Conservation property of symmetric jump processes

Jun Masamune[a] and Toshihiro Uemura[b]

[a]Department of Mathematics and Statistics, Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA. E-mail: jum35@psu.edu
[b]Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita, Osaka 564-8680, Japan.
E-mail: t-uemura@kansai-u.ac.jp

Abstract. Motivated by the recent development in the theory of jump processes, we investigate its conservation property. We will show that a jump process is conservative under certain conditions for the volume-growth of the underlying space and the jump rate of the process. We will also present examples of jump processes which satisfy these conditions.

Résumé. Motivés par les récents développements dans la théorie des processus de sauts, nous étudions leur propriété de conservation. Nous montrons qu'un processus de saut est conservatif sous certaines conditions sur la croissance du volume de l'espace sous-tendant et sur le taux de saut du processus. Nous donnons des examples de processus satisfaisant ces conditions.

1. Introduction

One of the most fundamental properties for a stochastic process is its *conservativeness*; that is, the process stays almost surely for all time $t > 0$ in the state space. Motivated by the recent development in the theory of *jump processes*, there have been some results for the conservation property of a jump process or of the associated *non-local operator*; Schilling [19] obtained a criteria for the conservation property via the symbol of the generator; the second named author established another criteria [12], which generalize the Oshima's conservativeness test [18] (see also Theorem 1.5.5 in [8]), and applied it to obtain the conservation property of a jump process. These two results are sharp, and therefore, in order to apply them, one needs certain information of the generator which is hard to determine for a given Dirichlet form. Other works are, for examples, [2,3,5], where they showed the conservation properties for some jump processes for their own purposes (see Section 4).

On the other hand, due to the corresponding researches for a *diffusion process* [6,9–11,14,21,22], we know that the main factor for the conservation property (for a diffusion process) is the volume growth of the underlying space. As far as the authors are concerned, there are no results which had addressed the relationships between the conservation property of a jump process and the volume growth of the underlying space. In this article we wish to understand what are the keys for a jump process to be conservative, in particular, in the connections with the volume growth of the space and the jump rate of the process without any knowledge of the generator. The suitable setting for our purpose is the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Let us now specify our space X and the Dirichlet form. Let (X, d) be a locally compact metric space with a positive Radon measure m with full support. We assume that any ball with respect to d

is relatively compact (however, we will relax this assumption later). Let $\mu(x, dy)$ be a kernel defined on the product space $X \times \mathcal{B}(X)$. Define the quadratic form $\mathcal{E}$ as

$$\mathcal{E}(u, v) := \iint_{x \neq y} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\mu(x, dy)m(dx)$$

for $u, v \in L^2 = L^2(X; m)$, whenever the right-hand side makes sense. We assume

Condition (C).

(i) $\mu(x, dy)m(dx)$ *is a symmetric measure on* $X \times X \backslash D$, *where* $D = \{(x, x) : x \in X\}$ *is the diagonal set*; *namely*,

$$\mu(x, dy)m(dx) = \mu(y, dx)m(dy).$$

(ii) $\quad M = \sup_{x \in X} \int_{y \neq x} \big(1 \wedge d^2(x, y)\big)\mu(x, dy) < \infty.$

Let $C_0^{\text{lip}} = C_0^{\text{lip}}(X)$ be the space of all Lipschitz continuous functions defined on X with compact support. Recall that Condition (C) implies that $(\mathcal{E}, C_0^{\text{lip}})$ is a closable Markovian symmetric form on L^2 (see [8,24]), and thus, there exists a symmetric Hunt process associated with its closure $(\mathcal{E}, \mathcal{F})$. Note that the corresponding process is of pure jump. Set

$$\mathcal{E}_1[u] = \mathcal{E}(u, u) + \|u\|^2 \quad \text{for } u \in \mathcal{F}.$$

Here $\|u\| = \sqrt{(u, u)}$ stands for the L^2-norm of u. Fix an arbitrary point $x_0 \in X$ and set $r(x) = d(x_0, x)$.

The principal purpose of this article is to show:

Main result. *Assume* (C). *If*

$$\mathrm{e}^{-ar(x)} \in L^1 \quad \text{for every } a > 0, \tag{1.1}$$

then the associated symmetric jump process is conservative.

This Main result shows that the conservativeness is controlled by the volume growth of the underlying space together with the jump rate; namely, even if the volume grows rapidly at least satisfying (1.1), the process should stay in the space, provided it jumps sufficiently rarely (at infinity of X) satisfying (C)-(ii). Since (1.1) is satisfied for any Euclidean space, this result implies the conservation properties in [2,5] (see Section 4). It should be noted that for the L^p-Liouville property of a jump process, we need a condition on the jump rate but not on the volume growth [17].

Condition (C) is so general that most of the typical examples of $\mu(x, dy)$ satisfy; the kernels $\mu(x, dy) = k(x, y)\,dy$ of symmetric α-stable processes; symmetric stable-like processes; and symmetric Lévy processes (see Section 4) on a Riemannian manifold with non-negative Ricci curvature outside a compact set. In general, (1.1) and (C)-(ii) do not imply each other (see Section 4). Comparing to the diffusion case, we need the additional condition (C)-(ii) and the stronger volume growth condition (1.1).

Our technique to prove the Main result is a perturbation method (see Section 4 in [20]); we will first establish the conservation property for the truncated process; that is, the process which jumps no more than distance $R > 0$ with some (uniform) $R > 0$, and secondly, show that there will be no explosion at infinity of X. More precisely, in order to prove the conservation property of the truncated process we apply both (1.1) and (C)-(ii). We will follow the standard steps (due to Gaffney [9] and Garding) but we need to develop and apply an integral derivation property (Proposition 2.2), which is interesting by its own (see, e.g., [17]). To show the conservation property of the original process, we will prove that under (C)-(ii), the generator B of the form $\mathcal{E}^2 = \mathcal{E} - \mathcal{E}^1$, where $\mathcal{E}^1$ is the form associated to the truncated process, is extended to a bounded operator on L^∞, and in particular, it becomes a (one order) integro-differential operator. This implies that $B1 = 0$. Then, combining this together with the previous result, we will obtain the Main result.

Let us point out that we may relax the assumptions for our Main result in the following two points: first, that the distance d on X may be replaced by a quasi-distance d'; that is, d' satisfies all of the properties of d but the constant in the triangle inequality may be an arbitrary positive constant; and secondly, that the balls in X do not need to be relatively compact but the Cauchy boundary (or b-boundary [7]) of X is polar (Corollary 3.3 in Section 3.1).

Therefore, our result is applicable also for singular spaces; for instance, algebraic varieties, cone-manifolds, orbifolds (Satake's V-manifolds), etc. Note that even for $X \subset \mathbf{R}^d$, its completion $\overline{X}$ does not need to be locally compact (see, e.g., [7], page 283), and if any ball in X is relatively compact, then X is complete, and, in particular, the Cauchy boundary is almost polar (see [16] when the space X is a Riemannian manifold).

We organize this article as follows. In Section 2 we prove the conservation property for the truncated process. We also prove the integral derivation property (Proposition 2.2). In Section 3, we will prove Main result (Theorem 3.1) and its extension to the incomplete space. Finally, we present some examples of the kernels k in Section 4.

2. Conservation property of truncated processes

In this section we consider the *truncated (jump) kernel* $\mu_1(x, dy)$:

$$\mu_1(x, A) = \mu\big(x, A \cap B(x, 1)\big) \quad \text{for } x \in X \text{ and } A \in \mathcal{B}(X),$$

where $B(x, 1) = \{y \in X : d(y, x) < 1\}$. We will establish the associated integral derivation property, and by applying it, we will prove the conservation property of truncated processes.

We can show that the following Dirichlet form $(\mathcal{E}^1, \mathcal{F})$ associated with μ_1 is regular:

$$\mathcal{E}^1(u, v) = \iint_{x \neq y} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\mu_1(x, dy)m(dx), \quad u, v \in \mathcal{F}.$$

Let C^{lip} be the space of uniformly Lipschitz continuous functions on X and $C_b^{\mathrm{lip}} := C^{\mathrm{lip}} \cap L^\infty$.

Lemma 2.1. *Assume* (C). *If* $u \in \mathcal{F}$ *and* $v \in C_b^{\mathrm{lip}}$, *then* $u \cdot v \in \mathcal{F}$.

Proof. Assume $u \in \mathcal{F}$ and $v \in C_b^{\mathrm{lip}}$. Then there exists a sequence $\{u_n\} \subset C_0^{\mathrm{lip}}$ such that u_n converges to u in $\sqrt{\mathcal{E}_1}$, and hence in $\sqrt{\mathcal{E}_1^1}$. Note that $u_n v \in C_0^{\mathrm{lip}}$ and it is easy to see that $u_n v$ converges to uv in L^2. To show $uv \in \mathcal{F}$, we need to estimate $\mathcal{E}^1[u_n v]$ as follows:

$$\mathcal{E}^1[u_n v] = \iint_{x \neq y} \big(u_n(x)v(x) - u_n(y)v(y)\big)^2 \mu_1(x, dy)m(dx)$$

$$\leq 2 \iint_{x \neq y} v^2(x)\big(u_n(x) - u_n(y)\big)^2 \mu_1(x, dy)m(dx)$$

$$+ 2 \iint_{x \neq y} u_n^2(x)\big(v(x) - v(y)\big)^2 \mu_1(x, dy)m(dx)$$

$$\leq 2\|v\|_\infty^2 \iint_{x \neq y} \big(u_n(x) - u_n(y)\big)^2 \mu_1(x, dy)m(dx)$$

$$+ 2\|v\|_{\mathrm{Lip}}^2 \int u_n^2(x) \int_{0 < d(x,y) < 1} d^2(x, y)\mu(x, dy)m(dx)$$

$$\leq 2\big(\|v\|_\infty^2 + M\|v\|_{\mathrm{Lip}}^2\big)\mathcal{E}^1[u_n].$$

Since this shows that $u_n v$ is a bounded sequence with respect to $\mathcal{E}_1^1$, we can conclude that $uv \in \mathcal{F}$. $\square$

Let $\Gamma : C_0^{\text{lip}} \times C_0^{\text{lip}} \to L^1$ be the relative carré du champ operator; namely,

$$\Gamma(u, v)(x) = \frac{1}{2} \int_{y \neq x} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\mu_1(x, \mathrm{d}y). \tag{2.1}$$

Proposition 2.2 (Integral derivation property). *Assume* (C).

$$\int \Gamma(u, v \cdot w) \, \mathrm{d}m = \int v \Gamma(u, w) \, \mathrm{d}m + \int w \Gamma(u, v) \, \mathrm{d}m \quad \forall u, v \in \mathcal{F}_b, \forall w \in C_b^{\text{lip}}. \tag{2.2}$$

Proof. Once the convergence of each integral in (2.2) is justified, the equality is easily seen as in [17]. The left-hand side makes sense because $u, vw \in \mathcal{F}$ by Lemma 2.1. Since w is bounded, it follows that the integral of the second term of the right-hand side converges by the Schwarz inequality. Thus, we only need to show that the first term of the right-hand side converges.

By applying the Schwarz inequality twice, we have

$$\left| \int v \Gamma(u, w) \, \mathrm{d}m \right|$$

$$\leq \int |v(x)| \int_{x \neq y} |u(x) - u(y)| \cdot |w(x) - w(y)| \mu_1(x, \mathrm{d}y) m(\mathrm{d}x)$$

$$\leq \int |v(x)| \left\{ \sqrt{\int_{x \neq y} |u(x) - u(y)|^2 \mu_1(x, \mathrm{d}y)} \sqrt{\int_{x \neq y} |w(x) - w(y)|^2 \mu_1(x, \mathrm{d}y)} \right\} m(\mathrm{d}x)$$

$$\leq \sqrt{\iint_{x \neq y} |u(x) - u(y)|^2 \mu_1(x, \mathrm{d}y) m(\mathrm{d}x)} \sqrt{\iint_{0 < d(x,y) < 1} v^2(x) |w(x) - w(y)|^2 \mu_1(x, \mathrm{d}y) m(\mathrm{d}x)}$$

$$\leq \|w\|_{\text{Lip}} \cdot \sqrt{\mathcal{E}(u, u)} \sqrt{\iint_{0 < d(x,y) < 1} v^2(x) d^2(x, y) \mu_1(x, \mathrm{d}y) m(\mathrm{d}x)}$$

$$\leq \sqrt{M} \|w\|_{\text{Lip}} \cdot \sqrt{\mathcal{E}(u, u)} \cdot \|v\| < \infty. \qquad \square$$

Set $g_a = \mathrm{e}^{-ar}$ with $a > 0$, where r is the distance $d(x_0, \cdot)$ from an arbitrary fixed point $x_0 \in X$.

Lemma 2.3. *If* (C) *and* (1.1), *then* $g_a \in \mathcal{F}$ *for any* $a > 0$.

Proof. Let χ_n be a sequence of cut-off functions on X defined as

$$\chi_n(x) := \frac{(n - r(x))}{n} \vee 0, \quad x, y \in X \text{ with } |r(x) - r(y)| \leq 1.$$

Fix $a > 0$ and set $g_n := g_a \chi_n$ for $n \in \mathbf{N}$. By condition (1.1) and the Lebesgue convergence theorem, we see that g_n converges to g_a in L^2 as $n \to \infty$. Utilizing the following inequality (which takes place of the chain rule for a local-operator [21]):

$$|g_a(x) - g_a(y)| \leq a \mathrm{e}^a g_a(x) |r(x) - r(y)|, \quad x, y \in X \text{ with } |r(x) - r(y)| \leq 1,$$

and taking into account that $|r(x) - r(y)| \leq d(x, y)$, we have

$$\mathcal{E}^1(g_a, g_a) \leq C \iint g_a^2(x) |r(x) - r(y)|^2 \mu_1(x, \mathrm{d}y) m(\mathrm{d}x) \leq CM \int g_a^2(x) m(\mathrm{d}x) < \infty,$$

where $C = ae^a$. Moreover, by setting $\eta_n = 1 - \chi_n$, it follows that

$$
\begin{aligned}
\mathcal{E}^1(g_a - g_n, g_a - g_n) &= \iint \big(g_a(x)\eta_n(x) - g_a(y)\eta_n(y)\big)^2 \mu_1(x, \mathrm{d}y)m(\mathrm{d}x) \\
&= \iint \big\{(g_a(x) - g_a(y))\eta_n(x) + g_a(y)\big(\eta_n(x) - \eta_n(y)\big)\big\}^2 \mu_1(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\le 2 \iint \big(g_a(x) - g_a(y)\big)^2 \eta_n^2(x)\mu_1(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\quad + 2 \iint g_a^2(y)\big(\eta_n(x) - \eta_n(y)\big)^2 \mu_1(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\le 2C \iint g_a^2(x)d^2(x, y)\eta_n^2(x)\mu_1(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\quad + \frac{2}{n^2} \iint g_a^2(y)d^2(x, y)\mu_1(x, \mathrm{d}y)m(\mathrm{d}x) \\
&\le 2CM \int g_a^2(x)\eta_n^2(x)m(\mathrm{d}x) + \frac{2M}{n^2}\|g_a\|^2,
\end{aligned}
$$

where the last line tends to 0 as $n \to \infty$ by the Lebesgue convergence theorem. Therefore, since $g_n \in \mathcal{F}$ (because every ball associated to d is relative compact), we have the assertion. $\square$

We are in a position to prove the conservation property of the truncated process.

Theorem 2.4. *If* (C) *and* (1.1) *hold, then the form* $(\mathcal{E}^1, \mathcal{F})$ *is conservative.*

Proof. Let $f \in C_0^{\mathrm{lip}}$ and $u_t = T_t f$, where T_t is the L^2-semigroup associated with $(\mathcal{E}^1, \mathcal{F})$. Set

$$
\theta = e^r \quad \text{and} \quad \theta_n = \theta \wedge n
$$

for every $n \in \mathbf{N}$. Taking into account Lemma 2.3, we have the equality:

$$
(u_t, g_a) - (f, g_a) = \int_0^t (\dot{u}_s, g_a)\,\mathrm{d}s, \tag{2.3}
$$

where $\dot{u}_s = \frac{\mathrm{d}}{\mathrm{d}s}u_s$. We will show that for any $t > 0$,

$$
\int_0^t (\dot{u}_s, g_a)\,\mathrm{d}s \to 0, \quad a \to 0. \tag{2.4}
$$

If this is the case, then by the dominated convergence theorem, we deduce

$$
(T_t f, 1) = (u_t, g_0) = (f, g_0) = (f, 1),
$$

which clearly implies the conservation property.

To the end we show (2.4). It follows that

$$
\begin{aligned}
\left|\int_0^t (\dot{u}_s, g_a)\,\mathrm{d}s\right| &= \left|\int_0^t \mathcal{E}^1(u_s, g_a)\,\mathrm{d}s\right| \\
&= \left|\int_0^t \iint \big(u_s(x) - u_s(y)\big)\big(g_a(x) - g_a(y)\big)\mu_1(x, \mathrm{d}y)m(\mathrm{d}x)\,\mathrm{d}s\right| \\
&\le ae^a \int_0^t \iint \big|u_s(x) - u_s(y)\big|e^{-ar(x)}\big|r(x) - r(y)\big|\mu_1(x, \mathrm{d}y)m(\mathrm{d}x)\,\mathrm{d}s
\end{aligned}
$$

$$\leq a e^a \int_0^t \sqrt{\iint \theta(x)\big(u_s(x) - u_s(y)\big)^2 \mu_1(x, \mathrm{d}y) m(\mathrm{d}x)} \, \mathrm{d}s$$

$$\times \sqrt{\iint \frac{e^{-2ar(x)}}{\theta(x)} d^2(x, y) \mu_1(x, \mathrm{d}y) m(\mathrm{d}x)}$$

$$\leq a e^a \int_0^t \sqrt{\int \theta \, \Gamma(u_s, u_s) \, \mathrm{d}m} \, \mathrm{d}s \sqrt{M \int e^{-(2a+1)r} \, \mathrm{d}m}$$

$$\leq a e^a \sqrt{t} \cdot \sqrt{M} \cdot \|g_{2a+1}\|_{L^1}^{1/2} \int_0^t \int \theta \, \Gamma(u_s, u_s) \, \mathrm{d}m \, \mathrm{d}s.$$

Therefore, by (1.1), it suffices to prove

$$\int_0^t \int \theta \, \Gamma[u_s] \, \mathrm{d}m \, \mathrm{d}s < \infty. \tag{2.5}$$

For any $\lambda > 0$ and $n \geq 1$, by applying Lemma 2.1 and Proposition 2.2, we have

$$0 = (\dot{u}_s, \theta_n u_s) + \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m + \int u \, \Gamma(u_s, \theta_n) \, \mathrm{d}m$$

$$\geq (\dot{u}_s, \theta_n u_s) + \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m - e \left| \iint u(x)\theta_n(x)d(x, y)\big(u(x) - u(y)\big)\mu_1(x, \mathrm{d}y) m(\mathrm{d}x) \right|$$

$$\geq (\dot{u}_s, \theta_n u_s) + \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m - e \sqrt{\iint u_s^2(x)\theta_n(x)d^2(x, y)\mu_1(x, \mathrm{d}y) m(\mathrm{d}x)} \sqrt{\int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m}$$

$$\geq (\dot{u}_s, \theta_n u_s) + \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m - e \sqrt{M \int u_s^2 \theta_n \, \mathrm{d}m} \sqrt{\int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m}$$

$$\geq (\dot{u}_s, \theta_n u_s) + \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m - \frac{eM}{2}\left(\lambda \int u_s^2 \theta_n + \frac{1}{\lambda} \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m\right)$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\|\sqrt{\theta_n} u_s\right\|^2 - \frac{\lambda eM}{2}\left\|\sqrt{\theta_n} u_s\right\|^2 + \left(1 - \frac{eM}{2\lambda}\right) \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\|\sqrt{\theta_n} u_s\right\|^2 \leq \lambda eM \left\|\sqrt{\theta_n} u_s\right\|^2 - \left(2 - \frac{eM}{\lambda}\right) \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m. \tag{2.6}$$

Now we specify λ so that $2 - (eM/\lambda) > 0$; that is, $\lambda > eM/2$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\|\sqrt{\theta_n} u_s\right\|^2 \leq \lambda eM \left\|\sqrt{\theta_n} u_s\right\|^2. \tag{2.7}$$

Solving this inequality, we have

$$\left\|\sqrt{\theta_n} u_s\right\|^2 \leq e^{\lambda eMs} \left\|\sqrt{\theta_n} u_0\right\|^2 = e^{\lambda eMs} \left\|\sqrt{\theta_n} f\right\|^2. \tag{2.8}$$

Integrating (2.6) on $[0, t]$,

$$\left\|\sqrt{\theta} u_t\right\|^2 - \left\|\sqrt{\theta_n} f\right\|^2 \leq \int_0^t \lambda eM \left\|\sqrt{\theta_n} u_s\right\|^2 \mathrm{d}s - \left(2 - \frac{eM}{\lambda}\right) \int_0^t \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m \, \mathrm{d}s,$$

hence, it follows by (2.8) that

$$\left(2 - \frac{\mathrm{e}M}{\lambda}\right) \int_0^t \int \theta_n \Gamma(u_s, u_s) \, \mathrm{d}m \, \mathrm{d}s \leq -\left\|\sqrt{\theta_n} u_t\right\|^2 + \left\|\sqrt{\theta_n} f\right\|^2 + \int_0^t \lambda \mathrm{e}M \left\|\sqrt{\theta} u_s\right\|^2 \mathrm{d}s$$

$$\leq \left\|\sqrt{\theta_n} f\right\|^2 + \lambda \mathrm{e}M \int_0^t \mathrm{e}^{\lambda \mathrm{e}Ms} \left\|\sqrt{\theta_n} f\right\|^2 \mathrm{d}s < \infty.$$

Letting $n \to \infty$ we obtain (2.5) and arrive at the conclusion. $\qquad\square$

3. Conservation property of general jump processes

In this section, we prove the Main result, and extend it to a singular space X.

Since both $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^1, \mathcal{F})$ are symmetric regular Dirichlet forms on L^2, there are associated sub-Markovian semigroups and resolvent operators, which we denote by T_t, T_t^1 and R_λ, R_λ^1, respectively. We also denote by A, A^1 the associated L^2-generators, respectively. Let B be the generator of the form

$$\mathcal{E}(u, v) - \mathcal{E}^1(u, v) = \iint_{d(x,y)>1} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\mu(x, \mathrm{d}y)m(\mathrm{d}x), \quad u, v \in \mathcal{F}.$$

Clearly, $B = A - A^1$. By (C)-(ii), we deduce that

$$Bu(x) = 2 \int_{d(x,y)>1} \big(u(y) - u(x)\big)\mu(x, \mathrm{d}y)$$

and

$$\|Bu\|_\infty \leq 4 \left(\sup_{x \in X} \int_{d(x,y)>1} \mu(x, \mathrm{d}y)\right)\|u\|_\infty \quad \text{for } u \in C_0^{\mathrm{lip}}.$$

This means that B extends to a bounded operator on L^∞, which we denote by the same symbol B. Note that since R_λ, R_λ^1 are sub-Markovian operators, these operators can be extended naturally to L^∞.

Theorem 3.1. *If $\mu(x, \mathrm{d}y)$ and m satisfy* (C) *and* (1.1), *then* $(\mathcal{E}, \mathcal{F})$ *is conservative.*

Proof. Recall that T_t is conservative if and only if so is λR_λ. For $\lambda > 0$,

$$R_\lambda - R_\lambda^1 = R_\lambda\big(A - A^1\big)R_\lambda^1 = R_\lambda B R_\lambda^1.$$

Thus, since $\lambda R_\lambda^1 1 = 1$ by Theorem 2.4 and $B1 = 0$, we have

$$\lambda R_\lambda 1 - 1 = \lambda R_\lambda 1 - \lambda R_\lambda^1 1 = R_\lambda B\big(\lambda R_\lambda^1\big)1 = R_\lambda B1 = R_\lambda 0 = 0.$$

This shows the conservativeness of the semigroup T_t. $\qquad\square$

Now we remove the assumption such that any ball in X is relatively compact. Note that this assumption implies the completeness of (X, d) (see the remark below).

Define the *Cauchy boundary* $\partial_C X$ of X as

$$\partial_C X := \overline{X} \setminus X,$$

where $\overline{X}$ is the completion of X with respect to d. We say $\partial_C X$ *is almost polar* if there exists a sequence of functions $e_n \in C^{\mathrm{lip}} \cap \mathcal{F}$ with $n \in \mathbf{N}$ such that

- $0 \leq e_n \leq 1$ for every $n \in \mathbf{N}$;

- There exists an open set $O_n \supset \partial_C X$ of $\overline{X}$ for each n such that $e_n = 1$ on $O_n \cap X$;
- $\mathcal{E}_1[e_n] \to 0$ as $n \to \infty$.

The key for this generalization is to extend Lemma 2.3 to

Lemma 3.2. *Assume that $\partial_C X$ is almost polar* (*but any ball in X is not necessarily a relatively compact*). *If* (C) *and* (1.1) *hold, then $g_a \in \mathcal{F}$ for every $a > 0$.*

Proof. Let $\mathcal{F}_N$ be the completion of $C^{\mathrm{lip}} \cap \{u \in L^2: \mathcal{E}(u, u) < \infty\}$ in $\mathcal{E}_1$-norm. It suffices to prove

$$\mathcal{F} = \mathcal{F}_N,$$

because $g_a \in \mathcal{F}_N$. Let $u \in \mathcal{F}_N$. Clearly we may assume that $u \in L^\infty$ without loss of generality. If e_n is the function which is in the definition of the almost polarity above, then $u_n = (1 - e_n)u \in \mathcal{F}$ and $u_n \to u$ in $\mathcal{E}_1$ as $n \to \infty$. This shows: $\mathcal{F} = \mathcal{F}_N$. $\qquad\square$

Then, since Theorems 2.4 and 3.1 extend to this setting, we have:

Corollary 3.3. *Assume* (C) *and* (1.1). *If the Cauchy boundary of X is almost polar, then $(\mathcal{E}, \mathcal{F})$ is conservative.*

Remark 3.1. *We can argue as follows the fact that if any ball in X is relatively compact, then X is complete. Let $x \in \overline{X}$ and $x_n \in X$ such that $\overline{d}(x, x_n) \to 0$ as $n \to \infty$, where $\overline{d}$ is the extended distance of d to $\overline{X}$. For any $r > 0$, there exists $n_0 \in \mathbf{N}$ such that $\overline{d}(x, x_n) < r$ for every $n > n_0$. Then $x_n \in B_{3r}(x_0) \subset X$ for $\forall n > n_0$, and due to the assumption, we conclude that $x = \lim_{n \to \infty} x_n \in \overline{B_{3r}(x_0)} \subset X$. Therefore $X \supset \overline{X}$, which says that X is complete.*

4. Examples

In this section, we present some examples. Let $d \geq 1$. We assume that X is a d-dimensional complete non-compact Riemannian manifold M without boundary and m is the associated Riemannian measure. We also assume that the Ricci curvature is non-negative outside a compact set.

In the following, we assume that the kernel $\mu(x, \mathrm{d}y)$ is absolutely continuous with respect to m for each x; namely, there exists a non-negative symmetric function $k(x, y) = k(y, x)$ such that

$$\mu(x, \mathrm{d}y) = k(x, y)m(\mathrm{d}y).$$

In this case, the condition (C)-(i) is always satisfied.

4.1. *Finite range jumping kernel*

Example 4.1. *Consider the following kernel $k(x, y)$:*

$$k(x, y) = k(y, x) = \frac{C(x, y)}{d(x, y)^{d+\alpha}} \mathbf{1}_{\{d(x, y) \leq \kappa\}}, \quad x, y \in M,$$

where $\kappa > 0$, $0 < \alpha < 2$ and $C(x, y)$ is a measurable function that is pinched by two positive constants. Applying the curvature condition, it is easy to verify (C)*-(ii).*

Example 4.2. *Let $0 < \alpha < \beta < 2$ and $C, c > 0$. If the kernel $k(x, y)$ is defined by*

$$\begin{cases} \dfrac{c}{d(x, y)^{d+\alpha}} \leq k(x, y) = k(y, x) \leq \dfrac{C}{d(x, y)^{d+\beta}} & \text{for } d(x, y) \leq 1, \\ k(x, y) = 0 & \text{for } d(x, y) > 1, \end{cases}$$

then, it satisfies (C)*-(ii).*

Remark 4.1. *It is proved in* [2,5] (*see also* [4]) *that the processes associated to* k *on* $\mathbf{R}^d$ *in Examples* 4.1 *and* 4.2 *are conservative, respectively. The approach to the conservation property in* [2] *is to make use of the estimates of the heat kernel corresponding to the semigroup, and to apply the Mosco convergence of the Dirichlet forms associated to the following kernels* $k^n(x, y)$ *to the Dirichlet form in the problem*:

$$k^n(x, y) = \begin{cases} k(x, y) & for |x - y| \geq 1/n, \\ C|x - y|^{-d-\beta} & for |x - y| < 1/n. \end{cases}$$

4.2. *Symmetric stable-like jumping kernel*

Let $\alpha(x)$ be a positive measurable function defined on M, which takes values in $(0, 2)$. Set

$$\tilde{k}(x, y) = \frac{C(\alpha(x))}{d(x, y)^{d+\alpha(x)}},$$

where $C(\alpha)$ satisfies $C(\alpha) \approx \alpha(2 - \alpha)$. Here, $\approx$ means that the ratio of the left and right-hand sides of $\approx$ is pinched by two positive constants.

Let us consider the following quadratic form on $L^2(M) = L^2(M; dm)$:

$$\mathcal{E}(u, v) = \iint_{x \neq y} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\tilde{k}(x, y)m(dx)m(dy),$$

$$\mathcal{D}[\mathcal{E}] = \big\{u \in L^2(M): \mathcal{E}(u, u) < \infty\big\}.$$

Then it is shown in [12] (see also [23]) that $\mathcal{D}[\mathcal{E}]$ contains $C_0^{\mathrm{lip}}(M)$ if and only if

$$\int_{r(x)>1} \frac{\alpha(x)}{r(x)^{d+\alpha(x)}}m(dx) < \infty. \tag{4.1}$$

This result is originally proven for $\mathbf{R}^d$ and it extends to M without any difficulties. So, under this condition, $(\mathcal{E}, C_0^{\mathrm{lip}}(M))$ is a symmetric closable Markovian form on $L^2(M)$; accordingly, there exists a symmetric Hunt process called a *symmetric stable-like process* corresponding to the closure $(\mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, C_0^{\mathrm{lip}}(M))$. Let us point out that (C)-(ii) implies (4.1).

Due to the Beurling–Deny formula (see [8]), $\mathcal{E}$ has the following alternative expression:

$$\mathcal{E}(u, v) = \frac{1}{2} \iint_{x \neq y} \big(\tilde{u}(x) - \tilde{u}(y)\big)\big(\tilde{v}(x) - \tilde{v}(y)\big)k(x, y)m(dx)m(dy), \quad u, v \in \mathcal{F},$$

where $k(x, y) = \tilde{k}(x, y) + \tilde{k}(y, x)$ and $\tilde{u}$ is a quasi-continuous modification of $u \in \mathcal{F}$.

Example 4.3. *Suppose that there exist* $0 < \alpha < \beta < 2$ *such that*

$$\alpha \leq \alpha(x) \leq \beta, \quad a.e. \ x \in M.$$

Then, as in the previous examples, we easily see that the kernel $k(x, y)$ *satisfies* (C)-(ii). *Thus, the associated symmetric stable-like process is conservative.*

In the next example, we consider the case $\alpha(x)$ tends to 2 as $r(x) \to \infty$.

Example 4.4. *Let* $0 < \beta < 2$. *Let* γ *be a decreasing function defined on* $[0, \infty)$, *which takes values in* $(0, \beta]$. *Set*

$$\alpha(x) = 2 - \gamma\big(r(x)\big), \quad x \in M.$$

Proposition 4.5. *If γ satisfies*

$$\lim_{t \to \infty} \gamma(t) = 0, \qquad \limsup_{t \to \infty} \frac{\gamma(t-1)}{\gamma(t+1)} < \infty, \tag{4.2}$$

then, *the kernel k satisfies* (C)-(ii), *therefore*, *the associated symmetric stable-like process is conservative.*

Proof. In the sequel, c and c' are positive constants which are independent on $x \in M$ and may differ from line to line. We check (C)-(ii) in all possible cases. Assume (4.2).

Case: $\tilde{k}(x, y) = d(x, y)^{-d-\alpha(x)}$. For any $x \in M$, it follows:

$$\int_{y \neq x} C\big(\alpha(x)\big)\big(d(x, y)^2 \wedge 1\big)d(x, y)^{-d-\alpha(x)}m(\mathrm{d}y)$$

$$\leq c\alpha(x)\big(2 - \alpha(x)\big) \int_{y \neq x} \big(d(x, y)^2 \wedge 1\big)d(x, y)^{-d-\alpha(x)}m(\mathrm{d}y)$$

$$= c\alpha(x)\big(2 - \alpha(x)\big)\bigg(\int_{0 < d(x,y) \leq 1} d(x, y)^{2-d-\alpha(x)}m(\mathrm{d}y) + \int_{d(x,y) > 1} d(x, y)^{-d-\alpha(x)}m(\mathrm{d}y)\bigg)$$

$$\leq c'\alpha(x)\big(2 - \alpha(x)\big)\bigg(\int_0^1 u^{1-\alpha(x)}\,\mathrm{d}u + \int_1^\infty u^{-1-\alpha(x)}\,\mathrm{d}u\bigg)$$

$$= c'\alpha(x)\big(2 - \alpha(x)\big)\bigg(\frac{1}{2 - \alpha(x)} + \frac{1}{\alpha(x)}\bigg)$$

$$= c'\big(\alpha(x) + \big(2 - \alpha(x)\big)\big) = 2c' < \infty.$$

Case: $\tilde{k}(y, x) = d(x, y)^{-d-\alpha(y)}$ when $d(x, y) > 1$. For any $x \in M$, it follows:

$$\int_{d(x,y) > 1} C\big(\alpha(y)\big)d(x, y)^{-d-\alpha(y)}m(\mathrm{d}y) \leq c \int_{d(x,y) > 1} d(x, y)^{-d-2+\gamma(r(y))}m(\mathrm{d}y)$$

$$\leq c \int_{d(x,y) > 1} d(x, y)^{-d-2+\beta}m(\mathrm{d}y)$$

$$\leq c' \int_1^\infty u^{-3+\beta}\,\mathrm{d}u = \frac{c'}{2 - \beta} < \infty.$$

Case: $\tilde{k}(y, x) = d(x, y)^{-d-\alpha(y)}$ when $0 < d(x, y) \leq 1$ and $r(x) \leq 2$. Noting that $r(y) \leq 3$ and γ is decreasing, so that $\gamma(r(y)) \geq \gamma(3)$, it follows:

$$\int_{0 < d(x,y) \leq 1} C\big(\alpha(y)\big)d(x, y)^{2-\alpha(y)-d}m(\mathrm{d}y) \leq c \int_{0 < d(x,y) \leq 1} d(x, y)^{\gamma(r(y))-d}m(\mathrm{d}y)$$

$$\leq c \int_{0 < d(x,y) \leq 1} d(x, y)^{\gamma(3)-d}m(\mathrm{d}y)$$

$$\leq c' \int_0^1 u^{\gamma(3)-1}\,\mathrm{d}u = \frac{c'}{\gamma(3)} < \infty.$$

Case: $\tilde{k}(y, x) = d(x, y)^{-d-\alpha(y)}$ when $0 < d(x, y) \leq 1$ and $r(x) > 2$. Since $\gamma(r(y)) \geq \gamma(r(x) + 1)$, by combining those calculus above, we conclude that

$$\sup_{x \in M} \int_{y \neq x} \big(d(x, y)^2 \wedge 1\big)k(x, y)m(\mathrm{d}y) < \infty.$$

Namely, the Dirichlet form generated by the kernel $k(x, y)$ is conservative.   □

Example 4.6. *For some $c_1, c_2 > 0$, set $\gamma(t) = c_1 e^{-c_2 t}$. Then γ satisfies* (4.2) *since*

$$\lim_{t \to \infty} e^{-c_2 t} = 0$$

and for any $t > 0$,

$$\frac{\gamma(t-1)}{\gamma(t+1)} = e^{-c_2(t-1) + c_2(t+1)} = e^{2c_2} < \infty.$$

In the next example, $\alpha(x) \to 0$ as $r(x) \to \infty$. Let us point out that this example confirms that our assumption is sharp in the sense stated below.

Example 4.7. *Put $\alpha(x) = (\ln(r(x) + 1))^{-\varepsilon}, x \in X$ for $\varepsilon > 0$. Then the condition* (C)*-(ii) is satisfied, namely, the associated process* (*exists and*) *is conservative if $\epsilon \neq 1$; and* (4.1) *does not need to be true for $\epsilon = 1$. For instance, if X is $\mathbf{R}^d$ with standard Lebesgue measure, then we can show that* (4.1) *is violated, whence $(\mathcal{E}, C_0^{\mathrm{Lip}}(\mathbf{R}^d))$ may not be closable and there are no associated Hunt processes. Moreover, since the symbol associated with the (L^2)-generator of the Dirichlet form is indeed singular* (*see* [20], *Corollary* 3.2) *even if the function α is smooth, we can not adopt the martingale theory to estimate some path properties of the processes different from the cases of diffusion processes or Lévy-type jump processes whose symbols are smooth* (*see, e.g.,* [13]).

Remark 4.2.

(i) *In the Riemannian manifold case, all examples we mentioned above seem to be new. Even when $X = \mathbf{R}^d$, the result in Example* 4.3 *still seems to be new if the function α is only assumed to be measurable* (*cf.* [20]).

(ii) (*e.g.,* [1]) *A classical example of $C(\alpha)$ on $\mathbf{R}^d$ is*

$$C(\alpha) = \frac{\Gamma(1 + \alpha/2)\Gamma((\alpha + d)/2) \sin((\pi\alpha)/2)}{2^{1-\alpha} \pi^{d/2+1}}.$$

4.3. *Non-Lebesgue measure cases*

Let $M = \mathbf{R}$ and $m(dx) = e^{2\lambda|x|} dx$, where $\lambda > 0$. Consider

$$k(x, y) = \left(e^{-\lambda(|x|+|y|)}\right) \mathbf{1}_{\{|x-y|<1\}}.$$

Then

$$\begin{aligned}
M(x) &= \int_{|x-y|<1} |x - y|^2 e^{-\lambda(|x|+|y|)} e^{2\lambda|y|} \, dy \\
&= e^{-\lambda|x|} \int_{|y|<1} y^2 e^{\lambda|x-y|} \, dy \\
&\leq e^{-\lambda|x|} \int_{|y|<1} y^2 e^{\lambda(|x|+|y|)} \, dy = \int_{|y|<1} y^2 e^{\lambda|y|} \, dy < 2e^{\lambda}.
\end{aligned}$$

Since $e^{-ar} \notin L^1(\mathbf{R}; m(dx))$ for every $a \leq 2\lambda$, this confirms the fact that (1.1) does not need to imply (C)-(ii). On the other hand, we can easily construct an example of the kernel on $\mathbf{R}^d$ which violates (C)-(ii) (of course (1.1) holds true on $\mathbf{R}^d$).

4.4. *Continuous-time Markov chains*

Let X be a countable set and m a measure (a function) on X with $m(i) > 0$ for every $i \in X$. Suppose we are given a function $q(i, j)$ defined on $X \times X$ satisfying

$$q(i, i) = \sum_{i \neq j} q(i, j) < \infty \quad \text{for each } i \in X.$$

It is well known that, considering the exponential holding time at each state, there exists a time homogeneous continuous-time Markov chain X_t satisfying

$$P(X_{t+h} = j | X_t = i) = q(i, j)h + \mathrm{o}(h) \quad \text{for every } i, j \in X \text{ and } h > 0.$$

Moreover, if m is a reversible measure with respect to $q(i, j)$; namely

$$q(i, j)m_j = q(j, i)m_i \quad \text{for every } i, j \in X,$$

we may associate a symmetric Dirichlet form as follows: for suitable functions u, v

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{i, j \in X} (u_i - u_j)(v_i - v_j)q(i, j)m_i = - \sum_{i, j \in X} q(i, j)u_i v_j m_i.$$

Then, by replacing the integral by the summation, our Main result says that X_t is conservative provided

$$\sup_{i \in X} \sum_{j \in X} q(i, j) < \infty \quad \text{and} \quad \sum_{i \in X} \mathrm{e}^{-ai} m_i < \infty \quad \text{for every } a > 0. \tag{4.3}$$

Example 4.8 (Birth-and-Death process). *Let us consider*

$$q(i, j) = \begin{cases} \lambda_i, & j = i + 1, \\ \mu_i, & j = i - 1, \\ 1 - (\lambda_i + \mu_1), & j = i, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4}$$

where $\lambda_i > 0$ for every $i \in \mathbf{Z}_+$, $\mu_0 = 0$ and $\mu_i > 0$ for $i \in \mathbf{Z}_+ i = 1, 2, \ldots$. This is the birth-and-death process X_t on $\mathbf{Z}_+$, with birth and death rates λ_i and μ_i, respectively. The associated reversible measure m_i is 2^{-i}. Due to [15], X_t is not conservative. Clearly, the second condition in (4.3) is satisfied but the first condition fails. It is possible to find, applying the property which the birth-and-death process shares with diffusion processes, a conservative birth-and-death process which does not satisfy (4.3). Of course, our result covers more general type of processes. (See also [3] for more general jump processes on a discrete space).

References

[1] N. Aronszajn and K. T. Smith. Theory of Bessel potentials I. *Ann. Inst. Fourier (Grenoble)* **11** (1961) 385–475. MR0143935

[2] M. T. Barlow, R. F. Bass, Z. Q. Chen and M. Kassmann. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.* **361** (2009) 1963–1999. MR2465826

[3] M. F. Chen. *From Markov Chains to Non-Equilibrium Particle Systems*, 2nd edition. World Scientific, River Edge, NJ, 2004. MR2091955

[4] Z. Q. Chen, P. Kim and T. Kumagai. Weighted Poincaré inequality and heat kernel estimates for finite range jump processes. *Math. Ann.* **342** (2008) 833–883. MR2443765

[5] Z. Q. Chen and T. Kumagai. Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Related Fields* **140** (2008) 277–317. MR2357678

[6] E. B. Davies. The heat kernel bounds, conservation of probability and the Feller property. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. *J. Anal. Math.* **58** (1992) 99–119. MR1226938

[7] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space–Time. Cambridge Monographs on Mathematical Physics* **1**. Cambridge Univ. Press, London, 1973. MR0424186

[8] M. Fukushima, Y. Oshima and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes. de Gruyter Studies in Mathematics* **19**. Walter de Gruyter, Berlin, 1994. MR1303354

[9] M. P. Gaffney. The conservation property of the heat equation on Riemannian manifolds. *Comm. Pure Appl. Math.* **12** (1959) 1–11. MR0102097

[10] A. Grigor'yan. Stochastically complete manifolds. *Dokl. Akad. Nauk SSSR* **290** (1986) 534–537 (Russian). MR0860324

[11] K. Ichihara. Explosion problems for symmetric diffusion processes. *Trans. Amer. Math. Soc.* **298** (1986) 515–536. MR0860378

[12] Y. Isozaki and T. Uemura. A family of symmetric stable-like processes and its global path properties. *Probab. Math. Statist.* **24** (2004) 145–164. MR2148226

[13] N. Jacob. *Pseudo differential operators and Markov processes*, Vol. 1–3. Imperial Colledge Press, London, 2001.

[14] L. Karp and P. Li. The heat equation on complete Riemannian manifold. Preprint, 1982.

[15] S. Karlin and J. L. McGregor. The differential equations of birth-and-death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.* **85** (1957) 489–546. MR0091566

[16] J. Masamune. Analysis of the Laplacian of an incomplete manifold with almost polar boundary. *Rend. Mat. Appl. (7)* **25** (2005) 109–126. MR2142127

[17] J. Masamune and T. Uemura. L^p-Liouville property for nonlocal operators. Preprint, 2008.

[18] Y. Oshima. On conservativeness and recurrence criteria for Markov processes. *Potential Anal.* **1** (1992) 115–131. MR1245880

[19] R. L. Schilling. Conservativeness of semigroups generated by pseudo-differential operators. *Potential Anal.* **9** (1998) 91–104. MR1644108

[20] R. L. Schilling and T. Uemura. On the Feller property of Dirichlet forms generated by pseudo-differential operator. *Tohoku Math. J. (2)* **59** (2007) 401–422. MR2365348

[21] K. T. Sturm. Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math.* **456** (1994) 173–196. MR1301456

[22] M. Takeda. On the conservativeness of the Brownian motion on a Riemannian manifold. *Bull. London Math. Soc.* **23** (1991) 86–88. MR1111541

[23] T. Uemura. On some path properties of symmetric stable-like processes for one dimension. *Potential Anal.* **16** (2002) 79–91. MR1880349

[24] T. Uemura. On symmetric stable-like processes: Some path properties and generators. *J. Theoret. Probab.* **17** (2004) 541–555. MR2091550