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Brownian motion with respect to time-changing Riemannian metrics, applications to Ricci flow

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Abstract. We generalize Brownian motion on a Riemannian manifold to the case of a family of metrics which depends on time. Such questions are natural for equations like the heat equation with respect to time dependent Laplacians (inhomogeneous diffusions). In this paper we are in particular interested in the Ricci flow which provides an intrinsic family of time dependent metrics. We give a notion of parallel transport along this Brownian motion, and establish a generalization of the Dohrn–Guerra or damped parallel transport, Bismut integration by part formulas, and gradient estimate formulas. One of our main results is a characterization of the Ricci flow in terms of the damped parallel transport. At the end of the paper we give a canonical definition of the damped parallel transport in terms of stochastic flows, and derive an intrinsic martingale which may provide information about singularities of the flow.

Résumé. Nous généralisons la notion de mouvement brownien sur une variété au cas du mouvement brownien dépendant d'une famille de métriques. Cette généralisation est naturelle quand on s'intéresse aux équations de la chaleur avec un laplacien qui dépend du temps, ou de manière générale dans le cadre de diffusions in-homogènes. Dans cet article, nous nous sommes particulièrement intéressés au flot de Ricci, flot géométrique fournissant une famille intrinsèque de métriques. Nous donnons une notion de transport parallèle le long d'un tel processus, puis nous généralisons celle du transport parallèle déformé, et donnons une formule d'intégration par parties à la Bismut dont nous tirons des formules de contrôle de norme de gradients de solutions d'équation de la chaleur in-homogène. Un des résultats principaux de cet article est une caractérisation probabiliste du flot de Ricci, en terme du transport parallèle déformé. Dans les dernières sections, nous donnons une définition canonique du transport parallèle déformé en utilisant le flot stochastique, et nous en dérivons une martingale intrinsèque, qui pourrait donner des informations sur les singularités du flot.

1. $g(t)$-Brownian motion

Let M be a compact connected n-dimensional manifold which carries a family of time-dependent Riemannian metrics $g(t)$. In this section we will give a generalization of the well known Brownian motion on M which will depend on the family of metrics. In other words, it will depend on the deformation of the manifold. Such a family of metrics comes naturally from geometric flows, like mean curvature flow or Ricci flow. The compactness assumption for the manifold is not essential. Let ∇^t be the Levi-Civita connection associated to the metric $g(t)$, Δ_t the associated Laplace–Beltrami operator. Let also $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying ordinary assumptions like right continuity and W be a $\mathbb{R}^n$-valued Brownian motion for this probability space.

Definition 1.1. *Let us take $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ and a $C^{1,2}$-family $g(t)_{t\in[0,T[}$ of Riemannian metrics on M. An M-valued process $X(x)$ defined on $\Omega \times [0,T[$ is called a $g(t)$-Brownian motion in M started at $x \in M$ if $X(x)$ is continuous, adapted, and if for every smooth function f,*

$$f\big(X_s(x)\big) - f(x) - \frac{1}{2}\int_0^s \Delta_t f\big(X_t(x)\big)\,\mathrm{d}t$$

is a local martingale.

We shall prove existence of this inhomogeneous diffusion and give a notion of parallel transport along this process.

Let $(e_i)_{i\in[1,\dots,d]}$ be an orthonormal basis of $\mathbb{R}^n$, $\mathcal{F}(M)$ the frame bundle over M, π the projection to M. For any $u \in \mathcal{F}(M)$, let $L_i(t,u) = h^t(ue_i)$ be the ∇^t horizontal lift of ue_i and $L_i(t)$ the associated vector field. Further let $V_{\alpha,\beta}$ be the canonical basis of vertical vector fields on $\mathcal{F}(M)$ defined by $V_{\alpha,\beta}(u) = Dl_u(E_{\alpha,\beta})$ where $E_{\alpha,\beta}$ is the canonical basis of $\mathcal{M}_n(\mathbb{R})$ and where

$$l_u : \mathrm{GL}_n(\mathbb{R}) \to \mathcal{F}(M)$$

is the left multiplication. Finally let $(\mathcal{O}(M), g(t))$ be the $g(t)$ orthonormal frame bundle.

Proposition 1.2. *Assume that $g(t)_{t\in[0,T[}$ is a $C^{1,2}(t,x)$-family of metrics over M, and*

$$A : [0,T[\times \mathcal{F}(M) \to \mathcal{M}_n(\mathbb{R}),$$
$$(t,U) \mapsto \big(A_{\alpha,\beta}(t,U)\big)_{\alpha,\beta}$$

is locally Lipschitz in U uniformly for each $[0,t]$ in $[0,T[$. Consider the Stratonovich differential equation in $\mathcal{F}(M)$:

$$\begin{cases} *\mathrm{d}U_t = \sum_{i=1}^n L_i(t,U_t) * \mathrm{d}W^i + \sum_{\alpha,\beta} A_{\alpha,\beta}(t,U_t) V_{\alpha,\beta}(U_t)\,\mathrm{d}t, \\ U_0 \in \mathcal{F}(M) \quad \text{such that} \quad U_0 \in \big(\mathcal{O}(M), g(0)\big). \end{cases} \tag{1.1}$$

Then there is a unique symmetric choice for A such that $U_t \in (\mathcal{O}(M), g(t))$. Moreover:

$$A(t,U) = -\frac{1}{2}\,\partial_1 G(t,U),$$

where $(\partial_1 G(t,U))_{i,j} = \langle Ue_i, Ue_j\rangle_{\partial_t g(t)}$.

Proof. Let us begin with curves. Let I be a real interval, $\pi : TM \to M$ the projection, V and C in $C^1(I, TM)$, two curves such that

$$x(t) := \pi\big(V(t)\big) = \pi\big(C(t)\big) \quad \text{for all } t \in I.$$

We want to compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\langle V(t), C(t)\rangle_{g(t,x(t))}\big)\Big|_{t=0}.$$

We write $\partial_1 g(t,x)$ for $\partial_s g(s,x)$ evaluated at t. Let us express the metric $g(t)$ in a coordinate system; without loss of generality we can differentiate at time 0. Let $(x^1,\dots,x^n)$ be a coordinate system at the point $x(0)$, in which we have:

$$V(t) = v^i(t)\,\partial_{x^i},$$
$$C(t) = c^i(t)\,\partial_{x^i},$$
$$g\big(t,x(t)\big) = g_{i,j}\big(t,x(t)\big)\,\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

In these local coordinates we get:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle V(t), C(t)\rangle_{g(t,x(t))}\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} g_{i,j}\big(t, x(t)\big)v^i(t)c^j(t)\Big|_{t=0} \\
&= (\partial_1 g_{i,j}(0,x)v^i(0)c^j(0) + \frac{\mathrm{d}}{\mathrm{d}t}\big(g_{i,j}\big(0,x(t)\big)v^i(t)c^j(t)\big)\Big|_{t=0} \\
&= \partial_1 g_{i,j}(0,x)v^i(0)c^j(0) + \big\langle \nabla^0_{\dot{x}(0)}V(0), C(0)\big\rangle_{g(0,x(0))} + \big\langle V(0), \nabla^0_{\dot{x}(0)}C(0)\big\rangle_{g(0,x(0))} \\
&= \big\langle V(0), C(0)\big\rangle_{\partial_1 g(0,x(0)} + \big\langle \nabla^0_{\dot{x}(0)}V(0), C(0)\big\rangle_{g(0,x(0))} + \big\langle V(0), \nabla^0_{\dot{x}(0)}C(0)\big\rangle_{g(0,x(0))}.
\end{aligned}$$

In order to compute the $g(t)$ norm of a tangent valued process we will use what Malliavin calls “the transfer principle,” as explained in [12,13].

Recall the equivalence between a given connection on a manifold M and a splitting on TTM, i.e. $TTM = H^{\nabla}TTM \oplus VTTM$ [19]. We have a bijection:

$$\begin{aligned}
&\mathcal{V}_v : T_{\pi(v)}M \longrightarrow V_vTTM, \\
&u \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}(v+tu)\Big|_{t=0}.
\end{aligned}$$

For $X, Y \in \Gamma(TM)$ we have:

$$\nabla_X Y(x) = \mathcal{V}^{-1}_{X(x)}\big(\big(\mathrm{d}Y(x)\big(X(x)\big)\big)^v\big),$$

where $(\cdot)^v$ is the projection of a vector in TTM onto the vertical subspace $VTTM$ parallel to $H^{\nabla}TTM$.

For a $T(M)$-valued process T_t, we define:

$$D^{S,t}T_t = (\mathcal{V}_{T_t})^{-1}\big((*\mathrm{d}T_t)^{v,t}\big), \tag{1.2}$$

where $(\cdot)^{v,t}$ is defined as before but for the connection ∇^t. The above generalization makes sense for a tangent valued process coming from a Stratonovich equation like U_te_i, where U_t is a solution of the Stratonovich differential equation (1.1).

For the solution U_t of Eq. (1.1) we get

$$\begin{aligned}
\mathrm{d}\big(\langle U_te_i, U_te_j\rangle_{g(t,\pi(U_t))}\big) &= \langle U_te_i, U_te_j\rangle_{\partial_1 g(t,\pi(U_t))}\,\mathrm{d}t \\
&\quad + \big\langle D^{S,t}U_te_i, U_te_j\big\rangle_{g(t,\pi(U_t))} + \big\langle U_te_i, D^{S,t}U_te_j\big\rangle_{g(t,\pi(U_t))}.
\end{aligned} \tag{1.3}$$

We would like to find a symmetric A such that the left hand side of the above equation vanishes for all time (i.e. $U_t \in (\mathcal{O}(M), g(t))$). Denote by $\mathrm{ev}_{e_i} : \mathcal{F}(M) \to TM$ the ordinary evaluation, and $\mathrm{d\,ev}_{e_i} : T\mathcal{F}(M) \to TTM$ its differential. It is easy to see that $\mathrm{d\,ev}_{e_i}$ sends $VT\mathcal{F}(M)$ to $VTTM$ and sends $H^{\nabla^h}T\mathcal{F}(M)$ to $H^{\nabla}TTM$. We obtain:

$$D^{S,t}U_te_i = \sum_{\alpha=1}^{n} A_{\alpha,i}(t, U_t)U_te_\alpha\,\mathrm{d}t. \tag{1.4}$$

For simplicity, we take for notation:

$$\big(\partial_1 G(t,U)\big)_{i,j} = \langle Ue_i, Ue_j\rangle_{\partial_t g(t)}$$

and

$$\big(G(t,U)\big)_{i,j} = \langle Ue_i, Ue_j\rangle_{g(t)}.$$

It is now easy to find the condition for A:

$$\big(G(t,U_t)A(t,U_t)\big)_{j,i} + \big(G(t,U_t)A(t,U_t)\big)_{i,j} = -\big(\partial_1 G(t,U_t)\big)_{i,j}. \tag{1.5}$$

Given orthogonality $G(t,U_t) = \mathrm{Id}$ and hence by Eq. (1.5) A differs from $-\frac{1}{2}\partial_1 G$ by a skew symmetric matrix, therefore will be equal to it if we demand symmetry. Conversely if $A = -\frac{1}{2}\partial_1 G$ then by Eqs (1.3) and (1.2) we see $G(t,U_t) = \mathrm{Id}$. □

Remark 1.3. *The SDE in Proposition 1.2 does not explode because on any compact time interval all coefficients and their derivatives up to order 2 in space and order 1 in time are bounded.*

Remark 1.4. *The condition of symmetry is linked to a good definition of parallel transport with moving metrics in some sense. To see where the condition of symmetry comes from we may observe what happens in the constant metric case. It is easy to see that the usual definition of parallel transport along a semi-martingale which depends on the vanishing of the Stratonovich integral of the connection form, is equivalent to isometry and the symmetry condition for the drift in the following SDE in $\mathcal{F}(M)$:*

$$\begin{cases} \mathrm{d}\tilde{U}_t = \sum_{i=1}^{d} L_i(\tilde{U}_t) * \mathrm{d}W^i + A(\tilde{U}_t)_{\alpha,\beta} V_{\alpha,\beta}(\tilde{U}_t)\,\mathrm{d}t, \\ \tilde{U}_0 \in \big(\mathcal{O}(M), g\big), \\ \tilde{U}_t \in \big(\mathcal{O}(M), g\big) \quad (\textit{isometry}), \\ A(\cdot,\cdot)_{\alpha,\beta} \in S(n) \quad (\textit{vertical evolution}). \end{cases}$$

Isometry of U_t forces A to be skew symmetric, see Eq. (1.5). An assumption of symmetry on A then forces $A = 0$. We then get the usual stochastic differential equation of the parallel transport in the constant metric case.

The next proposition is a direct adaptation of a proposition in [15], p. 42; hence the proof is omitted.

Proposition 1.5. *Let $\alpha \in \Gamma(T^*M)$ and $F_\alpha : \mathcal{F}(M) \to \mathbb{R}^d$, $F_\alpha^i(u) = \alpha_{\pi(u)}(ue_i)$ its scalarization. Then, for all $A \in \Gamma(TM)$,*

$$(\nabla_A \alpha)_{\pi(u)}(ue_i) = h(A_{\pi(u)}) F_\alpha^i.$$

Consequently, for all $u \in \mathcal{F}(M)$,

$$\big(\nabla_A^{g(t)} \mathrm{d}f\big)_{\pi(u)}(ue_i) = h^{g(t)}(A_{\pi(u)}) F_{\mathrm{d}f}^i,$$

and for $f \in C^\infty(M)$,

$$L_i(t)(f \circ \pi)(u) = \mathrm{d}(f \circ \pi) L_i(t,u) = F_{\mathrm{d}f}^i(u).$$

Thus we have the formula:

$$\begin{aligned} L_i(t)L_j(t)(f \circ \pi)(u) &= h^{g(t)}(ue_i) F_{\mathrm{d}f}^j \\ &= \big(\nabla_{ue_i}^{g(t)} \mathrm{d}f\big)(ue_j) = \nabla^{g(t)} \mathrm{d}f(ue_i, ue_j). \end{aligned}$$

Proposition 1.6. *Take $x \in M$ and the SDE in $\mathcal{F}(M)$:*

$$\begin{cases} *\mathrm{d}U_t = \sum_{i=1}^{n} L_i(t,U_t) * \mathrm{d}W^i - \frac{1}{2}\partial_1 G(t,U_t)_{\alpha,\beta} V_{\alpha,\beta}(U_t)\,\mathrm{d}t, \\ U_0 \in \mathcal{F}(M) \quad \textit{such that} \quad U_0 \in \big(\mathcal{O}_x(M), g(0)\big). \end{cases} \tag{1.6}$$

Then $X_t(x) = \pi(U_t)$ is a $g(t)$-Brownian motion, which we write $g(t)$-$\mathrm{BM}(x)$.

Proof. For $f \in C^\infty(M)$,

$$\begin{aligned} \mathrm{d}(f \circ \pi \circ U_t) &= \sum_{i=1}^n L_i(t)(f \circ \pi)(U_t) * \mathrm{d}W^i \\ &= \sum_{i=1}^n L_i(t)(f \circ \pi)(U_t)\,\mathrm{d}W^i + \frac{1}{2}\sum_{i,j=1}^n L_i(t)L_j(t)(f \circ \pi)\,\mathrm{d}W^i\,\mathrm{d}W^j \\ &\overset{\mathrm{d}\mathcal{M}}{\equiv} \frac{1}{2}\sum_{i=1}^n \nabla^{g(t)}\mathrm{d}f(U_t e_i, U_t e_i)\,\mathrm{d}t \\ &\overset{\mathrm{d}\mathcal{M}}{\equiv} \frac{1}{2}\Delta_t f(\pi \circ U_t)\,\mathrm{d}t. \end{aligned}$$

(Here we write $\overset{\mathrm{d}\mathcal{M}}{\equiv}$ to denote the equality modulo differentials of local martingales.) The last equality comes from the fact that $U_t \in (\mathcal{O}(M), g(t))$. □

Remark 1.7. *Recall that in the compact case the lifetime of Eq.* (1.6) *is deterministic and the same as the lifetime of the metrics family.*

Let U_t be the solution of Eq. (1.6). We will write $//_{0,t} = U_t \circ U_0^{-1}$ for the $g(t)$ parallel transport over a $g(t)$-Brownian motion (we call it parallel transport because it is a natural extension of the usual parallel transport in the constant metric case). As usual it is an isometry:

$$//_{0,t} : \big(T_{X_0}M, g(0)\big) \to \big(T_{X_t}M, g(t)\big).$$

We also get a development formula. Take an orthonormal basis $(v_1, \dots, v_n)$ of $(T_{X_0}M, g(0))$, and $X_t(x)$ a $g(t)$-Brownian motion of Proposition 1.6; then

$$* \mathrm{d}X_t(x) = //_{0,t} v_i * \mathrm{d}W_t^i.$$

For $f \in C^2(M)$ we get the Itô formula:

$$\mathrm{d}f\big(X_t(x)\big) = \big\langle \nabla^t f, //_{0,t} v_i \big\rangle_t \,\mathrm{d}W^i + \frac{1}{2}\Delta_t(f)\big(X_t(x)\big)\,\mathrm{d}t. \tag{1.7}$$

We will now give examples of $g(t)$-Brownian motions. Let $(S^n, g(0))$ be a sphere and the solution of the Ricci flow:

$$\frac{\partial}{\partial_t} g(t) = -2\,\mathrm{Ric}_t$$

that is $g(t) = (1 - 2(n-1)t)g(0)$ with explosion time $T_c = \frac{1}{2(n-1)}$. We will use the fact that all metrics are conformal to the initial metric to express the $g(t)$-Brownian motion in terms of the $g(0)$-Brownian motion. Let $f \in C^2(S^n)$, $X_t(x)$ be a $g(t)$-Brownian motion starting at $x \in S^n$. Then, for some real-valued Brownian motion B_t, and $\mathbb{B}_t(x)$ a S^n-valued $g(0)$-Brownian motion:

$$\mathrm{d}f\big(X_t(x)\big) = \big\|\nabla^t f\big(X_t(x)\big)\big\|_{g(t)}\,\mathrm{d}B_t + \frac{1}{2}\left(\frac{1}{1 - 2(n-1)t}\right)\Delta_0 f\big(X_t(x)\big)\,\mathrm{d}t.$$

We have:

$$\|\nabla^t f\|_{g(t)}^2 = \frac{1}{1 - 2(n-1)t}\|\nabla^0 f\|_0^2.$$

Let

$$\tau(t)=\int_0^t \frac{1}{1-2(n-1)s}\,\mathrm{d}s,$$

then

$$\tau(t)=\frac{\ln(1-2(n-1)t)}{-2(n-1)},\qquad \tau^{-1}(t)=\frac{\mathrm{e}^{-2(n-1)t}-1}{-2(n-1)}.$$

We have the equality in law:

$$\big(X_\cdot(x)\big)\overset{\mathcal{L}}{=}\big(\mathbb{B}_{\tau(\cdot)}(x)\big).$$

We have a similar result for the hyperbolic case: Let $(H^n(-1),g(0))$ be the hyperbolic space with constant curvature -1. Then $g(t)=(1+2(n-1)t)g(0)$ is the solution of the Ricci flow. Let $X_t(x)$ be a $g(t)$-Brownian motion starting at $x\in H^n$, and $\mathbb{B}_t(x)$ an H^n-valued $g(0)$-Brownian motion. Then:

$$\tau(t)=\int_0^t \frac{1}{1+2(n-1)s}\,\mathrm{d}s,$$

and in law:

$$\big(X_\cdot(x)\big)\overset{\mathcal{L}}{=}\big(\mathbb{B}_{\tau(\cdot)}(x)\big).$$

Let us look at what happens for some limit of the Ricci flow, the so called Hamilton cigar manifold [5]. Let on $\mathbb{R}^2$, $g(0,x)=\frac{1}{1+\|x\|^2}g_{\mathrm{can}}$ be the Hamilton cigar, where $\|\cdot\|$ is the Euclidean norm. Then the solution to the Ricci flow is given by

$$g(t,x)=\frac{1+\|x\|^2}{\mathrm{e}^{4t}+\|x\|^2}g(0,x).$$

Let $f\in C^2(\mathbb{R}^2)$, $X_t(x)$ be a $g(t)$-Brownian motion starting at $x\in\mathbb{R}^2$. Then, for some real-valued Brownian motion B_t, and $\mathbb{B}_t(x)$ some $\mathbb{R}^2$-valued $g(0)$-Brownian motion:

$$\mathrm{d}f\big(X_t(x)\big)=\big\|\nabla^t f\big(X_t(x)\big)\big\|_{g(t)}\mathrm{d}B_t+\frac{1}{2}\frac{\mathrm{e}^{4t}+\|X_t(x)\|^2}{1+\|X_t(x)\|^2}\Delta_0 f\big(X_t(x)\big)\,\mathrm{d}t.$$

We have:

$$\nabla^t f(x)=\frac{\mathrm{e}^{4t}+\|x\|^2}{1+\|x\|^2}\nabla^0 f(x),$$

$$\big\|\nabla^t f(x)\big\|_t^2=\frac{\mathrm{e}^{4t}+\|x\|^2}{1+\|x\|^2}\big\|\nabla^0 f(x)\big\|_0^2,$$

$$\Delta_t f=\frac{\mathrm{e}^{4t}+\|x\|^2}{1+\|x\|^2}\Delta_0 f.$$

We set:

$$\tau(t)=\int_0^t \frac{\mathrm{e}^{4s}+\|X_s(x)\|^2}{1+\|X_s(x)\|^2}\,\mathrm{d}s.$$

Then in law:

$$\big(X_\cdot(x)\big)\overset{\mathcal{L}}{=}\big(\mathbb{B}_{\tau(\cdot)}(x)\big).$$

Remark 1.8. *If $X_t(x)$ is a $g(t)$-Brownian motion associated to a Ricci flow started at $g(0)$ then $X_{t/c}(x)$ is a $cg(t/c)$-Brownian motion associated to a Ricci flow started at $cg(0)$, so it is compatible with the blow up.*

2. Local expression, evolution equation for the density, conjugate heat equation

We begin this section by expressing a $g(t)$-Brownian motion in local coordinates.

Proposition 2.1. *Let $x \in M$, $(x^1, \ldots, x^n)$ be local coordinates around x, and $X_t(x)$ a $g(t)$-Brownian motion. Before the exit time of the domain of coordinates, we have*:

$$dX_t^i(x) = \sqrt{g(t)^{i,j}}\, dB^j - \frac{1}{2} g^{k,l} \Gamma_{kl}^i\big(t, X_t(x)\big)\, dt,$$

where we denote by $\sqrt{g(t)^{i,j}} := \sqrt{g(t, X_t(x))^{i,j}}$ the unique positive square root of the inverse to the matrix $(g(t, \partial_{x^i}, \partial_{x^j}))_{i,j}(X_t(x))$. Here $\Gamma_{kl}^i(t, X_t(x))$ are the Christoffel symbols associated to $\nabla^{g(t)}$, and B^i are n independent Brownian motion.

Proof. From the Itô equation (1.7), we get:

$$dX_t^i(x) = \langle \nabla^t x^i, //_{0,t} v_l \rangle_{g(t)} dW^l + \frac{1}{2} \Delta_t x^i\big(X_t(x)\big)\, dt,$$

where $(v_1, \ldots, v_n)$ is a $g(0)$-orthogonal basis of $T_x M$. By the usual expression of the Laplacian in coordinates:

$$\Delta_t x^i\big(X_t(x)\big) = -g^{l,k} \Gamma_{kl}^i\big(t, X_t(x)\big),$$

and the gradient expression of the coordinates functions:

$$\nabla^t x_i = g(t)^{i,j} \frac{\partial}{\partial x_j},$$

we have:

$$\begin{aligned} dX_t^i(x) &= g(t)^{i,j} \left\langle \frac{\partial}{\partial x_j}, //_{0,t} v_l \right\rangle_{g(t)} dW^l - \frac{1}{2} g^{l,k} \Gamma_{kl}^i\big(t, X_t(x)\big)\, dt \\ &= \sum_m \sqrt{g(t)^{i,m}} \left\langle \sqrt{g(t)^{m,j}} \frac{\partial}{\partial x_j}, //_{0,t} v_l \right\rangle_{g(t)} dW^l - \frac{1}{2} g^{l,k} \Gamma_{kl}^i\big(t, X_t(x)\big)\, dt \\ &= \sqrt{g(t)^{i,m}}\, dB^m - \frac{1}{2} g^{l,k} \Gamma_{kl}^i\big(t, X_t(x)\big)\, dt, \end{aligned}$$

where $dB^m = \langle \sqrt{g(t)^{m,j}} \frac{\partial}{\partial x_j}, //_{0,t} v_l \rangle_{g(t)} dW^l$. By the isometry of the parallel transport and Lévy's Theorem $B = (B^1, \ldots, B^n)$ is a Brownian motion in $\mathbb{R}^n$. □

Remark 2.2. *The above equation is similar to the equation in the fixed metric case.*

Now we shall study the evolution equation for the density of the law of the $g(t)$-Brownian motion. Let $X_t(x)$ be a $g(t)$-BM(x), and $d\mu_t$ the Lebesgue measure on $(M, g(t))$. Since $X_t(x)$ is a diffusion with generator Δ_t, we have smoothness of the density (e.g. [22]). Let $h^x(t, y) \in C^\infty(]0, T[\times M)$ be such that:

$$\begin{cases} X_t(x) \overset{\mathcal{L}}{=} h^x(t, y)\, d\mu_t(y), & t > 0, \\ X_0(x) \overset{\mathcal{L}}{=} \delta_x. \end{cases}$$

By the continuity of $X_t(x)$ and the dominated convergence theorem we get the convergence in law:

$$\mathcal{L}\text{-}\lim_{t\downarrow 0} X_t(x) = \delta_x.$$

We write the expression of $\mathrm{d}\mu_t$ in terms of $\mathrm{d}\mu_0$ in a local chart, i.e.,

$$\mathrm{d}\mu_t = \frac{\sqrt{\det(g_{i,j}(t))}}{\sqrt{\det(g_{i,j}(0))}}\sqrt{\det\big(g_{i,j}(0)\big)}\big|\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n\big|$$

and set:

$$\mu_t(\mathrm{d}y) = \psi(t, y)\mu_0(\mathrm{d}y).$$

Proposition 2.3.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(h^x(t, y)\big) + h^x(t, y)\,\mathrm{Tr}\big(\frac{1}{2}\big(g^{-1}(t, y)\big)\frac{\mathrm{d}}{\mathrm{d}t}g(t, y)\big) = \frac{1}{2}\Delta_{g(t)}h^x(t, y), \\ \mathcal{L}\text{-}\lim_{t\downarrow 0} h^x(t, y)\,\mathrm{d}\mu_t = \delta_x. \end{cases}$$

Proof. For $f \in C^\infty(M)$, $t > 0$, by definition of $X_t(x)$ we have:

$$\mathbb{E}\big[f\big(X_t(x)\big)\big] - f(x) = \frac{1}{2}\mathbb{E}\bigg[\int_0^t \Delta_{g(s)} f\big(X_s(x)\big)\,\mathrm{d}s\bigg],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[f\big(X_t(x)\big)\big] = \frac{1}{2}\mathbb{E}\big[\Delta_{g(t)} f\big(X_t(x)\big)\big],$$

i.e.,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_M h^x(t, y) f(y)\mu_t(\mathrm{d}y) &= \frac{1}{2}\int_M \Delta_{g(t)} f(y) h^x(t, y)\mu_t(\mathrm{d}y) \\ &= \frac{1}{2}\int_M f(y)\Delta_{g(t)} h^x(t, y)\mu_t(\mathrm{d}y). \end{aligned}$$

The last equality comes from Green's Theorem and the compactness of the manifold. By setting $\mu_t(\mathrm{d}y) = \psi(t, y) \times \mu_0(\mathrm{d}y)$, we have:

$$\int_M f(y)\frac{\mathrm{d}}{\mathrm{d}t}\big(h^x(t, y)\psi(t, y)\big)\mu_0(\mathrm{d}y) = \frac{1}{2}\int_M f(y)\big(\Delta_{g(t)}h^x(t, y)\big)\psi(t, y)\mu_0(\mathrm{d}y),$$

and hence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(h^x(t, y)\psi(t, y)\big) = \frac{1}{2}\big(\Delta_{g(t)}h^x(t, y)\big)\psi(t, y). \tag{2.1}$$

We also have by determinant differentiation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, y) &= \frac{1}{2\sqrt{\det(g_{i,j}(0))}}\frac{1}{\sqrt{\det(g_{i,j}(t))}}\det\big(g_{i,j}(t)\big)\,\mathrm{Tr}\bigg(g^{-1}(t, y)\frac{\mathrm{d}}{\mathrm{d}t}g(t, y)\bigg) \\ &= \frac{1}{2}\psi(t, y)\,\mathrm{Tr}\bigg(g^{-1}(t, y)\frac{\mathrm{d}}{\mathrm{d}t}g(t, y)\bigg). \end{aligned}$$

Note that the part $\mathrm{Tr}(\frac{1}{2}g^{-1}(t, y)\frac{\mathrm{d}}{\mathrm{d}t}g(t, y))$ is intrinsic, it does not depend on the choice of the chart. Hence Eq. (2.1) gives the following inhomogeneous reaction-diffusion equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(h^x(t, y)\big) + h^x(t, y)\,\mathrm{Tr}\bigg(\frac{1}{2}g^{-1}(t, y)\frac{\mathrm{d}}{\mathrm{d}t}g(t, y)\bigg) = \frac{1}{2}\Delta_{g(t)}h^x(t, y).$$

□

We will give as example the evolution equation of the density in the case where the family of metrics comes from the forward (and resp. backward) Ricci flow. From now on Ricci flow will mean (probabilistic convention):

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{i,j} = -\operatorname{Ric}_{i,j}, \tag{2.2}$$

(respectively)

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{i,j} = \operatorname{Ric}_{i,j}. \tag{2.3}$$

Remark 2.4. *Hamilton* [14], *and later DeTurck* [7] *have shown existence in small times of such flows. In this section we don't care about the real existence time.*

For $x \in M$, we will denote by $S(t,x)$ the scalar curvature at the point x for the metric $g(t)$.

Corollary 2.5. *For the backward Ricci flow* (2.3), *we have*:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(h^x(t,y)\big) + \frac{1}{2} h^x(t,y) S(t,y) = \frac{1}{2}\Delta_{g(t)} h^x(t,y), \\ \mathcal{L}\text{-}\lim\limits_{t\downarrow 0} h^x(t,y)\,\mathrm{d}\mu_t = \delta_x. \end{cases}$$

For the forward Ricci flow (2.2), *we have*:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(h^x(t,y)\big) - \frac{1}{2} h^x(t,y) S(t,y) = \frac{1}{2}\Delta_{g(t)} h^x(t,y), \\ \mathcal{L}\text{-}\lim\limits_{t\downarrow 0} h^x(t,y)\,\mathrm{d}\mu_t = \delta_x. \end{cases}$$

Remark 2.6. *These equations are conservative. This is not the case for the ordinary heat equation with time depending Laplacian i.e.* $\Delta_{g(t)}$. *They are conjugate heat equations which are well known in the Ricci flow theory* (*e.g.* [24]).

3. Damped parallel transport, and Bismut formula for Ricci flow, applications to Ricci flow for surfaces

In this section, we will be interested in the heat equation under the Ricci flow. The principal fact is that under forward Ricci flow, the damped parallel transport or Dohrn–Guerra transport is the parallel transport defined before. The deformation of geometry under the Ricci flow compensates for the deformation of the parallel transport (i.e. the Ricci term in the usual formula for the damped parallel transport in constant metric case, see [8,9,23]). The isometry property of the damped parallel transport turns out to be an advantage for computations. In particular, for gradient estimate formulas, everything looks like the case of a Ricci flat manifold with constant metric. We begin with a general result independent of the fact that the flow is a Ricci flow. Let $g(t)_{[0,T_c[}$ be a $C^{1,2}$ family of metrics, and consider the heat equation:

$$\begin{cases} \partial_t f(t,x) = \frac{1}{2}\Delta_t f(t,x), \\ f(0,x) = f_0(x), \end{cases} \tag{3.1}$$

where f_0 is a function on M. We suppose that the solution of Eq. (3.1) exists until T_c. For $T < T_c$, let X_t^T be a $g(T-t)$-Brownian motion, $//_{0,t}^T$ the associated parallel transport.

Let $S \in \Gamma(T^*M \otimes T^*M)$ a 2-covariant tensor, g a metric on M and $v \in T_xM$, we will write $S^{\#g}(v)$ for the tangent vector in T_xM such that, for all $u \in T_xM$ we have

$$S(v,u) = \big\langle S^{\#g}(v), u\big\rangle_g.$$

Definition 3.1. *We define the damped parallel transport* $\mathbf{W}_{0,t}^T$ *as the solution of*:

$$*\mathrm{d}\big(\big(//_{0,t}^T\big)^{-1}\big(\mathbf{W}_{0,t}^T\big)\big) = -\frac{1}{2}\big(//_{0,t}^T\big)^{-1}\big(\operatorname{Ric}_{g(T-t)} - \partial_t\big(g(T-t)\big)\big)^{\#g(T-t)}\big(\mathbf{W}_{0,t}^T\big)\,\mathrm{d}t$$

with

$$\mathbf{W}^T_{0,t} : T_x M \longrightarrow T_{X^T_t(x)} M, \qquad \mathbf{W}^T_{0,0} = \mathrm{Id}_{T_x M}.$$

Theorem 3.2. *For every solution $f(t,\cdot)$ of Eq. (3.1), and for all $v \in T_x M$,*

$$\mathrm{d} f(T-t,\cdot)_{X^T_t(x)}\big(\mathbf{W}^T_{0,t} v\big)$$

is a local martingale.

Proof. Recall the equation of the parallel transport along the $g(T-t)$-Brownian motion $X^T_t(x)$:

$$\begin{cases} *\mathrm{d}U^T_t = \sum_{i=1}^d L_i\big(T-t, U^T_t\big) * \mathrm{d}W^i - \frac{1}{2}\,\partial_t\big(g(T-t)\big)\big(U^T_t e_\alpha, U^T_t e_\beta\big) V_{\alpha,\beta}\big(U^T_t\big)\,\mathrm{d}t, \\ U^T_0 \in \big(\mathcal{O}_x(M), g(T)\big). \end{cases} \tag{3.2}$$

For $f \in \mathcal{C}^\infty(M)$, its scalarization:

$$\widetilde{\mathrm{d}f} : \mathcal{F}(M) \longrightarrow \mathbb{R}^n,$$
$$U \longmapsto \big(\mathrm{d}f(Ue_1), \ldots, \mathrm{d}f(Ue_n)\big),$$

yields the following formula in $\mathbb{R}^n$:

$$\mathrm{d}f(T-t,\cdot)_{X^T_t(x)}\big(\mathbf{W}^T_{0,t} v\big) = \big\langle \widetilde{\mathrm{d}f}\big(T-t, U^T_t\big), \big(U^T_t\big)^{-1} \mathbf{W}^T_{0,t} v\big\rangle_{\mathbb{R}^n}$$

for every $v \in T_x M$. To recall the notation let:

$$\mathrm{ev}_{e_i} : \mathcal{F}(M) \longrightarrow TM,$$
$$U \longmapsto Ue_i,$$

and recall that U^T_t, as solution of Eq. (3.2), is a diffusion associated to the generator

$$\frac{1}{2}\Delta^H_{T-t} - \frac{1}{2}\,\partial_t\big(g(T-t)\big)\big(\mathrm{ev}_{e_i}(\cdot), \mathrm{ev}_{e_j}(\cdot)\big) V_{i,j}(\cdot),$$

where Δ^H_{T-t} is the horizontal Laplacian in $\mathcal{F}(M)$, associated to the metric $g(T-t)$. In the Itô sense, we get:

$$\begin{aligned}
&\mathrm{d}\big(\mathrm{d}f(T-t,\cdot)_{X^T_t(x)}\big(\mathbf{W}^T_{0,t}\big)v\big) \\
&\quad = \mathrm{d}\big\langle \widetilde{\mathrm{d}f}\big(T-t, U^T_t\big), \big(U^T_t\big)^{-1}\mathbf{W}^T_{0,t} v\big\rangle_{\mathbb{R}^n} \\
&\quad \overset{\mathrm{d}\mathcal{M}}{\equiv} \bigg\langle -\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathrm{d}f}\bigg)(T-t,\cdot)\big(U^T_t\big)\,\mathrm{d}t + \bigg[\frac{1}{2}\Delta^H_{T-t}\widetilde{\mathrm{d}f}(T-t,\cdot) \\
&\qquad - \frac{1}{2}\,\partial_t\big(g(T-t)\big)\big(\mathrm{ev}_{e_i}(\cdot), \mathrm{ev}_{e_j}(\cdot)\big) V_{i,j}(\cdot)\,\widetilde{\mathrm{d}f}(T-t,\cdot)\bigg]\big(U^T_t\big)\,\mathrm{d}t, \big(U^T_t\big)^{-1}\mathbf{W}^T_{0,t} v\bigg\rangle_{\mathbb{R}^n} \\
&\qquad + \big\langle\big(\widetilde{\mathrm{d}f}\big(T-t, U^T_t\big)\big), \big(U^T_0\big)^{-1}\mathrm{d}\big(\big(//^T_{0,t}\big)^{-1}\big(\mathbf{W}^T_{0,t}\big)\big)v\big\rangle_{\mathbb{R}^n} \\
&\quad \overset{\mathrm{d}\mathcal{M}}{\equiv} -\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}f\bigg)(T-t,\cdot)\big(\mathbf{W}^T_{0,t} v\big)\,\mathrm{d}t + \bigg\langle\bigg[\frac{1}{2}\Delta^H_{T-t}\widetilde{\mathrm{d}f}(T-t,\cdot) \\
&\qquad - \frac{1}{2}\,\partial_t\big(g(T-t)\big)\big(\mathrm{ev}_{e_i}(\cdot), \mathrm{ev}_{e_j}(\cdot)\big) V_{i,j}(\cdot)\,\widetilde{\mathrm{d}f}(T-t,\cdot)\bigg]\big(U^T_t\big)\,\mathrm{d}t, \big(U^T_t\big)^{-1}\mathbf{W}^T_{0,t} v\bigg\rangle_{\mathbb{R}^n} \\
&\qquad - \frac{1}{2}\big\langle\big(\widetilde{\mathrm{d}f}\big(T-t, U^T_t\big)\big), \big(U^T_0\big)^{-1}\big(//^T_{0,t}\big)^{-1}\big(\mathrm{Ric}_{g(T-t)} - \partial_t\big(g(T-t)\big)\big)^{\#g(T-t)}\big(\mathbf{W}^T_{0,t}\big)v\,\mathrm{d}t\big\rangle_{\mathbb{R}^n}.
\end{aligned}$$

We shall make separate computations for each term in the previous equation. Using the well known formula (e.g. [15], p. 193)

$$\Delta^H \widetilde{\mathrm{d}f} = \widetilde{\Delta \mathrm{d}f},$$

we first note that:

$$\begin{aligned}\left\langle \frac{1}{2}\Delta^H_{T-t}\widetilde{\mathrm{d}f}(T-t,\cdot)\big(U_t^T\big), \big(U_t^T\big)^{-1}\mathbf{W}_{0,t}^T v\,\mathrm{d}t\right\rangle_{\mathbb{R}^n} &= \frac{1}{2}\left\langle \widetilde{\Delta_{T-t}\mathrm{d}f}(T-t,\cdot)\big(U_t^T\big), \big(U_t^T\big)^{-1}\mathbf{W}_{0,t}^T v\right\rangle_{\mathbb{R}^n}\mathrm{d}t \\ &= \frac{1}{2}\Delta_{T-t}\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t.\end{aligned}$$

By definition:

$$\begin{aligned} V_{i,j}\widetilde{\mathrm{d}f}(u) &= \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathrm{d}f}\big(u(\mathrm{Id}+tE_{ij})\big)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{d}f\big(u(\mathrm{Id}+tE_{ij})e_s\big)\big)_{s=1,\dots,n}\Big|_{t=0} \\ &= \big(\mathrm{d}f\big(u\delta_i^s e_j\big)\big)_{s=1,\dots,n} \\ &= \big(0,\dots,0,\mathrm{d}f(ue_j),0,\dots,0\big) \quad i\text{th position,}\end{aligned}$$

so that:

$$\begin{aligned} &\sum_{ij}\partial_t\big(g(T-t)\big)\big(\mathrm{ev}_{e_i}(\cdot),\mathrm{ev}_{e_j}(\cdot)\big)V_{i,j}(\cdot)\,\widetilde{\mathrm{d}f}(T-t,\cdot)\big(U_t^T\big)\,\mathrm{d}t \\ &\quad= \sum_{ij}\partial_t\big(g(T-t)\big)\big(U_t^T e_i, U_t^T e_j\big)\,\mathrm{d}f\big(U_t^T e_j\big)e_i\,\mathrm{d}t \\ &\quad= \left(\left\langle \nabla^{T-t}f(T-t,\cdot), \sum_j \partial_t\big(g(T-t)\big)\big(U_t^T e_i, U_t^T e_j\big)U_t^T e_j\right\rangle_{T-t}\mathrm{d}t\right)_{i=1,\dots,n} \\ &\quad= \big(\mathrm{d}f\big(T-t,\partial_t\big(g(T-t)\big)^{\#g(T-t)}\big(U_t^T e_i\big)\big)\,\mathrm{d}t\big)_{i=1,\dots,n}.\end{aligned}$$

Then

$$\begin{aligned} &\mathrm{d}\big(\mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(\mathbf{W}_{0,t}^T v\big)\big) \\ &\quad\overset{\mathrm{d}\mathcal{M}}{\equiv} -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}f(T-t,\cdot)((\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t \\ &\qquad - \frac{1}{2}\big\langle\big(\mathrm{d}f\big(T-t,\partial_t\big(g(T-t)\big)^{\#g(T-t)}\big(U_t^T e_i\big)\big)\big)_{i=1,\dots,n}, \big(U_t^T\big)^{-1}\mathbf{W}_{0,t}^T v\big\rangle_{\mathbb{R}^n}\,\mathrm{d}t \\ &\qquad + \frac{1}{2}\Delta_{T-t}\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t \\ &\qquad - \frac{1}{2}\big\langle\big(\widetilde{\mathrm{d}f}\big(T-t,U_t^T\big)\big), \big(U_0^T\big)^{-1}\big(//_{0,t}^T\big)^{-1}\big(\mathrm{Ric}_{g(T-t)} - \partial_t\big(g(T-t)\big)\big)^{\#g(T-t)}\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t\big\rangle_{\mathbb{R}^n}.\end{aligned}$$

By the fact that U_t^T is a $g(T-t)$-isometry we have:

$$\begin{aligned} &\big\langle\big(\mathrm{d}f\big(T-t,\partial_t\big(g(T-t)\big)^{\#g(T-t)}\big(U_t^T e_i\big)\big)\big)_{i=1,\dots,n}, \big(U_t^T\big)^{-1}\mathbf{W}_{0,t}^T v\big\rangle_{\mathbb{R}^n} \\ &\quad= \left\langle \sum_i \partial_t\big(g(T-t)\big)\big(U_t^T e_i, \nabla^{T-t}f(T-t,\cdot)\big)e_i, \big(U_t^T\big)^{-1}\mathbf{W}_{0,t}^T v\right\rangle_{\mathbb{R}^n}\end{aligned}$$

$$= \left\langle \sum_i \partial_t \big(g(T-t)\big)\big(U_t^T e_i, \nabla^{T-t} f(T-t,\cdot)\big) U_t^T e_i, \mathbf{W}_{0,t}^T v \right\rangle_{g(T-t)}$$

$$= \big\langle \partial_t \big(g(T-t)\big)^{\# g(T-t)} \big(\mathbf{W}_{0,t}^T v\big), \nabla^{T-t} f(T-t,\cdot) \big\rangle_{g(T-t)}.$$

Consequently:

$$\begin{aligned}
&\mathrm{d}\big(\mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(\mathbf{W}_{0,t}^T v\big)\big) \\
&\overset{\mathrm{d}\mathcal{M}}{\equiv} -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t \\
&\quad -\frac{1}{2}\big\langle \nabla^{T-t} f(T-t,\cdot), \partial_t\big(g(T-t)\big)^{\# g(T-t)}\big(\mathbf{W}_{0,t}^T v\big)\big\rangle_{T-t}\,\mathrm{d}t \\
&\quad +\frac{1}{2}\Delta_{T-t}\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t \\
&\quad -\frac{1}{2}\big\langle \big(\widetilde{\mathrm{d}f}\big(T-t,U_t^T\big)\big), \big(U_t^T\big)^{-1}\big(\mathrm{Ric}_{g(T-t)} - \partial_t\big(g(T-t)\big)\big)^{\# g(T-t)}\big(\mathbf{W}_{0,t}^T\big) v\,\mathrm{d}t\big\rangle_{\mathbb{R}^n} \\
&\overset{\mathrm{d}\mathcal{M}}{\equiv} -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t + \frac{1}{2}\Delta_{T-t}\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t \\
&\quad -\frac{1}{2}\mathrm{d}f\big(T-t, \mathrm{Ric}_{g(T-t)}^{\# g(T-t)}\big(\mathbf{W}_{0,t}^T v\big)\big)\,\mathrm{d}t.
\end{aligned}$$

But recall that f is a solution of

$$\frac{\partial}{\partial t} f = \frac{1}{2}\Delta_t f,$$

so that

$$-\frac{\partial}{\partial t}\,\mathrm{d}f(T-t,\cdot) = -\frac{1}{2}\,\mathrm{d}\Delta_{T-t} f(T-t,\cdot).$$

We shall use the Hodge–de Rham Laplacian $\square_{T-t} = -(\mathrm{d}\delta_{T-t} + \delta_{T-t}\,\mathrm{d})$ which commutes with the de Rham differential, and we shall use the well-known Weitzenböck formula [16,17], which says that for θ a 1-form:

$$\square_{T-t}\theta = \Delta_{T-t}\theta - \mathrm{Ric}_{g(T-t)}\,\theta.$$

Here by duality we write $\theta^{\# g}(x)$ the element of $T_x M$ such that for all $v \in T_x M$ $\langle \theta^{\# g}(x), v\rangle_g = \theta(v)$ and $\mathrm{Ric}_{g(T-t)}\,\theta$ the 1-form such that for all $v \in T_x M$

$$\mathrm{Ric}_{g(T-t)}\,\theta(v) := \mathrm{Ric}_{g(T-t)}\big(\theta^{\# g(T-t)}(x), v\big).$$

We get:

$$\begin{aligned}
\mathrm{d}\Delta_{T-t} f(T-t,\cdot) &= \mathrm{d}\square_{T-t} f(T-t,\cdot) \\
&= \square_{T-t}\mathrm{d}f(T-t,\cdot) \\
&= \Delta_{T-t}\mathrm{d}f(T-t,\cdot) - \mathrm{Ric}_{g(T-t)}\,\mathrm{d}f(T-t,\cdot).
\end{aligned}$$

Finally:

$$\begin{aligned}
&\mathrm{d}\big(\mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(\mathbf{W}_{0,t}^T v\big)\big) \\
&\overset{\mathrm{d}\mathcal{M}}{\equiv} \frac{1}{2}\mathrm{Ric}_{g(T-t)}\,\mathrm{d}f(T-t,\cdot)\big(\mathbf{W}_{0,t}^T v\big)\,\mathrm{d}t - \frac{1}{2}\big\langle \nabla^{T-t} f(T-t,\cdot), \mathrm{Ric}_{g(T-t)}^{\# g(T-t)}\big(\mathbf{W}_{0,t}^T v\big)\big\rangle_{T-t}\,\mathrm{d}t \\
&\overset{\mathrm{d}\mathcal{M}}{\equiv} 0.
\end{aligned}$$

□

Remark 3.3. *For the forward Ricci flow, we have*:

$$//_{0,t}^T * \mathrm{d}\big((//_{0,t}^T)^{-1}\mathbf{W}_{0,t}^T\big) = 0.$$

For the backward Ricci flow, we have:

$$//_{0,t}^T * \mathrm{d}\big((//_{0,t}^T)^{-1}\mathbf{W}_{0,t}^T\big) = -\operatorname{Ric}_{g(T-t)}^{\#g(T-t)}\big(\mathbf{W}_{0,t}^T\big)\,\mathrm{d}t.$$

When the family of metrics is constant, we have the usual damped parallel transport which satisfies:

$$//_{0,t} * \mathrm{d}\big((//_{0,t})^{-1}\mathbf{W}_{0,t}\big) = -\frac{1}{2}\operatorname{Ric}^{\#}\big(\mathbf{W}_{0,t}\big)\,\mathrm{d}t.$$

Remark 3.4. *Roughly speaking, the result says that the deformation of the metric under the forward Ricci flow makes the damped parallel transport behaves like the damped parallel transport in the case of a constant metric with flat Ricci curvature.*

For the heat equation under the forward Ricci flow, we take the probabilistic convention:

$$\begin{cases} \partial_t f(t,x) = \frac{1}{2}\Delta_t f(t,x), \\ \frac{\mathrm{d}}{\mathrm{d}t} g_{i,j} = -\operatorname{Ric}_{i,j}, \\ f(0,x) = f_0(x). \end{cases} \tag{3.3}$$

We shall give a Bismut type formula and a gradient estimate formula for the above equation. For notation, let T_c be the maximal life time of the forward Ricci flow $g(t)_{t\in[0,T_c[}$, solution of Eq. (2.2). For $T < T_c$, X_t^T is a $g(T-t)$-Brownian motion and $//_{0,t}^T$ the associated parallel transport. In this case, for a solution $f(t,\cdot)$ of Eq. (3.3), $f(T-t, X_t^T(x))$ is a local martingale for any $x \in M$. When going back in time, one has to remember all deformations of the geometry.

We now recall a well known Lemma giving a Bismut type formula (e.g. [10]). Let $f(t,\cdot)$ and $g(t)$ be solution of Eq. (3.3), $T < T_c$, and $X_t^T(x)$ a $g(T-t)$-Brownian motion.

Lemma 3.5. *For any $\mathbb{R}^n$-valued process k such that $k \in L^2_{loc}(W)$ where W is the $\mathbb{R}^n$-valued Brownian motion (that appeared in the construction of $X_t^T(x)$), and for all $v \in T_xM$,*

$$N_t = \mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(U_t^T\big)\left[\big(U_0^T\big)^{-1}v - \int_0^t k_r\,\mathrm{d}r\right] + f\big(T-t, X_t^T(x)\big)\int_0^t \langle k_r, \mathrm{d}W\rangle_{\mathbb{R}^n}$$

is a local martingale.

Proof. The first remark after Theorem 3.2 yields that the first term is a semi-martingale. By Itô calculus we get:

$$\mathrm{d}\big(f\big(T-t, X_t^T(x)\big)\big) = \mathrm{d}f(T-t,\cdot)_{X_t^T(x)} U_t e_i\,\mathrm{d}W^i.$$

With $(l_i)_{i=1,\dots,n}$ a $g(T)$-orthonormal frame of T_xM, we write N_t as:

$$N_t = \sum_i \big(\mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(U_t^T\big(U_0^T\big)^{-1}\big)l_i\big)\left(v_i - \int_0^t \big\langle U_0^T(k_r), l_i\big\rangle_T\,\mathrm{d}r\right) + f\big(T-t, X_t^T(x)\big)\int_0^t \langle k_r, \mathrm{d}W\rangle_{\mathbb{R}^n},$$

and with Theorem 3.2:

$$\begin{aligned} \mathrm{d}N_t \overset{\mathrm{d}\mathcal{M}}{\equiv} & \sum_i \big(\mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(U_t^T\big(U_0^T\big)^{-1}\big)l_i\big)\big(-\big\langle U_0^T(k_t), l_i\big\rangle_T\,\mathrm{d}t\big) \\ & + \mathrm{d}\big(f\big(T-t, X_t^T(x)\big)\big)\langle k_t, \mathrm{d}W\rangle_{\mathbb{R}^n} \end{aligned}$$

$$\stackrel{\mathrm{d}\mathcal{M}}{\equiv} \sum_i \big(\mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(U_t^T\big(U_0^T\big)^{-1}\big)l_i\big)\big(-\langle U_0^T(k_t), l_i\rangle_T\,\mathrm{d}t\big)$$
$$+ \sum_i \mathrm{d}f(T-t,\cdot)_{X_t^T(x)}\big(U_t^T l_i\big)\,\mathrm{d}W^i\Big(\sum_j k_t^j\,\mathrm{d}W^j\Big)$$
$$\stackrel{\mathrm{d}\mathcal{M}}{\equiv} 0. \qquad \square$$

Remark 3.6. *Since T is smaller than the explosion time T_c, and by the compactness of M, N_t is clearly a true martingale, so we could use the martingale property for global estimates, or Doob's optional sampling Theorem for local estimates (e.g. [23]).*

Corollary 3.7. *Let $v \in T_x M$, and take for example*

$$k_r = \frac{(U_0^T)^{-1}v}{T}\mathbb{1}_{[0,T]}(r).$$

Then we have:

$$\mathrm{d}f(T,\cdot)_x v = \frac{1}{T}\sum_i \mathbb{E}\big[f_0\big(X_T^T(x)\big)\big\langle\big(U_0^T\big)^{-1}v, e_i\big\rangle_{\mathbb{R}^n} W_i(T)\big].$$

Proof. With the above remark, N_t is a martingale. The choice of k_r gives $(U_0^T)^{-1}v - \int_0^T k_r\,\mathrm{d}r = 0$; the result follows by taking expectation at time 0 and T. $\square$

We can give the following estimate for the gradient of the solution of Eq. (3.3):

Corollary 3.8. *Let $\|f\|_\infty = \sup_M |f_0|$. For $T < T_c$:*

$$\sup_{x\in M}\big\|\nabla^T f(T,x)\big\|_T \text{ is decreasing in time}$$

and:

$$\sup_{x\in M}\big\|\nabla^T f(T,x)\big\|_T \le \frac{\|f\|_\infty}{\sqrt{T}}.$$

Proof. Take $x \in M$ such that $\|\nabla^T f(T,x)\|_T$ is maximal. Using the damped parallel transport, by Theorem 3.2 we obtain that for all $v \in T_x M$:

$$\mathrm{d}f\big(T-t, X_t^T(x)\big)\mathbf{W}_{0,t}^T v$$

is a local martingale. By compactness, this is a true martingale. Taking $v = \nabla^T f(T,x)$ and averaging the previous martingale at time 0 and t we get:

$$\big\|\nabla^T f(T,x)\big\|_T^2 = \mathbb{E}\big[\big\langle\nabla^{T-t} f\big(T-t, X_t^T(x)\big), \mathbf{W}_{0,t}^T v\big\rangle_{T-t}\big].$$

Using Theorem 3.2 and the fact that the family of metrics involves according to forward Ricci flow we obtain $\mathbf{W}_{0,t}^T = //_{0,t}^T$, hence the isometry property of $\mathbf{W}_{0,t}^T$, i.e.

$$\big\|\mathbf{W}_{0,t}^T v\big\|_{T-t} = \|v\|_T.$$

So we obtain the first result.

If we choose

$$k_r = \frac{(U_0^T)^{-1} v}{T} \mathbb{1}_{[0,T]}(r)$$

in Lemma 3.5, then N_t is a martingale. Taking expectations at times 0 and T, we obtain

$$\mathrm{d} f(T,\cdot)_x v = \frac{1}{T}\mathbb{E}\bigg[f_0\big(X_T^T(x)\big) \int_0^T \big\langle \big(U_0^T\big)^{-1} v, \mathrm{d}W\big\rangle_{\mathbb{R}^n} \bigg].$$

For $x \in M$ and $v = \nabla^T f(T,x)$, Schwartz inequality gives

$$\big\|\nabla^T f(T,x)\big\|_T^2 \le \frac{\|f\|_\infty}{T}\mathbb{E}\bigg[\bigg|\int_0^T \big\langle \big(U_0^T\big)^{-1} v, \mathrm{d}W\big\rangle_{\mathbb{R}^n}\bigg|^2\bigg]^{1/2}.$$

We have:

$$\mathbb{E}\bigg[\bigg|\int_0^T \big\langle \big(U_0^T\big)^{-1} v, \mathrm{d}W\big\rangle_{\mathbb{R}^n}\bigg|^2\bigg] = T\|v\|_T^2.$$

The result follows. □

For geometric interpretation, let us give an example of normalized Ricci flow for surfaces (which is completely understood, e.g. [5]). We are interested in this example because the equation for the scalar curvature under this flow is a reaction-diffusion equation which is quite similar to the heat equation under Ricci flow. We will give a gradient estimate formula for the scalar curvature under normalized Ricci flow which gives in the case $\chi(M) < 0$ (the easiest case) the convergence of the metric to a metric of constant curvature.

The normalized Ricci flow of surfaces comes from normalizing the metric by some time dependent function to preserve the volume. Let M be a 2-dimensional manifold, $R(t)$ the scalar curvature, $r = \int_M R_t \,\mathrm{d}\mu_t/\mu_t(M)$ its average (which will be constant in time, being a topological constant, e.g. by Gauss–Bonnet Theorem). We get the following equation for the normalized Ricci flow:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{i,j}(t) = \big(r - R(t)\big) g_{i,j}(t).$$

Remark 3.9. *Hamilton gives a proof of the existence of solutions to this equation*, *defined for all time* (*e.g.* [5]).

Recall that (e.g. [5]) the equation for the scalar curvature R under this normalised flow is:

$$\frac{\partial}{\partial t} R = \Delta_t R + R(R - r).$$

Proposition 3.10. *Let* $T \in \mathbb{R}$, $X_t^T(x)$ *be a* $\frac{1}{2}g(T-t)$-$\mathrm{BM}(x)$, $//_{0,t}^T$ *the parallel transport*, $v \in T_x M$ *and* $\varphi_t v$ *the solution of the following equation*:

$$//_{0,t}^T \mathrm{d}\big(\big(//_{0,t}^T\big)^{-1} \varphi_t v\big) = -\bigg(\frac{3}{2} r - 2R\big(T-t, X_t^T(x)\big)\bigg)\varphi_t v \,\mathrm{d}t,$$
$$\varphi_0 = \mathrm{Id}_{T_x M}.$$

Then $\mathrm{d}R(T-t,\cdot)_{X_t^T(x)} \varphi_t v$ *is a martingale and*:

$$\big\|\nabla^T R(T,x)\big\|_T \le \sup_M \big\|\nabla^0 R(0,x)\big\|_0 \mathrm{e}^{-(3/2) r T}\mathbb{E}\bigg[\exp\bigg(\int_0^T 2R\big(T-t, X_t^T(x)\big)\,\mathrm{d}t\bigg)\bigg]. \tag{3.4}$$

Proof. The proof is similar to the one in Theorem 3.2, the difference is the reaction term: $R(R-r)$. For notations and some details see the proof of Theorem 3.2. Take $F: x \mapsto x(x-r)$, then:

$$\frac{\partial}{\partial t} R = \Delta_t R + F(R).$$

We write:

$$\mathrm{d}R(T-t,\cdot)|_{X_t^T(x)} \varphi_t v = \bigl\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr\rangle_{\mathbb{R}^2},$$

where U_t^T is a diffusion on $\mathcal{F}(M)$ with generator

$$\Delta_{T-t}^H + \frac{1}{4}\bigl(r - R(T-t,\pi(\cdot)\bigr) g(T-t)\bigl(\mathrm{ev}_{e_i}(\cdot), \mathrm{ev}_{e_j}(\cdot)\bigr) V_{i,j}(\cdot).$$

Using Theorem 3.2, we have:

$$\begin{aligned}
&\mathrm{d}\bigl\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr\rangle_{\mathbb{R}^2} \\
&\quad = \bigl\langle \mathrm{d}\bigl(\tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr)\bigr), \bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr\rangle_{\mathbb{R}^2} + \bigl\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \mathrm{d}\bigl(\bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr)\bigr\rangle_{\mathbb{R}^2} \\
&\quad \overset{\mathrm{d}\mathcal{M}}{\equiv} \left[\frac{\partial}{\partial_t}\bigl(\mathrm{d}R(T-t,\cdot)\bigr) + \Delta_{T-t} \mathrm{d}R(t-t,\cdot) + \frac{1}{2}\bigl(r - R(T-t,\pi(\cdot)\bigr)\, \mathrm{d}R(T-t,\cdot)\right](\varphi_t v)\, \mathrm{d}t \\
&\qquad + \bigl\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \mathrm{d}\bigl(\bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr)\bigr\rangle_{\mathbb{R}^2}.
\end{aligned}$$

Using the Weitzenböck formula and the equation for R we have:

$$\frac{\partial}{\partial t} \mathrm{d}R(T-t,\cdot) = -\bigl[\Delta_{T-t} \mathrm{d}R(T-t,\cdot) - \mathrm{Ric}_{g(T-t)}\, \mathrm{d}R(T-t,\cdot) + F'\bigl(R(T-t,\cdot)\bigr)\, \mathrm{d}R(T-t,\cdot)\bigr].$$

Recall that for the surface:

$$\mathrm{Ric}_{g(T-t)}\, \mathrm{d}R(T-t,\cdot) = \frac{1}{2} R(T-t,\cdot)\, \mathrm{d}R(T-t,\cdot).$$

Consequently, we have

$$\begin{aligned}
&\mathrm{d}\bigl\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr\rangle_{\mathbb{R}^2} \\
&\quad \overset{\mathrm{d}\mathcal{M}}{\equiv} \left(\frac{1}{2} r - F'\bigl(R(T-t,\cdot)\bigr)\, \mathrm{d}R(T-t,\cdot)\right)(\varphi_t v)\, \mathrm{d}t + \bigl\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \mathrm{d}\bigl(\bigl(U_t^T\bigr)^{-1} \varphi_t v\bigr)\bigr\rangle_{\mathbb{R}^2} \\
&\quad \overset{\mathrm{d}\mathcal{M}}{\equiv} \left(\frac{1}{2} r - 2R(T-t,\cdot) + r\right) \mathrm{d}R(T-t,\cdot)(\varphi_t v)\, \mathrm{d}t \\
&\qquad + \left\langle \tilde{\mathrm{d}} R\bigl(T-t, U_t^T\bigr), \bigl(U_t^T\bigr)^{-1}\left(-\frac{3}{2} r + 2R(T-t,\cdot)\varphi_t v\right)\right\rangle_{\mathbb{R}^2} \\
&\quad \overset{\mathrm{d}\mathcal{M}}{\equiv} 0,
\end{aligned}$$

where we used the equation of $\varphi_t v$ in the last step.

For the second part of the proposition, with the equation for $\varphi_t v$ we find:

$$\mathrm{d}\bigl(\|\varphi_t v\|_{T-t}^2\bigr) = \bigl(4R\bigl(T-t, X_t^T(x)\bigr) - 3r\bigr)\|\varphi_t v\|_{T-t}^2\, \mathrm{d}t,$$

so that

$$\|\varphi_T v\|_0^2 = \|\varphi_0 v\|_T^2 \mathrm{e}^{-3rT} \exp\left(\int_0^T 4R\bigl(T-s, X_s^T(x)\bigr)\, \mathrm{d}s\right).$$

Take $v=\nabla_T R(T,x)$ and average at time 0 and T (it is a true martingale because all coefficients are bounded) to get:

$$\left\|\nabla^T R(T,x)\right\|_T \le \sup_M \left\|\nabla^0 R(0,x)\right\|_0 \mathrm{e}^{-(3/2)rT}\mathbb{E}\left[\exp\int_0^T 2R\big(T-s,X_s^T(x)\big)\,\mathrm{d}s\right]. \qquad \square$$

Remark 3.11. *For reaction-diffusion equations we can find by this calculation the correction to the parallel transport leading to a Bismut type formula for the gradient of the equation*:

$$\frac{\partial}{\partial t}f=\Delta_t f+F(f), \tag{3.5}$$

where Δ_t is a Laplace Beltrami operator associated to a family of metrics $g(t)$, and $F:\mathbb{R}\mapsto\mathbb{R}$ is a C^1 function. Let $X_t^T(x)$ be a $\frac{1}{2}g(T-t)$-BM(x), $//_{0,t}^T$ the associated parallel transport and $v\in T_xM$. Consider the covariant equation:

$$//_{0,t}^T\mathrm{d}\big(//_{0,t}^T\big)^{-1}\Theta_t v=-\left(\mathrm{Ric}_{g(T-t)}^{\#g(T-t)}-\frac{1}{2}\left[\frac{\partial}{\partial t}\big(g(T-t)\big)\right]^{\#g(T-t)}-F'(f)\right)\Theta_t v\,\mathrm{d}t.$$

Then for f a solution of Eq. (3.5) *and $v\in T_xM$ we obtain that*:

$$\mathrm{d}f(T-t,\cdot)\Theta_t v$$

is a local martingale.

Corollary 3.12. *For $\chi(M)<0$, there exists $C>0$ depending only on $g(0)$, such that*:

$$\left\|\nabla^T R(T,x)\right\|_T \le \sup_M \left\|\nabla^0 R(0,x)\right\|_0 \mathrm{e}^{rT/2}\exp\left(2C\frac{\mathrm{e}^{rT}-1}{r}\right).$$

Proof. We use Proposition 5.18 in [5]. In this case we have $r<0$ and a constant $C>0$ depending only on the initial metric such that $R(t,\cdot)\le r+C\mathrm{e}^{rt}$ and the estimate follows from previous proposition. $\square$

Remark 3.13. *For the case $\chi(M)<0$ we obtained an estimate which decreases exponentially. For the case $\chi(M)>0$ one could control the expectation in Eq.* (3.4) *with the same consequences*.

4. The point of view of the stochastic flow

Let $g(t)_{[0,T_c[}$ be a $C^{1,2}$ family of metrics, and consider the heat equation:

$$\begin{cases}\partial_t f(t,x)=\frac{1}{2}\Delta_t f(t,x),\\ f(0,x)=f_0(x),\end{cases} \tag{4.1}$$

where f_0 is a function on M. We suppose that the solution of this equation exists until T_c. For $T<T_c$, let X_t^T be a $g(T-t)$-Brownian motion and $//_{0,t}^T$ the associated parallel transport.

We will build (cf. Eq. (4.2)) a family of semimartingales $(T-t,X_t^T(x))$ such that $X_t^T(x)$ is a $g(T-t)$-BM(x) for all x nearby x_0 and such that the family of martingales $f(T-t,X_t^T(x))_x$ is differentiable at x_0 with respect to the parameter x. However, in this section, we will not do it directly using stochastic flows in the sense of [20]. Instead, we will use differentiation of families of martingales defined as a limit in some semi-martingale space (the topology is as in [11] which has been extended by Arnaudon, Thalmaier to the manifold case [1–4]).

We work in the space–time $I\times M$, its tangent bundle being identified to $TI\times TM$ endowed with the cross connection $\tilde{\nabla}=\overline{\nabla}\otimes\nabla_{T-t}$ where $\overline{\nabla}$ is the flat connection. Let $X_t^T(x_0)$ be a $g(T-t)$-BM started at x_0, and define $Y_t(x_0)=(t,X_t^T(x_0))$ a $I\times M$-valued semi martingale. From now on $P_{X,Y}^{\tilde{\nabla}}$ stands for the parallel transport along the shortest $\tilde{\nabla}$-geodesic between nearby points $X\in I\times M$ and $Y\in I\times M$ for the connection $\tilde{\nabla}$.

Let $\tilde{c}$ a curve in $I \times M$, we write $P^{\tilde{\nabla}}_{\tilde{c}}$ for the $\tilde{\nabla}$ parallel transport along $\tilde{c}$ and for a curve c in M we denote by $//^{T-s}_{c}$ the ∇^{T-s} parallel transport along c. We also denote $\pi : I \times M \to M$ the natural projection.

For a curve $\gamma : t \longrightarrow (s, x_t)$ in $I \times M$, where s is a fixed time, we have the following observation:

$$P^{\tilde{\nabla}}_{\gamma} = \big(\mathrm{Id}, //^{T-s}_{\pi(\gamma)}\big).$$

Define the Itô stochastic equation in the sense of [13]:

$$\mathrm{d}^{\tilde{\nabla}} Y_t(x) = P^{\tilde{\nabla}}_{Y_t(x_0), Y_t(x)} \mathrm{d}^{\tilde{\nabla}} Y_t(x_0). \tag{4.2}$$

Remark 4.1. *The above equation is well defined, for x sufficiently close to* x_0, *because* $\mathrm{d}_{T-t}(Y_t(x), Y_t(x_0))$ *is a finite variation process, with bounded derivative (by a short computation and* [6,18]).

Let $\tilde{//}_{0,t}$ be the parallel transport, associated to the connection $\tilde{\nabla}$, along the semi-martingale $Y_t(x_0)$. In the next Lemma, we will explain the relationship between the two parallel transport $\tilde{//}_{0,t}$ and $//^{T}_{0,t}$.

Lemma 4.2. *Let* $(e_i)_{i=1,\dots,n}$ *be a orthonormal base of* $(T_{x_0}M, g(T))$ *then*

$$\mathrm{d}\big(\big(//^{T}_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big)(0, e_i) = \frac{1}{2} \big(//^{T}_{0,t}\big)^{-1} \left(\frac{\partial}{\partial_t} g(T-t)\right)^{\#g(T-t)} \big(\mathrm{d}\pi\, \tilde{//}_{0,t}(0, e_i)\big)\, \mathrm{d}t.$$

Proof. The parallel transport $\tilde{//}_{0,t}$ does not modify the time vector, i.e.,

$$\tilde{//}^{-1}_{(t,X_t)}(0, \dots) = (0, \dots)$$

as can be shown for every curve, and hence for the semi-martingale Y_t by the transfer principle.

We identify $\tilde{T} = \{(0, v) \in T_{(0,x_0)} I \times M\}$ and $T_{x_0}M$ with the help of $(0, v) \mapsto v$. Hence

$$\big(//^{T}_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t} : \tilde{T} \to T_{x_0}M$$

becomes an element in $\mathcal{M}_{n,n}(\mathbb{R})$.

Recall that $//^{T}_{0,t} = U^T_t U^{T,-1}_0$. We use the definition of $D^{S,t}$ given in Eq. (1.2). Then we get using the shorthand $e_i = U^T_0 \tilde{e}_i$, with $(\tilde{e}_i)_{i=1,\dots,n}$ an orthonormal frame of $\mathbb{R}^n$,

$$\begin{aligned} *\mathrm{d}\big(\big(//^{T}_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big) &= *\mathrm{d}\big(\big\langle \big(//^{T}_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t} e_i, e_j \big\rangle_{g(T)}\big)_{i,j} \\ &= *\mathrm{d}\big(\big\langle \mathrm{d}\pi\, \tilde{//}_{0,t} e_i, //^{T}_{0,t} e_j \big\rangle_{g(T-t)}\big)_{i,j} \\ &= \Big(\big\langle D^{S,T-t} \mathrm{d}\pi\, \tilde{//}_{0,t} e_i, U^T_t \tilde{e}_j \big\rangle_{g(T-t)} + \frac{\partial}{\partial_t}\big(g(T-t)\big)\big(\mathrm{d}\pi\, \tilde{//}_{0,t} e_i, U^T_t \tilde{e}_j\big)\, \mathrm{d}t \\ &\qquad + \big\langle \mathrm{d}\pi\, \tilde{//}_{0,t} e_i, D^{S,T-t} U^T_t \tilde{e}_j \big\rangle_{g(T-t)}\Big)_{i,j}. \end{aligned}$$

We also have:

$$\begin{aligned} D^{S,T-t} \mathrm{d}\pi\, \tilde{//}_{0,t} e_i &= \mathcal{V}^{-1}_{\mathrm{d}\pi \tilde{//}_{0,t} e_i} \big(\big(*\mathrm{d}(\mathrm{d}\pi\, \tilde{//}_{0,t} e_i)\big)^{v_{T-t}}\big) \\ &= \mathcal{V}^{-1}_{\mathrm{d}\pi \tilde{//}_{0,t} e_i} \big(\big(\mathrm{d}\,\mathrm{d}\pi\, \mathrm{d}\,\mathrm{ev}_{e_i}(*\mathrm{d}\tilde{//}_{0,t})\big)^{v_{T-t}}\big) \\ &= \mathcal{V}^{-1}_{\mathrm{d}\pi \tilde{//}_{0,t} e_i} \big(\mathrm{d}\,\mathrm{d}\pi\big(\mathrm{d}\,\mathrm{ev}_{e_i}\big(*\mathrm{d}\tilde{//}_{0,t}\big)\big)^{\tilde{v}}\big) = 0. \end{aligned}$$

Where we used in the last equality the fact that $\tilde{//}_{0,t}$ is the $\tilde\nabla$ horizontal lift of Y_t. The third one may be seen for curves, it comes from the definition of $\tilde\nabla$.

Following computations similar to ones in the first section, we have by Eq. (1.4):

$$\begin{aligned}
*\mathrm{d}\big(\big(//_{0,t}^T\big)^{-1}\mathrm{d}\pi\,\tilde{//}_{0,t}\big)_{i,j} &= \frac{\partial}{\partial_t}g(T-t)\big(\mathrm{d}\pi\,\tilde{//}_{0,t}e_i, U_t^T\tilde e_j\big)\,\mathrm{d}t \\
&\quad+\big\langle \mathrm{d}\pi\,\tilde{//}_{0,t}e_i, D^{S,T-t}U_t^T\tilde e_j\big\rangle_{g(T-t)} \\
&= \frac{\partial}{\partial_t}g(T-t)\big(\mathrm{d}\pi\,\tilde{//}_{0,t}e_i, U_t^T\tilde e_j\big)\,\mathrm{d}t \\
&\quad+\left\langle \mathrm{d}\pi\,\tilde{//}_{0,t}e_i, -\frac{1}{2}\sum_{\alpha=1}^d\frac{\partial}{\partial_t}g(T-t)\big(U_t^T\tilde e_j, U_t^T\tilde e_\alpha\big)U_t^T\tilde e_\alpha\right\rangle_{g(T-t)}\mathrm{d}t \\
&= \frac{\partial}{\partial_t}g(T-t)\big(\mathrm{d}\pi\,\tilde{//}_{0,t}e_i, U_t^T\tilde e_j\big)\,\mathrm{d}t \\
&\quad-\frac{1}{2}\sum_{\alpha=1}^d\frac{\partial}{\partial_t}g(T-t)\big(U_t^T\tilde e_j, U_t^T\tilde e_\alpha\big)\big\langle \mathrm{d}\pi\,\tilde{//}_{0,t}e_i, U_t^T\tilde e_\alpha\big\rangle_{g(T-t)}\,\mathrm{d}t \\
&= \frac{1}{2}\frac{\partial}{\partial_t}g(T-t)\big(\mathrm{d}\pi\,\tilde{//}_{0,t}e_i, U_t^T\tilde e_j\big)\,\mathrm{d}t.
\end{aligned}$$

In the general case, and by the previous identification:

$$\begin{aligned}
\mathrm{d}\big(\big(//_{0,t}^T\big)^{-1}\mathrm{d}\pi\,\tilde{//}_{0,t}\big)(0,e_i) &= \frac{1}{2}\sum_j\frac{\partial}{\partial_t}g(T-t)\big(\mathrm{d}\pi\,\tilde{//}_{0,t}e_i, U_t^T\tilde e_j\big)e_j\,\mathrm{d}t \\
&= \frac{1}{2}\left(\big(//_{0,t}^T\big)^{-1}\frac{\partial}{\partial_t}g(T-t)\right)^{\#g(T-t)}\big(\mathrm{d}\pi\,\tilde{//}_{0,t}(0,e_i)\big)\,\mathrm{d}t. \qquad\square
\end{aligned}$$

Differentiating (4.2) along a geodesic curve beginning at $(0,x_0)$ with velocity $(a,v)\in T_0I\times T_{x_0}M$ and using Corollary 3.17 in [2] we get:

$$\tilde{//}_{0,t}\mathrm{d}\big(\tilde{//}_{0,t}^{-1}TY_t(a,v)\big) = -\frac{1}{2}\tilde R\big(TY_t(a,v),\mathrm{d}Y_t(x_0)\big)\,\mathrm{d}Y_t(x_0),$$

where $\tilde R$ is the curvature tensor. Let $v\in T_xM$ we write:

$$TX_tv := \mathrm{d}\pi\; TY_t(0,v).$$

In a more canonical way than Definition 3.1, we have the following proposition.

Proposition 4.3. *For all $v\in T_xM$ we have*:

$$\mathrm{d}\big(\big(//_{0,t}^T\big)^{-1}TX_tv\big) = -\frac{1}{2}\big(//_{0,t}^T\big)^{-1}\big(\mathrm{Ric}_{g(T-t)} - \partial_t\big(g(T-t)\big)\big)^{\#g(T-t)}(TX_tv)\,\mathrm{d}t.$$

Proof. For a triple of tangent vectors $(L_t,L),(A_t,A),(Z_t,Z)\in TI\times TM$, we have:

$$\tilde R\big((L_t,L),(A_t,A)\big)(Z_t,Z) = \big(0, R_{T-t}(L,A)Z\big).$$

Hence, according to the relation $\mathrm{d}Y(x_0) = (\mathrm{d}t, *\mathrm{d}X_t) = (\mathrm{d}t, //_{0,t}^Te_i * \mathrm{d}W^i)$ and the definition of the Ricci tensor:

$$\tilde{//}_{0,t}\mathrm{d}\big(\tilde{//}_{0,t}^{-1}TY_t(0,v)\big) = -\frac{1}{2}\big(0, \mathrm{Ric}_{g(T-t)}^{\#g(T-t)}(TX_tv)\big)\,\mathrm{d}t. \tag{4.3}$$

In order to compute in $\mathbb{R}^n$, we write:

$$\big(//^T_{0,t}\big)^{-1} T X_t v = \big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big)\big(\tilde{//}^{-1}_{0,t} T Y_t(0,v)\big). \tag{4.4}$$

By Eq. (4.3), we have $\mathrm{d}(\tilde{//}^{-1}_{0,t} T Y_t(0,v)) \in \mathrm{d}\mathcal{A}$ where $\mathcal{A}$ is the space of finite variation processes. We get:

$$\mathrm{d}\big(\big(//^T_{0,t}\big)^{-1} T X_t v\big) = \mathrm{d}\big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big)\big(\tilde{//}^{-1}_{0,t} T Y_t(0,v)\big) + \big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big)\, \mathrm{d}\big(\tilde{//}^{-1}_{0,t} T Y_t(0,v)\big).$$

By Eq. (4.4) and Lemma 4.2 we get:

$$\begin{aligned}
\mathrm{d}\big(\big(//^T_{0,t}\big)^{-1} T X_t v\big) &= *\mathrm{d}\big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big)\big(\tilde{//}^{-1}_{0,t} T Y_t(0,v)\big) + \big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big) * \mathrm{d}\big(\tilde{//}^{-1}_{0,t} T Y_t(0,v)\big) \\
&= *\mathrm{d}\big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\, \tilde{//}_{0,t}\big)\big(\tilde{//}^{-1}_{0,t} T Y_t(0,v)\big) - \frac{1}{2}\big(\big(//^T_{0,t}\big)^{-1} \mathrm{d}\pi\big)\big(0, \mathrm{Ric}^{\#g(T-t)}_{g(T-t)}(T X_t v)\big)\,\mathrm{d}t \\
&= \frac{1}{2}\big(//^T_{0,t}\big)^{-1}\Big(\frac{\partial}{\partial_t} g(T-t)\Big)^{\#g(T-t)}(T X_t v)\,\mathrm{d}t - \frac{1}{2}\big(//^T_{0,t}\big)^{-1} \mathrm{Ric}^{\#g(T-t)}_{g(T-t)}(T X_t v)\,\mathrm{d}t. \qquad \square
\end{aligned}$$

For all $f_0 \in C^\infty(M)$ and for $f(t,\cdot)$ a solution of Eq. (3.3), where $g(t)$ evolves along a forward Ricci flow, $f(T-t, X^T_t(x))$ is a martingale, where $(T-t, X^T_t(x)) = Y_t(x)$ is built as in Eq. (4.2). We have the following corollary in agreement with Theorem 3.2.

Corollary 4.4. *For all $v \in T_x M$:*

$$\mathrm{d}f\big(T-t, X^T_t(\cdot)\big)v = \mathrm{d}f(T-t,\cdot)_{X^T_t(x)} //^T_{0,t} v$$

is a martingale.

Proof. By differentiation with respect to x of $f(T-t, X^T_t(x))$, we get a local martingale. According to [2] and by the chain rule for differentials we have:

$$\mathrm{d}f\big(T-t, X^T_t(\cdot)\big)v = \mathrm{d}f(T-t,\cdot)_{X^T_t(x)} T X_t v.$$

Using the evolution of the metric under forward Ricci flow and Proposition 4.3, we get the corollary after replacing $T X_t$ by $//^T_{0,t}$. $\square$

In an canonical way, we have the following result.

Theorem 4.5. *The following conditions are equivalent for a $C^{1,2}$ family $g(t)$ of metrics*:

(i) *$g(t)$ evolves under the forward Ricci flow.*
(ii) *For all $T < T_c$ we have $//^T_{0,t} = \mathbf{W}^T_{0,t} = T X_t$.*
(iii) *For all $T < T_c$, the damped parallel transport $\mathbf{W}^T_{0,t}$ is an isometry.*

Proof. Here, the forward Ricci flow has probabilistic convention (2.2). The result follows by the equation of $g(t)$ and by Proposition 4.3 and Theorem 3.2. $\square$

5. Second derivative of the stochastic flow

We take the differential of the stochastic flow in order to obtain a intrinsic martingale. We take the same notation as the previous section, and $g(t)$ is a family of metrics coming from a forward Ricci flow. Let $X^T_t(x)$ be the $g(T-t)$-BM started at x, constructed as in Eq. (4.2) in the previous section by the parallel coupling of a $g(T-t)$-BM started at x_0, $\tilde{\nabla}$ and $Y_t(x)=(t, X_t^T(x))$ as before, define the intrinsic trace (that does not depend on the choice of E_i as below):

$$\operatorname{Tr}\nabla.TX_t(x_0)(\cdot) := \mathrm{d}\pi\left(\sum_i \tilde{\nabla}_{(0,e_i)}\big(TY_t(0,E_i)\big) - TY_t\tilde{\nabla}_{(0,e_i)}(0,E_i)\right),$$

where (e_i) is a $(T_{x_0}M, g(T))$ orthonormal basis, E_i are C^1 vectors fields such that $E_i(x_0)=e_i$ and $\tilde{\nabla}_{(0,e_i)}(TY_t(0, E_i)) := \tilde{\nabla}_{(0,e_i)}(TY_t(\cdot)(0,E_i(\cdot)))$ is a derivative of a bundle-valued semi-martingale in the sense of [1–3]. By Theorem 4.5:

$$\operatorname{Tr}\nabla.TX_t(x_0)(\cdot) = \mathrm{d}\pi\sum_i \tilde{\nabla}_{(0,e_i)}\big(TY_t(0,E_i)\big) - //_{0,t}^T\mathrm{d}\pi\left(\sum_i \tilde{\nabla}_{(0,e_i)}(0,E_i)\right).$$

Theorem 5.1. *Let $L_t := (//_{0,t}^T)^{-1}\operatorname{Tr}\nabla.TX_t(x_0)(\cdot)$, a $T_{x_0}M$-valued process, started at* 0. *Then*:

(i) *L_t is a $(T_{x_0}M, g(T))$-valued martingale, independent of the choice of E_i, and we have the following equation*:

$$L_t = \int_0^t \sum_i //_{0,s}^{T,-1}\operatorname{Ric}_{g(T-s)}^{\#g(T-s)}\big(//_{0,s}^T e_i\big)\,\mathrm{d}W_s^i.$$

(ii) *The $g(T)$-quadratic variation of L is given by*:

$$d[L,L]_t = \big\|\big|\operatorname{Ric}_{g(T-t)}\big(X_t(x_0)\big)\big|\big\|_{g(T-t)}^2\,\mathrm{d}t,$$

where $\||\cdot|\|$ is the usual Hilbert–Schmidt norm of linear operator.

Proof. Recall that by the same construction as in the previous section:

$$\tilde{D}\big(TY_t(x)\big(0,E_i(x)\big)\big) = -\frac{1}{2}\tilde{R}\big(TY_t(x)\big(0,E_i(x)\big),\mathrm{d}Y_t(x)\big)\,\mathrm{d}Y_t(x).$$

By the general commutation formula (e.g. Theorem 4.5 in [3]), and by the previous equation which cancels two terms in this formula, we get:

$$\begin{aligned}\tilde{D}\tilde{\nabla}_{(0,e_i)}\big(TY_t(x)\big(0,E_i(x)\big)\big) &= \tilde{\nabla}_{(0,e_i)}\tilde{D}\big(TY_t(x)\big(0,E_i(x)\big)\big)\\ &\quad + \tilde{R}\big(\mathrm{d}^{\tilde{\nabla}}Y_t(x_0), TY_t(x_0)(0,e_i)\big)TY_t(x_0)(0,e_i)\\ &\quad - \frac{1}{2}\tilde{\nabla}\tilde{R}\big(\mathrm{d}Y_t(x_0), TY_t(x_0)(0,e_i),\mathrm{d}Y_t(x_0)\big)TY_t(x_0)(0,e_i)\\ &= -\frac{1}{2}\tilde{\nabla}_{(0,e_i)}\big(\tilde{R}\big(TY_t(x)\big(0,E_i(x)\big),\mathrm{d}Y_t(x)\big)\,\mathrm{d}Y_t(x)\big)\\ &\quad + \tilde{R}\big(\mathrm{d}^{\tilde{\nabla}}Y_t(x_0), TY_t(x_0)(0,e_i)\big)TY_t(x_0)(0,e_i)\\ &\quad - \frac{1}{2}\big(\tilde{\nabla}_{\mathrm{d}Y_t(x_0)}\tilde{R}\big)\big(TY_t(x_0)(0,e_i),\mathrm{d}Y_t(x_0)\big)TY_t(x_0)(0,e_i).\end{aligned}$$

Taking trace in the previous equation we can go one step further. Recall that $(e_i)_{i=1,\dots,n}$ is a orthogonal basis of $(T_{x_0}M, g(T))$, and write for notation:

$$\tilde{\operatorname{Ric}}^{\#}_{(t,x)}(V) = \big(0, \operatorname{Ric}_{g(T-t)}{}^{\#g(T-t)}(\mathrm{d}\pi V)\big),$$

then:

$$\begin{aligned}
&\sum_i \tilde{D}\tilde{\nabla}_{(0,e_i)}\big(TY_t(x)\big(0, E_i(x)\big)\big)\\
&\quad = -\frac{1}{2}\sum_i \tilde{\nabla}_{(0,e_i)}\big(\tilde{\mathrm{Ric}}^{\#}_{Y_t(x)}\big(TY_t(x)E_i(x)\big)\big)\,\mathrm{d}t\\
&\qquad + \sum_i \tilde{R}\big(\mathrm{d}^{\tilde{\nabla}} Y_t(x_0), TY_t(x_0)(0,e_i)\big)TY_t(x_0)(0,e_i)\\
&\qquad - \frac{1}{2}\sum_i\big(\tilde{\nabla}_{\mathrm{d}Y_t(x_0)}\tilde{R}\big)\big(TY_t(x_0)(0,e_i), \mathrm{d}Y_t(x_0)\big)TY_t(x_0)(0,e_i)\\
&\quad = -\frac{1}{2}\sum_i\big(\tilde{\nabla}_{(TY_t(x_0)(0,e_i))}\tilde{\mathrm{Ric}}^{\#}\big)\big(TY_t(x_0)(0,e_i)\big)\,\mathrm{d}t\\
&\qquad - \frac{1}{2}\tilde{\mathrm{Ric}}^{\#}_{Y_t(x_0)}\bigg(\sum_i \tilde{\nabla}_{(0,e_i)}\big(TY_t(x)\big(0, E_i(x)\big)\big)\bigg)\,\mathrm{d}t + \tilde{\mathrm{Ric}}^{\#}_{Y_t(x_0)}\big(\mathrm{d}^{\tilde{\nabla}} Y_t(x_0)\big)\\
&\qquad - \frac{1}{2}\sum_i\big(\tilde{\nabla}_{\mathrm{d}Y_t(x_0)}\tilde{R}\big)\big(TY_t(x_0)(0,e_i), \mathrm{d}Y_t(x_0)\big)TY_t(x_0)(0,e_i).
\end{aligned}$$

In the last equality, we use the chain derivative formula, and derivation is taken with respect to x. We will make an independent computation for the last term in the previous equation. Let Tr stand for the usual trace:

$$\begin{aligned}
&\sum_i\big(\tilde{\nabla}_{\mathrm{d}Y_t(x_0)}\tilde{R}\big)\big(TY_t(x_0)(0,e_i), \mathrm{d}Y_t(x_0)\big)TY_t(x_0)(0,e_i)\\
&\quad = \sum_i\big(0, \big(\nabla^{T-t}_{\mathrm{d}X_t} R^{T-t}\big)\big(TX_t(x_0)e_i, \mathrm{d}X_t\big)TX_t(x_0)e_i\big)\\
&\quad = \sum_{i,j}\big(0, \big(\nabla^{T-t}_{//^T_{0,t}e_j} R^{T-t}\big)\big(TX_t(x_0)e_i, //^T_{0,t}e_j\big)TX_t(x_0)e_i\big)\,\mathrm{d}t\\
&\quad = \sum_j\big(0, \mathrm{Tr}_{1,3}\big(\nabla^{T-t}_{//^T_{0,t}e_j} R^{T-t}\big)\big(//^T_{0,t}e_j\big)\big)\,\mathrm{d}t\\
&\quad = \sum_j\big(0, \big(\nabla^{T-t}_{//^T_{0,t}e_j}\,\mathrm{Tr}_{1,3}\,R^{T-t}\big)\big(//^T_{0,t}e_j\big)\big)\,\mathrm{d}t\\
&\quad = -\sum_j\big(0, \big(\nabla^{T-t}_{//^T_{0,t}e_j}\,\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big)\big(//^T_{0,t}e_j\big)\big)\,\mathrm{d}t,
\end{aligned}$$

where we used in the second equality the fact that in case of the forward Ricci flow $//^T_{0,t}$ is a $g(T-t)$ isometry and $\mathrm{d}X = //^T_{0,t}e_j\,\mathrm{d}W^j$. In the last equality we use the commutation between trace and covariant derivative (for example [21], or [19]). Note that:

$$\begin{aligned}
&\sum_i\big(\tilde{\nabla}_{(TY_t(x_0)(0,e_i))}\tilde{\mathrm{Ric}}^{\#}\big)\big(TY_t(x_0)(0,e_i)\big)\,\mathrm{d}t\\
&\quad = \sum_i\big(0, \big(\nabla^{T-t}_{TX_t(x_0)e_i}\,\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big)\big(TX_t(x_0)e_i\big)\big)\,\mathrm{d}t.
\end{aligned}$$

Hence, using Theorem 4.5:

$$\tilde{D}\Big(\sum_i \tilde{\nabla}_{(0,e_i)}\big(TY_t(x)\big(0,E_i(x)\big)\big)\Big)$$
$$= -\frac{1}{2}\tilde{\mathrm{Ric}}^{\#}_{Y_t(x_0)}\Big(\sum_i \tilde{\nabla}_{(0,e_i)}\big(TY_t(x)\big(0,E_i(x)\big)\big)\Big)\,\mathrm{d}t + \tilde{\mathrm{Ric}}^{\#}_{Y_t(x_0)}\big(\mathrm{d}^{\tilde{\nabla}}Y_t(x_0)\big).$$

Write, for simplicity, B for $\sum_i \tilde{\nabla}_{(0,e_i)}(TY_t(x)(0,E_i(x)))$. We compute:

$$\begin{aligned}
\mathrm{d}\big(//_{0,t}^{T,-1}\mathrm{d}\pi B\big) &= \mathrm{d}\big(\big[//_{0,t}^{T,-1}\mathrm{d}\pi\,\tilde{//}_{0,t}\big]\big[\big(\tilde{//}_{0,t}\big)^{-1}B\big]\big)\\
&= \frac{1}{2}//_{0,t}^{T,-1}\big(\partial_t g(T-t)\big)^{\#g(T-t)}(\mathrm{d}\pi\, B)\,\mathrm{d}t\\
&\quad + //_{0,t}^{T,-1}\Big(-\frac{1}{2}\mathrm{d}\pi\big(\tilde{\mathrm{Ric}}^{\#}(B)\big)\,\mathrm{d}t + \mathrm{d}\pi\big(\tilde{\mathrm{Ric}}^{\#}_{Y_t(x_0)}\big(\mathrm{d}^{\tilde{\nabla}}Y_t(x_0)\big)\big)\Big)\\
&= //_{0,t}^{T,-1}\big(\mathrm{d}\pi\tilde{\mathrm{Ric}}^{\#}_{Y_t(x_0)}\big(\mathrm{d}^{\tilde{\nabla}}Y_t(x_0)\big)\big)\\
&= \sum_i //_{0,t}^{T,-1}\,\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big(//_{0,t}^{T}e_i\big)\,\mathrm{d}W^i,
\end{aligned}$$

where we used Lemma 4.2 in the first equality. We get a intrinsic martingale that does not depend on E_i, starting at 0. By the definition in Theorem 5.1 and by the formula preceding Theorem 5.1, the above calculations yield:

$$L_t = \int_0^t \sum_i //_{0,s}^{T,-1}\,\mathrm{Ric}^{\#g(T-s)}_{g(T-s)}\big(//_{0,s}^{T}e_i\big)\,\mathrm{d}W^i_s.$$

For the $g(T)$-quadratic variation of L_t we use the isometry property of the parallel transport; we compute the quadratic variation:

$$\begin{aligned}
d[L,L]_t &= \big\langle //_{0,t}^{T,-1}\,\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big(//_{0,t}^{T}e_i\big), //_{0,t}^{T,-1}\,\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big(//_{0,t}^{T}e_i\big)\big\rangle_T\,\mathrm{d}t\\
&= \sum_i \big\|\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big(//_{0,t}^{T}e_i\big)\big\|^2_{g(T-t)}\,\mathrm{d}t\\
&= \big\|\big\|\mathrm{Ric}^{\#g(T-t)}_{g(T-t)}\big(X_t^T(x_0)\big)\big\|\big\|^2_{g(T-t)}\,\mathrm{d}t. \qquad \square
\end{aligned}$$

Remark 5.2. *By the independence of the choice of the orthonormal basis (e_i) we can express this norm in terms of the eigenvalues of the Ricci operator*:

$$d[L,L]_t = \sum_i \lambda_i^2\big(T-t, X_t^T(x)\big)\,\mathrm{d}t.$$

Remark 5.3. *We could choose E_i such that $\tilde{\nabla}_{(0,e_i)}(0,E_i(x)) = 0$. That does not change the martingale L, but gives a simpler version.*

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References

[1] M. Arnaudon and A. Thalmaier. Complete lifts of connections and stochastic Jacobi fields. *J. Math. Pures Appl. (9)* **77** (1998) 283–315. MR1618537

[2] M. Arnaudon and A. Thalmaier. Stability of stochastic differential equations in manifolds. In *Séminaire de Probabilités, XXXII* 188–214. *Lecture Notes in Math.* **1686**. Springer, Berlin, 1998. MR1655151

[3] M. Arnaudon and A. Thalmaier. Horizontal martingales in vector bundles. In *Séminaire de Probabilités, XXXVI* 419–456. *Lecture Notes in Math.* **1801**. Springer, Berlin, 2003. MR1971603

[4] M. Arnaudon, R. O. Bauer and A. Thalmaier. A probabilistic approach to the Yang–Mills heat equation. *J. Math. Pures Appl. (9)* **81** (2002) 143–166. MR1994607

[5] B. Chow and D. Knopf. *The Ricci Flow: An Introduction*. *Mathematical Surveys and Monographs* **110**. Amer. Math. Soc., Providence, RI, 2004. MR2061425

[6] M. Cranston. Gradient estimates on manifolds using coupling. *J. Funct. Anal.* **99** (1991) 110–124. MR1120916

[7] D. M. DeTurck. Deforming metrics in the direction of their Ricci tensors. *J. Differential Geom.* **18** (1983) 157–162. MR0697987

[8] K. D. Elworthy and X.-M. Li. Formulae for the derivatives of heat semigroups. *J. Funct. Anal.* **125** (1994) 252–286. MR1297021

[9] K. D. Elworthy and M. Yor. Conditional expectations for derivatives of certain stochastic flows. In *Séminaire de Probabilités, XXVII* 159–172. *Lecture Notes in Math.* **1557**. Springer, Berlin, 1993. MR1308561

[10] K. D. Elworthy, Y. Le Jan and X.-M. Li. *On the Geometry of Diffusion Operators and Stochastic Flows*. *Lecture Notes in Math.* **1720**. Springer, Berlin, 1999. MR1735806

[11] M. Emery. Une topologie sur l'espace des semimartingales. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)* 260–280. *Lecture Notes in Math.* **721**. Springer, Berlin, 1979. MR0544800

[12] M. Émery. *Stochastic Calculus in Manifolds: With an Appendix by P.-A. Meyer*. Springer, Berlin, 1989. MR1030543

[13] M. Émery. On two transfer principles in stochastic differential geometry. In *Séminaire de Probabilités, XXIV, 1988/89* 407–441. *Lecture Notes in Math.* **1426**. Springer, Berlin, 1990. MR1071558

[14] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982) 255–306. MR0664497

[15] E. P. Hsu. *Stochastic Analysis on Manifolds Graduate Studies in Mathematics* **38**. Amer. Math. Soc., Providence, RI, 2002. MR1882015

[16] J. Jost. *Harmonic Mappings between Riemannian Manifolds*. *Proceedings of the Centre for Mathematical Analysis, Australian National University* **4**. Australian National University Centre for Mathematical Analysis, Canberra, 1984. MR0756629

[17] J. Jost. *Riemannian Geometry and Geometric Analysis*, 4th edition. Springer, Berlin, 2005. MR2165400

[18] W. S. Kendall. Nonnegative Ricci curvature and the Brownian coupling property. *Stochastics* **19** (1986) 111–129. MR0864339

[19] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry. Vol. I. Wiley Classics Library*. Wiley, New York, 1996. Reprint of the 1963 original, Wiley. MR1393940

[20] H. Kunita. *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge, 1990. MR1070361

[21] J. M. Lee. *Riemannian Manifolds: An Introduction to Curvature*. *Graduate Texts in Mathematics* **176**. Springer, New York, 1997. MR1468735

[22] D. W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional Diffusion Processes. Classics in Mathematics*. Springer, Berlin, 2006. Reprint of the 1997 edition. MR2190038

[23] A. Thalmaier and F.-Y. Wang. Gradient estimates for harmonic functions on regular domains in Riemannian manifolds. *J. Funct. Anal.* **155** (1998) 109–124. MR1622800

[24] P. Topping. *Lectures on the Ricci Flow*. *London Mathematical Society Lecture Note Series* **325**. Cambridge Univ. Press, Cambridge, 2006. MR2265040