation applies equally well to convert solutions to (6) into travelling wave solutions associated to (7).

Restricting ourselves to the particular form (1) allows us to approach our analysis entirely through a probabilistic treatment of super-Brownian motion. In this respect, our main results, Theorems 2.1 and 2.6 below, are not new from the point of PDEs. (We shall elaborate further on the overlap of some of our conclusions with the PDE literature later on.) Indeed this article is concerned with the probabilistic structures embedded within super-Brownian motion that can be seen to explain the known travelling wave results, albeit for a smaller class of forcing terms ψ than existing literature would allow.

There exists a suite of literature which gives a probabilistic handling of (7) and its travelling waves; see [3,4,6, 21,25,35,36]. Key to all of these papers is the relationship between the travelling wave equation and two types of martingales found in branching Brownian motion commonly referred to as *additive* and *multiplicative* martingales. Our objective in this paper is to show that many of the known probabilistic ideas can be adapted, subject to the use of appropriate alternative technologies, to handle (6). In particular we shall largely work with Dynkin exit measures as well a new *pathwise* version of Evans' immortal particle decomposition of our ψ-super-Brownian motion.

The remainder of the paper is structured as follows. In the next section we state our main results. These pertain to a complete existence, uniqueness and asymptotic result for the travelling wave equation. In special cases, it is possible to give more explicit details concerning the form of the solutions to the travelling wave equation in terms of martingale limits. For this reason, part of our main results includes some martingale convergence theorems. One of our martingale results, concerning the question of convergence to a non-trivial limit of the so-called derivative martingale, offers a moment dichotomy which has not been previously achieved for the analogous martingales in the case of branching Brownian motion and branching random walks. In Section 3 we examine certain Dynkin exit measures which will be key to later analysis. The remaining sections are dedicated to the proofs of the main results with the exception of Section 5 which provides the new pathwise spine (or immortal particle) decomposition that features heavily in the proofs.

On a final note, we mention that the condition (5) appears to be a natural sufficient condition under which to perform all of our analysis. This will become clear later on through several of the preparatory results. We refrain from imposing this condition throughout the paper however (in favour of stating it when required) as a number of the mathematical tools we appeal to, which are of intrinsic interest on their own, still have meaning without it.

2. Main results

Our first result gives us a very general characterisation of the existence, uniqueness and asymptotics of non-negative travelling waves solving (6). Subsequently we give moment conditions under which some of the quantities involved can be explicitly identified. For convenience we write $\underline{\lambda} = \sqrt{-2\psi'(0^+)}$ and for each $\lambda \in \mathbb{R}$ define

$$c_\lambda = -\psi'(0^+)/\lambda + \lambda/2. \tag{10}$$

Note that for $\lambda \in (0, \underline{\lambda}]$, c_λ has range $[\underline{\lambda}, \infty)$. In particular $c_{\underline{\lambda}} = \underline{\lambda}$. We shall also write $\mathbb{P}$ as shorthand for $\mathbb{P}_{\delta_0}$ with corresponding expectation operator given by $\mathbb{E}$.

Theorem 2.1.

(i) *If (5) holds then no travelling waves exist with wave speed c if $c \in [0, c_{\underline{\lambda}})$.*

(ii) *A travelling wave exists with wave speed c if $c \geq c_{\underline{\lambda}}$. In particular for $\lambda \in (0, \underline{\lambda}]$ there exists a travelling wave with wave speed c_λ which may be written in the form*

$$\Phi_{c_\lambda}(x) = -\log \mathbb{E}\big[e^{-e^{-\lambda x} \Delta(\lambda)}\big], \tag{11}$$

where $\Delta(\lambda)$ is a non-negative random variable such that $\{\Delta(\lambda) = 0\}$ agrees with $\mathcal{E}$, $\mathbb{P}$-almost surely.

(iii) *Suppose that $\lambda \in (0, \underline{\lambda}]$. Then, up to an additive shift in its argument, there is a unique travelling wave at speed c_λ.*

(iv) *Moreover, when $\lambda \in (0, \underline{\lambda}]$, there exists some constant $k_\lambda \in (0, \infty)$ and a slowly varying function $L_\lambda : (0, \infty) \mapsto (0, \infty)$ such that*

$$\lim_{x \to \infty} \frac{\Phi_{c_\lambda}(x)}{e^{-\lambda x} L_\lambda(e^{-\lambda x})} = k_\lambda. \tag{12}$$

The conclusions given in the above theorem conform to what is already understood about the existence, uniqueness and asymptotic decay of travelling wave solutions to the classical FKPP equation (7) and hence by the transformation described in the Introduction, the non-linear equation (3). Indeed Theorem 1.1 in [41], for which the non-linear function F in (7) belongs to a more general class than the one that the ψ we consider here corresponds to, covers the content of Theorem 2.1, perhaps with the exception of the representation (11). Under further assumptions the function $L_\lambda(z)$ is known to behave as $-\log z$ as $z \downarrow 0$. This conclusion will also appear shortly in the forthcoming Theorem 2.6 with a comparison to existing literature.

As alluded to above, the next two main theorems make a clearer statement about the quantities $\Delta(\lambda)$ and L_λ when we impose additional assumptions. To do this, we need to introduce two families of $\mathbb{P}$-martingales with respect to the natural filtration $\mathcal{F}_t := \sigma(X_u; u \le t)$. The first such family of martingales is identified in the following lemma.

Lemma 2.2. *The process* $W(\lambda) = \{W_t(\lambda)\colon t \ge 0\}$ *where* $\lambda \in \mathbb{R}$ *and*

$$W_t(\lambda) := \mathrm{e}^{-\lambda c_\lambda t}\langle \mathrm{e}^{-\lambda \cdot}, X_t\rangle, \quad t \ge 0, \tag{13}$$

is a martingale.

Proof. The proof appeals to a classical technique which we briefly outline. Define for each $x \in \mathbb{R}$, $g \in C_b^+(\mathbb{R})$ and $\theta, t \ge 0$, $u_g^\theta(x,t) = -\log \mathbb{E}_{\delta_x}(\mathrm{e}^{-\theta\langle g, X_t\rangle})$ and note that, with limits understood as $\theta \downarrow 0$, $u_g(x,t)|_{\theta=0} = 0$ and $v_g(x,t) := \mathbb{E}_{\delta_x}(\langle g, X_t\rangle) = \partial u_g^\theta(x,t)/\partial\theta|_{\theta=0}$. Differentiating in θ in (3) shows that v_g solves the equation

$$\frac{\partial}{\partial t} v_g(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v_g(x,t) - \psi'(0^+) v_g(x,t), \tag{14}$$

with $v_g(x,0) = g(x)$. Note that classical Feynman–Kac theory tells us that (14) has a unique solution and it is necessarily equal to $\Pi_x(\mathrm{e}^{-\psi'(0^+)t} g(\xi_t))$ where $\{\xi_t\colon t \ge 0\}$ is a Brownian motion issued from $x \in \mathbb{R}$ under the measure Π_x. The above procedure also works for $g(x) = \mathrm{e}^{-\lambda x}$ in which case we easily conclude that for all $x \in \mathbb{R}$ and $t \ge 0$, $\mathrm{e}^{-\lambda c_\lambda t}\mathbb{E}_{\delta_x}(\langle \mathrm{e}^{-\lambda\cdot}, X_t\rangle) = \mathrm{e}^{-\lambda x}$. Finally, the martingale property follows using the previous equality together with the Markov branching property associated with X. □

Note that $W(\lambda)$ is a non-negative martingale and therefore converges almost surely. As a corollary to the above lemma, we may describe the second family of martingales we are interested in by taking the negative derivative in λ of $W(\lambda)$. Note that this produces a signed martingale which does not necessarily converge almost surely.

Corollary 2.3. *The process* $\partial W(\lambda) := \{\partial W_t(\lambda), t \ge 0\}$, *where* $\lambda \in \mathbb{R}$ *and*

$$\partial W_t(\lambda) := -\frac{\partial}{\partial \lambda} W_t(\lambda) = \big\langle (\lambda t + \cdot)\mathrm{e}^{-\lambda(c_\lambda t + \cdot)}, X_t\big\rangle, \quad t \ge 0, \tag{15}$$

is also a martingale.

It turns out that the convergence of both these martingales in the appropriate sense is important to give a more precise characterization of the limit $\Delta(\lambda)$ and the normalizing sequence L_λ in Theorem 2.1. The following theorem contains the necessary information.

Theorem 2.4.

(i) *The almost sure limit of* $W(\lambda)$, *denoted by* $W_\infty(\lambda)$, *is also an* $L^1(\mathbb{P})$*-limit if and only if* $|\lambda| < \underline{\lambda}$ *and*

$$\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty.$$

When $W_\infty(\lambda)$ *is an* $L^1(\mathbb{P})$*-limit the event* $\{W_\infty(\lambda) > 0\}$ *agrees with* $\mathcal{E}^c$, $\mathbb{P}$*-almost surely. Otherwise, when it is not an* $L^1(\mathbb{P})$*-limit, its limit is identically zero.*

(ii) *Assume that* (5) *holds. The martingale* $\partial W(\lambda)$ *has an almost sure non-negative limit when* $|\lambda| \geq \underline{\lambda}$ *which is identically zero when* $|\lambda| > \underline{\lambda}$ *and when* $|\lambda| = \underline{\lambda}$ *its limit is almost surely strictly positive on* $\mathcal{E}^c$ *if and only if*

$$\int_{[1,\infty)} r(\log r)^2 \nu(\mathrm{d}r) < \infty.$$

Remark 2.5. *Note that other similar theorems exist for derivative martingales in the branching random walk*, [2], *and branching Brownian motion*, [25]. *In those cases however, an exact dichotomy for convergence to a non-zero limit in the critical regime was not achieved unlike the case here.*

We may now turn to our final main theorem which is a refinement of Theorem 2.1 under additional assumptions. For convenience we write $W_\infty(\lambda)$ and $\partial W_\infty(\lambda)$ for the martingale limits (when it exists in the latter case).

Theorem 2.6. *Assume* (5).

(i) *Suppose that* $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty$ *and* $\lambda \in (0, \underline{\lambda})$. *Then, up to an additive constant in its argument, the travelling wave solution* Φ_{c_λ} *to* (6) *is given by*

$$\Phi_{c_\lambda}(x) = -\log \mathbb{E}\big[\mathrm{e}^{-\mathrm{e}^{-\lambda x} W_\infty(\lambda)}\big],$$

and there is a constant $k_\lambda \in (0, \infty)$ *such that*

$$\lim_{x\to\infty} \frac{\Phi_{c_\lambda}(x)}{\mathrm{e}^{-\lambda x}} = k_\lambda.$$

(ii) *Suppose that* $\int_{[1,\infty)} r(\log r)^2 \nu(\mathrm{d}r) < \infty$ *and* $\lambda = \underline{\lambda}$. *Then, the critical travelling wave solution* $\Phi_{\underline{\lambda}}$ *to* (6) *is given by*

$$\Phi_{\underline{\lambda}}(x) = -\log \mathbb{E}\big[\mathrm{e}^{-\mathrm{e}^{-\underline{\lambda} x} \partial W_\infty(\underline{\lambda})}\big].$$

Moreover, there is a constant $k_\lambda \in (0, \infty)$ *such that*

$$\lim_{x\to\infty} \frac{\Phi_{c_\lambda}(x)}{x\mathrm{e}^{-\lambda x}} = k_\lambda. \tag{16}$$

Remark 2.7. *Note that*

$$\int_0^1 r^{-2}\psi(r)\,\mathrm{d}r < \infty \iff \int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty,$$

$$\int_0^1 r^{-2}|\log r|\psi(r)\,\mathrm{d}r < \infty \iff \int_{[1,\infty)} r(\log r)^2\nu(\mathrm{d}r) < \infty.$$

Using these equivalences we see that our Theorem 2.6 *agrees precisely with Theorems* 2.1, 2.2 *and* 2.3 *in Uchiyama* [41], *where again, through the transformation described in the Introduction, we alert the reader to the fact that Uchiyama allows for a more general setting. Such asymptotics are similarly prescribed in equations* (1.13) *and* (1.14) *in Bramson* [3] *under slightly different assumptions. We can also use these equivalences to provide some examples in which the moment conditions appearing in Theorem* 2.6 *hold or fail.*

Firstly, we provide an example where (5) *holds true but* $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) = \infty$. *According to* [38], $\psi_1(\lambda) = \lambda^2 \log^{-1}(1+\lambda)$, $\lambda \geq 0$ *is a branching mechanism. By some elementary calculations, we can check that* ψ_1 *satisfies* (5) *but* $\int_0^1 r^{-2}\psi_1(r)\,\mathrm{d}r = \infty$. *Let* ν_1 *be the measure* ν *in* (1) *corresponding to* ψ_1. *Then* $\int_{[1,\infty)} r\log r\,\nu_1(\mathrm{d}r) = \infty$.

Secondly, we give an example where $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty$ *and* $\int_{[1,\infty)} r(\log r)^2\nu(\mathrm{d}r) = \infty$. *According to* [38], $\psi_2(\lambda) = \lambda(\lambda\log\lambda - \lambda + 1)/(\log\lambda)^2$, $\lambda > 0$, *is a branching mechanism. We can check that* ψ_2 *satisfies* (5) *and* $\int_0^1 r^{-2}\psi_2(r)\,\mathrm{d}r < \infty$, *but* $\int_0^1 r^{-2}|\log r|\psi_2(r)\,\mathrm{d}r = \infty$. *Let* ν_2 *be the measure* ν *in* (1) *corresponding to* ψ_2. *Than* $\int_{[1,\infty)} r\log r\,\nu_2(\mathrm{d}r) < \infty$ *but* $\int_{[1,\infty)} r(\log r)^2\nu_2(\mathrm{d}r) = \infty$.

3. Branching exit Markov systems and embedded continuous state branching processes

For each $y, t \geq 0$, define the space–time domain $D_y^t = \{(x,u)\colon x < y, u < t\}$ and for each $c \in \mathbb{R}$ let $X^c = \{X_t^c\colon t \geq 0\}$ be the sequence of measures which satisfies $\langle f, X_t^c\rangle = \langle f(ct+\cdot), X_t\rangle$ for all $t \geq 0$ and $f \in C_b^+(\mathbb{R})$. It is straightforward to deduce that for each $\mu \in \mathcal{M}_F(\mathbb{R})$, $(X^c, \mathbb{P}_\mu)$ is a superprocess with general branching mechanism ψ whose movement component corresponding to a Brownian motion with drift c. According to Dynkin's theory of exit measures [10] it is possible to describe the mass in the superprocess X^c as it first exits the growing family of domains $\{D_y^t\colon t \geq 0, y \geq 0\}$ as a sequence of random measures on $\mathbb{R} \times [0,\infty)$, known as *branching Markov exit measures*, which we denote by $\{X^c_{D_y^t}\colon t \geq 0, y \geq 0\}$. In particular, according to the characterisation for branching Markov exit measures given in Section 1.1 of [12], each of the random measures $X^c_{D_y^t}$ is supported on $\partial D_y^t = (\{y\} \times [0,t)) \cup ((-\infty, y] \times \{t\})$ and has the following defining Markov branching property. Let $\mathcal{F}^c_{D_y^t} = \sigma(X^c_{D_x^u}\colon u \leq t, x \leq y)$. For all $t \geq r$, $y \geq z$, $\mu \in \mathcal{M}_F(\mathbb{R})$ with $\operatorname{supp}\mu \subset (-\infty, z]$ and $f \in C_b^+(D_y^t)$ such that $f(x,t) = f(x,0) =: f(x)$ for all $t \geq 0$,

$$\mathbb{E}_\mu\big(\mathrm{e}^{-\langle f, X^c_{D_y^t}\rangle} | \mathcal{F}^c_{D_z^r}\big) = \mathrm{e}^{-\langle u_f^y(\cdot, t-\cdot), X^c_{D_z^r}\rangle}, \tag{17}$$

where, for all (x,s) in D_y^t, u_f^y is the unique positive solution of the partial differential equation

$$\frac{\partial}{\partial s} u_f^y(x,s) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u_f^y(x,s) + c\frac{\partial}{\partial x} u_f^y(x,s) - \psi\big(u_f^y(x,s)\big), \tag{18}$$

with boundary conditions $u_f^y(y,s) = f(y)$ for $0 \leq s \leq t$ and $u_f^y(x,0) = f(x)$ for $x \leq y$. We may similarly consider the branching Markov property of the exit measures $X^c_{D_{-z}^t}$ where $D_{-z}^t = \{(x,r)\colon r < t, -z < x\}$ with $z \geq 0$. Moreover, define for convenience $D_y = D_y^\infty$ and by monotonicity one may also define $X^c_{D_y} = \lim_{t\uparrow\infty} X^c_{D_y^t}|_{\{y\}\times[0,t)}$.

An important consequence of the Markov branching property above is the following theorem which will feature crucially in our proof of Theorem 2.1.

Theorem 3.1. *Define for each $y \geq 0$ and $c \geq 0$, $Z_y^c := \langle 1, X^c_{D_y}\rangle = \|X^c_{D_y}\|$ and $Z^c_{-y} := \langle 1, X^c_{D_{-y}}\rangle = \|X^c_{D_{-y}}\|$. For all $x \in \mathbb{R}$ and $\lambda \in (0, \underline{\lambda}]$ the following statements hold $\mathbb{P}_{\delta_x}$-almost surely.*

(i) *The process $\{Z_y^{c_\lambda}\colon y \geq x\}$ is a conservative supercritical continuous state branching process with growth rate λ. Moreover, the process Z^{c_λ} becomes extinct with positive probability if and only if (5) holds.*

(ii) *The process $\{Z_y^{c_\lambda}\colon y \leq -x\}$ is a subcritical continuous state branching process with growth rate $-\lambda$. Moreover, there is almost sure extinction if and only if (5) holds.*

Proof. First part of (i). For $x \leq y$ and $f \in C_b^+(\mathbb{R}\times[0,\infty))$ such that $f(x,t) = f(x,0) =: f(x)$ for all $t \geq 0$, let $v_f^y(x,t) := \mathbb{E}_{\delta_x}(\langle f, X^c_{D_y^t}\rangle)$. By performing a similar linearisation to the linearisation (14) of (3), we have that

$$\frac{\partial}{\partial t} v_f^y(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v_f^y(x,t) + c\frac{\partial}{\partial x} v_f^y(x,t) - \psi'\big(0^+\big) v_f^y(x,t), \tag{19}$$

with $v_f^y(y,t) = f(y)$ for $t \geq 0$ and $v_f^y(x,0) = f(x)$ for $x \leq y$. The classical Feynman–Kac formula allows us to write the unique solution to (19) as

$$v_f^y(x,t) = \Pi_x^c\big[\mathrm{e}^{-\psi'(0^+)(t\wedge\tau_y^+)} f(\xi_{t\wedge\tau_y^+})\big], \tag{20}$$

where $\tau_y^+ = \inf\{t > 0\colon \xi_t > y\}$ and under Π_x^c, $\{\xi_t\colon t \geq 0\}$ is a Brownian motion with drift c issued from x. By means of an increasing sequence of continuous functions which are valued 0 at y and which converge pointwise to $\mathbf{1}_{(-\infty,y]}(\cdot)$, it is now possible to deduce by monotone convergence that

$$\mathbb{E}_{\delta_x}\big(X^c_{D_y^t}\big((-\infty, y]\times\{t\}\big)\big) = \mathrm{e}^{-\psi'(0^+)t}\Pi_x^c\big(\tau_y^+ > t\big). \tag{21}$$

For $x \leq y$ it is known that the density of τ_y^+ is given by

$$\frac{y-x}{\sqrt{2\pi t^3}} \exp\left(-\frac{(y-ct)^2}{2t}\right), \quad t > 0. \tag{22}$$

Now let $c = c_\lambda$ for $\lambda \in (0, \underline{\lambda}]$. From (21) and (22), an application of L'Hôpital's rule shows that

$$\lim_{t\uparrow\infty} \mathbb{E}_{\delta_x}\big(X^{c_\lambda}_{D_y^t}\big((-\infty, y] \times \{t\}\big)\big) = 0. \tag{23}$$

It now follows from (20) with $f = 1$, (21) and (23) that for all $x \in (-\infty, y]$,

$$\mathbb{E}_{\delta_x}\big(\big\|X^{c_\lambda}_{D_y}\big\|\big) = \lim_{t\uparrow\infty} \mathbb{E}_{\delta_x}\big(\big\|X^{c_\lambda}_{D_y^t}\big\|\big) = \Pi_x^{c_\lambda}\big[\mathrm{e}^{-\psi'(0^+)\tau_y^+}; \tau_y^+ < \infty\big] = \mathrm{e}^{\lambda(y-x)}.$$

Note also from (17) we have that for all $a, b, \theta \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}_{(a+b)\delta_x}\big(\mathrm{e}^{-\theta\langle 1, X^{c_\lambda}_{D_y}\rangle}\big) = \mathbb{E}_{a\delta_x}\big(\mathrm{e}^{-\theta\langle 1, X^{c_\lambda}_{D_y}\rangle}\big)\mathbb{E}_{b\delta_x}\big(\mathrm{e}^{-\theta\langle 1, X^{c_\lambda}_{D_y}\rangle}\big), \tag{24}$$

showing that Z^{c_λ} is a conservative continuous state branching process.

First part of (ii). By symmetry, it suffices to prove that for $\lambda \in (0, \underline{\lambda}]$, the process $\{Z_x^{-c_\lambda}\colon\ x \geq 0\}$ is a subcritical continuous state branching process with growth rate $-\lambda$. This conclusion follows from a similar analysis to the proof of part (i), noting in particular that

$$\mathbb{E}_{\delta_x}\big(\big\|X^{-c_\lambda}_{D_y}\big\|\big) = \Pi_x^{-c_\lambda}\big[\mathrm{e}^{-\psi'(0^+)\tau_y^+}; \tau_y^+ < \infty\big] = \mathrm{e}^{-\lambda(y-x)}.$$

The details are left to the reader.

Second part of (ii). For any $y \geq z$, $\mu \in \mathcal{M}(\mathbb{R})$ with $\operatorname{supp}\mu \subset (-\infty, z]$ and $\theta > 0$,

$$\mathbb{E}_\mu\big(\mathrm{e}^{-\langle\theta, X^{-c_\lambda}_{D_y}\rangle}|\mathcal{F}^{-c_\lambda}_{D_z}\big) = \mathrm{e}^{-\langle u_\theta^y, X^{-c_\lambda}_{D_z}\rangle}, \tag{25}$$

where, by taking limits as t and then r tend to infinity in (18), we have that u_θ^y is the unique positive solution to the equation

$$0 = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_\theta^y(x) - c_\lambda\frac{\partial}{\partial x}u_\theta^y(x) - \psi\big(u_\theta^y(x)\big), \tag{26}$$

on $(-\infty, y]$ with boundary value $u_\theta^y(y) = \theta$.

This tells us that for each fixed $\theta \geq 0$,

$$\mathbb{E}\big(\mathrm{e}^{-\langle\theta, X^{-c_\lambda}_{D_x}\rangle}\big) = \mathrm{e}^{-u_\theta^x(0)} = \mathrm{e}^{-u_\theta^0(-x)},$$

where u_θ^0 solves

$$0 = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_\theta^0(x) - c_\lambda\frac{\partial}{\partial x}u_\theta^0(x) - \psi\big(u_\theta^0(x)\big),$$

on $(-\infty, 0)$ with boundary value $u_\theta^0(0) = \theta$. Written yet another way, this tells us that $u_\theta^x := u_\theta^x(0)$ satisfies

$$0 = \frac{1}{2}\frac{\partial^2}{\partial x^2}u_\theta^x + c_\lambda\frac{\partial}{\partial x}u_\theta^x - \psi\big(u_\theta^x\big),$$

on $(0, \infty)$ with boundary value $u_\theta^0 = \theta$.

On the other hand, if ψ_{-c_λ} is the branching mechanism of $\{Z_x^{-c_\lambda}\colon\ x \geq 0\}$, then we also know that

$$\frac{\partial}{\partial x} u_\theta^x + \psi_{-c_\lambda}\big(u_\theta^x\big) = 0$$

for $x \geq 0$. Combining the previous two differential equations, we easily deduce that

$$\frac{1}{2}\psi'_{-c_\lambda}\big(u_\theta^x\big)\psi_{-c_\lambda}\big(u_\theta^x\big) - c_\lambda \psi_{-c_\lambda}\big(u_\theta^x\big) = \psi\big(u_\theta^x\big).$$

As $\{Z_x^{-c_\lambda}\colon\ x \geq 0\}$ is subcritical, we know that $u_\theta^\infty = 0$. Thus by continuity, for each fixed $\theta > 0$, the range of $\{u_\theta^x\colon\ x \geq 0\}$ contains $[0, \theta]$. As θ may be made arbitrarily large, it follows that

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}u}\psi^2_{-c_\lambda}(u) - c_\lambda \psi_{-c_\lambda}(u) = \psi(u), \quad u \geq 0.$$

Subcriticality also implies that $\psi_{-c_\lambda}(0) = 0$.

Next note that

$$\frac{1}{4}\big(\psi^2_{-c_\lambda}(u) - \psi^2_{-c_\lambda}\big(\lambda^*\big)\big) - c_\lambda \int_{\lambda^*}^u \psi_{-c_\lambda}(s)\,\mathrm{d}s = \int_{\lambda^*}^u \psi(s)\,\mathrm{d}s. \tag{27}$$

As ψ_{-c_λ} tends to infinity at infinity, we may apply L'Hôpital's rule to deduce that

$$\lim_{u\uparrow\infty} \frac{\int_{\lambda^*}^u \psi_{-c_\lambda}(s)\,\mathrm{d}s}{\psi^2_{-c_\lambda}(u)} = \frac{1}{2}\lim_{u\uparrow\infty} \frac{1}{\psi'_{-c_\lambda}(u)}. \tag{28}$$

Note it follows in a straightforward manner form the Lévy–Khintchine formula that the limit on the right-hand side above exists (and may possibly equal zero). The limit (28) when combined with (27) now allows us to conclude that

$$\int^\infty \frac{1}{\psi_{-c_\lambda}(\xi)}\,\mathrm{d}\xi < \infty \quad \Longleftrightarrow \quad \int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^\xi \psi(u)\,\mathrm{d}u}}\,\mathrm{d}\xi < \infty. \tag{29}$$

As $\{Z_x^{-c_\lambda}\colon\ x \geq 0\}$ is subcritical, this is equivalent to saying that there is almost sure extinction if and only if (5) holds.

Second part of (i). Using exactly the same proof we can show that (29) holds with ψ_{-c_λ} replaced by ψ_{c_λ}. The desired result follows by recalling that $\int^\infty 1/\psi_{c_\lambda}(\xi)\,\mathrm{d}\xi < \infty$ is the necessary and sufficient condition in the current context for the event of extinction to agree with the event of becoming extinguished. □

Corollary 3.2. *Suppose that* (5) *holds. Fix* $x \in \mathbb{R}$. *For each* $c \geq 0$, *let* $\mathcal{R}^{-c}$ *be the smallest closed set containing* $\operatorname{supp} X_t^{-c}$ *for all* $t \geq 0$. *Then for all* $c \geq c_{\underline{\lambda}} = \underline{\lambda}$ *we have that* $\mathbb{P}_{\delta_x}(\sup \mathcal{R}^{-c} < \infty) = 1$ *and for all* $c < c_{\underline{\lambda}}$ *we have* $\mathbb{P}_{\delta_x}(\sup \mathcal{R}^{-c} = \infty | \mathcal{E}^c) = 1$. *In particular if* $R_t = \sup\{y \in \mathbb{R}\colon\ X_t(y, \infty) > 0\}$ *then*

$$\lim_{t\uparrow\infty} \frac{R_t}{t} = \underline{\lambda}, \tag{30}$$

$\mathbb{P}_{\delta_x}$*-almost surely on* $\mathcal{E}^c$.

Proof. From Theorem 3.1(ii), under the assumption of (5), the process $\{Z_x^{-c_\lambda}\colon\ x \geq 0\}$ is subcritical and becomes extinct with probability 1. This implies that for all $c \geq c_{\underline{\lambda}} = \underline{\lambda}$, $\mathbb{P}_{\delta_x}(\sup \mathcal{R}^{-c} < \infty) = 1$ and hence

$$\limsup_{t\uparrow\infty} \frac{R_t}{t} \leq \underline{\lambda},$$

where $R_t = \sup\{y \in \mathbb{R}\colon\ X_t(y, \infty) > 0\}$.

Next we want to show

$$\liminf_{t\uparrow\infty} \frac{R_t}{t} \geq \underline{\lambda}, \tag{31}$$

on $\mathcal{E}^c$.

We shall use the conclusion of Theorem 2.4 to prove (31). The reader should note that the proof of Theorem 2.4, which appears later in this paper, does not depend on the result we are currently proving. We also make use of an argument which is essentially taken from Git et al. [18]. For $0 < \epsilon < \underline{\lambda}/2$ and $\gamma = \underline{\lambda} - \epsilon$ note that $\mathrm{e}^{\gamma x}\mathbf{1}_{(x\leq(\gamma-\epsilon)t)} \leq \mathrm{e}^{(\gamma-\epsilon)x}\mathrm{e}^{\epsilon(\gamma-\epsilon)t}$, and hence

$$\limsup_{t\uparrow\infty} \mathrm{e}^{-(\gamma^2/2-\psi'(0^+))t}\big\langle \mathrm{e}^{\gamma\cdot}\mathbf{1}_{(\cdot\leq(\gamma-\epsilon)t)}, X_t\big\rangle \leq \limsup_{t\uparrow\infty} \mathrm{e}^{-\epsilon^2 t/2} W_t(-\gamma+\epsilon) = 0, \tag{32}$$

$\mathbb{P}_{\delta_x}$-almost surely. It follows that

$$\lim_{t\uparrow\infty} \mathrm{e}^{-(\gamma^2/2-\psi'(0^+))t}\big\langle \mathrm{e}^{\gamma\cdot}\mathbf{1}_{(\cdot>(\gamma-\epsilon)t)}, X_t\big\rangle = W_\infty(-\gamma+\epsilon), \tag{33}$$

$\mathbb{P}_{\delta_x}$-almost surely. Note that by Theorem 2.4(i) the event $\{W_\infty(-\gamma+\epsilon) > 0\}$ agrees with $\mathcal{E}^c$. Hence as ϵ can be made arbitrarily small, (31) follows on $\mathcal{E}^c$.

Together with (31) this implies the strong law of large numbers, (30), on $\mathcal{E}^c$ and all other claims in the corollary follow immediately. □

Theorem 3.3. *Suppose that $\lambda \in (0, \underline{\lambda}]$ and $Z^{c_\lambda} = \{Z^{c_\lambda}_y\colon\ y \geq 0\}$. Then $\mathbb{P}$-almost surely, $\{Z^{c_\lambda}$ extinguishes$\}$ agrees with the event $\mathcal{E}$.*

Proof. First we establish that $\widetilde{\mathcal{E}} := \{Z^{c_\lambda}$ extinguishes$\} \subseteq \mathcal{E}$. Begin by noting that, thanks to monotonicity,

$$\lim_{y\uparrow\infty} X^{c_\lambda}_{D^t_y}\big((-\infty, y]\times\{t\}\big) = \|X_t\|. \tag{34}$$

Next note that since Z^{c_λ} is a supercritical conservative branching process, it follows that there is a $\lambda_0 > 0$ such that $\mathbb{P}(\widetilde{\mathcal{E}}) = \mathrm{e}^{-\lambda_0}$. Using the Markov branching property for exit measures we have,

$$\mathbb{E}\big(\mathbf{1}_{\widetilde{\mathcal{E}}}|\mathcal{F}^{c_\lambda}_{D^t_y}\big) = \mathrm{e}^{-\lambda_0\|X^{c_\lambda}_{D^t_y}\|} \leq \mathrm{e}^{-\lambda_0 X^{c_\lambda}_{D^t_y}((-\infty,y]\times\{t\})}.$$

Hence

$$\begin{aligned}
\mathbb{E}\Big[\lim_{t\uparrow\infty}\lim_{y\uparrow\infty}\mathbb{E}\big(\mathbf{1}_{\widetilde{\mathcal{E}}}|\mathcal{F}^{c_\lambda}_{D^t_y}\big)\mathbf{1}_{\mathcal{E}^c}\Big] &\leq \mathbb{E}\Big[\lim_{t\uparrow\infty}\lim_{y\uparrow\infty}\mathrm{e}^{-\lambda_0 X^{c_\lambda}_{D^t_y}((-\infty,y]\times\{t\})}\mathbf{1}_{\mathcal{E}^c}\Big] \\
&= \mathbb{E}\Big[\lim_{t\uparrow\infty}\mathrm{e}^{-\lambda_0\|X_t\|}\mathbf{1}_{\mathcal{E}^c}\Big] \\
&= 0.
\end{aligned} \tag{35}$$

Note that in the first equality we have used the fact that $\|X^{c_\lambda}_t\| = \|X_t\|$. Our objective is to show that

$$\mathbb{E}(\mathbf{1}_{\mathcal{E}^c\cap\widetilde{\mathcal{E}}}) = 0,$$

and hence $\widetilde{\mathcal{E}} \subseteq \mathcal{E}$, $\mathbb{P}$-almost surely. To this end, in light of (35), it suffices to prove that $\mathcal{E}^c \in \sigma(\bigcup_{t>0}\bigcup_{y>0}\mathcal{F}^{c_\lambda}_{D^t_y})$. Note however that, by (34), $\|X_t\| \in \sigma(\bigcup_{y>0}\mathcal{F}^{c_\lambda}_{D^t_y})$, which implies that $\mathcal{E} \in \sigma(\bigcup_{t>0}\bigcup_{y>0}\mathcal{F}^{c_\lambda}_{D^t_y})$.

Now fix $t > 0$. Note that the Markov branching property applied to the exit measure $X^{c_\lambda}_{D^t_y}$ implies that

$$\mathbb{P}(\mathcal{E}) = \mathbb{E}\big(\mathbb{P}\big(\mathcal{E}|\mathcal{F}^{c_\lambda}_{D^t_y}\big)\big) = \mathbb{E}\big[\mathrm{e}^{-\lambda^*\|X^{c_\lambda}_{D^t_y}\|}\big] \leq \mathbb{E}\big[\mathrm{e}^{-\lambda^* X^{c_\lambda}_{D^t_y}(\{y\}\times[0,t))}\big].$$

Now taking limits as $t \uparrow \infty$ we have with the help of both monotone and dominated convergence that

$$\mathbb{P}(\mathcal{E}) \leq \mathbb{E}\big[\mathrm{e}^{-\lambda^* \| X^{c_\lambda}_{D_y} \|}\big] = \mathbb{E}\big(\mathrm{e}^{-\lambda^* Z^{c_\lambda}_y}\big).$$

Taking limits again as $y \uparrow \infty$ we find that $\mathbb{P}(\mathcal{E}) \leq \mathbb{P}(Z^{c_\lambda}$ extinguishes). Together with the conclusion of the previous paragraph, we are forced to conclude that $\mathcal{E} = \{Z^{c_\lambda}$ extinguishes$\}$, $\mathbb{P}$-almost surely, as required. □

4. Proof of Theorem 2.1

Proof of Theorem 2.1(i). Suppose that there exists a travelling wave at speed $c \in [0, c_{\underline{\lambda}})$ which we shall denote by Φ. For all $x \in \mathbb{R}$ and $t \geq 0$, we have $\mathbb{E}_{\delta_x}(\mathrm{e}^{-\langle \Phi, X^c_t \rangle}) = \mathrm{e}^{-u^c_\Phi(x,t)}$, where u^c_Φ solves

$$\frac{\partial}{\partial t} u^c_\Phi(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u^c_\Phi(x,t) + c \frac{\partial}{\partial x} u^c_\Phi(x,t) - \psi\big(u^c_\Phi(x,t)\big), \tag{36}$$

with initial condition $u(x,0) = \Phi(x)$. This partial differential equation has a unique positive solution for the same reasons that (3) has a unique solution. Since $\Phi(x)$ also solves (36), it follows that

$$\mathbb{E}_{\delta_x}\big(\mathrm{e}^{-\langle \Phi, X^c_t \rangle}\big) = \mathrm{e}^{-\Phi(x)}.$$

Together with the branching property, this in turn implies that $\{\mathrm{e}^{-\langle \Phi, X^c_t \rangle}\colon\ t \geq 0\}$ is a uniformly integrable martingale. Its almost sure and $L^1(\mathbb{P}_{\delta_x})$ limit is denoted by M_∞. The Markov branching property applied to the exit measure $X^c_{D^t_y}$ implies that for all $x \leq y$,

$$\mathbb{E}_{\delta_x}\big(\mathrm{e}^{-\langle \Phi, X^c_{D^t_y} \rangle}\big) = \mathbb{E}_{\delta_x}\big[\mathbb{E}\big(M_\infty | \mathcal{F}^c_{D^t_y}\big)\big] = \mathrm{e}^{-\Phi(x)}.$$

Note however that for all $z \in \operatorname{supp} X^c_{D^t_y}$ we have by monotonicity, $\Phi(z) \geq \Phi(y)$. Moreover, as a measure, we also have $X^c_{D^t_y} \geq X^c_{D^t_y}|_{(-\infty,y)\times\{t\}}$. It follows that

$$\mathrm{e}^{-\Phi(x)} \leq \mathbb{E}_{\delta_x}\big(\mathrm{e}^{-\Phi(y) X^c_{D^t_y}((-\infty,y)\times\{t\})}\big). \tag{37}$$

Our next objective is to show that for any $y > x$,

$$\mathbb{P}_{\delta_x}\Big(\liminf_{n\uparrow\infty} X^c_{D^n_y}\big((-\infty, y) \times \{n\}\big) > 0\Big) > 0. \tag{38}$$

Suppose now that the probability in (38) is equal to zero for a given $y > x$. Let $X(n) := X^c_{D^n_y}((-\infty, y) \times \{n\})$ for $n \geq 0$. Then

$$\liminf_{n\to\infty} X(n) = 0, \quad \mathbb{P}_{\delta_x}\text{-a.s.} \tag{39}$$

Note that, under (5), 0 is an absorbing state for the sequence $\{X(n)\colon\ n \geq 0\}$ in the sense that $X(m) = 0$ implies that $X(m+k) = 0$ for all $k \geq 0$. Note that since $\{X^c_{D^t_y}|_{(-\infty,y)\times\{t\}}\colon\ t \geq 0\}$ is a superprocess with branching mechanism ψ and underlying motion which is that of a Brownian motion with drift c killed on hitting y, and therefore Markovian, then we have the estimate

$$\begin{aligned}
&\mathbb{P}_{\delta_x}\big(\exists m \text{ s.t. } X(m) = 0 | X(0), \ldots, X(n)\big) \\
&\quad \geq \inf_{\mu:\|\mu\|=X(n)} \mathbb{P}_\mu\big(\exists m \text{ s.t. } X(m) = 0\big) \\
&\quad \geq \inf_{\mu:\|\mu\|=X(n)} \mathbb{P}_\mu(\mathcal{E}) \\
&\quad = \mathrm{e}^{-\lambda^* X(n)}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by (39), we obtain that $P_{\delta_x}(\exists m \text{ s.t. } X(m) = 0) = 1$. It follows that $\liminf_{t\uparrow\infty} L_t^c \geq y$ $\mathbb{P}_{\delta_x}$-almost surely where $L_t^c = \inf\{z\colon\ X_t^c(-\infty, z] > 0\}$. However, from Corollary 3.2, we also deduce that under (5),

$$\lim_{t\uparrow\infty} \frac{L_t^c}{t} = c - \underline{\lambda} < 0,$$

which constitutes a contradiction. Therefore (38) holds for any $y > x$.

It follows by (38) and the Reverse Fatou Lemma, that

$$\mathrm{e}^{-\Phi(x)} \leq \limsup_{n\uparrow\infty} \mathbb{E}_{\delta_x}\big(\mathrm{e}^{-\Phi(y)X^c_{D_y^n}((-\infty,y]\times\{n\})}\big) < 1$$

for all sufficiently large $y > x$. As x may be chosen arbitrarily in this argument, it follows that there exists a constant $C > 0$ such that $\Phi(x) > C$ for all $x \in \mathbb{R}$. This leads to a contradiction of the assumption that Φ is a travelling wave. □

Proof of Theorem 2.1(ii). Grey [19] solves the classical Seneta–Heyde norming problem for continuous state branching processes. In particular he shows that for all $\lambda \in (0, \underline{\lambda}]$, taking account of the fact that Z^{c_λ} is a continuous state branching process with growth rate λ, there exists a slowly varying function at 0, L_λ such that

$$\lim_{x\uparrow\infty} \mathrm{e}^{-\lambda x} L_\lambda\big(\mathrm{e}^{-\lambda x}\big) Z_x^{c_\lambda} = \Delta(\lambda), \tag{40}$$

where $\Delta(\lambda) \geq 0$ is non-degenerate and the event $\{\Delta(\lambda) = 0\}$ agrees with the event that Z^{c_λ} becomes extinguished which in turn, by Theorem 3.3 agrees with the event $\mathcal{E}$. Note from (40) and the fact that L_λ is slowly varying, it is straightforward to show that for all $\mu \in \mathcal{M}_F(\mathbb{R})$

$$\mathbb{E}_\mu\big[\exp\{-\Delta(\lambda)\}\big] = \exp\{-\langle \Phi, \mu\rangle\},$$

where for all $x \in \mathbb{R}$,

$$\mathrm{e}^{-\Phi(x)} = \mathbb{E}_{\delta_x}\big[\exp\{-\Delta(\lambda)\}\big] = \mathbb{E}\big[\exp\{-\mathrm{e}^{-\lambda x}\Delta(\lambda)\}\big]. \tag{41}$$

Note in particular that Φ is a monotone decreasing function which is twice continuously differentiable on $(0, \infty)$ and moreover satisfies $\Phi(\infty) = 0$ and $\Phi(-\infty) = \lambda^*$.

The Markov branching property for Z^{c_λ} implies that by conditioning on $\mathcal{F}^{c_\lambda}_{D_x}$, where $x, z \in \mathbb{R}$, we get

$$\mathrm{e}^{-\Phi(z)} = \mathbb{E}\big[\exp\{-\mathrm{e}^{-\lambda z}\Delta(\lambda)\}\big] = \mathbb{E}\big[\exp\{-\Phi(x+z)Z_x^{c_\lambda}\}\big]. \tag{42}$$

Setting $z = 0$, $\mu = \delta_0$, $f = \Phi$, $c = c_\lambda$ in (25) and (26) we see that Φ necessarily solves

$$\frac{1}{2}\Phi'' + c_\lambda \Phi' - \psi(\Phi) = 0, \quad \text{on } \mathbb{R}, \tag{43}$$

as required. □

Proof of Theorem 2.1(iii) and (iv). Let $\lambda \in (0, \underline{\lambda}]$ and assume that Φ_{c_λ} is a travelling wave solution to the equation (6) with speed c_λ. From (2) and (3) it follows that for all $z \in \mathbb{R}$

$$\mathrm{e}^{-\Phi_{c_\lambda}(z)} = \mathbb{E}_{\delta_z}\big[\exp\{-\langle \Phi_{c_\lambda}(\cdot + c_\lambda t), X_t\rangle\}\big] = \mathbb{E}_{\delta_z}\big[\exp\{-\langle \Phi_{c_\lambda}, X_t^{c_\lambda}\rangle\}\big] = \mathbb{E}\big[\exp\{-\langle \Phi_{c_\lambda}(z+\cdot), X_t^{c_\lambda}\rangle\}\big]$$

and hence, together with the branching Markov property, we have that for all $z \in \mathbb{R}$

$$M_t^{\lambda,z} := \exp\{-\langle \Phi_{c_\lambda}(z+\cdot), X_t^{c_\lambda}\rangle\}, \quad t \geq 0, \tag{44}$$

is a positive uniformly integrable $\mathbb{P}$-martingale. From (23) we may deduce that there exists a deterministic subsequence $\{t_n\colon n \geq 0\}$ which increases to infinity (and may depend on x) such that

$$\lim_{n\uparrow\infty} X^{c_\lambda}_{D_x^{t_n}}\big((-\infty, x] \times \{t_n\}\big) = 0. \tag{45}$$

Now fix $z \in \mathbb{R}$ and $x \geq 0$. Let $M^{\lambda,z}_\infty := \lim_{t\uparrow\infty} M^{\lambda,z}_t$ then the branching Markov property applied to the exit measure $X^{c_\lambda}_{D_z^{t_n}}$ gives us

$$\mathbb{E}\big[M^{\lambda,z}_\infty | \mathcal{F}^{c_\lambda}_{D_x}\big] = \lim_{n\uparrow\infty} \mathbb{E}\big[M^{\lambda,z}_\infty | \mathcal{F}^{c_\lambda}_{D_x^{t_n}}\big] = \lim_{n\uparrow\infty} \exp\big\{-\big\langle \Phi_{c_\lambda}, X^{c_\lambda}_{D_x^{t_n}}\big\rangle\big\} = \exp\big\{-\Phi_{c_\lambda}(x+z) Z^{c_\lambda}_x\big\}.$$

From (44) we see that the event $\{M^{\lambda,z}_\infty = 1\}$ contains in the event that X becomes extinguished which in turn, from the proof of the previous part of the theorem, agrees with the event that $\{\Delta(\lambda) = 0\}$. Recalling (40), it follows that there exist a set of positive $\mathbb{P}$-probability on which Z^{c_λ} has a strictly positive normalised limit, such that the normalising sequence may be taken to be either $\mathrm{e}^{-\lambda x} L_\lambda(\mathrm{e}^{-\lambda x})$ or $\Phi_\lambda(x+z)$ as $x \uparrow \infty$. It must therefore follow that there exists a constant $k_{\lambda,z} \in (0, \infty)$ such that

$$\lim_{x\uparrow\infty} \frac{\Phi_\lambda(x+z)}{\mathrm{e}^{-\lambda x} L_\lambda(\mathrm{e}^{-\lambda x})} = k_{\lambda,z}.$$

As L_λ is slowly varying it is easy to see that $k_{\lambda,z} = \mathrm{e}^{-\lambda z} k_\lambda$ where $k_\lambda := k_{\lambda,0}$.

This also tells us that

$$M^{\lambda,z}_\infty = \exp\big\{-k_\lambda \mathrm{e}^{-\lambda z} \Delta(\lambda)\big\},$$

and hence taking expectations with respect to $\mathbb{P}$ we see that

$$\mathrm{e}^{-\Phi_{c_\lambda}(z)} = \mathbb{E}\big[\exp\big\{-k_\lambda \mathrm{e}^{-\lambda z} \Delta(\lambda)\big\}\big],$$

thus establishing uniqueness up to an additive constant. □

Reviewing the proof of Theorem 2.1 we obtain the following corollary, a simpler version of which has appeared in Neveu [36] and in parallel to writing of this paper, a similar result for branching Brownian motion has been described in [34].

Corollary 4.1. *For $\lambda \in (0, \underline{\lambda}]$, the continuous state branching process Z^{c_λ} has branching mechanism*

$$\psi_{c_\lambda}(\theta) = \Phi'_{c_\lambda}\big(\Phi^{-1}_{c_\lambda}(\theta)\big)$$

for $\theta \in [0, \lambda^*]$ where Φ_{c_λ} is any version of the unique travelling wave at speed c_λ. Alternatively ψ_{c_λ} solves the differential equation

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}u} f^2(u) + c_\lambda f(u) = \psi(u), \quad u \in \big(0, \lambda^*\big),$$

with boundary conditions $f(0) = 0$ and $f(\lambda^*) = 0$.

Proof. Since Z^{c_λ} is a continuous time continuous state branching process with branching mechanism, say ψ_{c_λ}, equation (42) implies that $\Phi(z) = u_x(\Phi(x+z))$, where for $\theta \geq 0$, $u_x(\theta)$ satisfies the semi-group equation

$$\frac{\partial u_x(\theta)}{\partial x} + \psi_{c_\lambda}\big(u_x(\theta)\big) = 0, \tag{46}$$

with initial condition $u_0(\theta) = \theta$. Differentiating the equality $\Phi(z) = u_x(\Phi(x+z))$ with respect to x we get

$$0 = \frac{\partial u_x(\Phi(z+x))}{\partial x} + \frac{\partial u_x(\theta)}{\partial \theta}\bigg|_{\theta=\Phi(z+x)} \Phi'(z+x).$$

By setting $x = 0$ in the previous equality, making use of (46) and the fact that $\partial u_0(\theta)/\partial\theta = 1$, we obtain that

$$\Phi'(z) = \psi_{c_\lambda}\big(\Phi(z)\big). \tag{47}$$

The first part follows directly from (47). For the second part, one may differentiate (47) and obtain

$$\Phi''(z) = \psi'_{c_\lambda}\big(\Phi(z)\big)\psi_{c_\lambda}\big(\Phi(z)\big). \tag{48}$$

Combining (47) and (48) with (43) and noting the domain of $u := \Phi(x)$, $x \in \mathbb{R}$ is $(0, \lambda^*)$, we get

$$\frac{1}{2}\psi'_{c_\lambda}(u)\psi_{c_\lambda}(u) + c_\lambda\psi_{c_\lambda}(u) - \psi(u) = 0,$$

which is the claimed differential equation.

To show the boundary conditions, first note that in all cases, since, by Theorem 3.1, Z^{c_λ} is conservative, we necessarily have $\psi_{c_\lambda}(0) = 0$. For the other boundary condition recall from the beginning and the end of the proof of Theorem 2.1(ii) that the event $\{\Delta(\lambda) = 0\}$ agrees both with the event $\mathcal{E}$ as well as the event that Z^{c_λ} becomes extinguished. Hence we have $\mathbb{P}(Z^{c_\lambda}$ becomes extinguished$) = \mathrm{e}^{-\lambda^*}$ and then necessarily λ^* must be the largest root in $[0, \infty)$ of the equation $\psi_{c_\lambda}(\theta) = 0$. This justifies the boundary condition $\psi_{c_\lambda}(\lambda^*) = 0$. □

5. Pathwise spine decomposition

The convergence of the martingales $W(\lambda)$ and $\partial W(\lambda)$ to non-trivial limits will ultimately allow us to identify the limiting variable $\Delta(\lambda)$ as either $W_\infty(\lambda)$ or $\partial W_\infty(\lambda)$. There is a well understood technique for branching particle processes, due to Lyons et al. [33] and Lyons [32], which can be used to establish in a straightforward way conditions under which the latter limits are non-trivial. This involves looking at how the given martingales (or variants of them) perform as changes of measure. In the case of superprocesses, this technique can be seen under the pretext of Evans' immortal particle decomposition; see for example Evans [14] and Engländer and Kyprianou [13]. In the latter references, the decomposition has only been explored through the semi-group of the underlying superprocess which has its limitations when using it to analyse the martingales $W(\lambda)$ and $\partial W(\lambda)$ in the spirit of pathwise spine decompositions for branching particle processes. In the analysis below, we give a new immortal particle decomposition for our class of superprocesses which is defined in a pathwise sense and therefore lends itself to the aforementioned classical martingale analysis. The feature which is in particular new to our spine decomposition is the use of the Dynkin–Kuznetsov $\mathbb{N}$-measure to describe a Poisson point process of immigration along the immortal particle. For branching mechanism without quadratic term (i.e., $\beta = 0$ in the definition of ψ given by (1)), a similar pathwise spine decomposition was given by Liu et al. [31] when dealing with another martingale which, like $W(\lambda)$, was constructed from a positive eigen-function to the linear semi-group of the underlying superprocess.

Let us move to our new spine decomposition and hencewith we start by defining some martingale changes of measure. For each $\lambda \in \mathbb{R}$ and $\mu \in \mathcal{M}_F(\mathbb{R})$, let $\mathbb{P}_\mu^{-\lambda}$ be defined by

$$\left.\frac{\mathrm{d}\mathbb{P}_\mu^{-\lambda}}{\mathrm{d}\mathbb{P}_\mu}\right|_{\mathcal{F}_t} = \frac{W_t(\lambda)}{W_0(\lambda)}, \quad t \geq 0,$$

where $\mathcal{F}_t = \sigma(X_s\colon\ s \leq t)$. Note in particular that $W_0(\lambda) = \langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle$. Next recall that for each $x \in \mathbb{R}$ we defined the process $\xi := \{\xi_t\colon\ t \geq 0\}$ under Π_x to be a Brownian motion issued from x. If $\Pi_x^{-\lambda}$ is the law under which ξ is a Brownian motion with drift $-\lambda \in \mathbb{R}$ issued from $x \in \mathbb{R}$, then for each $t \geq 0$,

$$\left.\frac{\mathrm{d}\Pi_x^{-\lambda}}{\mathrm{d}\Pi_x}\right|_{\mathcal{G}_t} = \mathrm{e}^{-\lambda(\xi_t - x) - \lambda^2 t/2}, \tag{49}$$

where $\mathcal{G}_t = \sigma(\xi_s, s \leq t)$. For convenience we shall also write

$$\Pi_\mu^{-\lambda}(\cdot) = \frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle}\int_{\mathbb{R}} \mathrm{e}^{-\lambda x}\mu(\mathrm{d}x)\Pi_x^{-\lambda}(\cdot). \tag{50}$$

Theorem 5.1. *Suppose that $\lambda \in \mathbb{R}$, $\mu \in \mathcal{M}_F(\mathbb{R})$ and $g \in C_b^+(\mathbb{R})$. Then*

$$\mathbb{E}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g, X_t\rangle}\big] = \mathbb{E}_\mu\big[\mathrm{e}^{-\langle g, X_t\rangle}\big]\Pi_\mu^{-\lambda}\left[\exp\left\{-\int_0^t \phi\big(u_g(\xi_{t-s}, s)\big)\,\mathrm{d}s\right\}\right], \tag{51}$$

where

$$\phi(\lambda) = \psi'(\lambda) - \psi'\big(0^+\big) = 2\beta\lambda + \int_0^\infty \big(1 - \mathrm{e}^{-\lambda x}\big)x\,\nu(\mathrm{d}x), \tag{52}$$

and u_g is the unique solution of (3).

Proof. By the definition of $\mathbb{E}_\mu^{-\lambda}$, we get

$$\begin{aligned}\mathbb{E}_\mu^{-\lambda}\big(\mathrm{e}^{-\langle g, X_t\rangle}\big) &= \frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle}\mathbb{E}_\mu\big(\mathrm{e}^{-\lambda c_\lambda t}\big\langle \mathrm{e}^{-\lambda\cdot}, X_t\big\rangle \mathrm{e}^{-\langle g, X_t\rangle}\big)\\ &= -\frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle}\mathrm{e}^{-\lambda c_\lambda t}\mathbb{E}_\mu\left(\frac{\partial}{\partial\theta}\mathrm{e}^{-\langle g_\theta, X_t\rangle}\Big|_{\theta=0^+}\right),\end{aligned}$$

with $g_\theta(x) := g(x) + \theta\mathrm{e}^{-\lambda x}$. Interchanging the expectation and differentiation, we get

$$\mathbb{E}_\mu^{-\lambda}\big(\mathrm{e}^{-\langle g, X_t\rangle}\big) = -\frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle}\mathrm{e}^{-\lambda c_\lambda t}\frac{\partial}{\partial\theta}\mathrm{e}^{-\langle u_{g_\theta}(\cdot,t), \mu\rangle}\Big|_{\theta=0^+},$$

where u_{g_θ} satisfies (3) with g replaced by g_θ. Note that, $u_{g_0} = u_g$. Hence,

$$\mathbb{E}_\mu^{-\lambda}\big(\mathrm{e}^{-\langle g, X_t\rangle}\big) = \frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle}\mathrm{e}^{-\lambda c_\lambda t}\mathrm{e}^{-\langle u_{g_\theta}(\cdot,t), \mu\rangle}\frac{\partial}{\partial\theta}\big\langle u_{g_\theta}(\cdot, t), \mu\big\rangle\Big|_{\theta=0^+}. \tag{53}$$

Now let $m^g(x,t) := \frac{\partial}{\partial\theta}u_{g_\theta}(x,t)|_{\theta=0^+}$ for all $x \in \mathbb{R}$. Taking derivatives in (3) with g replaced by g_θ and then taking the limit as θ goes to zero, we obtain the differential equation

$$\begin{cases}\frac{\partial}{\partial t}m^g(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}m^g(x,t) - \psi'\big(u_g(x,t)\big)m^g(x,t),\\ m^g(x,0) = \mathrm{e}^{-\lambda x}.\end{cases}$$

The classical Feynman–Kac formula gives

$$m^g(x,t) = \Pi_x\left[\mathrm{e}^{-\lambda\xi_t}\exp\left\{-\int_0^t \psi'\big(u_g(\xi_{t-s}, s)\big)\,\mathrm{d}s\right\}\right].$$

Plugging back into (53) yields the following equality,

$$\begin{aligned}&\mathbb{E}_\mu^{-\lambda}\big(\mathrm{e}^{-\langle g, X_t\rangle}\big)\\ &\quad= \frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle}\mathbb{E}_\mu\big[\mathrm{e}^{-\langle g, X_t\rangle}\big]\int_{\mathbb{R}}\mathrm{e}^{-\lambda x}\mu(\mathrm{d}x)\Pi_x\left[\mathrm{e}^{-\lambda(\xi_t - x) - \lambda c_\lambda t}\exp\left\{-\int_0^t \psi'\big(u_g(\xi_{t-s}, s)\big)\,\mathrm{d}s\right\}\right]\\ &\quad= \mathbb{E}_{\delta_x}\big[\mathrm{e}^{-\langle g, X_t\rangle}\big]\Pi_\mu^{-\lambda}\left[\exp\left\{-\int_0^t \phi\big(u_g(\xi_{t-s}, s)\big)\,\mathrm{d}s\right\}\right],\end{aligned}$$

where in the final equality we have used (49) and the fact that $\lambda c_\lambda = -\psi'(0^+) + \lambda^2/2$.

To complete the proof, note that the expression given for $\phi(\lambda)$ on the right-hand side of (52) is obtained by straightforward differentiation of (1). □

Equation (51) suggests that, under the measure $\mathbb{P}_\mu^{-\lambda}$, the superprocess X can be decomposed into two parts. The first one is a copy of the original superprocess and the second one can be related to an independent process of immigration. As we shall demonstrate next, the process of immigration is governed by a spine or immortal particle along which two independent Poisson point processes of immigration occur. We need first to introduce some more notation.

Associated to the laws $\{\mathbb{P}_{\delta_x}\colon x \in \mathbb{R}\}$ are the measures $\{\mathbb{N}_x\colon x \in \mathbb{R}\}$, defined on the same measurable space, which satisfy

$$\mathbb{N}_x\big(1 - \mathrm{e}^{-\langle f, X_t\rangle}\big) = -\log \mathbb{E}_{\delta_x}\big(\mathrm{e}^{-\langle f, X_t\rangle}\big) \tag{54}$$

for all $f \in C_b^+(\mathbb{R})$ and $t \geq 0$. Such measures are formally defined and explored in detail in [12]. See also [30]. Note that from the definition (54) it follows that

$$\mathbb{N}_x\big(\langle f, X_t\rangle\big) = \mathbb{E}_{\delta_x}\big(\langle f, X_t\rangle\big),$$

whenever $f \in C_b^+(\mathbb{R})$.

The measures $\{\mathbb{N}_x\colon x \in \mathbb{R}\}$ will play a crucial role in the forthcoming analysis. Intuitively speaking, the branching property implies that $\mathbb{P}_{\delta_x}$ is an infinitely divisible measure on the path space of X and (54) is a 'Lévy–Khinchine' formula in which $\mathbb{N}_x$ plays the role of its 'Lévy measure.' In this sense, $\mathbb{N}_x$ can be considered as the 'rate' at which superprocesses 'with zero initial mass' contribute to a unit mass at position x. It is important to note that $\mathbb{N}_x$ is not a probability measure as such.

With this measure in hand, let us now proceed to the definition of a measure-valued process of immigration, which we denote by $\Lambda = \{\Lambda_t\colon t \geq 0\}$. Fix $x \in \mathbb{R}$ and $\mu \in \mathcal{M}_F(\mathbb{R})$.

(i) *Spine*: We take a copy of the process $\xi = \{\xi_t\colon t \geq 0\}$ under $\Pi_x^{-\lambda}$ and henceforth refer to it as the *spine*.
(ii) *Continuum immigration*: Suppose that $\mathbf{n}$ is a Poisson point process such that, for $t \geq 0$, given the spine ξ, $\mathbf{n}$ issues a superprocess $X^{\mathbf{n},t}$ at space–time position (ξ_t, t) with rate $\mathrm{d}t \times 2\beta\,\mathrm{d}\mathbb{N}_{\xi_t}$.
(iii) *Jump immigration*: Suppose that $\mathbf{m}$ is a Poisson point process such that, independently of $\mathbf{n}$, given the spine ξ, $\mathbf{m}$ issues a superprocess $X^{\mathbf{m},t}$ at space–time point (ξ_t, t) with initial mass r at rate $\mathrm{d}t \times r\nu(\mathrm{d}r) \times \mathrm{d}\mathbb{P}_{r\delta_{\xi_t}}$.

We now define for $t \geq 0$,

$$\Lambda_t = X'_t + X_t^{(\mathbf{n})} + X_t^{(\mathbf{m})}, \tag{55}$$

where $\{X'_t\colon t \geq 0\}$ is an independent copy of $(X, \mathbb{P}_\mu)$,

$$X_t^{(\mathbf{n})} = \sum_{s \leq t:\mathbf{n}} X_{t-s}^{\mathbf{n},s}, \quad t \geq 0 \quad \text{and} \quad X_t^{(\mathbf{m})} = \sum_{s \leq t:\mathbf{m}} X_{t-s}^{\mathbf{m},s}, \quad t \geq 0.$$

In the last two equalities we understand the first sum to be over times at which the process $\mathbf{n}$ has a point and the second sum is understood similarly. Note that since the processes $X^{(\mathbf{n})}$ and $X^{(\mathbf{m})}$ are initially zero valued it is clear that since $X'_0 = \mu$ then $\Lambda_0 = \mu$. In that case, we use the notation $\widetilde{\mathbb{P}}_{\mu,x}^{-\lambda}$ to denote the law of the pair (Λ, ξ). Note also that the pair (Λ, ξ) are a time-homogenous Markov process. We are interested in the case that the initial position of the spine ξ is randomised using the measure μ via (50). In that case we shall write

$$\widetilde{\mathbb{P}}_\mu^{-\lambda}(\cdot) = \frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \mu\rangle} \int_{\mathbb{R}} \mathrm{e}^{-\lambda x}\mu(\mathrm{d}x)\widetilde{\mathbb{P}}_{\mu,x}^{-\lambda}(\cdot)$$

for short. The next theorem identifies the process Λ as the *pathwise* spine decomposition of $(X, \mathbb{P}_\mu^{-\lambda})$ and in particular it shows that as a process on its own Λ is Markovian.

Theorem 5.2. *For all $\lambda \in \mathbb{R}$ and $\mu \in \mathcal{M}_F(\mathbb{R})$, $(X, \mathbb{P}_\mu^{-\lambda})$ and $(\Lambda, \widetilde{\mathbb{P}}_\mu^{-\lambda})$ are equal in law.*

Proof. Fix $\lambda \in \mathbb{R}$. Firstly we must prove that for any $t \geq 0$ and $\mu \in \mathcal{M}_F(\mathbb{R})$, we have

$$\mathbb{E}_\mu^{-\lambda}\big(\mathrm{e}^{-\langle g, X_t\rangle}\big) = \widetilde{\mathbb{E}}_\mu^{-\lambda}\big(\mathrm{e}^{-\langle g, \Lambda_t\rangle}\big), \tag{56}$$

where $g \in C_b^+(\mathbb{R})$, and for this it suffices to show that

$$\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g, X_t^{(\mathbf{n})}+X_t^{(\mathbf{m})}\rangle}\big] = \Pi_\mu^{-\lambda}\bigg[\exp\bigg\{-\int_0^t \phi\big(u_g(\xi_{t-s}, s)\big)\,\mathrm{d}s\bigg\}\bigg]. \tag{57}$$

Secondly we must show that $(\Lambda, \widetilde{\mathbb{P}}_\mu^{-\lambda})$ is a Markov process.

To this end note that for $g \in C_b^+(\mathbb{R})$,

$$\begin{aligned}
&\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g, X_t^{(\mathbf{n})}+X_t^{(\mathbf{m})}\rangle}\big] \\
&\quad= \widetilde{\mathbb{E}}_\mu^{-\lambda}\big\{\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g, X_t^{(\mathbf{n})}\rangle}\mathrm{e}^{-\langle g, X_t^{(\mathbf{m})}\rangle}|\xi\big]\big\} \\
&\quad= \Pi_\mu^{-\lambda}\bigg\{\widetilde{\mathbb{E}}_\mu^{-\lambda}\bigg[\exp\bigg\{-\sum_{s\le t:\mathbf{n}}\langle g, X_{t-s}^{\mathbf{n},s}\rangle\bigg\}\Big|\xi\bigg]\widetilde{\mathbb{E}}_\mu^{-\lambda}\bigg[\exp\bigg\{-\sum_{s\le t:\mathbf{m}}\langle g, X_{t-s}^{\mathbf{m},s}\rangle\bigg\}\Big|\xi\bigg]\bigg\},
\end{aligned} \tag{58}$$

where we have used the independence of $X^{(\mathbf{m})}$ and $X^{(\mathbf{n})}$. Applying Campbell's formula to the first inner expectation and using (54), we get

$$\begin{aligned}
\widetilde{\mathbb{E}}_\mu^{-\lambda}\bigg[\exp\bigg\{-\sum_{s\le t:\mathbf{n}}\langle g, X_{t-s}^{\mathbf{n},s}\rangle\bigg\}\Big|\xi\bigg] &= \exp\bigg\{-2\beta\int_0^t\int\big(1-\mathrm{e}^{-\langle g, X_{t-s}\rangle}\big)\,\mathrm{d}\mathbb{N}_{\xi_s}\,\mathrm{d}s\bigg\} \\
&= \exp\bigg\{-2\beta\int_0^t -\log\mathbb{E}_{\delta_{\xi_s}}\big(\mathrm{e}^{-\langle g, X_{t-s}\rangle}\big)\,\mathrm{d}s\bigg\} \\
&= \exp\bigg\{-2\beta\int_0^t u_g(\xi_s, t-s)\,\mathrm{d}s\bigg\} \\
&= \exp\bigg\{-2\beta\int_0^t u_g(\xi_{t-s}, s)\,\mathrm{d}s\bigg\}.
\end{aligned} \tag{59}$$

To deal with the second expectation in (58) first note that,

$$\widetilde{\mathbb{E}}_\mu^{-\lambda}\bigg[\exp\bigg\{-\sum_{s\le t:\mathbf{m}}\langle g, X_{t-s}^{\mathbf{m},s}\rangle\bigg\}\Big|\xi\bigg] = \widetilde{\mathbb{E}}_\mu^{-\lambda}\bigg[\exp\bigg\{-\sum_{s\le t:\mathbf{m}} m_s u_g(\xi_s, t-s)\bigg\}\Big|\xi\bigg], \tag{60}$$

where for $s \ge 0$, $m_s = \|X_0^{\mathbf{m},s}\|$. In particular note that the process $\{m_t\colon t \ge 0\}$ is a Poisson point process on $(0,\infty)^2$, independent of ξ, with intensity $\mathrm{d}t \times r\nu(\mathrm{d}r)$. Hence, putting (59) and (60) into (58) and again appealing to Campbell's formula yields

$$\begin{aligned}
&\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\exp\{-\langle g, X_t^{(\mathbf{n})} + X_t^{(\mathbf{m})}\rangle\}\big] \\
&\quad= \Pi_\mu^{-\lambda}\bigg\{\exp\bigg\{-2\beta\int_0^t u_g(\xi_{t-s}, s)\,\mathrm{d}s\bigg\}\widetilde{\mathbb{E}}_\mu^{-\lambda}\bigg[\exp\bigg\{-\sum_{s\le t:\mathbf{m}} m_s u_g(\xi_s, t-s)\bigg\}\Big|\xi\bigg]\bigg\} \\
&\quad= \Pi_\mu^{-\lambda}\bigg\{\exp\bigg\{-2\beta\int_0^t u_g(\xi_{t-s}, s)\,\mathrm{d}s\bigg\}\exp\bigg\{-\int_0^t\int_{(0,\infty)}\big(1-\mathrm{e}^{-r u_g(\xi_{t-s}, s)}\big) r\nu(\mathrm{d}r)\,\mathrm{d}s\bigg\}\bigg\}.
\end{aligned}$$

Taking note of (52) we have in conclusion that (57) follows.

Next we turn our attention to showing that $(\Lambda, \widetilde{\mathbb{P}}_\mu^{-\lambda})$ is a Markov process. To this end, it suffices to show that for all $x \in \mathbb{R}$,

$$\widetilde{\mathbb{P}}_\mu^{-\lambda}(\xi_t \in \mathrm{d}x|\Lambda_t) = \frac{1}{\langle \mathrm{e}^{-\lambda\cdot}, \Lambda_t\rangle}\mathrm{e}^{-\lambda x}\Lambda_t(\mathrm{d}x). \tag{61}$$

Indeed, in that case it follows that for all $g \in C_b^+(\mathbb{R})$

$$\begin{aligned}
\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}|\Lambda_r\colon r\le s\big] &= \widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}|(\xi_r,\Lambda_r)\colon r\le s\big]\big|\Lambda_r\colon r\le s\big]\\
&= \widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\widetilde{\mathbb{E}}_{(\mu',x')}^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}\big]\big|_{\mu'=\Lambda_s,x'=\xi_s}\big|\Lambda_r\colon r\le s\big]\\
&= \widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\widetilde{\mathbb{E}}_{(\mu',x')}^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}\big]\big|_{\mu'=\Lambda_s,x'=\xi_s}\big|\Lambda_s\big]\\
&= \frac{1}{\langle \mathrm{e}^{-\lambda\cdot},\Lambda_s\rangle}\int_{\mathbb{R}}\mathrm{e}^{-\lambda x}\Lambda_s(\mathrm{d}x)\widetilde{\mathbb{E}}_{\mu',x}^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_{t-s}\rangle}\big]\Big|_{\mu'=\Lambda_s}\\
&= \widetilde{\mathbb{E}}_{\mu'}^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_{t-s}\rangle}\big]\big|_{\mu'=\Lambda_s},
\end{aligned}$$

where in the third equality we have used that (ξ,Λ) is Markovian.

To show (61) it suffices to show that for all $\theta\in\mathbb{R}$

$$\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\theta\xi_t}|\Lambda_t\big]\big] = \widetilde{\mathbb{E}}_\mu^{-\lambda}\left[\mathrm{e}^{-\langle g,\Lambda_t\rangle}\frac{\langle \mathrm{e}^{-(\lambda+\theta)\cdot},\Lambda_t\rangle}{\langle \mathrm{e}^{-\lambda\cdot},\Lambda_t\rangle}\right]. \tag{62}$$

Note however that the left-hand side of (62) is equal to

$$\begin{aligned}
\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}\mathrm{e}^{-\theta\xi_t}\big] &= \mathrm{e}^{\gamma t}\frac{1}{\langle \mathrm{e}^{-\lambda\cdot},\mu\rangle}\int_{\mathbb{R}}\mu(\mathrm{d}x)\mathrm{e}^{-(\lambda+\theta)x}\Pi_x^{-\lambda}\big[\mathrm{e}^{-\theta(\xi_t-x)-\gamma t}\widetilde{\mathbb{E}}_\mu^{-\lambda}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}|\xi_t\big]\big]\\
&= \mathrm{e}^{\gamma t}\frac{1}{\langle \mathrm{e}^{-\lambda\cdot},\mu\rangle}\int_{\mathbb{R}}\mu(\mathrm{d}x)\mathrm{e}^{-(\lambda+\theta)x}\Pi_x^{-(\lambda+\theta)}\big[\widetilde{\mathbb{E}}_\mu^{-(\lambda+\theta)}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}|\xi_t\big]\big]\\
&= \mathrm{e}^{\gamma t}\frac{\langle \mathrm{e}^{-(\lambda+\theta)\cdot},\mu\rangle}{\langle \mathrm{e}^{-\lambda\cdot},\mu\rangle}\widetilde{\mathbb{E}}_\mu^{-(\lambda+\theta)}\big[\mathrm{e}^{-\langle g,\Lambda_t\rangle}\big],
\end{aligned} \tag{63}$$

where $\gamma=(\lambda+\theta)^2/2-\lambda^2/2$.

On the other hand, by (56), the right-hand side of (62) is equal to

$$\begin{aligned}
\mathbb{E}_\mu\left[\mathrm{e}^{-\lambda c_\lambda t}\frac{\langle \mathrm{e}^{-\lambda\cdot},X_t\rangle}{\langle \mathrm{e}^{-\lambda\cdot},\mu\rangle}\mathrm{e}^{-\langle g,X_t\rangle}\frac{\langle \mathrm{e}^{-(\lambda+\theta)\cdot},X_t\rangle}{\langle \mathrm{e}^{-\lambda\cdot},X_t\rangle}\right] &= \mathrm{e}^{\gamma t}\mathbb{E}_\mu\left[\mathrm{e}^{-(\lambda+\theta)c_{\lambda+\theta}t}\frac{\langle \mathrm{e}^{-(\lambda+\theta)\cdot},X_t\rangle}{\langle \mathrm{e}^{-\lambda\cdot},\mu\rangle}\mathrm{e}^{-\langle g,X_t\rangle}\right]\\
&= \mathrm{e}^{\gamma t}\frac{\langle \mathrm{e}^{-(\lambda+\theta)\cdot},\mu\rangle}{\langle \mathrm{e}^{-\lambda\cdot},\mu\rangle}\mathbb{E}_\mu^{-(\lambda+\theta)}\big[\mathrm{e}^{-\langle g,X_t\rangle}\big].
\end{aligned} \tag{64}$$

Again appealing to (56) we see that both (63) and (64) agree and the proof is complete. □

6. Proof of Theorem 2.4(i)

For reasons of symmetry, it sufficies to prove the result with $\lambda\ge 0$. Now that we are in possession of a pathwise spine decomposition, we may pursue a classical approach due to Lyons [32], see also Kyprianou [25], to prove Theorem 2.4(i). The key element to the reasoning is the following measure theoretic result (see for example p. 242 of Durrett [5]).

Let $\overline{W}_\infty(\lambda):=\limsup_{t\uparrow\infty}W_t(\lambda)$. Then

$$\overline{W}_\infty(\lambda)=\infty,\quad \mathbb{P}^{-\lambda}\text{-a.s.}\quad\Longleftrightarrow\quad \overline{W}_\infty(\lambda)=0,\quad \mathbb{P}\text{-a.s.}, \tag{65}$$

$$\overline{W}_\infty(\lambda)<\infty,\quad \mathbb{P}^{-\lambda}\text{-a.s.}\quad\Longleftrightarrow\quad \mathbb{E}\big(\overline{W}_\infty(\lambda)\big)=1. \tag{66}$$

Write for convenience $\widetilde{\mathbb{P}}^{-\lambda}$ instead of $\widetilde{\mathbb{P}}_{\delta_0}^{-\lambda}$. Thanks to the spine decomposition (Theorem 5.2) we may replace on the left-hand sides of (65) and (66) $\mathbb{P}^{-\lambda}$ by $\widetilde{\mathbb{P}}^{-\lambda}$ and $\overline{W}_\infty(\lambda)$ by $\overline{W}_\infty^{\Lambda}(\lambda):=\limsup_{t\uparrow\infty}W_t^{\Lambda}(\lambda)$ where

$$W_t^{\Lambda}(\lambda):=\mathrm{e}^{-\lambda c_\lambda t}\big\langle \mathrm{e}^{-\lambda\cdot},\Lambda_t\big\rangle.$$

We shall study $\overline{W}^{\Lambda}_{\infty}(\lambda)$ with the help of the martingale decomposition (which follows from the spine decomposition) under $\widetilde{\mathbb{P}}^{-\lambda}$,

$$W^{\Lambda}_t(\lambda) = W'_t(\lambda) + \sum_{s\le t:\mathbf{n}} \mathrm{e}^{-\lambda c_\lambda s} W^{\mathbf{n},s}_{t-s}(\lambda) + \sum_{s\le t:\mathbf{m}} \mathrm{e}^{-\lambda c_\lambda s} W^{\mathbf{m},s}_{t-s}(\lambda), \tag{67}$$

where $W'(\lambda)$ is an independent copy of $W(\lambda)$ under $\mathbb{P}$, $W^{\mathbf{n},s}_{t-s}(\lambda) = \mathrm{e}^{-\lambda c_\lambda (t-s)}\langle \mathrm{e}^{-\lambda\cdot}, X^{\mathbf{n},s}_{t-s}\rangle$ and $W^{\mathbf{m},s}_{t-s}(\lambda) = \mathrm{e}^{-\lambda c_\lambda (t-s)}\langle \mathrm{e}^{-\lambda\cdot}, X^{\mathbf{m},s}_{t-s}\rangle$.

Suppose that $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty$ and $\lambda \in (0, \underline{\lambda})$. First note that since $W(\lambda)$ is a $\mathbb{P}$-martingale, it follows as a standard result that $W(\lambda)^{-1}$ is a $\mathbb{P}^{-\lambda}$-supermartingale (see for example [22]) and hence $\lim_{t\uparrow\infty} W_t(\lambda)$ exists $\mathbb{P}^{-\lambda}$-almost surely, and hence $\lim_{t\uparrow\infty} W^{\Lambda}_t(\lambda)$ exists $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely and is equal to $W^{\Lambda}_{\infty}(\lambda)$. Our objective is to show that

$$\limsup_{t\uparrow\infty} \widetilde{\mathbb{E}}^{-\lambda}\big[W^{\Lambda}_t(\lambda)|\xi, \mathbf{m}\big] < \infty, \tag{68}$$

in which case Fatou's lemma and the existence of $\lim_{t\uparrow\infty} W^{\Lambda}_t(\lambda) = \overline{W}^{\Lambda}_{\infty}(\lambda)$ implies that, $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely, $\overline{W}^{\Lambda}_{\infty}(\lambda) < \infty$. This in turn implies that $\mathbb{P}^{-\lambda}$-almost surely, $\overline{W}_{\infty}(\lambda) < \infty$ and hence, by (66), $W_{\infty}(\lambda)$ is an $L^1(\mathbb{P})$ limit.

To this end note that, since $\widetilde{\mathbb{E}}^{-\lambda}[W'_t(\lambda)] = 1$, it suffices to prove that

$$\limsup_{t\uparrow\infty} \widetilde{\mathbb{E}}^{-\lambda}\bigg[\sum_{s\le t:\mathbf{n}} \mathrm{e}^{-\lambda c_\lambda s} W^{\mathbf{n},s}_{t-s}(\lambda) + \sum_{s\le t:\mathbf{m}} \mathrm{e}^{-\lambda c_\lambda s} W^{\mathbf{m},s}_{t-s}(\lambda)\bigg|\xi, \mathbf{m}\bigg] < \infty.$$

First note that, $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely,

$$\begin{aligned}
\limsup_{t\uparrow\infty} \widetilde{\mathbb{E}}^{-\lambda}\bigg[\sum_{s\le t:\mathbf{n}} \mathrm{e}^{-\lambda c_\lambda s} W^{\mathbf{n},s}_{t-s}(\lambda)\bigg|\xi, \mathbf{m}\bigg] &= \limsup_{t\uparrow\infty} 2\beta \int_0^t \mathrm{e}^{-\lambda c_\lambda s}\mathbb{N}_{\xi_s}\big[W_{t-s}(\lambda)\big]\,\mathrm{d}s \\
&= \limsup_{t\uparrow\infty} 2\beta \int_0^t \mathrm{e}^{-\lambda c_\lambda s}\mathbb{E}_{\delta_{\xi_s}}\big[W_{t-s}(\lambda)\big]\,\mathrm{d}s \\
&= 2\beta \int_0^{\infty} \mathrm{e}^{-\lambda(\xi_s + c_\lambda s)}\,\mathrm{d}s \\
&< \infty,
\end{aligned} \tag{69}$$

where the second identity holds by (54) and the final inequality is a result of the strong law of large numbers for linear Brownian motion together with the fact that $\xi_t + c_\lambda t$ has drift $c_\lambda - \lambda$ which is strictly positive when $\lambda \in (0, \underline{\lambda})$.

Next, recalling that for all $x \in \mathbb{R}$, $t \ge 0$, $r > 0$, $\mathbb{E}_{r\delta_x}(W_t(\lambda)) = r\mathrm{e}^{-\lambda x}\mathbb{E}(W_t(\lambda)) = r\mathrm{e}^{-\lambda x}$, we have that

$$\widetilde{\mathbb{E}}^{-\lambda}\bigg[\sum_{s\le t:\mathbf{m}} \mathrm{e}^{-\lambda c_\lambda s} W^{\mathbf{m},s}_{t-s}(\lambda)\bigg|\xi, \mathbf{m}\bigg] = \widetilde{\mathbb{E}}^{-\lambda}\bigg[\sum_{s\le t:\mathbf{m}} m_s \mathrm{e}^{-\lambda(\xi_s + c_\lambda s)}\bigg|\xi, \mathbf{m}\bigg] \le \sum_{s\ge 0;\mathbf{m}} m_s \mathrm{e}^{-\lambda(\xi_s + c_\lambda s)}, \tag{70}$$

where we recall that the process $\{m_t\colon t \ge 0\}$ is a Poisson point process, independent of ξ and with intensity $\mathrm{d}t \times r\nu(\mathrm{d}r)$ However, on the one hand we have that $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely

$$\begin{aligned}
\widetilde{\mathbb{E}}^{-\lambda}\bigg[\sum_{s\ge 0;\mathbf{m}} m_s \mathbf{1}_{\{m_s<1\}}\mathrm{e}^{-\lambda(\xi_s + c_\lambda s)}\bigg|\xi\bigg] &= \int_{(0,1)}\int_0^{\infty} r\mathrm{e}^{-\lambda(\xi_s + c_\lambda s)} r\,\mathrm{d}s\nu(\mathrm{d}r) \\
&= \int_{(0,1)} r^2\,\mathrm{d}\nu(\mathrm{d}r)\int_0^{\infty} \mathrm{e}^{-\lambda(\xi_s + c_\lambda s)}\,\mathrm{d}s \\
&< \infty.
\end{aligned} \tag{71}$$

On the other hand, define $\tau_0 = 0$ and $\tau_i = \inf\{t \geq \tau_{i-1} \colon m_t \geq 1\}$, $i = 1, 2, \ldots$. Note that, $\{\tau_i \colon i \geq 0\}$ are the times of arrival in a Poisson process with arrival rate $\int_{[1,\infty)} r\nu(\mathrm{d}r)$. Recall that if $\{Z_i \colon i \geq 1\}$ is a sequence of i.i.d. random variables with probability measure Q then it can easily be shown with the help of the Borel–Cantelli Lemma that, Q-a.s.

$$\limsup_{i\to\infty} i^{-1} \log Z_i = \begin{cases} 0, & \text{if } Q\big(\log^+ Z_1\big) < \infty, \\ \infty, & \text{if } Q\big(\log^+ Z_1\big) = \infty. \end{cases} \tag{72}$$

In this instance we would like to take $Z_i = m_{\tau_i}$ which has common probability measure $Q(\mathrm{d}r) = r\nu(\mathrm{d}r)/\int_{[1,\infty)} r\nu(\mathrm{d}r)$ on $[1, \infty)$. We have $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely that

$$\sum_{s\geq 0;\mathbf{m}} m_s \mathbf{1}_{\{m_s \geq 1\}} \mathrm{e}^{-\lambda(\xi_s + c_\lambda s)} = \sum_{i\geq 1} m_{\tau_i} \mathrm{e}^{-\lambda(\xi_{\tau_i} + c_\lambda \tau_i)} < \infty, \tag{73}$$

where the equality follows by virtue of the fact that entries in the first sum arrive at rate $\int_{[1,\infty)} r\nu(\mathrm{d}r)$, which is finite, and the final inequality follows from (72), the assumption $\int_{[1,\infty)} r \log r \nu(\mathrm{d}r) < \infty$, the fact that

$$\lim_{i\uparrow\infty} \frac{\tau_i}{i} = \frac{1}{\int_{[1,\infty)} r\nu(\mathrm{d}r)}, \tag{74}$$

and the strong law of large numbers for linear Brownian motion. In conclusion, (71) and (73) show that (70) is finite and hence $W_\infty(\lambda)$ is an $L^1(\mathbb{P})$-limit.

Finally we prove that $\{W_\infty(\lambda) = 0\}$ agrees with the event that X becomes extinguished. To this end let $q = \mathbb{P}_{\delta_x}(W_\infty(\lambda) = 0) < 1$. Note that q does not depend on x as $W_\infty(\lambda)$ under $\mathbb{P}_{\delta_x}$ has the same law as $\mathrm{e}^{-\lambda x} W_\infty(\lambda)$ under $\mathbb{P}_{\delta_0}$ thanks to the definition of $W(\lambda)$ and the fact that the branching mechanism is not spatially dependent. Taking conditional expectations and using the Markov branching property we have for all $t \geq 0$,

$$\mathbb{E}(\mathbf{1}_{\{W_\infty(\lambda)=0\}} | \mathcal{F}_t) = q^{\|X_t\|}.$$

Therefore, on the one hand, taking expectations across this last equality and then limits as $t \uparrow \infty$, we easily deduce with the help of the Dominated Convergence Theorem that $\mathbb{P}(W_\infty(\lambda) = 0) = \mathrm{e}^{-\lambda^*}$. On the other hand, noting that $\mathcal{E}^c \in \sigma(\bigcup_{t>0} \mathcal{F}_t)$, we also have that

$$\mathbb{P}\big(\mathcal{E}^c \cap \big\{W_\infty(\lambda) = 0\big\}\big) = \mathbb{E}\Big[\mathbf{1}_{\mathcal{E}^c} \lim_{t\uparrow\infty} \mathbb{E}(\mathbf{1}_{\{W_\infty(\lambda)=0\}} | \mathcal{F}_t)\Big] = \mathbb{E}\Big[\mathbf{1}_{\mathcal{E}^c} \lim_{t\uparrow\infty} q^{\|X_t\|}\Big] = 0.$$

Hence it follows that $\{W_\infty(\lambda) = 0\} = \mathcal{E}$, $\mathbb{P}$-almost surely.

Next we deal with the cases that $\lambda \geq \underline{\lambda}$ or $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) = \infty$. Recall that, given ξ, the Poisson point process $\mathbf{m}$ initiates a superprocess at time t, $X^{\mathbf{m},t}$, with initial mass r at rate $\mathrm{d}t \times r\nu(\mathrm{d}r) \times \mathrm{d}\mathbb{P}_{r\delta_{\xi_t}}$. For each τ_i (defined in the previous part of the proof) we have

$$W^{\Lambda}_{\tau_i}(\lambda) \geq m_{\tau_i} \mathrm{e}^{-\lambda(\xi_{\tau_i} + c_\lambda \tau_i)}, \quad \widetilde{\mathbb{P}}^{-\lambda}\text{-a.s.} \tag{75}$$

Under $\widetilde{\mathbb{P}}$, ξ is a Brownian motion with drift $-\lambda$ and is independent of $\mathbf{m}$. Note also that $c_\lambda \leq \lambda$ when $\lambda \geq \underline{\lambda}$. Hence, $\xi_t + c_\lambda t$ is a Brownian motion with non-positive drift. Then, from (74) and (75) we conclude that when $\lambda \geq \underline{\lambda}$,

$$\limsup_{t\to\infty} W^{\Lambda}_t(\lambda) \geq \limsup_{i\to\infty} \mathrm{e}^{-\lambda(\xi_{\tau_i} + c_\lambda \tau_i)} = \infty, \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

It follows from (65) that $W_\infty(\lambda) = 0$, $\mathbb{P}$-almost surely.

Now suppose that $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) = \infty$. Recall that under $\widetilde{\mathbb{P}}^{-\lambda}$, ξ is a Brownian motion with drift $-\lambda$. Then, the strong law of large numbers gives $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely, $\lim_{t\to\infty} t^{-1}(\xi_t + c_\lambda t) = c_\lambda - \lambda$. We thus have from (72), (74) and (75) that $\widetilde{\mathbb{P}}^{-\lambda}$-a.s.,

$$\limsup_{i\to\infty} i^{-1} \log W^{\Lambda}_{\tau_i}(\lambda) \geq \limsup_{i\to\infty} i^{-1} \log m_{\tau_i} - \lambda \limsup_{i\to\infty} \frac{(\xi_{\tau_i} + c_\lambda \tau_i)}{\tau_i} \frac{\tau_i}{i} = \infty. \tag{76}$$

Therefore, $\overline{W}^{\Lambda}_{\infty}(\lambda) = \infty$, $\widetilde{\mathbb{P}}^{-\lambda}$-almost surely, and (65) implies that $W_{\infty}(\lambda) = \overline{W}_{\infty}(\lambda) = 0$, $\mathbb{P}$-almost surely.

Remark 6.1. *One can easily go further with the analysis of $W(\lambda)$ through the pathwise spine decomposition. Indeed following arguments in Hardy and Harris* [20], *sufficient conditions for $L^p(\mathbb{P})$ convergence for $p \in (1, 2]$ are given in* [28].

7. Proof of Theorem 2.4(ii)

As with the case of $W(\lambda)$ we only give the proof for the case that $\lambda \geq 0$ noting that reasoning involving symmetry covers the case that $\lambda \leq 0$. Recall that $\partial W(\lambda)$ is a signed martingale and therefore does not necessarily converge almost surely. A technique used by Kyprianou [25] to get round this problem in the case of a branching Brownian motion is to consider a truncated form of the derivative martingale which is a positive martingale. In order to describe the aforementioned martingale in the current context we need more notation.

Recall that the superprocess X^c was defined as the superprocess whose movement component is that of a Brownian motion with drift c but whose branching mechanism is still ψ. Consider the domain $D^t_{-y} = \{(z, s)\colon z > -y, s \in (0, t)\}$. Dynkin's theory of exit measures may still be applied in this context and we denote the exit measure associated with the domain D^t_{-y} for the process X^{λ} by $X^{\lambda}_{D^t_{-y}}$ for $\lambda \geq 0$. Next define for all $\lambda \geq 0$,

$$b_{\lambda} = c_{\lambda} - \lambda = -\psi'\big(0^+\big)/\lambda - \lambda/2,$$

and note that $b_{\lambda} > 0$ for $\lambda \in (0, \underline{\lambda})$ and $b_{\lambda} \leq 0$ for $\lambda \geq \underline{\lambda}$.

In the spirit of [25] we introduce a new martingale, for each $y > 0$,

$$V^{-y}_t(\lambda) = \mathrm{e}^{-\lambda b_{\lambda} t}\frac{1}{y}\big\langle (y + \cdot)\mathrm{e}^{-\lambda \cdot}, X^{\lambda}_{D^t_{-y}}\big\rangle, \quad t \geq 0. \tag{77}$$

To show that $V^{-y}(\lambda) := \{V^{-y}_t(\lambda)\colon t \geq 0\}$ is a $\mathbb{P}$-unit mean martingale, let $\mathcal{H}^{\lambda}_t = \sigma(X^{\lambda}_{D^s_{-x}}\colon x \leq y, s \leq t) \subseteq \mathcal{K}_t = \sigma(X^{\lambda}_{D^s_{-x}}\colon x < \infty, s \leq t)$ and note that

$$\mathbb{E}\big(yW_t(\lambda) + \partial W_t(\lambda)|\mathcal{H}^{\lambda}_t\big) = yV^{-y}_t(\lambda),$$

and that $\mathcal{H}^{\lambda}_{t+s} \cap \mathcal{K}_t = \mathcal{H}^{\lambda}_t$. Hence

$$\begin{aligned}
\mathbb{E}\big(yV^{-y}_{t+s}(\lambda)|\mathcal{K}_t\big) &= \mathbb{E}\big(yW_{t+s}(\lambda) + \partial W_{t+s}(\lambda)|\mathcal{H}^{\lambda}_{t+s}|\mathcal{K}_t\big)\\
&= \mathbb{E}\big(yW_{t+s}(\lambda) + \partial W_{t+s}(\lambda)|\mathcal{H}^{\lambda}_t\big)\\
&= \mathbb{E}\big(yW_{t+s}(\lambda) + \partial W_{t+s}(\lambda)|\mathcal{K}_t|\mathcal{H}^{\lambda}_t\big)\\
&= \mathbb{E}\big(yW_t(\lambda) + \partial W_t(\lambda)|\mathcal{H}^{\lambda}_t\big)\\
&= yV^{-y}_t(\lambda).
\end{aligned}$$

It is clear that $V^{-y}(\lambda)$ is positive and hence there always exists an almost sure limit which we denote by $V^{-y}_{\infty}(\lambda)$. From Corollary 3.2 we know that, when $\lambda \geq \underline{\lambda}$, $\mathbb{P}(-\inf\mathcal{R}^{\lambda} < \infty) = 1$. It follows that on the event $\{\inf\mathcal{R}^{\lambda} \geq -y\}$, for this regime of λ,

$$\mathbb{E}\big(yW_t(\lambda) + \partial W_t(\lambda)|\mathcal{H}^{\lambda}_t\big) = yW_t(\lambda) + \partial W_t(\lambda),$$

and hence, letting $t \uparrow \infty$, $yV^{-y}_{\infty}(\lambda) = yW_{\infty}(\lambda) + \partial W_{\infty}(\lambda)$, where implicitly we understand the limit of $\partial W(\lambda)$ to exist in the last equality because the limit $V^{-y}_{\infty}(\lambda)$ exists. Note however from Theorem 2.4(i) that $W_{\infty}(\lambda) = 0$ when $\lambda \geq \underline{\lambda}$ so that in fact

$$yV^{-y}_{\infty}(\lambda) = \partial W_{\infty}(\lambda) \quad \text{on } \big\{\inf\mathcal{R}^{\lambda} \geq -y\big\}. \tag{78}$$

As y may be taken arbitrarily large, it follows that $\partial W_\infty(\lambda) \geq 0$.

Remaining in the regime $\lambda \geq \underline{\lambda}$, the proof that $\{\partial W_\infty(\lambda) = 0\} = \mathcal{E}$, $\mathbb{P}$-almost surely goes along almost the same lines as the earlier proof that $\{W_\infty(\lambda) = 0\} = \mathcal{E}$, $\mathbb{P}$-almost surely.

Taking account of the relationship between $\partial W(\lambda)$ and $V^{-y}(\lambda)$ for $\lambda \geq \underline{\lambda}$, the proof of Theorem 2.4(ii) would now follow directly from parts (ii) and (iii) of the following theorem; which itself plays the analogous role of Theorem 13 in Kyprianou [26].

Theorem 7.1. *Fix $y > 0$.*

(i) *If $\lambda > \underline{\lambda}$ then $V_\infty^{-y}(\lambda) = 0$ $\mathbb{P}$-almost surely.*
(ii) *If $\lambda = \underline{\lambda}$ then $V_\infty^{-y}(\lambda)$ is an $L^1(\mathbb{P})$ limit if and only if $\int_{[1,\infty)} r(\log r)^2 \nu(\mathrm{d}r) < \infty$ otherwise $V_\infty^{-y}(\lambda) = 0$ $\mathbb{P}$-almost surely.*
(iii) *If $\lambda \in (0, \underline{\lambda})$ then $V_\infty^{-y}(\lambda)$ is an $L^1(\mathbb{P})$ limit if and only if $\int_{[1,\infty)} r(\log r)\nu(\mathrm{d}r) < \infty$ otherwise $V_\infty^{-y}(\lambda) = 0$ $\mathbb{P}$-almost surely.*

Below we only give the proof of part (ii) of Therorem 7.1. Once this has been done, the proof of parts (i) and (iii) should be apparent given the proof of Theorem 13 in Kyprianou [26] and we leave the details to the reader.

Proof of Theorem 7.1(ii). We shall again appeal to classical techniques based around using a martingale change of measure. Specifically we are interested in understanding the change of measure

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}^{-y}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := V_t^{-y}(\underline{\lambda}), \quad t \geq 0, \tag{79}$$

where $y > 0$. Similarly to Theorem 5.2 the change of measure induces a spine decomposition. In order to describe it, recall that under Π_x the process $\xi := \{\xi_t\colon\ t \geq 0\}$ is a Brownian motion issued from $x \in \mathbb{R}$. If we let $\tau_{-y} = \inf\{t \geq 0\colon\ y + \xi_t + \underline{\lambda} t \leq 0\}$ then another well known change of measure for Brownian motion is the following. For $y \geq 0$,

$$\left.\frac{\mathrm{d}\widehat{\Pi}^{-y}}{\mathrm{d}\Pi}\right|_{\mathcal{G}_t} := \frac{y + \xi_t + \underline{\lambda} t}{y} \mathrm{e}^{-\underline{\lambda}\xi_t - \underline{\lambda}^2 t/2} \mathbf{1}_{\{t < \tau_{-y}\}}, \quad t \geq 0,$$

where $\mathcal{G}_t = \sigma(\xi_s\colon\ s \leq t)$ and $\Pi = \Pi_0$. Under $\widehat{\Pi}^{-y}$ the process $\{y + \xi_t + \underline{\lambda} t\colon\ t \geq 0\}$ has the law of a standard Brownian motion issued from y and conditioned never to enter the half line $(-\infty, 0)$. Otherwise said, the process $y + \xi_t + \underline{\lambda} t$ is a Bessel-3 process issued from y. Bearing this last change of measure in mind, we have the following result which describes the effect of the change of measure (79).

Theorem 7.2. *Fix $y \geq 0$. Consider the process Λ as defined in (55) with the exception that the spine ξ is assigned the measure $\widehat{\Pi}^{-y}$ such that $\{y + \xi_t + \underline{\lambda} t\colon\ t \geq 0\}$ is a Bessel-3 process issued from y and x is chosen specifically equal to 0. Denote its law by $\widetilde{\mathbb{P}}^{-y}$. Then $(X, \widehat{\mathbb{P}}^{-y}) = (\Lambda, \widetilde{\mathbb{P}}^{-y})$.*

For the sake of brevity we omit the proof mentioning instead that it requires very similar computations, with obvious differences, to those of Theorems 5.1 and 5.2 combined.

Theorem 7.2 allows us to conclude that the process $V^{-y}(\lambda)$ under $\widehat{\mathbb{P}}^{-y}$ is equal in law to

$$\begin{aligned} V_t^{\Lambda,-y}(\underline{\lambda}) := V_t'^{-y}(\underline{\lambda}) &+ \sum_{s \leq t:\mathbf{n}} \frac{(y + \xi_s + \underline{\lambda} s)}{y} \mathrm{e}^{-\underline{\lambda}(\xi_s + \underline{\lambda} s)} V_{t-s}^{\mathbf{n},s,-(y+\xi_s+\underline{\lambda} s)}(\underline{\lambda}) \\ &+ \sum_{s \leq t:\mathbf{m}} \frac{(y + \xi_s + \underline{\lambda} s)}{y} \mathrm{e}^{-\underline{\lambda}(\xi_s + \underline{\lambda} s)} V_{t-s}^{\mathbf{m},s,-(y+\xi_s+\underline{\lambda} s)}(\underline{\lambda}), \quad t \geq 0, \end{aligned} \tag{80}$$

under $\widetilde{\mathbb{P}}^{-y}$, where $V'^{,-y}(\underline{\lambda})$ plays the role of $V^{-y}(\underline{\lambda})$ for the process X' and, given $(\xi, \mathbf{m})$, $V^{\mathbf{m},s,-(y+\xi_s+\underline{\lambda} s)}(\underline{\lambda})$ and $V^{\mathbf{n},s,-(y+\xi_s+\underline{\lambda} s)}(\underline{\lambda})$ play the role of $V^{-(y+\xi_s+\underline{\lambda} s)}(\underline{\lambda})$ for the processes $X^{\mathbf{m},s}$ and $X^{\mathbf{n},s}$, respectively under the laws $\mathbb{P}_{m_s \delta_0}$ and $\mathbb{N}_0$.

A similar statement to (66) tells us that if we can show $\widetilde{\mathbb{P}}^{-y}(\limsup_{t\uparrow\infty} V_t^{\Lambda,-y}(\underline{\lambda}) < \infty) = 1$ then $V_\infty^{-y}(\underline{\lambda})$ is an $L^1(\mathbb{P})$ limit. Similar reasoning to the proof of Theorem 2.4(i) tells us that it now suffices to prove that

$$\limsup_{t\uparrow\infty} \widetilde{\mathbb{E}}^{-y}\big(V_t^{\Lambda,-y}(\underline{\lambda})|\xi, \mathbf{m}\big) < \infty, \tag{81}$$

almost surely. To this end, first recall that, given ξ, the Poisson point process $\mathbf{n}$ of immigration has intensity $\mathrm{d}s \times 2\beta\,\mathrm{d}\mathbb{N}_{\xi_s}$. It follows that

$$\begin{aligned}
&\limsup_{t\uparrow\infty} \widetilde{\mathbb{E}}^{-y}\bigg[\sum_{s\le t:\mathbf{n}} (y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)} V_{t-s}^{\mathbf{n},s,-(y+\xi_s+\underline{\lambda}s)}(\underline{\lambda})\bigg|\xi\bigg] \\
&\quad= \limsup_{t\uparrow\infty} 2\beta\int_0^t (y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)}\mathbb{N}_0\big(V_{t-s}^{\mathbf{n},s,-(y+\xi_s+\underline{\lambda}s)}(\underline{\lambda})|\xi\big)\,\mathrm{d}s \\
&\quad= \limsup_{t\uparrow\infty} 2\beta\int_0^t (y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)}\mathbb{E}\big(V_{t-s}^{-(y+\xi_s+\underline{\lambda}s)}(\underline{\lambda})|\xi\big)\,\mathrm{d}s \\
&\quad= \int_0^\infty (y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)}\,\mathrm{d}s \\
&\quad< \infty,
\end{aligned}$$

$\widetilde{\mathbb{P}}^{-y}$-almost surely, where the final equality follows by virtue of the fact that $\{y+\xi_t+\underline{\lambda}t\colon t\ge 0\}$ is a Bessel-3 process issued from y and hence eventually grows no slower than $t^{1/2-\epsilon}$ for any $1/2 > \epsilon > 0$.

Next we note that

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{-y}\bigg[\sum_{s\le t:\mathbf{m}} (y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)} V_{t-s}^{\mathbf{m},s,-(y+\xi_s+\underline{\lambda}s)}(\underline{\lambda})\bigg|\xi,\mathbf{m}\bigg] \qquad (82)\\
&\quad= \sum_{s\le t:\mathbf{m}} (y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)}\mathbb{E}_{m_s\delta_0}\big(V_{t-s}^{\mathbf{m},s,-(y+\xi_s+\underline{\lambda}s)}(\underline{\lambda})|\xi,\mathbf{m}\big) \\
&\quad= \sum_{s\le t:\mathbf{m}} m_s(y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)} \\
&\quad= \sum_{\{m_s<\mathrm{e}^{\varepsilon(\xi_s+\underline{\lambda}s)}:\mathbf{m}\}} m_s(y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)} \\
&\qquad+ \sum_{\{m_s\ge\mathrm{e}^{\varepsilon(\xi_s+\underline{\lambda}s)}:\mathbf{m}\}} m_s(y+\xi_s+\underline{\lambda}s)\mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)} \\
&\quad=: I + II. \qquad (83)
\end{aligned}$$

We want to show that I and II are both $\widetilde{\mathbb{P}}^{-y}$-almost surely finite. For I, choose $0 < \varepsilon < \underline{\lambda}$, we have that

$$\begin{aligned}
\widetilde{\mathbb{E}}^{-y}(I|\xi) &\le \int_{(0,1)} r^2\nu(\mathrm{d}r)\int_0^\infty (y+\xi_t+\underline{\lambda}t)\mathrm{e}^{-\underline{\lambda}(\xi_t+\underline{\lambda}t)}\,\mathrm{d}t \\
&\quad+ \int_{[1,\infty)} r\nu(\mathrm{d}r)\int_0^\infty (y+\xi_t+\underline{\lambda}t)\mathrm{e}^{-(\underline{\lambda}-\varepsilon)(\xi_t+\underline{\lambda}t)}\,\mathrm{d}t \\
&< \infty.
\end{aligned}$$

Note that we have again used the fact that the assumption $\psi'(0^+) \in (-\infty, 0)$ implies that $\int_{[1,\infty)} r\nu(\mathrm{d}r) < \infty$.

To show that II is $\widetilde{\mathbb{P}}^{-y}$-almost surely finite, it suffices to note that

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{-y}\Bigg[\sum_{\{m_s\geq \mathrm{e}^{\varepsilon(\xi_s+\underline{\lambda}s)}:\mathbf{m}\}} 1\Bigg]\\
&= \widehat{\Pi}_0^{-y}\int_0^\infty \mathrm{d}t\int_{[1,\infty)} r\nu(\mathrm{d}r)\mathbf{1}_{\{r>\mathrm{e}^{\varepsilon(\xi_t+\underline{\lambda}t)}\}}\\
&= \widehat{\Pi}_0^{-y}\int_{[1,\infty)} r\nu(\mathrm{d}r)\int_0^\infty \mathrm{d}t\mathbf{1}_{\{x+\xi_t+\underline{\lambda}t\leq x+\varepsilon^{-1}\log r\}}\\
&= \int_{[1,\infty)} r\nu(\mathrm{d}r)\int_{\{|\mathbf{y}|\leq x+\varepsilon^{-1}\log r\}}\mathrm{d}\mathbf{y}\int_0^\infty p(t,\hat{\mathbf{x}},\mathbf{y})\,\mathrm{d}t\\
&= C\int_{[1,\infty)} r\nu(\mathrm{d}r)\int_{\mathbb{R}^3}\frac{\mathbf{1}_{\{|\mathbf{y}|\leq x+\varepsilon^{-1}\log r\}}}{|\mathbf{y}-\hat{\mathbf{x}}|}\,\mathrm{d}\mathbf{y}\\
&\leq C\int_{[1,\infty)} r\nu(\mathrm{d}r)\int_0^{2x+\varepsilon^{-1}\log r} u\,\mathrm{d}u\\
&= \frac{C}{2}\int_{[1,\infty)} r\big(2x+\varepsilon^{-1}\log r\big)^2\nu(\mathrm{d}r)\\
&< \infty,
\end{aligned}$$

where $\hat{\mathbf{x}}=(x,0,0)$, $p(t,\hat{\mathbf{x}},\mathbf{y})$ is the probability density function of a three dimensional Brownian motion starting from $\hat{\mathbf{x}}$, and C is a positive constant. This tells us that, $\widetilde{\mathbb{P}}^{-y}$-almost surely, II is a summation over a finite set, and therefore II is $\widetilde{\mathbb{P}}^{-y}$-almost surely finite.

A similar statement to (65) tells us that if we can show $\widetilde{\mathbb{P}}^{-y}(\limsup_{t\uparrow\infty} V_t^{\Lambda,-y}(\underline{\lambda})=\infty)=1$ then $V_\infty^{-y}(\underline{\lambda})=0$ $\mathbb{P}$-almost surely. Since each term in the identity (80) is non-negative, we just consider the $\mathbf{m}$-immigration. If we prove the supremum limit of the third term in (80) is infinity, then we are done. Let N be any positive number and define the stochastic time sequence

$$\tau_1=\inf\big\{t\geq 0\colon m_t>1\vee \mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\big\},\qquad \tau_{i+1}=\inf\big\{t>\tau_i\colon m_t>1\vee \mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\big\},\quad i=1,\ldots.$$

Under $\widetilde{\mathbb{P}}^{-y}(\cdot|\xi)$, $\mathbf{m}$ is a Poisson point process. Thus the process $\sum_{s\leq t}\mathbf{1}_{\{m_s\geq 1\vee \mathrm{e}^{N(\xi_s+\underline{\lambda}s)}\}}$ is a Poisson process with instant intensity $\int_1^\infty r\nu(\mathrm{d}r)\mathbf{1}_{\{r>\mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\}}\,\mathrm{d}t$ under $\widetilde{\mathbb{P}}^{-y}(\cdot|\xi)$ and its domain is $\{\tau_i\colon i=1,2,\ldots\}$. Therefore,

$$\begin{aligned}
&\sum_{\tau_i<\infty}\mathbf{1}_{\{m_{\tau_i}\geq \mathrm{e}^{N(\xi_{\tau_i}+\underline{\lambda}\tau_i)}\}}<\infty,\quad \widetilde{\mathbb{P}}^{-y}(\cdot|\xi)\text{-a.s.}\\
&\iff\quad \int_0^\infty \mathrm{d}t\int_1^\infty r\nu(\mathrm{d}r)\mathbf{1}_{\{r>\mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\}}<\infty,\quad \widetilde{\mathbb{P}}^{-y}(\cdot|\xi)\text{-a.s.}
\end{aligned}$$

Let a be some constant and define the set $C=\{\int_0^\infty \mathrm{d}t\int_1^\infty r\nu(\mathrm{d}r)\mathbf{1}_{\{r>\mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\}}<a\}$. Recall that $y+\xi_t+\underline{\lambda}t$ is a $\mathrm{BES}^3(y)$ process under the probability $\widetilde{\mathbb{P}}^{-y}$. It is well known that $\mathrm{BES}^3(y)$ is identically distributed to the modulus process of $B_t+\hat{y}$, where $(B_t,\mathbb{Q})$ is a three dimensional Brownian motion starting at 0 and $\hat{y}$ is a point in $\mathbb{R}^3$ with norm y. Denote the modulus process by $|B_t+\hat{y}|$. We still use C to denote the same set corresponding to $(B_t,\mathbb{Q})$.

$$\begin{aligned}
&\widetilde{\mathbb{E}}^{-y}\Bigg[\mathbf{1}_C\int_0^\infty \mathrm{d}t\int_1^\infty r\nu(\mathrm{d}r)\mathbf{1}_{\{r>\mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\}}\Bigg]\\
&= \int_0^\infty \mathrm{d}t\int_1^\infty r\nu(\mathrm{d}r)\widetilde{\mathbb{E}}^{-y}[\mathbf{1}_C\mathbf{1}_{\{r>\mathrm{e}^{N(\xi_t+\underline{\lambda}t)}\}}]\\
&= \int_1^\infty r\nu(\mathrm{d}r)\int_0^\infty \widetilde{\mathbb{E}}^{-y}[\mathbf{1}_C\mathbf{1}_{\{\xi_t+\underline{\lambda}t\leq N^{-1}\log r\}}]\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty r\nu(\mathrm{d}r) \int_0^\infty \mathbb{Q}[\mathbf{1}_C \mathbf{1}_{\{|B_t+\hat{y}|\le y+N^{-1}\log r\}}]\,\mathrm{d}t \\
&\ge \mathbb{Q}\bigg[\mathbf{1}_C \int_1^\infty r\nu(\mathrm{d}r) \int_0^\infty \mathbf{1}_{\{|B_t|\le N^{-1}\log r\}}\,\mathrm{d}t\bigg].
\end{aligned}
$$

Then under $\mathbb{Q}$, $|B_t|$ is a BES$^3(0)$ process. Let l_∞^a be the local time of $|B_t|$. Exercise (2.5) in [39] tells us l_∞^a is a BESQ$^2(0)$. Then $l_\infty^a \overset{d}{=} a l_\infty^1$ and $\mathbb{Q}(l_\infty^1 = 0) = 0$. For the given set C,

$$
\begin{aligned}
&\mathbb{Q}\bigg[\mathbf{1}_C \int_1^\infty r\nu(\mathrm{d}r) \int_0^\infty \mathbf{1}_{\{|B_t|\le N^{-1}\log r\}}\,\mathrm{d}t\bigg] \\
&= \mathbb{Q}\bigg[\mathbf{1}_C \int_1^\infty r\nu(\mathrm{d}r) \int_0^{N^{-1}\log r} l_\infty^a\,\mathrm{d}a\bigg] \\
&= \mathbb{Q}\bigg[\mathbf{1}_C \int_0^\infty l_\infty^a\,\mathrm{d}a \int_{\mathrm{e}^{Na}}^\infty r\nu(\mathrm{d}r)\bigg] \\
&= \mathbb{Q}\bigg[\mathbf{1}_C \int_0^\infty a\,\mathrm{d}a \int_0^{a^{-1}l_\infty^a} \mathrm{d}u \int_{\mathrm{e}^{Na}}^\infty r\nu(\mathrm{d}r)\bigg] \\
&= \int_0^\infty a\,\mathrm{d}a \int_{\mathrm{e}^{Na}}^\infty r\nu(\mathrm{d}r) \int_0^\infty \mathbb{Q}[\mathbf{1}_C \mathbf{1}_{\{l_\infty^a > au\}}]\,\mathrm{d}u \\
&\ge \int_0^\infty a\,\mathrm{d}a \int_{\mathrm{e}^{Na}}^\infty r\nu(\mathrm{d}r) \int_0^\infty \big[\mathbb{Q}(C) - \mathbb{Q}\big(a^{-1}l_\infty^a > u\big)\big]^+\,\mathrm{d}u \\
&= \int_0^\infty a\,\mathrm{d}a \int_{\mathrm{e}^{Na}}^\infty r\nu(\mathrm{d}r) \int_0^\infty \big[\mathbb{Q}(C) - \mathbb{Q}\big(l_\infty^1 < u\big)\big]^+\,\mathrm{d}u.
\end{aligned}
$$

Note that

$$
\int_0^\infty a\,\mathrm{d}a \int_{\mathrm{e}^{Nt}}^\infty r\nu(\mathrm{d}r) = \int_1^\infty r\nu(\mathrm{d}r) \int_0^{N^{-1}\log r} a\,\mathrm{d}a = \frac{1}{2N^2}\int_1^\infty r(\log r)^2\nu(\mathrm{d}r).
$$

Therefore, if $\widetilde{\mathbb{P}}^{-y}(C) > 0$, then $\int_1^\infty r(\log r)^2\nu(\mathrm{d}r) < \infty$, which means $\int_1^\infty r(\log r)^2\nu(\mathrm{d}r) = \infty$ implies $m_{\tau_i} > \mathrm{e}^{N(\xi_{\tau_i}+\underline{\lambda}\tau_i)}$ infinitely times $\widetilde{\mathbb{P}}^{-y}$-a.s. The process BES$^3(y)$ is transient, so $\lim_{t\to\infty} \xi_t + \underline{\lambda}t = \infty$ $\widetilde{\mathbb{P}}^{-y}$-a.s. We reach the conclusion that for any $N > 0$, there exist an increasing sequence of stochastic time $\{\tau_i\colon i = 1, 2, \ldots\}$ such that

$$
\limsup_{i\to\infty} m_{\tau_i}(\xi_{\tau_i} + \underline{\lambda}\tau_i)\mathrm{e}^{-N(\xi_{\tau_i}+\underline{\lambda}\tau_i)} = \infty. \tag{84}
$$

Consider the process $V_t^{\Lambda,-y}(\underline{\lambda})$ in (80). We deduce from (84) that

$$
\begin{aligned}
\limsup_{t\to\infty} V_t^{\Lambda,-y}(\underline{\lambda}) &\ge \limsup_{i\to\infty} V_{\tau_i}^{\Lambda,-y}(\underline{\lambda}) \\
&\ge \limsup_{i\to\infty} \sum_{s\le\tau_i:\mathbf{m}} \frac{(y+\xi_s+\underline{\lambda}s)}{y} \mathrm{e}^{-\underline{\lambda}(\xi_s+\underline{\lambda}s)} V_{\tau_i-s}^{\mathbf{m},s,-(y+\xi_s+\underline{\lambda}s)}(\underline{\lambda}) \\
&\ge \limsup_{i\to\infty} y^{-1} m_{\tau_i}(y+\xi_{\tau_i}+\underline{\lambda}\tau_i)\mathrm{e}^{-\underline{\lambda}(\xi_{\tau_i}+\underline{\lambda}\tau_i)} \\
&= \infty.
\end{aligned}
$$

□

8. Proof of Theorem 2.6

(i) Under the given conditions, we know that there exists an $L^1(\mathbb{P})$-limit $W_\infty(\lambda)$ for the martingale $W(\lambda)$. In light of Corollary 3.2 we have through, a now familiar projection, that

$$\mathbb{E}\big(W_\infty(\lambda)|\mathcal{F}_{D_x}^{c_\lambda}\big) = \lim_{t\uparrow\infty} \mathbb{E}\big(W_\infty(\lambda)|\mathcal{F}_{D_x^t}^{c_\lambda}\big) = \mathrm{e}^{-\lambda x} Z_x^{c_\lambda}.$$

This has the implication that the normalizing sequence discussed in the proof of Theorem 2.1(ii) must satisfy $L_\lambda(\mathrm{e}^{-\lambda x}) \sim 1$ as $x \uparrow \infty$ and, up to a non-negative constant, $\Delta(\lambda) = W_\infty(\lambda)$.

(ii) In a similar fashion, we note that under the given conditions of the theorem, for fixed $y > 0$, V_∞^{-y} is an $L^1(\mathbb{P})$-limit and hence

$$\mathbb{E}\big(y V_\infty^{-y}|\mathcal{F}_{D_x}^{c_{\underline{\lambda}}}\big) = (y+x)\mathrm{e}^{-\underline{\lambda} x} Z_x^{c_{\underline{\lambda}}}.$$

It follows that the normalizing sequence discussed in the proof of Theorem 2.1(ii) must instead satisfy $L_\lambda(\mathrm{e}^{-\underline{\lambda} x}) \sim x$ as $x \uparrow \infty$ and, taking account of (78), we have, up to a non-negative constant, $\Delta(\underline{\lambda}) = \partial W_\infty(\underline{\lambda})$.

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