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Disorder relevance for the random walk pinning model in dimension 3

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Abstract. We study the continuous time version of the *random walk pinning model*, where conditioned on a continuous time random walk $(Y_s)_{s\geq 0}$ on $\mathbb{Z}^d$ with jump rate $\rho > 0$, which plays the role of disorder, the law up to time t of a second independent random walk $(X_s)_{0\leq s\leq t}$ with jump rate 1 is Gibbs transformed with weight $e^{\beta L_t(X,Y)}$, where $L_t(X,Y)$ is the collision local time between X and Y up to time t. As the inverse temperature β varies, the model undergoes a localization–delocalization transition at some critical $\beta_c \geq 0$. A natural question is whether or not there is disorder relevance, namely whether or not β_c differs from the critical point β_c^{ann} for the annealed model. In [3], it was shown that there is disorder irrelevance in dimensions $d = 1$ and 2, and disorder relevance in $d \geq 4$. For $d \geq 5$, disorder relevance was first proved in [2]. In this paper, we prove that if X and Y have the same jump probability kernel, which is irreducible and symmetric with finite second moments, then there is also disorder relevance in the critical dimension $d = 3$, and $\beta_c - \beta_c^{\mathrm{ann}}$ is at least of the order $e^{-C(\zeta)/\rho^{\zeta}}$, $C(\zeta) > 0$, for any $\zeta > 2$. Our proof employs coarse graining and fractional moment techniques, which have recently been applied by Lacoin [13] to the directed polymer model in random environment, and by Giacomin, Lacoin and Toninelli [10] to establish disorder relevance for the random pinning model in the critical dimension. Along the way, we also prove a continuous time version of Doney's local limit theorem [5] for renewal processes with infinite mean.

Résumé. Nous étudions la version à temps continu du *modèle de marche aléatoire avec accrochage*, où conditionné sur une marche aléatoire à temps continu $(Y_s)_{s\geq 0}$ sur $\mathbb{Z}^d$ avec taux de saut $\rho > 0$, qui joue le rôle de désordre, la loi jusqu'au temps t d'une seconde marche aléatoire indépendante $(X_s)_{0\leq s\leq t}$ avec taux de saut 1 est la transformée de Gibbs avec poids $e^{\beta L_t(X,Y)}$, où $L_t(X,Y)$ est le temps local de collision entre X et Y jusqu'au temps t. Lorsque la température inverse β varie, le modèle subit une transition de localisation-délocalisation à un $\beta_c \geq 0$ critique. Une question naturelle est de savoir s'il y a pertinence du désordre ou pas, i.e., si β_c diffère ou pas du point critique β_c^{ann} pour le modèle moyenné. Dans [3], il a été montré qu'il y avait non pertinence du désordre en dimensions $d = 1$ et 2, et pertinence du désordre lorsque $d \geq 4$. Pour $d \geq 5$, la pertinence du désordre fût d'abord prouvée dans [2]. Dans ce papier, nous prouvons que si X et Y ont le même noyau de probabilité de saut, qui est irréductible et symétrique avec des moments du second ordre finis, alors il y a également pertinence du désordre en dimension critique $d = 3$, et $\beta_c - \beta_c^{\mathrm{ann}}$ est au moins de l'ordre $e^{-C(\zeta)/\rho^{\zeta}}$, $C(\zeta) > 0$, pour tout $\zeta > 2$. Notre preuve utilise des techniques de coarse graining et de moment fractionnaire, qui ont été récemment appliquées par Lacoin [13] au modèle de polymère dirigé en milieu aléatoire, et par Giacomin, Lacoin et Toninelli [10] pour établir la pertinence du désordre pour le modèle d'accrochages aléatoires en dimension critique. En chemin, nous prouvons également une version en temps continu du théorème limite local de Doney [5] pour des processus de renouvellement avec moyenne infinie.

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1. Model and result

Let us recall the continuous time *random walk pinned to random walk model* studied in [3], which we will abbreviate from now on as the *random walk pinning model* (RWPM). Let X and Y be two continuous time random walks on $\mathbb{Z}^3$ starting from the origin, such that X and Y have respectively jump rates 1 and $\rho \geq 0$, and identical irreducible symmetric jump probability kernels on $\mathbb{Z}^3$ with finite second moments. Let μ_t denote the law of $(X_s)_{0\leq s\leq t}$. Then given $\beta \in \mathbb{R}$ and conditioned on $(Y_s)_{s\geq 0}$, which we interpret as a random environment or disorder, we define a Gibbs transform $\mu^\beta_{t,Y}$ of the path measure μ_t via Radon–Nikodym derivative

$$\frac{\mathrm{d}\mu^\beta_{t,Y}}{\mathrm{d}\mu_t}(X) = \frac{\mathrm{e}^{\beta L_t(X,Y)}}{Z^\beta_{t,Y}}, \tag{1.1}$$

where $L_t(X,Y) = \int_0^t 1_{\{X_s=Y_s\}}\,\mathrm{d}s$, and

$$Z^\beta_{t,Y} = \mathbb{E}^X_0\bigl[\mathrm{e}^{\beta L_t(X,Y)}\bigr] \tag{1.2}$$

is the *quenched partition function* with $\mathbb{E}^X_x[\cdot]$ denoting expectation w.r.t. X starting from $x \in \mathbb{Z}^3$. We can interpret X as a polymer which is attracted to a random defect line Y. A more commonly studied model is to consider a constant defect line $Y \equiv 0$, but with random strength of interaction between X and Y at different time points. This is known as the *random pinning model* (RPM), the discrete time analogue of which was the subject of many recent papers (see, e.g., [4,9,10]), as well as a book [8].

A common variant of the Gibbs measure $\mu^\beta_{t,Y}$ is to introduce pinning of path at the end point t, i.e., we define the Gibbs measure $\mu^{\beta,\mathrm{pin}}_{t,Y}$ with

$$\frac{\mathrm{d}\mu^{\beta,\mathrm{pin}}_{t,Y}}{\mathrm{d}\mu_t}(X) = 1_{\{X_t=Y_t\}}\frac{\mathrm{e}^{\beta L_t(X,Y)}}{Z^{\beta,\mathrm{pin}}_{t,Y}} \tag{1.3}$$

with $Z^{\beta,\mathrm{pin}}_{t,Y} = \mathbb{E}^X_0[\mathrm{e}^{\beta L_t(X,Y)}1_{\{X_t=Y_t\}}]$. It was shown in [3] that, almost surely w.r.t. Y, the limit

$$F(\beta,\rho) = \lim_{t\to\infty}\frac{1}{t}\log Z^\beta_{t,Y} = \lim_{t\to\infty}\frac{1}{t}\log Z^{\beta,\mathrm{pin}}_{t,Y} \tag{1.4}$$

exists and is independent of the disorder Y, which we call the *quenched free energy* of the model. There exists a critical inverse temperature $\beta_\mathrm{c} = \beta_\mathrm{c}(\rho)$, such that $F(\beta,\rho) > 0$ if $\beta > \beta_\mathrm{c}$ and $F(\beta) = 0$ if $\beta < \beta_\mathrm{c}$. The supercritical region $\beta \in (\beta_\mathrm{c},\infty)$ is the localized phase where given Y, and with respect to either $\mu^\beta_{t,Y}$ or $\mu^{\beta,\mathrm{pin}}_{t,Y}$, the contact fraction $L_t(X,Y)/t$ between X and Y up to time t typically remains positive as $t\to\infty$, so that the walk X is pinned to Y. In fact, by the convexity of $\log Z^\beta_{t,Y}$ in β and (1.4), it is not hard to see that almost surely,

$$\begin{aligned}
\liminf_{t\to\infty}\mu^\beta_{t,Y}\bigl(t^{-1}L_t(X,Y)\bigr) &= \liminf_{t\to\infty}\frac{\partial(t^{-1}\log Z^\beta_{t,Y})}{\partial\beta} \geq \frac{\partial F(\beta,\rho)}{\partial_-\beta},\\
\limsup_{t\to\infty}\mu^\beta_{t,Y}\bigl(t^{-1}L_t(X,Y)\bigr) &= \limsup_{t\to\infty}\frac{\partial(t^{-1}\log Z^\beta_{t,Y})}{\partial\beta} \leq \frac{\partial F(\beta,\rho)}{\partial_+\beta},
\end{aligned}$$

where $\frac{\partial}{\partial_-\beta}$ and $\frac{\partial}{\partial_+\beta}$ denote respectively the left and right derivative w.r.t. β. The convexity of $F(\beta,\rho)$ in β implies that $\frac{\partial F(\beta,\rho)}{\partial_-\beta} > 0$ for all $\beta > \beta_\mathrm{c}$. In contrast, the subcritical region $\beta \in (-\infty,\beta_\mathrm{c})$ is the de-localized phase, where $\partial F(\beta,\rho)/\partial\beta = 0$ and the contact fraction $L_t(X,Y)/t$ is typically of order $\mathrm{o}(1)$ as $t\to\infty$, so that X becomes delocalized from Y.

An important tool in the study of models with disorder is to compare the quenched free energy with the annealed free energy, which is defined by

$$F_{\mathrm{ann}}(\beta,\rho) := \lim_{t\to\infty}\frac{1}{t}\log Z^{\beta}_{t,\mathrm{ann}} = \lim_{t\to\infty}\frac{1}{t}\log Z^{\beta,\mathrm{pin}}_{t,\mathrm{ann}}, \tag{1.5}$$

where

$$Z^{\beta}_{t,\mathrm{ann}} = \mathbb{E}^{X,Y}_{0,0}\left[\mathrm{e}^{\beta L_t(X,Y)}\right] \quad \text{and} \quad Z^{\beta,\mathrm{pin}}_{t,\mathrm{ann}} = \mathbb{E}^{X,Y}_{0,0}\left[\mathrm{e}^{\beta L_t(X,Y)}1_{\{X_t=Y_t\}}\right]$$

are the free (resp., constrained) versions of the annealed partition function for the RWPM. Since $X-Y$ is also a random walk, we see that $Z^{\beta}_{t,\mathrm{ann}}$ and $Z^{\beta,\mathrm{pin}}_{t,\mathrm{ann}}$ are the partition functions of a RWPM where the random walk $X-Y$ is attracted to the constant defect line 0. This defines the annealed model. In particular, there also exists a critical point $\beta^{\mathrm{ann}}_{\mathrm{c}} = \beta^{\mathrm{ann}}_{\mathrm{c}}(\rho)$ such that $F_{\mathrm{ann}}(\beta,\rho) > 0$ when $\beta > \beta^{\mathrm{ann}}_{\mathrm{c}}$ and $F_{\mathrm{ann}}(\beta,\rho) = 0$ when $\beta < \beta^{\mathrm{ann}}_{\mathrm{c}}$. It is easy to show that $\beta^{\mathrm{ann}}_{\mathrm{c}} = (1+\rho)/G$, where G is the Green function of X, see end of Section 2, while no explicit expression for β_{c} is known. By Jensen's inequality, it is easily seen that $F(\beta,\rho) \le F_{\mathrm{ann}}(\beta,\rho)$, and hence $\beta_{\mathrm{c}} \ge \beta^{\mathrm{ann}}_{\mathrm{c}}$. A fundamental question is then to determine whether the disorder is sufficient to shift the critical point of the model so that $\beta_{\mathrm{c}} > \beta^{\mathrm{ann}}_{\mathrm{c}}$, which is called *disorder relevance*. If $\beta_{\mathrm{c}} = \beta^{\mathrm{ann}}_{\mathrm{c}}$, then we say there is *disorder irrelevance*, and it is generally believed that the quenched model's behavior in this case is similar to that of the annealed model. It turns out that disorder relevance/irrelevance has an interesting dependence on the spatial dimension d.

In [3], it was shown that if X and Y are continuous time simple random walks, then the RWPM is disorder irrelevant in $d=1$ and 2, and disorder relevant in $d \ge 4$. Furthermore, it was shown that in $d \ge 5$, there exists $a > 0$ such that $\beta_{\mathrm{c}} - \beta^{\mathrm{ann}}_{\mathrm{c}} > a\rho$ for all $\rho \in [0,1]$; while in $d=4$, for any $\delta > 0$, there exists $a_\delta > 0$ such that $\beta_{\mathrm{c}} - \beta^{\mathrm{ann}}_{\mathrm{c}} \ge a_\delta \rho^{1+\delta}$ for all $\rho \in [0,1]$. It is easy to check that the proof of these results in [3] apply equally well to continuous time random walks X and Y with the same irreducible symmetric jump probability kernel with finite second moments. In this paper, we resolve the marginal case $d=3$ and show that there is disorder relevance.

Theorem 1.1 (Annealed vs quenched critical points). *Let X and Y be two continuous time random walks with respective jump rates 1 and $\rho > 0$ and identical irreducible symmetric jump probability kernel $q(\cdot)$ on $\mathbb{Z}^3$ with finite second moments. Assume $X_0 = Y_0 = 0$. Then for the associated RWPM, $\beta_{\mathrm{c}}(\rho) > \beta^{\mathrm{ann}}_{\mathrm{c}}(\rho)$ for all $\rho > 0$, and for any $\zeta > 2$, there exists $C(\zeta) > 0$ such that for all $\rho \in (0,1]$,*

$$\beta_{\mathrm{c}}(\rho) - \beta^{\mathrm{ann}}_{\mathrm{c}}(\rho) \ge \mathrm{e}^{-C(\zeta)\rho^{-\zeta}}. \tag{1.6}$$

Remark. *It is intriguing that our lower bound for the critical point shift is of the same form as for the RPM in the marginal case, where a lower bound of $\mathrm{e}^{-C(\zeta)\beta^{-\zeta}}$ was obtained in [10] for any $\zeta > 2$, and $\zeta = 2$ is known to provide an upper bound. For the RWPM, there has been no heuristics or results so far on the upper bound.*

Let

$$\beta^*_c(\rho) = \sup\Big\{\beta \in \mathbb{R}\colon \sup_{t>0} Z^{\beta}_{t,Y} < \infty \text{ a.s. w.r.t. } Y\Big\}. \tag{1.7}$$

Note that $\beta^*_{\mathrm{c}}(\rho) \le \beta_{\mathrm{c}}(\rho)$. We will in fact prove the following stronger version of Theorem 1.1.

Theorem 1.2 (Non-coincidence of critical points strengthened). *Assuming the same conditions as in Theorem 1.1, then the conclusions therein also hold with $\beta_{\mathrm{c}}(\rho)$ replaced by $\beta^*_{\mathrm{c}}(\rho)$.*

Remark. *The question whether $\beta^*_{\mathrm{c}} = \beta_{\mathrm{c}}$ or $\beta^*_{\mathrm{c}} < \beta_{\mathrm{c}}$ remains open, and so is the analogous question for the RPM. Note that when $Z^{\beta}_{t,Y}$ is uniformly bounded in $t > 0$, the distribution of $L_t(X,Y)$ under the measure $\mu^{\beta}_{t,Y}$ remains tight as $t \to \infty$, while $Z^{\beta}_{t,Y} \to \infty$ if and only if $L_t(X,Y)$ under $\mu^{\beta}_{t,Y}$ tends to ∞ in probability. If $\beta^*_{\mathrm{c}} < \beta_{\mathrm{c}}$, then there exists a phase in the delocalized regime where $L_t(X,Y)$ under $\mu^{\beta}_{t,Y}$ tends to ∞ at a rate that is $\mathrm{o}(t)$, which would be very surprising.*

Theorem 1.2 confirms a conjecture of Greven and den Hollander [11], Conjecture 1.8, that in $d = 3$, the Parabolic Anderson Model (PAM) with Brownian noise could admit an equilibrium measure with an infinite second moment. We refer to [3], Section 1.4, for a more detailed discussion on the connection between the RWPM and the PAM, as well as the connection of the discrete time RPMs and RWPMs with the directed polymer model in random environment.

Our proof of Theorem 1.2 will follow the general approach developed by Giacomin, Lacoin and Toninelli in [9,10] for proving the marginal relevance of disorder for the random pinning model (RPM), as well as by Lacoin in [13] for the study of the directed polymer model in random environment. The basic ingredients are change of measure arguments for bounding fractional moments of the partition function $Z^{\beta}_{t,Y}$, coupled with a coarse grain splitting of $Z^{\beta}_{t,Y}$. These techniques have proven to be remarkably powerful, and they apply to a wide range of models: in particular, to weighted renewal processes in random environments, including the random pinning, the random walk pinning, and the copolymer models (see [3], Section 1.4, for a more detailed discussion), as well as to weighted random walks in random environments, including the directed polymer model [13] and random walk in random environments [16]. We will recall in detail the fractional moment techniques and the coarse graining procedure and formulate them for the RWPM, which will constitute the model independent part of our analysis. A key element of the fractional moment technique involves a change of measure, and more generally, the choice of a suitable test function. This is the model dependent part of the analysis, which in general is far from trivial since disorder relevance in the critical dimension is a rather subtle effect. The bulk of this paper is thus dedicated to the choice of a suitable test function for the RWPM and its analysis. Compared to the RPM and the directed polymer model, new complications arise due to the different nature of the disorder of the RWPM.

We also include here a result on the monotonicity of $\beta_c(\rho) - \beta_c^{\text{ann}}(\rho)$, resp. $\beta_c^*(\rho) - \beta_c^{\text{ann}}(\rho)$, in ρ, which was pointed out to us by the referee along with an elegant proof.

Theorem 1.3 (Monotonicity of critical point shift). *Assuming the same conditions as in Theorem 1.1 for a RWPM in $\mathbb{Z}^d$ with $d \geq 3$, we have*

$$\left.\begin{aligned} &\frac{\beta_c(\rho')}{1+\rho'} \geq \frac{\beta_c(\rho)}{1+\rho}, \\ &\frac{\beta_c^*(\rho')}{1+\rho'} \geq \frac{\beta_c^*(\rho)}{1+\rho} \end{aligned}\right. \qquad \text{for all } \rho' > \rho \geq 0. \tag{1.8}$$

In particular,

$$\left.\begin{aligned} &\beta_c(\rho') - \beta_c^{\text{ann}}(\rho') > \beta_c(\rho) - \beta_c^{\text{ann}}(\rho), \\ &\beta_c^*(\rho') - \beta_c^{\text{ann}}(\rho') > \beta_c^*(\rho) - \beta_c^{\text{ann}}(\rho) \end{aligned}\right. \qquad \text{for all } \rho' > \rho \geq 0. \tag{1.9}$$

We defer its proof to Appendix C. We remark that proving the strict inequalities in (1.9) requires Theorem 1.2 and its analogue in dimensions $d \geq 4$.

Outline. The rest of the paper is organized as follows. In Section 2, we recall from [2] and [3] a representation of $Z^{\beta}_{t,Y}$ as the partition function of a weighted renewal process in random environment. In Section 3, we recall the coarse graining procedure and fractional moment techniques developed in [9,10,13]. To prove Theorem 1.2, we apply the coarse graining procedure to $Z^{\beta}_{t,Y}$ instead of the constrained partition function $Z^{\beta,\text{pin}}_{t,Y}$ as done in [9,10]. The proof of disorder relevance is then reduced in Section 3 to two key propositions: Proposition 3.1, which is model dependent and needs to be proved for any new weighted renewal process in random environment one is interested in, and Proposition 3.2, which is model independent. Compared to analogues of Proposition 3.1 formulated previously for the RPM (see [10], Lemma 3.1), our weaker formulation here (more precisely its reduction to Proposition 5.1 in Section 5) allows a more direct comparison with renewal processes without boundary constraints, which conceptually simplifies subsequent analysis. In Section 4, we identify a crucial test function $H_L(Y)$ for the disorder Y and state some essential properties. Assuming these properties, we then prove in Section 5 the key Proposition 3.1, which is further reduced to a model dependent Proposition 5.1 by extracting some model independent renewal calculations. In Section 6, we deduce Proposition 3.2 from Proposition 3.1, which is again model independent. The properties of the test function H_L are then established in Sections 7 and 8. In Appendices A and B, we prove some renewal and random walk estimates which we need for our proof. In particular, we prove in Lemma A.1 a continuous time version of Doney's local limit theorem [5], Theorem 3, for renewal processes with infinite mean. Finally, in Appendix C, we include a proof of Theorem 1.3 shown to us by the referee.

Note. During the preparation of this manuscript, we became aware of a preprint by Berger and Toninelli [1], in which they proved disorder relevance for the *discrete time* RWPM in dimension 3 under the assumption that the random walk increment is symmetric with sub-Gaussian tails. An inspection shows that the main difference between our two approaches lies in the choice of the test function $H_L(\cdot)$ in (4.2), which results in completely different model dependent analysis as well as different assumptions on the model. In principle, both approaches should be applicable to both discrete and continuous time models. Most results in this paper carry over directly to the discrete time case. The only exception is Lemma B.5, for which we do not have a proof for its discrete time analogue. Lemma B.5 is used to prove Lemma 4.2, (4.8)–(4.9). In light of [1], we will not pursue this further in this paper.

Notation. *Throughout the rest of this paper, unless stated otherwise, we will use C, C_1 and C_2 to denote generic constants whose precise values may change from line to line. However, their values all depend only on the jump rate ρ and the jump probability kernel $q(\cdot)$, and are uniform in $\rho \in (0, 1]$.*

2. Representation as a weighted renewal process in random environment

First we recall from [2] and [3] a representation of $Z^{\beta}_{t,Y}$ as the partition function of a weighted renewal process in random environment. Let $p_s(\cdot) = p^X_s(\cdot)$ denote the transition probability kernel of X at time s. Then Y and $X-Y$ have respective transition kernels $p^Y_s(\cdot) := p_{\rho s}(\cdot)$ and $p^{X-Y}_s(\cdot) := p_{(1+\rho)s}(\cdot)$. Let

$$G = \int_0^\infty p_s(0)\,\mathrm{d}s, \qquad G^{X-Y} = \int_0^\infty p_{(1+\rho)s}(0)\,\mathrm{d}s = \frac{G}{1+\rho}, \qquad K(t) = \frac{p^{X-Y}_t(0)}{G^{X-Y}} = \frac{(1+\rho)p_{(1+\rho)t}(0)}{G},$$

where $K(t)\,\mathrm{d}t$ is to be interpreted as the renewal time distribution of a recurrent renewal process $\sigma = \{\sigma_0 = 0 < \sigma_1 < \cdots\} \subset [0,\infty)$. Let $z = \beta G^{X-Y} = \beta G/(1+\rho)$ and $\mathcal{Z}^z_{t,Y} := Z^{\beta}_{t,Y}$. Then

$$\begin{aligned}
\mathcal{Z}^z_{t,Y} &= \mathbb{E}^X_0\bigl[\mathrm{e}^{\beta L_t(X,Y)}\bigr] = \mathbb{E}^X_0\left[1 + \sum_{m=1}^{\infty} \frac{\beta^m}{m!}\left(\int_0^t 1_{\{X_s = Y_s\}}\,\mathrm{d}s\right)^m\right] \\
&= \mathbb{E}^X_0\left[1 + \sum_{m=1}^{\infty} \beta^m \int\cdots\int_{0<\sigma_1<\cdots<\sigma_m<t} 1_{\{X_{\sigma_1}=Y_{\sigma_1},\ldots,X_{\sigma_m}=Y_{\sigma_m}\}}\,\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_m\right] \\
&= 1 + \sum_{m=1}^{\infty} \beta^m \int\cdots\int_{0<\sigma_1<\cdots<\sigma_m<t} p_{\sigma_1}(Y_{\sigma_1})\,p_{\sigma_2-\sigma_1}(Y_{\sigma_2}-Y_{\sigma_1})\cdots \\
&\qquad \times p_{\sigma_m-\sigma_{m-1}}(Y_{\sigma_m}-Y_{\sigma_{m-1}})\,\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_m \\
&= 1 + \sum_{m=1}^{\infty} z^m \int\cdots\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m<t} \prod_{i=1}^{m} K(\sigma_i-\sigma_{i-1})W(\sigma_i-\sigma_{i-1}, Y_{\sigma_i}-Y_{\sigma_{i-1}})\,\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_m,
\end{aligned} \tag{2.1}$$

where

$$W(\sigma_i-\sigma_{i-1}, Y_{\sigma_i}-Y_{\sigma_{i-1}}) = \frac{p^X_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})}{p^{X-Y}_{\sigma_i-\sigma_{i-1}}(0)}. \tag{2.2}$$

We can thus interpret $\mathcal{Z}^z_{t,Y}$ as the partition function of a weighted renewal process σ in the random environment Y, where the renewal time distribution is given by $K(\cdot)$, and the ith renewal return incurs a weight factor of $zW(\sigma_i - \sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}})$.

Similarly, for any $0 \le U \le V$, we can define $\mathcal{Z}^{z,\mathrm{pin}}_{[U,V],Y} := 1$ when $U = V$, and otherwise

$$\mathcal{Z}^{z,\mathrm{pin}}_{[U,V],Y} := \sum_{m=1}^{\infty} \int \cdots \int_{\sigma_0 = U < \sigma_1 < \cdots < \sigma_m = V} z^m \prod_{i=1}^{m} K(\sigma_i - \sigma_{i-1}) \\ \times W(\sigma_i - \sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}})\, \mathrm{d}\sigma_1 \cdots \mathrm{d}\sigma_{m-1}, \tag{2.3}$$

where the term for $m = 1$ is defined to be $zK(V-U)W(V-U, Y_V - Y_U)$. Note that $\mathcal{Z}^{z,\mathrm{pin}}_{[0,t],Y} = \beta Z^{\beta,\mathrm{pin}}_{t,Y}$, which we will simply denote by $\mathcal{Z}^{z,\mathrm{pin}}_{t,Y}$.

Since K is the renewal time distribution of a recurrent renewal process σ on $[0,\infty)$, and note that $\mathbb{E}^Y_0[W(v-u, Y_v - Y_u)] = 1$ for any $u < v$, the critical point $z^{\mathrm{ann}}_{\mathrm{c}}$ of the annealed model with partition function $\mathbb{E}^Y_0[\mathcal{Z}^z_{t,Y}]$ is exactly 1. By the mapping $z = \beta G^{X-Y}$, we deduce that $\beta^{\mathrm{ann}}_{\mathrm{c}} = 1/G^{X-Y} = (1+\rho)/G$. The mapping to a weighted renewal process in random environment casts the RWPM in the same framework as the RPM, which paves the way for the application of general approaches developed in [9,10].

3. Fractional moment techniques and coarse graining

We now recall the fractional moment techniques and the coarse graining procedure, which were developed in a series of papers for the RPM that culminated in [9,10], where marginal relevance of disorder was established, as well as in [13] where the same techniques were applied to the directed polymer model in random environment.

By (1.2), $Z^{\beta}_{t,Y} = \mathcal{Z}^z_{t,Y}$ is monotonically increasing in t for every realization of Y. Therefore, to prove Theorem 1.2, it suffices to show that for some $\gamma \in (0,1)$, and some $z > 1$ depending suitably on ρ, we have

$$\sup_{t>0} \mathbb{E}^Y_0\big[\big(\mathcal{Z}^z_{t,Y}\big)^{\gamma}\big] < \infty. \tag{3.1}$$

We will choose below a coarse graining scale L and show that for each $\rho > 0$, (3.1) holds for all $z \in (1, 1+1/L]$ if L is sufficiently large, which implies $z_{\mathrm{c}} \ge 1 + 1/L$ and $\beta^*_{\mathrm{c}} - \beta^{\mathrm{ann}}_{\mathrm{c}} \ge (1+\rho)/GL$. For $\rho \in (0,1]$, we will let $L = \mathrm{e}^{B_1/\rho^{\zeta}}$ for any fixed $\zeta > 2$, and prove that (3.1) holds for all $z \in (1, 1+1/L]$ uniformly in $\rho \in (0,1]$ if B_1 is large enough. This would then imply the lower bound on $\beta^*_{\mathrm{c}} - \beta^{\mathrm{ann}}_{\mathrm{c}}$ in Theorem 1.2.

Note that by using more refined large deviation estimates for the renewal process with waiting time density $K(\cdot)$, it seems possible to extend (3.1) to $z \in (1, 1+1/L^{\eta}]$ with a suitable $\eta \in (1/2, 1)$. By the relation between L and ρ in our coarse-graining scheme, this would only affect the (unspecified) constant $C(\zeta)$ in (1.6), not the exponent ζ itself.

To bound the fractional moment $\mathbb{E}^Y_0[(\mathcal{Z}^z_{t,Y})^{\gamma}]$, we apply the inequality

$$\left(\sum_{i=1}^{n} a_i\right)^{\gamma} \le \sum_{i=1}^{n} a_i^{\gamma} \quad \text{for } a_i \ge 0,\ 1 \le i \le n \text{ and } \gamma \in (0,1). \tag{3.2}$$

This seemingly trivial inequality turns out to be exceptionally powerful in bounding fractional moments. However, the success of such a bound depends crucially on how $\mathcal{Z}^z_{t,Y}$ is split into a sum of terms. This is where coarse graining comes in, which was used in [9,10,13]. We remark that in the earlier paper [4] on the RPM, and later in the analysis [3] of the RWPM in $d \ge 4$, $\mathcal{Z}^z_{t,Y}$ is partitioned according to the values of the pair of consecutive renewal times $\sigma_i < \sigma_{i+1}$ which straddle a fixed time $L > 0$. The coarse graining procedure we recall below uses a more refined partition of $\mathcal{Z}^z_{t,Y}$.

Fix a large constant $L > 0$, which will be the coarse graining scale. Assume that $t = mL$ for some $m \in \mathbb{N}$. Then we partition $(0,t]$ into m blocks $\Lambda_1, \ldots, \Lambda_m$ with $\Lambda_i := ((i-1)L, iL]$. The coarse graining procedure simply groups terms in (2.1) according to which blocks Λ_i does the renewal configuration $\sigma := \{\sigma_0 = 0 < \sigma_1 < \cdots\}$ intersect. More precisely, the set of blocks in $\{\Lambda_i\}_{1 \le i \le m}$ which σ intersects can be represented by a set $I \subset \{1, \ldots, m\}$. Then we can decompose $\mathcal{Z}^z_{t,Y}$ in (2.1) as

$$\mathcal{Z}^z_{t,Y} = \sum_{I \subset \{1,\ldots,m\}} \mathcal{Z}^{z,I}_{t,Y},$$

where $\mathcal{Z}^{z,\varnothing}_{t,Y} := 1$, and for each $I = \{1 \le i_1 < i_2 < \cdots < i_k \le m\} \neq \varnothing$,

$$\mathcal{Z}^{z,I}_{t,Y} = \int_{\substack{a_1<b_1\\ a_1,b_1\in\Lambda_{i_1}}} \cdots \int_{\substack{a_k<b_k\\ a_k,b_k\in\Lambda_{i_k}}} \prod_{j=1}^{k} K(a_j - b_{j-1}) z W(a_j - b_{j-1}, Y_{a_j} - Y_{b_{j-1}}) \mathcal{Z}^{z,\mathrm{pin}}_{[a_j,b_j],Y} \prod_{j=1}^{k} \mathrm{d}a_j\,\mathrm{d}b_j, \tag{3.3}$$

where $b_0 := 0$. By (3.2), for any $\gamma \in (0,1)$, we have

$$\mathbb{E}^Y_0\big[\big(\mathcal{Z}^z_{t,Y}\big)^\gamma\big] \le \sum_{I\subset\{1,\ldots,m\}} \mathbb{E}^Y_0\big[\big(\mathcal{Z}^{z,I}_{t,Y}\big)^\gamma\big]. \tag{3.4}$$

We will prove (3.1) by comparing $\mathbb{E}^Y_0[(\mathcal{Z}^{z,I}_{t,Y})^\gamma]$ with the probability that a subcritical renewal process on $\mathbb{N}\cup\{0\}$ intersects $\{1,\ldots,m\}$ exactly at I.

To bound $\mathbb{E}^Y_0[(\mathcal{Z}^{z,I}_{t,Y})^\gamma]$, one introduces a change of measure. Let $f_I(Y)$ be a non-negative function of the disorder Y. By Hölder's inequality,

$$\begin{aligned}\mathbb{E}^Y_0\big[\big(\mathcal{Z}^{z,I}_{t,Y}\big)^\gamma\big] &= \mathbb{E}^Y_0\big[f_I(Y)^\gamma f_I(Y)^{-\gamma}\big(\mathcal{Z}^{z,I}_{t,Y}\big)^\gamma\big] \\ &\le \mathbb{E}^Y_0\big[f_I(Y)^{-\gamma/(1-\gamma)}\big]^{1-\gamma}\mathbb{E}^Y_0\big[f_I(Y)\mathcal{Z}^{z,I}_{t,Y}\big]^\gamma.\end{aligned} \tag{3.5}$$

To decouple different blocks Λ_i, we will let $f_I(Y) = \prod_{i\in I} f((Y_s - Y_{(i-1)L})_{s\in\Lambda_i})$ with

$$\mathbb{E}^Y_0\big[f\big((Y_s)_{0\le s\le L}\big)^{-\gamma/(1-\gamma)}\big] \le 2. \tag{3.6}$$

To make $\mathbb{E}^Y_0[(\mathcal{Z}^{z,I}_{t,Y})^\gamma]$ small, f should be chosen to make $\mathbb{E}^Y_0[f_I(Y)\mathcal{Z}^{z,I}_{t,Y}]$ small. There have been two approaches in bounding $\mathbb{E}^Y_0[f_I(Y)\mathcal{Z}^{z,I}_{t,Y}]$ in the literature.

The first approach is to choose $f_I(Y)$ to be a probability density so that $\mathbb{E}^Y_0[f_I(Y)\mathcal{Z}^{z,I}_{t,Y}]$ becomes the annealed partition function of a RWPM with a new law for the disorder Y. This approach was used in [4] to prove disorder relevance for the RPM, where the laws of the disorder at different time points are independently tilted to favor delocalization. It was later adapted to the RWPM in dimensions $d \ge 4$ in [3], where the change of measure for Y increases its jump rate, which turns out to favor delocalization. To prove disorder relevance for the RPM at the critical dimension, the so-called *marginal disorder relevance*, which borderlines the known disorder relevance/irrelevance regimes, a more sophisticated change of measure was introduced in [9] for the RPM with Gaussian disorder, which induces negative correlation between the disorder at different time points, and Gaussian calculations are used to estimate the annealed partition function under the new disorder. For the RWPM in the critical dimension $d = 3$, the analogue would be to introduce correlation between the increments of Y at different time steps. However the presence of correlation makes it unfeasible to estimate the annealed partition function under the new disorder.

A variant approach to estimate $\mathbb{E}^Y_0[f_I(Y)\mathcal{Z}^{z,I}_{t,Y}]$ was then introduced in [13] for the directed polymer model in random environment, and later in [10] for the RPM at the critical dimension with general disorder. The function f_I will be taken to be a test function on the disorder Y instead of as a probability density that changes the law of Y. For simplicity, f in (3.6) is taken to be of the form

$$f\big((Y_j)_{0\le s\le L}\big) = 1_{\{H_L(Y)\le M\}} + \varepsilon_M 1_{\{H_L(Y)>M\}}, \tag{3.7}$$

where $H_L(Y)$ is a functional of the disorder Y, positively correlated with $\mathcal{Z}^{z,I}_{t,Y}$, and we choose

$$\varepsilon_M = \mathbb{P}^Y_0\big(H_L(Y) > M\big)^{(1-\gamma)/\gamma} \tag{3.8}$$

to guarantee that (3.6) holds. We will make ε_M small by choosing M large. To bound $\mathbb{E}_0^Y[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}]$, we use the representation (3.3) to write

$$f_I(Y)\mathcal{Z}_{t,Y}^{z,I} = \int_{\substack{a_1<b_1\\ a_1,b_1\in\Lambda_{i_1}}} \cdots \int_{\substack{a_k<b_k\\ a_k,b_k\in\Lambda_{i_k}}} \prod_{j=1}^{k} K(a_j-b_{j-1})zW(a_j-b_{j-1}, Y_{a_j}-Y_{b_{j-1}})$$

$$\times \mathcal{Z}_{[a_j,b_j],Y}^{z,\mathrm{pin}} f\big((Y_s-Y_{(i_j-1)L})_{s\in\Lambda_{i_j}}\big)\prod_{j=1}^{k}\mathrm{d}a_j\,\mathrm{d}b_j,$$

where $b_0:=0$. By Lemma B.1, the local central limit theorem for X and $X-Y$, there exists $C>0$ such that uniformly in $t>0$ and Y, we have

$$W(t,Y_t-Y_0)=\frac{p_t^X(Y_t-Y_0)}{p_t^{X-Y}(0)}=\frac{p_t(Y_t-Y_0)}{p_{(1+\rho)t}(0)}\le C. \tag{3.9}$$

Therefore

$$\begin{aligned}&\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big]\\ &\quad\le \int_{\substack{a_1<b_1\\ a_1,b_1\in\Lambda_{i_1}}} \cdots \int_{\substack{a_k<b_k\\ a_k,b_k\in\Lambda_{i_k}}} (Cz)^k\prod_{j=1}^{k}K(a_j-b_{j-1})\mathbb{E}_0^Y\big[\mathcal{Z}_{[a_j,b_j],Y}^{z,\mathrm{pin}} f\big((Y_s-Y_{(i_j-1)L})_{s\in\Lambda_{i_j}}\big)\big]\prod_{j=1}^{k}\mathrm{d}a_j\,\mathrm{d}b_j,\end{aligned} \tag{3.10}$$

where we used the independence of $(Y_s-Y_{(i-1)L})_{s\in\Lambda_i}$, $i\in\mathbb{N}$. The proof of (3.1), and hence Theorem 1.2, then hinges on the following key proposition.

Proposition 3.1. *Let $\rho>0$, and let $L=\mathrm{e}^{B_1\rho^{-\zeta}}$ for any fixed $\zeta>2$. Then for every $\varepsilon>0$ and $\delta>0$, we can find suitable choices of $H_L(\cdot)$ and $M=M(L)$ in (3.7), such that for $B_1=B_1(\rho)$ sufficiently large, which can be chosen uniformly for $\rho\in(0,1]$, and for all $z\in(1,1+L^{-1}]$, $a\in[0,(1-3\varepsilon)L]$ and $c>L$, we have*

$$\int_{a+\varepsilon L}^{(1-\varepsilon)L}\mathbb{E}_0^Y\big[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}} f\big((Y_s)_{s\in[0,L]}\big)\big]\,\mathrm{d}b\le\delta\int_a^L P(b-a)\,\mathrm{d}b, \tag{3.11}$$

$$\int_{a+\varepsilon L}^{(1-\varepsilon)L}\mathbb{E}_0^Y\big[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}} f\big((Y_s)_{s\in[0,L]}\big)\big]K(c-b)\,\mathrm{d}b\le\delta\int_a^L P(b-a)K(c-b)\,\mathrm{d}b, \tag{3.12}$$

where

$$P(t)=\sum_{m=1}^{\infty}\int\cdots\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=t}\prod_{i=1}^{m}K(\sigma_i-\sigma_{i-1})\prod_{i=1}^{m-1}\mathrm{d}\sigma_i, \tag{3.13}$$

with term for $m=1$ defined to be $K(t)$, is the renewal density associated with $K(\cdot)$.

We will show that Proposition 3.1 implies the following:

Proposition 3.2. *Let ρ, B_1, ζ, L, H_L and $M(L)$ be as in Proposition 3.1. Then for every $\eta>0$, we can choose $B_1(\rho)$ sufficiently large, which can be chosen uniformly for $\rho\in(0,1]$, such that for all $z\in(1,1+L^{-1}]$, $m\in\mathbb{N}$, and $I=\{i_1<i_2<\cdots<i_k\}\subset\{1,\dots,m\}$, we have*

$$\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big]\le C_L\prod_{j=1}^{k}\frac{\eta}{(i_j-i_{j-1})^{3/2}} \tag{3.14}$$

for some $C_L>1$ depending only on L.

By (3.4) and (3.5), Proposition 3.2 implies that uniformly in $t = mL$, $m \in \mathbb{N}$, we have

$$\mathbb{E}_0^Y\big[\big(\mathcal{Z}_{t,Y}^z\big)^\gamma\big] \le \sum_{k=0}^{\infty}\sum_{\substack{I\subset\mathbb{N}\\|I|=k}} \mathbb{E}_0^Y\big[\big(\mathcal{Z}_{t,Y}^{z,I}\big)^\gamma\big] \le \sum_{k=0}^{\infty}\sum_{I=\{i_1<\cdots<i_k\}} C_L^\gamma \prod_{j=1}^{k} \frac{\eta^\gamma 2^{(1-\gamma)}}{(i_j-i_{j-1})^{3\gamma/2}} \le C_L^\gamma \sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{\eta^\gamma 2^{1-\gamma}}{n^{3\gamma/2}}\right)^k,$$

which is finite if we choose $\gamma \in (2/3, 1)$, and $\eta > 0$ sufficiently small such that $\sum_{n=1}^\infty \frac{\eta^\gamma 2^{1-\gamma}}{n^{3\gamma/2}} < 1$. By the monotonicity of $\mathcal{Z}_{t,Y}^z$ in t, this implies (3.1), and hence Theorem 1.2 by the discussion following (3.1).

The key is therefore Proposition 3.1, which is the model dependent part and whose proof will be the focus of the rest of this paper. The new idea developed in [13] and [10] to bound quantities like $\mathbb{E}_0^Y[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}} f((Y_s)_{s\in[0,L]})]$ is to use the renewal representation (2.3) to write

$$\begin{aligned}
&\mathbb{E}_0^Y\big[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}} f\big((Y_s)_{s\in[0,L]}\big)\big]\\
&\quad= \sum_{k=1}^{\infty}\int\cdots\int_{\sigma_0=a<\cdots<\sigma_k=b} z^k \prod_{i=1}^{k} K(\sigma_i-\sigma_{i-1})\\
&\qquad\times \mathbb{E}_0^Y\left[f\big((Y_s)_{s\in[0,L]}\big)\prod_{i=1}^{k} W(\sigma_i-\sigma_{i-1}, Y_{\sigma_i}-Y_{\sigma_{i-1}})\right]\prod_{i=1}^{k-1}\mathrm{d}\sigma_i\\
&\quad= \sum_{k=1}^{\infty}\int\cdots\int_{\sigma_0=a<\cdots<\sigma_k=b} z^k \prod_{i=1}^{k} K(\sigma_i-\sigma_{i-1})\mathbb{E}_0^{Y^\sigma}\big[f\big(\big(Y_s^\sigma\big)_{s\in[0,L]}\big)\big]\prod_{i=1}^{k-1}\mathrm{d}\sigma_i,
\end{aligned}\tag{3.15}$$

where $\prod_{i=1}^k W(\sigma_i-\sigma_{i-1}, Y_{\sigma_i}-Y_{\sigma_{i-1}})$ has been interpreted as the density for a change of measure for Y, and $\mathbb{E}_0^{Y^\sigma}[\cdot]$ denotes expectation with respect to a random path $(Y_s^\sigma)_{0\le s\le t}$ whose law is absolutely continuous with respect to that of $(Y_s)_{0\le s\le t}$ with density $\prod_{i=1}^k W(\sigma_i-\sigma_{i-1}, Y_{\sigma_i}-Y_{\sigma_{i-1}})$. Recall the form of f in (3.7): the key point is to choose the functional H_L such that for typical realizations of σ and Y^σ, $H_L((Y_s^\sigma)_{s\in[0,L]})$ is much larger than typical values of $H_L((Y_s)_{s\in[0,L]})$. Then in (3.7), we can choose M large such that $\varepsilon_M \ll 1$ and $\mathbb{E}_0^{Y^\sigma}[f((Y_s^\sigma)_{s\in[0,1]})] \ll 1$. The factor z^k in (3.15) can be bounded by a constant of order 1 if $z \in (1, 1+L^{-1}]$, since conditioned on the renewal process σ with $a < b \in \sigma$, the number of renewal returns in $[a,b]$ is typically of the order $\sqrt{b-a} \le \sqrt{L}$.

Remark. *The above procedure applies to general weighted renewal processes in random environments, whose partition functions can be represented in the form of* (2.3) *and* (2.1), *where given a random environment $(\Omega_s)_{s\ge 0}$ with stationary independent increments and a renewal configuration $\sigma := \{\sigma_0 = 0 < \sigma_1 < \cdots\}$, each two consecutive renewal times $\sigma_i < \sigma_{i+1}$ give rise to a weight factor $zW(\sigma_{i+1}-\sigma_i, (\Omega_s-\Omega_{\sigma_i})_{\sigma_i<s\le\sigma_{i+1}})$. See, e.g.,* [3], *Section* 1.3, *for a more detailed exposition on how random pinning, random walk pinning, and copolymer models can all be seen as renewal processes in random environments with different weight factors W. With proper normalization, $W(\sigma_{i+1}-\sigma_i, (\Omega_s-\Omega_{\sigma_i})_{\sigma_i<s\le\sigma_{i+1}})$ can always be interpreted as a change of measure for the disorder Ω.*

4. Mean and variance of $H_L(Y)$ and $H_L(Y^\sigma)$

We will now choose the functional $H_L(\cdot)$ in (3.7), state its essential properties, and briefly outline how these properties may lead to Proposition 3.1. Given a renewal configuration $\sigma := \{\sigma_0 = a < \cdots < \sigma_k = b\}$, the new disorder random walk Y^σ introduced in (3.15) has heuristically smaller fluctuations than Y due to the density $\prod W(\sigma_i-\sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}}) = \prod \frac{p^X(Y_{\sigma_i}-Y_{\sigma_{i-1}})}{p^{X-Y}(\sigma_i-\sigma_{i-1})}$ which favors values of Y_{σ_i} that are close to $Y_{\sigma_{i-1}}$. A natural choice for H_L in (3.7) is then the following. Fix $A_1 := \mathrm{e} < A_2 < \infty$, where later in the proof of Proposition 3.1 we will set $A_2 = L^{1/8}$. The reason for this choice of A_2 will be explained briefly at the end of this section. Given $\zeta > 2$ as in Theorem 1.1, let

$$\xi := 1 - \zeta^{-1} > \frac{1}{2}. \tag{4.1}$$

Then we define

$$H_L(Y) = H_L\big((Y_s - Y_0)_{0 \le s \le L}\big) := \int\int_{\substack{0<r<s<L \\ A_1<s-r<A_2}} \frac{1_{\{Y_r=Y_s\}}}{(\log(s-r))^{\xi}}\, \mathrm{d}r\, \mathrm{d}s. \tag{4.2}$$

We have the following bound on the mean and variance of $H_L(Y)$.

Lemma 4.1. *Let $H_L(Y)$ be defined as in (4.2). Then*

$$\mathbb{E}_0^Y\big[H_L(Y)\big] = \int\int_{\substack{0<r<s<L \\ A_1<s-r<A_2}} \frac{p_{\rho(s-r)}(0)}{(\log(s-r))^{\xi}}\, \mathrm{d}r\, \mathrm{d}s \le (A_2 - A_1)L, \tag{4.3}$$

and there exists some $0 < C < \infty$ such that uniformly for all $A_1 = \mathrm{e} < A_2 < \infty$ and $\rho > 0$,

$$\mathrm{Var}\big(H_L(Y)\big) \le C\rho^{-3}L. \tag{4.4}$$

Remark. *The condition $\xi > \frac{1}{2}$ guarantees the validity of (4.4). The technical reason for the relation $\xi = 1 - 1/\zeta$, hence $\zeta > 2$, will become evident in the proof Lemma 5.1, see especially (5.15), below.*

To show that in (3.15), $\mathbb{E}_0^{Y^\sigma}[f((Y_s^\sigma)_{s\in[0,L]})]$ is small for typical realizations of σ, it then suffices to show that for typical realizations of σ and $(Y_s^\sigma)_{s\in[0,L]}$, $H_L(Y^\sigma) > \mathbb{E}_0^Y[H_L(Y)] + D\rho^{-3/2}\sqrt{L}$, where D can be made arbitrarily large by choosing B_1 large in $L = \mathrm{e}^{B_1\rho^{-\zeta}}$, with B_1 uniform for $\rho \in (0,1]$. Thus we need to bound the mean and variance of $H_L(Y^\sigma)$ conditioned on σ. Recall that given $\sigma = \{\sigma_0 = a < \sigma_1 < \cdots < \sigma_k = b\} \subset [0,L]$, the law of $(Y_s^\sigma)_{s\ge 0}$ is absolutely continuous with respect to the law of $(Y_s)_{s\ge0}$ with density $\prod_{i=1}^k \frac{p^X_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})}{p^{X-Y}_{\sigma_i-\sigma_{i-1}}(0)}$. Due to the dependency structure of Y^σ, we will decompose $H_L(Y^\sigma)$ in (4.2) according to whether or not the variables of integration $r < s$ satisfy $(r,s) \cap \{\sigma_0 = a < \cdots < \sigma_k = b\} = \varnothing$ in order to extract some independence. Namely,

$$H_L\big(Y^\sigma\big) = H^{\mathrm{int}}_{[0,a]}\big(Y^\sigma\big) + H^{\mathrm{int}}_{[b,L]}\big(Y^\sigma\big) + \sum_{i=1}^k H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big) + H^{\mathrm{ext}}_{[0,a]}\big(Y^\sigma\big) + \sum_{i=1}^k H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big) - C_{\sigma,Y^\sigma}, \tag{4.5}$$

where for any $s < t$,

$$\begin{aligned} H^{\mathrm{int}}_{[s,t]}\big(Y^\sigma\big) &:= \int\int_{\substack{s<s_1<s_2<t \\ A_1<s_2-s_1<A_2}} \frac{1_{\{Y^\sigma_{s_1}=Y^\sigma_{s_2}\}}}{(\log(s_2-s_1))^{\xi}}\, \mathrm{d}s_2\, \mathrm{d}s_1, \\ H^{\mathrm{ext}}_{[s,t]}\big(Y^\sigma\big) &:= \int\int_{\substack{s<s_1<s_2<t \\ A_1<s_2-s_1<A_2}} \frac{1_{\{Y^\sigma_{s_1}=Y^\sigma_{s_2}\}}}{(\log(s_2-s_1))^{\xi}}\, \mathrm{d}s_2\, \mathrm{d}s_1 \end{aligned} \tag{4.6}$$

and

$$C_{\sigma,Y^\sigma} := \int\int_{\substack{0<s_1<\sigma_k, s_2>L \\ A_1<s_2-s_1<A_2}} \frac{1_{\{Y^\sigma_{s_1}=Y^\sigma_{s_2}\}}}{(\log(s_2-s_1))^{\xi}}\, \mathrm{d}s_2\, \mathrm{d}s_1$$

arises because $H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}$ may include pair correlation terms $1_{\{Y^\sigma_{s_1}=Y^\sigma_{s_2}\}}$ with $s_1 < L < s_2$, which is excluded in the definition of H_L. Note that

$$C_{\sigma,Y^\sigma} \le A_2^2 \quad \text{and} \quad H^{\mathrm{ext}}_{[s,t]}\big(Y^\sigma\big) \le A_2^2 \quad \text{for all } \sigma, Y^\sigma, s<t \text{ and } \rho > 0. \tag{4.7}$$

Conditional on σ, for any two consecutive renewal times $\sigma_{i-1} < \sigma_i$, we then have the following bounds on the mean of $H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ and $H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$, and the variance of $H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$.

Lemma 4.2. *For any* $A_1 := \mathrm{e} < A_2 < \infty$ *and* $\sigma := \{\sigma_0 = a < \sigma_1 < \cdots < \sigma_k = b\} \subset (0, L]$, *we have*

$$\mathbb{E}_0^{Y^\sigma}\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big)\big] - \mathbb{E}_0^Y\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] > 0, \quad 1 \le i \le k, \tag{4.8}$$

$$\mathbb{E}_0^{Y^\sigma}\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big)\big] - \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] > 0, \quad 1 \le i \le k. \tag{4.9}$$

Furthermore,

$$\mathbb{E}_0^{Y^\sigma}\big[H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}\big(Y^\sigma\big)\big] - \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}(Y)\big] > \frac{C\sqrt{\sigma_1 - \sigma_0}}{\sqrt{\rho}(\log(\sigma_1 - \sigma_0))^\xi} 1_{\{2A_1 < \sigma_1 - \sigma_0 < A_2\}} \tag{4.10}$$

and

$$\mathrm{Var}\big(H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}\big(Y^\sigma\big)\big|\sigma\big) \le C\rho^{-3}(\sigma_1 - \sigma_0), \tag{4.11}$$

where $\mathrm{Var}(\cdot|\sigma)$ *denotes variance w.r.t.* Y^σ *conditioned on* σ, *and the* C*s in* (4.10)–(4.11) *are uniform in* A_2 *and* $\rho > 0$.

Let us sketch briefly how Lemma 4.2 can be used to deduce Proposition 3.1. Let σ be a renewal process conditioned on $\sigma_0 = a$, and let Y^σ be defined accordingly by changing the measure of Y independently on each renewal interval (σ_{i-1}, σ_i). By the discussion following (3.15), the key to proving Proposition 3.1 is to show that for typical σ and Y^σ, $H_L(Y^\sigma)$ is much larger than typical values of $H_L(Y)$. Using the decomposition (4.5), this is achieved by controlling the mean and variance of $\sum_{i=1}^k H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ and $\sum_{i=1}^k H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ conditional on σ, which is facilitated by Lemma 4.2. If we only want to establish disorder relevance for a fixed $\rho > 0$, which amounts to proving Proposition 3.1 (more precisely, its reduction, Proposition 5.1) by choosing L sufficiently large for a fixed ρ, we can avoid quantitative estimates by simply applying the law of large numbers to the i.i.d. sequence $(\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)])_{i\in\mathbb{N}}$ and the ergodic theorem to the ergodic sequence $(H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma))_{i\in\mathbb{N}}$ to show that, if L is sufficiently large, then for typical σ, the conditional mean of

$$\sum_{i=1}^k H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big) - \sum_{i=1}^k \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] \tag{4.12}$$

far exceeds its conditional standard deviation as well as the standard deviation of $H_L(Y)$, which are of the order $C\rho^{-3/2}\sqrt{L}$ by (4.4) and (4.11); while for typical σ and Y^σ,

$$\sum_{i=1}^k H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big) - \sum_{i=1}^k \mathbb{E}_0^Y\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] > 0.$$

To get quantitative bounds on the gap between the annealed and the quenched critical points, we need to get bounds on L, and this is achieved by replacing the law of large numbers above with a quantitative estimate on $\sum_{i=1}^k \mathbb{E}_0^{Y^\sigma}[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)]$, and replacing the ergodic theorem with a quantitative bound on the conditional variance of $\sum_{i=1}^k H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ and then applying the Markov inequality. The details will be given in Section 5.

The reason for choosing $A_2 = L^{1/8}$ in the definition of $H_L(Y)$ is the following. When we lower bound the conditional mean in (4.12) (conditional on σ) using (4.10), we need to choose A_2 as large as possible. It turns out that any power of L will suffice, as will be seen in the proof of Lemma 5.3. On the other hand, when we upper bound the conditional variance of $\sum_{i=1}^k H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$, we need to choose A_2 to be a sufficiently small power of L, which can be seen from the bounds in (4.7) as well as in the proof of Lemma 5.2. The choice $A_2 = L^{1/8}$ turns out to be sufficient for our purposes.

5. Proof of Proposition 3.1

We now prove Proposition 3.1 using the functional H_L defined in (4.2) and Lemmas 4.1 and 4.2. We remark that Proposition 3.1 is the analogue of [10], Lemma 3.1, formulated for the discrete time random pinning model. The main difference is that [10], Lemma 3.1, involves a comparison of the integrands on both sides of (3.12) for each b with $b-a \geq \varepsilon L$. Our integral formulation of (3.12) allows us to reduce more easily estimates involving renewal configurations pinned at two points $a<b$ to renewal configurations pinned only at a. More precisely, we reduce Proposition 3.1 to the following proposition by extracting some model independent renewal calculations.

Proposition 5.1. *Let $A_1 = \mathrm{e}$ and $A_2 = L^{1/8}$ in the definition of H_L in (4.2), with $L = \mathrm{e}^{B_1\rho^{-\zeta}}$ as in Proposition 3.1. Then for every $\varepsilon > 0$ and $\delta > 0$, we can find $D > 0$ and set $M = \mathbb{E}_0^Y[H_L(Y)] + D\rho^{-3/2}\sqrt{L}$ in (3.7), such that for all $B_1 = B_1(\rho)$ sufficiently large, which can be chosen uniformly for $\rho \in (0,1]$, and for all $z \in (1, 1+L^{-1}]$ and $a \in [0,(1-3\varepsilon)L]$, we have*

$$\int_a^L \mathbb{E}_0^Y\big[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f\big((Y_s)_{s\in[0,L]}\big)\big] K\big([L-b,\infty)\big)\,\mathrm{d}b \leq \delta, \tag{5.1}$$

where $K([x,\infty)) = \int_x^\infty K(t)\,\mathrm{d}t = \int_x^\infty (1+\rho)G^{-1}p_{(1+\rho)t}(0)\,\mathrm{d}t$.

Proof of Proposition 3.1. The deduction of Proposition 3.1 from Proposition 5.1 is model independent and depends only on $K(\cdot)$. Since $K(t) \sim \frac{C}{(1+\rho)^{1/2}t^{3/2}}$, we have $K([t,\infty)) = \int_t^\infty K(s)\,\mathrm{d}s \sim \frac{2C}{\sqrt{(1+\rho)t}}$. By Lemma A.1, we also have $P(t) \sim \frac{C}{\sqrt{(1+\rho)t}}$, where $P(t)$ is defined in (3.13). Therefore, given $\varepsilon > 0$ and for B_1 (and hence L) large, there exist C_1 and C_2 depending only on $\varepsilon > 0$, such that uniformly for all $\varepsilon L \leq a + \varepsilon L \leq b_1, b_2 \leq (1-\varepsilon)L$ and $c > L$, we have

$$C_1 \leq \frac{P(b_1-a)}{P(b_2-a)} \leq C_2, \qquad C_1 \leq \frac{K(c-b_1)}{K(c-b_2)} \leq C_2. \tag{5.2}$$

Under the assumptions of Proposition 3.1, by Proposition 5.1, we have

$$\begin{aligned}
\delta &\geq \int_{a+\varepsilon L}^{(1-\varepsilon)L} \frac{\mathbb{E}_0^Y[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})]}{P(b-a)} P(b-a)K\big([L-b,\infty)\big)\,\mathrm{d}b \\
&\geq \frac{C}{L}\int_{a+\varepsilon L}^{(1-\varepsilon)L} \frac{\mathbb{E}_0^Y[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})]}{P(b-a)}\,\mathrm{d}b,
\end{aligned}$$

where C depends only on ε and ρ and is uniform for $\rho \in (0,1]$. Together with (5.2), this implies

$$\begin{aligned}
&\int_{a+\varepsilon L}^{(1-\varepsilon)L} \mathbb{E}_0^Y\big[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f\big((Y_s)_{s\in[0,L]}\big)\big] K(c-b)\,\mathrm{d}b \\
&\quad = \int_{a+\varepsilon L}^{(1-\varepsilon)L} \frac{\mathbb{E}_0^Y[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})]}{P(b-a)} P(b-a)K(c-b)\,\mathrm{d}b \\
&\quad = \int_{a+\varepsilon L}^{(1-\varepsilon)L} \frac{\mathbb{E}_0^Y[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})]}{P(b-a)} \left(\int_{a+\varepsilon L}^{(1-\varepsilon)L} \frac{P(b_2-a)K(c-b_2)}{P(b-a)K(c-b)}\,\mathrm{d}b_2\right)^{-1}\mathrm{d}b \\
&\qquad \times \int_{a+\varepsilon L}^{(1-\varepsilon)L} P(b-a)K(c-b)\,\mathrm{d}b \\
&\quad \leq \frac{1}{C_1^2}\frac{1}{(1-\varepsilon)L-(a+\varepsilon L)}\frac{\delta L}{C}\int_a^L P(b-a)K(c-b)\,\mathrm{d}b \leq \frac{\delta}{\varepsilon C C_1^2}\int_a^L P(b-a)K(c-b)\,\mathrm{d}b,
\end{aligned}$$

where we used the assumption that $a \leq (1-3\varepsilon)L$. Since given $\varepsilon > 0$, we can choose $\delta > 0$ arbitrarily small by Proposition 5.1, (3.12) then follows. The proof of (3.11) is similar. Note that we need (3.11) because we are studying $\mathcal{Z}^z_{t,Y}$, instead of the constrained partition function $\mathcal{Z}^{z,\mathrm{pin}}_{t,Y}$ as in [10]. □

The proof of Proposition 5.1 is based on two lemmas, for which we first introduce some notation. Let $\sigma := \{\sigma_0 = a < \sigma_1 < \cdots\}$ be a renewal process on $[a,\infty)$ with renewal time distribution $K(t)\,\mathrm{d}t = (1+\rho)G^{-1}p_{(1+\rho)t}(0)\,\mathrm{d}t$, and let $k(\sigma, L) := |\sigma \cap (a, L]|$. Let $\mathbb{P}_a^\sigma$ and $\mathbb{E}_a^\sigma$ denote respectively probability and expectation for σ. Let $(\tilde{Y}_s^\sigma)_{s\ge 0}$ be defined analogously to Y^σ, where conditional on σ, for each $n \in \mathbb{N}$, the law of $(\tilde{Y}_s^\sigma)_{0\le s\le \sigma_n}$ is absolutely continuous w.r.t. the law of $(Y_s)_{0\le s\le\sigma_n}$ with Radon–Nikodym derivative

$$\prod_{i=1}^n \frac{p_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})}{p_{(1+\rho)(\sigma_i-\sigma_{i-1})}(0)}.$$

Then we have

Lemma 5.1. *Let $L = \mathrm{e}^{B_1\rho^{-\zeta}}$, $A_1 = \mathrm{e}$ and $A_2 = L^{1/8}$. Let $a \in [0, (1-\varepsilon)L]$ for some $0 < \varepsilon < 1$. For any $D_1 > 0$ and $\delta > 0$, if $B_1 = B_1(\rho)$ is sufficiently large, which can be chosen uniformly for $\rho \in (0, 1]$, then for all $a \in [0, (1-\varepsilon)L]$, we have*

$$\mathbb{P}_a^\sigma \mathbb{P}_0^{\tilde{Y}^\sigma}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}\big(\tilde{Y}^\sigma\big) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] < D_1\rho^{-3/2}\sqrt{\varepsilon L}\right) \le \delta. \tag{5.3}$$

Lemma 5.2. *Let L, A_1, A_2, ε and a be as in Lemma 5.1. For any $\delta > 0$, if $B_1 = B_1(\rho)$ is sufficiently large, which can be chosen uniformly for $\rho \in (0, 1]$, then for all $a \in [0, (1-\varepsilon)L]$, we have*

$$\mathbb{P}_a^\sigma \mathbb{P}_0^{\tilde{Y}^\sigma}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\big(\tilde{Y}^\sigma\big) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] < -\sqrt{L}\right) \le \delta. \tag{5.4}$$

We defer the proofs of Lemmas 5.1 and 5.2 and first deduce Proposition 5.1.

Proof of Proposition 5.1. Let σ, $k := k(\sigma, L)$ and $\tilde{Y}^\sigma$ be as introduced before Lemma 5.1. Then we define $Y_s^\sigma = \tilde{Y}_s^\sigma$ for $s \in [0, \sigma_k]$, and $Y_s^\sigma - Y_{\sigma_k}^\sigma = Y_s - Y_{\sigma_k}$ for $s \ge \sigma_k$. By (3.15), uniformly in $z \in (1, 1 + 1/L]$, we may rewrite the left-hand side of (5.1) as

$$\begin{aligned}
&\int_a^L \mathbb{E}_0^Y\big[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f\big((Y_s)_{s\in[0,L]}\big)\big] K\big([L-b,\infty)\big)\,\mathrm{d}b \\
&\quad = \mathbb{E}_a^\sigma\big[1_{\{k\ge 1\}} z^k \mathbb{E}_0^{Y^\sigma}\big[f\big(\big(Y_s^\sigma\big)_{s\in[0,L]}\big)\big]\big] \\
&\quad \le \mathbb{E}_a^\sigma\big[\big(1+L^{-1}\big)^k 1_{\{k>L\}}\big] + \mathrm{e}\mathbb{E}_a^\sigma\big[\mathbb{E}_0^{Y^\sigma}\big[f\big(\big(Y_s^\sigma\big)_{s\in[0,L]}\big)\big]\big] \\
&\quad \le C_1\mathrm{e}^{-C_2L} + s\mathrm{e}\mathbb{P}_a^\sigma\mathbb{P}_0^{Y^\sigma}\big(H_L\big(Y^\sigma\big) \le M\big) + \mathrm{e}\varepsilon_M,
\end{aligned} \tag{5.5}$$

where the bound for $\mathbb{E}_a^\sigma[(1+L^{-1})^k 1_{\{k>L\}}]$ follows from standard large deviation estimates for the i.i.d. random variables $(\sigma_i - \sigma_{i-1})_{i\in\mathbb{N}}$. In (5.5), the first and the last terms can both be made arbitrarily small by choosing B_1 large enough, and D large enough in $M = \mathbb{E}_0^Y[H_L(Y)] + D\rho^{-3/2}\sqrt{L}$, which follows from (3.8) and (4.4).

Recall the decomposition of $H_L(Y^\sigma)$ in (4.5). Fix any $\delta > 0$. Since $\tilde{Y}_s^\sigma = Y_s^\sigma$ on $[0, \sigma_k]$, by Lemma 5.1, we can choose B_1 large enough (uniformly for $\rho \in (0, 1]$) such that for all $a \in [0, (1-3\varepsilon)L]$, we have

$$\mathbb{P}_a^\sigma \mathbb{P}_0^{Y^\sigma}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] < 4D\rho^{-3/2}\sqrt{L}\right) \le \delta. \tag{5.6}$$

By the same reasoning as in (4.7), we note that $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$ and $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ differ by at most $A_2^2 = L^{1/4}$. Therefore by Lemma 5.2, we can choose B_1 large enough such that

$$\mathbb{P}_a^\sigma \mathbb{P}_0^{Y^\sigma}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\big(Y^\sigma\big) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] < -2\sqrt{L}\right) \le \delta. \tag{5.7}$$

In the decomposition of $H_L(Y^\sigma)$ in (4.5), again by (4.7), we have

$$H^{\mathrm{ext}}_{[0,a]}\big(Y^\sigma\big) - \mathbb{E}^Y_0\big[H^{\mathrm{ext}}_{[0,a]}(Y)\big] - C_{\sigma,Y^\sigma} + \mathbb{E}^Y_0[C_{\sigma,Y}] \geq -2A_2^2 = -2L^{1/4}, \tag{5.8}$$

where $C_{\sigma,Y}$ is defined exactly as C_{σ,Y^σ} with Y^σ replaced by Y. The same calculation as in the proof of Lemma 4.1, (4.4), to appear in Section 7, shows that $\mathrm{Var}(H^{\mathrm{int}}_{[0,a]}(Y)) \leq C\rho^{-3}L$ and $\mathrm{Var}(H^{\mathrm{int}}_{[\sigma_k,L]}(Y)) \leq C\rho^{-3}L$. Since by construction, Y^σ have the same increments as Y on $[0,a]$ and $[\sigma_k, L]$, we can choose D large enough such that

$$\begin{aligned}
&\mathbb{P}^\sigma_a \mathbb{P}^{Y^\sigma}_0\big(H^{\mathrm{int}}_{[0,a]}\big(Y^\sigma\big) - \mathbb{E}^Y_0\big[H^{\mathrm{int}}_{[0,a]}(Y)\big] \leq -D\rho^{-3/2}\sqrt{L}\big) \leq \delta, \\
&\mathbb{P}^\sigma_a \mathbb{P}^{Y^\sigma}_0\big(H^{\mathrm{int}}_{[\sigma_k,L]}\big(Y^\sigma\big) - \mathbb{E}^Y_0\big[H^{\mathrm{int}}_{[\sigma_k,L]}(Y)\big] \leq -D\rho^{-3/2}\sqrt{L}\big) \leq \delta.
\end{aligned} \tag{5.9}$$

If we first choose D large and then B_1 large, and let $M = \mathbb{E}^Y_0[H_L(Y)] + D\rho^{-3/2}\sqrt{L}$, then by the decomposition of $H_L(Y^\sigma)$ in (4.5) and (5.6)–(5.9), we find that in (5.5), we have

$$\mathbb{P}^\sigma_a \mathbb{P}^{Y^\sigma}_0\big(H_L\big(Y^\sigma\big) \leq M\big) \leq 4\delta.$$

Since $\delta > 0$ can be made arbitrarily small, Proposition 5.1 then follows. $\square$

We now prove Lemmas 5.1 and 5.2 by controlling the mean and variance of $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$ and $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$, conditional on σ. Our bound on the conditional mean of $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$ is based on the following lemma, which also leads to our choice of $L = \mathrm{e}^{B_1\rho^{-\zeta}}$.

Lemma 5.3. *Let $(\Delta_i)_{i\in\mathbb{N}}$ be i.i.d. with common distribution $K(t)\,\mathrm{d}t$ on $[0,\infty)$, where we have $K(t) = (1+\rho)p_{(1+\rho)t}(0)/G$. Let $Z_i^L = \frac{\sqrt{\Delta_i}}{(\log\Delta_i)^\xi}1_{\{2\mathrm{e}<\Delta_i<L^{1/8}\}}$ with $L = \mathrm{e}^{B_1\rho^{-\zeta}}$, where $(1-\xi)\zeta = 1$ and $\zeta > 2$. Then there exists $B_2 > 0$ such that for any $h > 0$ and $\delta > 0$, if $B_1 = B_1(\rho)$ is sufficiently large, which can be chosen uniformly for $\rho \in (0,1]$, then we have*

$$\mathbb{P}\Bigg(\sum_{i=1}^{h\sqrt{L}} Z_i^L < B_2 h\sqrt{L}(\log L)^{1-\xi}\Bigg) \leq \delta. \tag{5.10}$$

Proof. If B_1 is sufficiently large, which can be chosen uniformly for $\rho \in (0,1]$, we have

$$\mu_L := \mathbb{E}\big[Z_i^L\big] = \int_{2\mathrm{e}}^{L^{1/8}} \frac{\sqrt{\Delta}}{(\log\Delta)^\xi}\frac{(1+\rho)p_{(1+\rho)\Delta}(0)}{G}\,\mathrm{d}\Delta \geq C\int_{2\mathrm{e}}^{L^{1/8}} \frac{\mathrm{d}\Delta}{\Delta(\log\Delta)^\xi} \geq 4B_2(\log L)^{1-\xi}$$

for some $B_2 > 0$ independent of B_1, $\xi \in (\frac{1}{2}, 1)$ and $\rho \in (0,1]$. We then prove (5.10) by a large deviation estimate. Namely, if we let $M(\lambda) = \log\mathbb{E}[\mathrm{e}^{\lambda Z_1^L}]$, then for any $\lambda < 0$, we have

$$\mathbb{P}\Bigg(\sum_{i=1}^{h\sqrt{L}} Z_i^L < B_2 h\sqrt{L}(\log L)^{1-\xi}\Bigg) \leq \exp\big\{h\sqrt{L}\big(M(\lambda) - \lambda B_2(\log L)^{1-\xi}\big)\big\}. \tag{5.11}$$

Let $\lambda = -L^{1/8}$. Then for B_1 sufficiently large, uniformly in $\rho \in (0,1]$, we have

$$\begin{aligned}
M(\lambda) &= \log\bigg(\int_0^\infty \mathrm{e}^{-\frac{\sqrt{\Delta}}{L^{1/8}(\log\Delta)^\xi}1_{\{2\mathrm{e}<\Delta<L^{1/8}\}}}\frac{(1+\rho)p_{(1+\rho)\Delta}(0)}{G}\,\mathrm{d}\Delta\bigg) \\
&< \log\bigg(\int_0^\infty \bigg(1 - \frac{\sqrt{\Delta}}{2L^{1/8}(\log\Delta)^\xi}1_{\{2\mathrm{e}<\Delta<L^{1/8}\}}\bigg)\frac{(1+\rho)p_{(1+\rho)\Delta}(0)}{G}\,\mathrm{d}\Delta\bigg) \\
&= \log\bigg(1 - \frac{\mu_L}{2L^{1/8}}\bigg) < -\frac{\mu_L}{2L^{1/8}} \leq 2\lambda B_2(\log L)^{1-\xi}.
\end{aligned}$$

Therefore the right-hand side of (5.11) is bounded by $\exp\{-\frac{h\sqrt{L}B_2(\log L)^{1-\xi}}{L^{1/8}}\}$, which tends to 0 uniformly in $\rho \in (0,1]$ as $B_1 \uparrow \infty$, thus implying (5.10). □

Proof of Lemma 5.1. For $i \in \mathbb{N}$, let $\Delta_i = \sigma_i - \sigma_{i-1}$, which are i.i.d. with common distribution $K(t)\,\mathrm{d}t = (1+\rho)p_{(1+\rho)t}(0)\,\mathrm{d}t/G$. By Lemma A.2, for any $\delta > 0$, we can find $C_1 > 0$ small enough such that for all L sufficiently large and uniformly in $a \in [0, (1-\varepsilon)L]$, we have

$$\mathbb{P}_a^\sigma\big(k(\sigma, L) < C_1\sqrt{\varepsilon L}\big) \le \frac{\delta}{4}. \tag{5.12}$$

By Lemma 4.2, (4.10), almost surely with respect to σ,

$$\sum_{i=1}^{k(\sigma,L)} \big(\mathbb{E}_0^{\tilde{Y}^\sigma}\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\big] - \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big]\big) > \sum_{i=1}^{k(\sigma,L)} \frac{C\sqrt{\Delta_i}}{\sqrt{\rho}(\log\Delta_i)^\xi} 1_{\{2\mathrm{e}<\Delta_i<L^{1/8}\}}, \tag{5.13}$$

while by Lemma 5.3, given $h = C_1\sqrt{\varepsilon}$ and $\delta > 0$, we can find $B_2 > 0$ such that for all B_1 sufficiently large, we have

$$\mathbb{P}_a^\sigma\left(\sum_{i=1}^{C_1\sqrt{\varepsilon L}} \frac{C\sqrt{\Delta_i}}{\sqrt{\rho}(\log\Delta_i)^\xi} 1_{\{2\mathrm{e}<\Delta_i<L^{1/8}\}} < \frac{CB_2}{\sqrt{\rho}} C_1\sqrt{\varepsilon L}(\log L)^{1-\xi}\right) \le \frac{\delta}{4}. \tag{5.14}$$

Therefore for a set of σ with probability at least $1 - \frac{\delta}{2}$, conditional on σ, we have

$$\sum_{i=1}^{k(\sigma,L)} \big(\mathbb{E}_0^{\tilde{Y}^\sigma}\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\big] - \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big]\big) \ge CC_1B_2\frac{\sqrt{\varepsilon L}}{\sqrt{\rho}}(\log L)^{1-\xi} = D_2\rho^{-3/2}\sqrt{\varepsilon L}, \tag{5.15}$$

where $D_2 = CC_1B_2B_1^{1-\xi}$ and we used $L = \mathrm{e}^{B_1\rho^{-\zeta}} = \mathrm{e}^{B_1\rho^{-1/(1-\xi)}}$. For any σ, by Lemma 4.2, (4.11),

$$\mathrm{Var}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\Big|\sigma\right) \le C\rho^{-3}(L-a) \le C\rho^{-3}L.$$

$D_2\sqrt{\varepsilon}$ can be made arbitrarily large by choosing B_1 large. Therefore for any $D_1 > 0$ and for all σ satisfying (5.15), by making B_1 sufficiently large, we have by Markov inequality

$$\mathbb{P}_0^{\tilde{Y}^\sigma}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y\big[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] < D_1\rho^{-3/2}\sqrt{\varepsilon L}\right) \le \frac{\delta}{2}.$$

Since the set of σ that violates (5.15) has probability at most $\frac{\delta}{2}$, this implies (5.3). □

Proof of Lemma 5.2. By Lemma 4.2, (4.8), for all σ, we have

$$\sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^{\tilde{Y}^\sigma}\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\big] - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y\big[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)\big] > 0.$$

Therefore to establish (5.4), by Markov inequality, it suffices to show that: For any $\delta > 0$, if B_1 in $L = \mathrm{e}^{B_1\rho^{-\zeta}}$ is sufficiently large (uniform for $\rho \in (0,1]$), then for all $a \in [0, (1-\varepsilon)L]$, we have

$$\mathbb{P}_a^\sigma\left(\mathrm{Var}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\Big|\sigma\right) \ge \delta L\right) < \delta. \tag{5.16}$$

We will decompose the sum $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$ to extract some independence.

Given $\sigma := \{\sigma_0 = a < \sigma_1 < \cdots\}$, let $\tau_0 = 0$, and for $j \in \mathbb{N}$, define inductively

$$\tau_j := \min\bigl\{i > \tau_{j-1}\colon \sigma_i - \sigma_{i-1} \ge A_2 = L^{1/8}\bigr\}.$$

Let $J = J(\sigma, L) := \max\{j \in \mathbb{N}\colon \sigma_{\tau_j} \le L\}$. Then we have the decomposition

$$\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr) = \sum_{j=1}^{J}\sum_{i=\tau_{j-1}+1}^{\tau_j-1} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr) + \sum_{i=\tau_J+1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr) + \sum_{j=1}^{J} H^{\mathrm{ext}}_{[\sigma_{\tau_j-1},\sigma_{\tau_j}]}\bigl(\tilde{Y}^\sigma\bigr).$$

Now note that conditional on σ, $\sum_{i=\tau_{j-1}+1}^{\tau_j-1} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$, $1 \le j \le J$, and $\sum_{i=\tau_J+1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$ are all independent. Similarly, for even (resp., odd) $1 \le j \le J$, $H^{\mathrm{ext}}_{[\sigma_{\tau_j-1},\sigma_{\tau_j}]}(\tilde{Y}^\sigma)$ are all independent. Therefore using independence and the fact that $\mathrm{Var}(X + Y + Z) \le 3(\mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Var}(Z))$ (with $Y := \sum_{j\,\mathrm{even}}^{J} H^{\mathrm{ext}}_{[\sigma_{\tau_j-1},\sigma_{\tau_j}]}(\tilde{Y}^\sigma)$, $Z := \sum_{j\,\mathrm{odd}}^{J} H^{\mathrm{ext}}_{[\sigma_{\tau_j-1},\sigma_{\tau_j}]}(\tilde{Y}^\sigma)$, $X := \sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma) - Y - Z$), we have

$$\begin{aligned}
V(\sigma, L) &:= \mathrm{Var}\left(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr)\Big|\sigma\right)\\
&\le 3\sum_{j=1}^{J}\mathrm{Var}\left(\sum_{i=\tau_{j-1}+1}^{\tau_j-1} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr)\Big|\sigma\right)\\
&\quad + 3\,\mathrm{Var}\left(\sum_{i=\tau_J+1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr)\Big|\sigma\right) + 3\sum_{j=1}^{J}\mathrm{Var}\bigl(H^{\mathrm{ext}}_{[\sigma_{\tau_j-1},\sigma_{\tau_j}]}\bigl(\tilde{Y}^\sigma\bigr)\big|\sigma\bigr).
\end{aligned}\tag{5.17}$$

By (4.7), $H^{\mathrm{ext}}_{[\sigma_{\tau_j-1},\sigma_{\tau_j}]}(\tilde{Y}^\sigma) \le A_2^2$ for each $j \in \mathbb{N}$. Similarly, using the definition of H^{ext} in (4.6),

$$\sum_{i=\tau_{j-1}+1}^{\tau_j-1} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}\bigl(\tilde{Y}^\sigma\bigr) \le (\sigma_{\tau_j-1} - \sigma_{\tau_{j-1}})A_2 \quad \text{for } j \in \mathbb{N}.$$

Therefore we obtain from (5.17)

$$\mathbb{E}^\sigma_a\bigl[V(\sigma, L)\bigr] \le 3\mathbb{E}^\sigma_a\left[\sum_{j=1}^{J+1}(\sigma_{\tau_j-1} - \sigma_{\tau_{j-1}})^2 A_2^2\right] + 3\mathbb{E}^\sigma_a\bigl[J A_2^4\bigr].\tag{5.18}$$

Note that $\sigma_{\tau_{J+1}-1} \ge \sigma_{k(\sigma,L)}$, in particular, the second term on the right-hand side of (5.17) is accounted for in (5.18).

Let $\Delta_i = \sigma_i - \sigma_{i-1}$, which are i.i.d. Then $k(\sigma, L) + 1$ is a stopping time w.r.t. the sequence $(\Delta_i)_{i\in\mathbb{N}}$, and by Wald's equation [6], Section 3.1,

$$\begin{aligned}
\mathbb{E}^\sigma_a[J] &\le \mathbb{E}^\sigma_a\left[\sum_{i=1}^{k(\sigma,L)+1} 1_{\{\Delta_i \ge A_2\}}\right] = \mathbb{E}^\sigma_a\bigl[1 + k(\sigma, L)\bigr]\mathbb{P}^\sigma_a(\Delta_1 \ge A_2)\\
&\le \bigl(1 + C_1\sqrt{L}\bigr)\int_{A_2}^{\infty}\frac{(1+\rho)p_{(1+\rho)t}(0)\,\mathrm{d}t}{G} \le C\sqrt{\frac{L}{A_2}},
\end{aligned}$$

where we used Lemma A.2 and the discussion following it to deduce $\mathbb{E}_a^\sigma[k(\sigma,L)] = C_1\sqrt{L}$, and we bounded $p_t(0)$ by $Ct^{-3/2}$. Note that $(\sigma_{\tau_j-1} - \sigma_{\tau_{j-1}}, \sigma_{\tau_j} - \sigma_{\tau_j-1})$, $j \in \mathbb{N}$, is an i.i.d. sequence of $\mathbb{R}^2$-valued random variables, and $J+1$ is a stopping time with respect to this sequence. So again by Wald's equation,

$$\mathbb{E}_a^\sigma\left[\sum_{j=1}^{J+1}(\sigma_{\tau_j-1} - \sigma_{\tau_{j-1}})^2\right] = \mathbb{E}_a^\sigma[J+1]\mathbb{E}_a^\sigma\big[(\sigma_{\tau_1-1} - \sigma_{\tau_0})^2\big] \le \big(1 + C\sqrt{L/A_2}\big)\mathbb{E}_a^\sigma\left[\left(\sum_{i=1}^{\tau_1}\Delta_i 1_{\{\Delta_i < A_2\}}\right)^2\right],$$

where if we denote $Z_i := \Delta_i 1_{\{\Delta_i < A_2\}}$ and $\mu := \mathbb{E}_a^\sigma[Z_i] = \int_0^{A_2}(1+\rho)G^{-1}tp_{(1+\rho)t}(0)\,\mathrm{d}t \le C\sqrt{A_2}$, then

$$\begin{aligned}
\mathbb{E}_a^\sigma\left[\left(\sum_{i=1}^{\tau_1}\Delta_i 1_{\{\Delta_i < A_2\}}\right)^2\right] &= \mathbb{E}_a^\sigma\left[\left(\sum_{i=1}^{\tau_1} Z_i\right)^2\right] \le 2\mathbb{E}_a^\sigma\left[\left(\sum_{i=1}^{\tau_1}(Z_i - \mu)\right)^2\right] + 2\mu^2\mathbb{E}_a^\sigma\big[\tau_1^2\big] \\
&= 2\mathbb{E}_a^\sigma[\tau_1]\mathbb{E}_a^\sigma\big[(Z_1-\mu)^2\big] + 2\mu^2\mathbb{E}_a^\sigma\big[\tau_1^2\big] \\
&\le 2\mathbb{E}_a^\sigma[\tau_1]\mathbb{E}_a^\sigma\big[Z_1^2\big] + 2\mu^2\mathbb{E}_a^\sigma\big[\tau_1^2\big] \\
&\le \frac{2\int_0^{A_2}(1+\rho)G^{-1}t^2p_{(1+\rho)t}(0)\,\mathrm{d}t}{\mathbb{P}_a^\sigma(\Delta_1 \ge A_2)} + \frac{4C^2A_2}{\mathbb{P}_a^\sigma(\Delta_1 \ge A_2)^2} \\
&\le CA_2^2,
\end{aligned}$$

where we have used Wald's second equation [6], Section 3.1, and the fact that τ_1 is a stopping time for $(\Delta_i)_{i\in\mathbb{N}}$, which is geometrically distributed with $\mathbb{E}_a^\sigma[\tau_1] = p^{-1}$ and $\mathbb{E}_a^\sigma[\tau_1^2] = (2-p)/p^2 \le 2/p^2$ for $p = \mathbb{P}_a^\sigma(\Delta_1 \ge A_2) \ge C/\sqrt{A_2}$. Collecting the above estimates and substituting them in (5.18) then yields

$$\mathbb{E}_a^\sigma\big[V(\sigma,L)\big] \le C\sqrt{L}A_2^{7/2} = CL^{15/16},$$

which by Markov inequality implies (5.16) if B_1, and hence $L = \mathrm{e}^{B_1\rho^{-\zeta}}$, is sufficiently large. □

6. Proof of Proposition 3.2

The deduction of Proposition 3.2 from Proposition 3.1 is model independent. Part of the proof is similar to its discrete time analogue (see, e.g., the proofs of Proposition 2.3 and Lemma 2.4 in [10]), with the main difference being that in the integrals in (3.11) and (3.12), we have excluded not only contributions from $b \in [a, a+\varepsilon L]$, but also from $b \in [L-\varepsilon L, L]$. The latter requires new bounds.

First note that by (2.3), for any $0 < a < b < L$ and $z \in [1, 1+L^{-1}]$, we have

$$\begin{aligned}
\mathbb{E}_0^Y\big[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}}\big] &= \sum_{m=1}^\infty \int\cdots\int_{\sigma_0=a<\sigma_1<\cdots<\sigma_m=b} z^m\prod_{i=1}^m K(\sigma_i - \sigma_{i-1})\,\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_{m-1} \\
&= z\big(G^{X-Y}\big)^{-1}\mathbb{E}_{0,0}^{X,Y}\big[\mathrm{e}^{z(G^{X-Y})^{-1}\int_a^b 1_{\{X_s=Y_s\}}\,\mathrm{d}s}1_{\{X_b=Y_b\}}|X_a = Y_a\big] \\
&\le z\big(G^{X-Y}\big)^{-1}\mathbb{E}_{0,0}^{X,Y}\big[\mathrm{e}^{(G^{X-Y})^{-1}(1+\int_a^b 1_{\{X_s=Y_s\}}\,\mathrm{d}s)}1_{\{X_b=Y_b\}}|X_a = Y_a\big] \\
&= z\mathrm{e}^{(G^{X-Y})^{-1}}P(b-a) \le CP(b-a),
\end{aligned} \tag{6.1}$$

where $P(t)$ is defined in (3.13), and $C > 0$ is uniform in $L \ge 1$ and Y's jump rate $\rho \in [0,1]$. For the inequality in the third line, we used $z \le 1 + L^{-1}$ and $b - a \le L$, while the equality in the last line follows by Taylor expanding as in (2.1) and setting $z = 1$, i.e. $\beta = 1/G^{X-Y}$, in (2.3) and then averaging over Y. Let $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1,\dots,m\}$

for some $m \in \mathbb{N}$. Then by (3.10), Proposition 3.1 and (6.1),

$$\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big] \le \int \cdots \int_{\substack{a_j < b_j \in \Lambda_{i_j} \\ 1 \le j \le k}} (Cz)^k \prod_{j=1}^k K(a_j - b_{j-1}) \times \big(C1_{\{b_j - a_j < \varepsilon L\}} + C1_{\{b_j \ge (i_j - \varepsilon)L\}} + \delta\big) P(b_j - a_j) \prod_{j=1}^k \mathrm{d}a_j\, \mathrm{d}b_j. \tag{6.2}$$

We will show that for any $\eta > 0$, if $\varepsilon > 0$ and $\delta > 0$ are chosen sufficiently small, then we have

$$\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big] \le \int \cdots \int_{\substack{a_j < b_j \in \Lambda_{i_j} \\ 1 \le j \le k}} (Cz\eta)^k \prod_{j=1}^k K(a_j - b_{j-1}) P(b_j - a_j) \prod_{j=1}^k \mathrm{d}a_j\, \mathrm{d}b_j. \tag{6.3}$$

Proposition 3.2 then follows from the bound

$$P_L(I) := \int \cdots \int_{\substack{a_j < b_j \in \Lambda_{i_j} \\ 1 \le j \le k}} \prod_{j=1}^k K(a_j - b_{j-1}) P(b_j - a_j) \prod_{j=1}^k \mathrm{d}a_j\, \mathrm{d}b_j \le C_L \prod_{j=1}^{|I|} \frac{C}{(i_j - i_{j-1})^{3/2}}, \tag{6.4}$$

where C_L depends only on L.

First we give a proof of (6.4), which is similar to its discrete time counterpart, [10], Lemma 2.4. We partition I into blocks of consecutive integers $\{u_1, u_1+1, \ldots, v_1\}$, $\{u_2, u_2+1, \ldots, v_2\}$, $\ldots$, $\{u_l, u_l+1, \ldots, v_l\}$, where $u_j - v_{j-1} \ge 2$ for all $2 \le j \le l$. When substituting the definition of $P(b_j - a_j)$ in (3.13) into (6.4), the resulting multifold expansion is the probability of a set of renewal configurations, where the variables of integration constitute the renewal configuration σ. By only retaining the constraint that σ intersects Λ_{u_i} and Λ_{v_i} for $1 \le i \le l$, we obtain

$$P_L(I) \le \int_{\substack{a_1 < b_1 \\ a_1 \in \Lambda_{u_1}, b_1 \in \Lambda_{v_1}}} \cdots \int_{\substack{a_l < b_l \\ a_l \in \Lambda_{u_l}, b_l \in \Lambda_{v_l}}} \prod_{j=1}^l K(a_j - b_{j-1}) P(b_j - a_j) \prod_{j=1}^l \mathrm{d}a_j\, \mathrm{d}b_j, \tag{6.5}$$

where $b_0 := 0$. We integrate out one pair of variables a_j, b_j at a time. For $j = l$,

$$\begin{aligned} &\int_{a_l \in \Lambda_{u_l}, b_l \in \Lambda_{v_l}} \int_{a_l < b_l} K(a_l - b_{l-1}) P(b_l - a_l)\, \mathrm{d}b_l\, \mathrm{d}a_l \\ &\quad \le \frac{C}{(u_l - v_{l-1})^{3/2} L^{3/2}} \int_{a_l \in \Lambda_{u_l}, b_l \in \Lambda_{v_l}} \int_{a_l < b_l} P(b_l - a_l)\, \mathrm{d}b_l\, \mathrm{d}a_l, \end{aligned} \tag{6.6}$$

since $a_l - b_{l-1} \ge (u_l - v_{l-1} - 1)L \ge L$, and hence $K(a_l - b_{l-1}) \le \frac{C_1}{(u_l - v_{l-1} - 1)^{3/2} L^{3/2}} \le \frac{C}{(u_l - v_{l-1})^{3/2} L^{3/2}}$. Regardless of whether $u_l = v_l$ or $v_l > u_l$, by Lemma A.1, we have

$$\int_{a_l \in \Lambda_{u_l}, b_l \in \Lambda_{v_l}} \int_{a_l < b_l} P(b_l - a_l)\, \mathrm{d}b_l\, \mathrm{d}a_l \le \int_{a_l \in \Lambda_{u_l}, b_l \in \Lambda_{v_l}} \int_{a_l < b_l} \frac{C}{\sqrt{b_l - a_l}}\, \mathrm{d}b_l\, \mathrm{d}a_l \le C L^{3/2}.$$

Integrating out a_l, b_l in (6.5) thus gives a factor $C(u_l - v_{l-1})^{-3/2}$. Iterating this procedure then gives the bound in (6.4), where a prefactor $C_L = L^{3/2}$ arises when we integrate out a_1 and b_1 in the case $u_1 = 1$. This proves (6.4).

To deduce (6.3) from (6.2), we first bound the contributions from $C1_{\{b_j-a_j<\varepsilon L\}}$. We claim that there exists some $C>0$ depending only on $K(\cdot)$ and uniform in $\rho\in[0,1]$, such that for all L sufficiently large, $\varepsilon\in(0,1/4)$, and $a\le 0<L\le b$, we have

$$\int_{\substack{0\le s<t\le L\\ t-s<\varepsilon L}} K(s-a)P(t-s)K(b-t)\,\mathrm{d}t\,\mathrm{d}s\le C\sqrt{\varepsilon}\int_{0\le s<t\le L} K(s-a)P(t-s)K(b-t)\,\mathrm{d}t\,\mathrm{d}s, \tag{6.7}$$

$$\int_{\substack{0\le s<t\le L\\ t-s<\varepsilon L}} K(s-a)P(t-s)\,\mathrm{d}t\,\mathrm{d}s\le C\sqrt{\varepsilon}\int_{0\le s<t\le L} K(s-a)P(t-s)\,\mathrm{d}t\,\mathrm{d}s. \tag{6.8}$$

To prove (6.7), note that either $s\le L/2$ or $s>L/2$ in the integral. Using the fact that $K(t)\sim\frac{C}{t^{3/2}}$ by the local central limit theorem and the fact that $\int_0^t P(s)\,\mathrm{d}s\sim C\sqrt{t}$ by Lemma A.1, we have

$$\begin{aligned}\int_{\substack{0\le s<t\le L\\ s\le L/2, t-s<\varepsilon L}} K(s-a)P(t-s)K(b-t)\,\mathrm{d}t\,\mathrm{d}s&\le\frac{C\sqrt{\varepsilon L}}{(b-3L/4)^{3/2}}\int_{0\le s\le L/2}K(s-a)\,\mathrm{d}s\\ &\le\frac{C\sqrt{\varepsilon L}}{b^{3/2}}\int_{0\le s\le L/2}K(s-a)\,\mathrm{d}s,\end{aligned}$$

where we used $b-t>b-3L/4$ and $b\ge L$. On the other hand,

$$\begin{aligned}\int_{0\le s<t\le L} K(s-a)P(t-s)K(b-t)\,\mathrm{d}t\,\mathrm{d}s&\ge\int_{\substack{0\le s\le L/2\\ 0<t-s\le L/4}} K(s-a)P(t-s)K(b-t)\,\mathrm{d}t\,\mathrm{d}s\\ &\ge\frac{C\sqrt{L}}{b^{3/2}}\int_{0\le s\le L/2}K(s-a)\,\mathrm{d}s.\end{aligned}$$

Together with a similar bound for the left-hand side of (6.7) integrated over $s>L/2$, this implies (6.7). The proof of (6.8) is similar and will be omitted. Substituting (6.7) and (6.8) into (6.2) then gives

$$\begin{aligned}&\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big]\\ &\quad\le(Cz)^k\int\cdots\int_{\substack{a_j<b_j\in\Lambda_{i_j}\\ 1\le j\le k}}\prod_{j=1}^k K(a_j-b_{j-1})\big(C1_{\{b_j\ge(i_j-\varepsilon)L\}}+\tilde\eta\big)P(b_j-a_j)\prod_{j=1}^k\mathrm{d}a_j\,\mathrm{d}b_j,\end{aligned} \tag{6.9}$$

where $\tilde\eta=C\sqrt{\varepsilon}+\delta$, which can be made arbitrarily small by choosing ε and δ small. By expanding the product $\prod_{j=1}^k(C1_{\{b_j\ge(i_j-\varepsilon)L\}}+\tilde\eta)$, we note that (6.3) follows once we show that there exists some C such that for any $J\subset\{1,\dots,k\}$, we have

$$\int\cdots\int_{\substack{1\le j\le k: a_j<b_j\in\Lambda_{i_j}\\ j\in J: b_j\ge(i_j-\varepsilon)L}}\prod_{j=1}^k K(a_j-b_{j-1})P(b_j-a_j)\prod_{j=1}^k\mathrm{d}a_j\,\mathrm{d}b_j\le\big(C\sqrt{\varepsilon}\big)^{|J|}P_L(I), \tag{6.10}$$

where $P_L(I)$ was defined in (6.4).

If $J=\varnothing$, then (6.10) is trivial; otherwise, let l be the largest element in J. It suffices to show that we can replace the indicator $1_{\{b_l\ge(i_l-\varepsilon)L\}}$ by the factor $C\sqrt{\varepsilon}$. We can then apply the argument inductively to deduce (6.10). There are three cases: either (1) $l = k$; or (2) $i_{l+1} - i_l \geq 2$; or (3) $i_{l+1} - i_l = 1$. For the case $l = k$, it suffices to show that uniformly in $b_{k-1} \in \Lambda_{i_{k-1}}$, we have

$$\iint_{\substack{a_k < b_k \in \Lambda_{i_k} \\ b_k \geq (i_k - \varepsilon)L}} K(a_k - b_{k-1}) P(b_k - a_k) \, \mathrm{d}a_k \, \mathrm{d}b_k \leq C\sqrt{\varepsilon} \iint_{a_k < b_k \in \Lambda_{i_k}} K(a_k - b_{k-1}) P(b_k - a_k) \, \mathrm{d}a_k \, \mathrm{d}b_k. \tag{6.11}$$

Note that by Lemma A.1, uniformly in $u > 0$, we have $\int_u^{u+\varepsilon L} P(s) \, \mathrm{d}s \leq C\sqrt{\varepsilon L}$ for L large. Uniformly in $u > 0$, we also have $\int_u^{u+L} K(s) \, \mathrm{d}s \leq 2 \int_u^{u+L/2} K(s) \, \mathrm{d}s$. It follows that

$$\iint_{\substack{a_k < b_k \in \Lambda_{i_k} \\ b_k \geq (i_k - \varepsilon)L}} K(a_k - b_{k-1}) P(b_k - a_k) \, \mathrm{d}a_k \, \mathrm{d}b_k \leq 2C\sqrt{\varepsilon L} \int_{(i_k - 1)L \leq a_k \leq (i_k - 1/2)L} K(a_k - b_{k-1}) \, \mathrm{d}a_k.$$

On the other hand, by Lemma A.1, $\int_0^t P(s) \, \mathrm{d}s \sim C\sqrt{t}$. Therefore for L sufficiently large,

$$\begin{aligned}
\iint_{a_k < b_k \in \Lambda_{i_k}} K(a_k - b_{k-1}) P(b_k - a_k) \, \mathrm{d}a_k \, \mathrm{d}b_k &\geq \int_{\substack{(i_k - 1)L \leq a_k \leq (i_k - 1/2)L \\ 0 \leq b_k - a_k \leq L/2}} K(a_k - b_{k-1}) P(b_k - a_k) \, \mathrm{d}a_k \, \mathrm{d}b_k \\
&\geq C\sqrt{L} \int_{(i_k - 1)L \leq a_k \leq (i_k - 1/2)L} K(a_k - b_{k-1}) \, \mathrm{d}a_k.
\end{aligned}$$

The above two estimates together imply (6.11).

For case (2), $i_{l+1} - i_l \geq 2$, it suffices to show that uniformly in $b_{l-1} \in \Lambda_{i_{l-1}}$ and $a_{l+1} \in \Lambda_{i_{l+1}}$, we have

$$\begin{aligned}
&\iint_{\substack{a_l < b_l \in \Lambda_{i_l} \\ b_l \geq (i_l - \varepsilon)L}} K(a_l - b_{l-1}) P(b_l - a_l) K(a_{l+1} - b_l) \, \mathrm{d}a_l \, \mathrm{d}b_l \\
&\quad \leq C\sqrt{\varepsilon} \iint_{a_l < b_l \in \Lambda_{i_l}} K(a_l - b_{l-1}) P(b_l - a_l) K(a_{l+1} - b_l) \, \mathrm{d}a_l \, \mathrm{d}b_l.
\end{aligned}$$

This follows from the same proof as for (6.11) once we note that, because $a_{l+1} - b_l \geq L$, uniformly in $s_1, s_2 \in \Lambda_{i_l}$ and $t_1, t_2 \in \Lambda_{i_{l+1}}$, we have $C \leq \frac{K(t_1 - s_1)}{K(t_2 - s_2)} \leq C^{-1}$ for some $C \in (0, \infty)$ depending only on $K(\cdot)$.

For case (3), $i_{l+1} - i_l = 1$, there are two subcases: either $l + 1 = k$ or $l + 1 < k$. We only examine the case $l + 1 < k$, since the case $l + 1 = k$ is similar and simpler. To simplify notation, we will shift coordinates and assume $l = 1$ and $i_l = 1$. Since l is the largest element in J, it suffices to show that uniformly in $b_0 \leq 0$ and $a_3 \geq 2L$, we have

$$\begin{aligned}
&\int \cdots \int_{\substack{b_1 \geq (1 - \varepsilon)L \\ 0 < a_1 < b_1 < L < a_2 < b_2 < 2L}} K(a_1 - b_0) P(b_1 - a_1) K(a_2 - b_1) P(b_2 - a_2) K(a_3 - b_2) \, \mathrm{d}a_1 \, \mathrm{d}b_1 \, \mathrm{d}a_2 \, \mathrm{d}b_2 \\
&\quad \leq C\sqrt{\varepsilon} \int \cdots \int_{0 < a_1 < b_1 < L < a_2 < b_2 < 2L} K(a_1 - b_0) P(b_1 - a_1) K(a_2 - b_1) \\
&\qquad \times P(b_2 - a_2) K(a_3 - b_2) \, \mathrm{d}a_1 \, \mathrm{d}b_1 \, \mathrm{d}a_2 \, \mathrm{d}b_2.
\end{aligned} \tag{6.12}$$

By restricting the region of integration to $a_1 \in [0, L/4]$, $b_1 \in [3L/4, L]$, $a_2 \in [L, 5L/4]$ and $b_2 \in [7L/4, 2L]$, and using the fact that $P(t) \sim \frac{C}{\sqrt{t}}$, $K(t) \sim \frac{C}{t^{3/2}}$, $\int_t^\infty K(s) \, \mathrm{d}s \sim \frac{C}{\sqrt{t}}$, we find

$$\begin{aligned}
&\int \cdots \int_{0 < a_1 < b_1 < L < a_2 < b_2 < 2L} K(a_1 - b_0) P(b_1 - a_1) K(a_2 - b_1) P(b_2 - a_2) K(a_3 - b_2) \, \mathrm{d}a_1 \, \mathrm{d}b_1 \, \mathrm{d}a_2 \, \mathrm{d}b_2 \\
&\quad \geq \frac{C}{\sqrt{L}} \int_0^{L/4} K(a_1 - b_0) \, \mathrm{d}a_1 \int_{7L/4}^{2L} K(a_3 - b_2) \, \mathrm{d}b_2.
\end{aligned} \tag{6.13}$$

To upper bound the left-hand side of (6.12), we claim that uniformly in all $b_1 \le L < 2L \le a_3$, we have

$$\begin{aligned}\int_L^{2L}\int_{a_2}^{2L} & K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)\,\mathrm{d}b_2\,\mathrm{d}a_2 \\ & \le \frac{C}{\sqrt{L}}\int_L^{5L/4} K(a_2-b_1)\,\mathrm{d}a_2 \int_{7L/4}^{2L} K(a_3-b_2)\,\mathrm{d}b_2, \end{aligned} \tag{6.14}$$

and uniformly for all $b_0 \le 0$ and $(1-\varepsilon)L < b_1 < L$, we have

$$\int_0^{b_1} K(a_1-b_0)P(b_1-a_1)\,\mathrm{d}a_1 \le \frac{C}{\sqrt{L}}\int_0^{L/4} K(a_1-b_0)\,\mathrm{d}a_1, \tag{6.15}$$

which when substituted into the left-hand side of (6.12) imply that

$$\begin{aligned}\int\cdots\int_{\substack{b_1\ge(1-\varepsilon)L \\ 0<a_1<b_1<L<a_2<b_2<2L}} & K(a_1-b_0)P(b_1-a_1)K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)\,\mathrm{d}a_1\,\mathrm{d}b_1\,\mathrm{d}a_2\,\mathrm{d}b_2 \\ & \le \frac{C}{L}\int_{(1-\varepsilon)L}^{L}\int_L^{5L/4} K(a_2-b_1)\,\mathrm{d}a_2\,\mathrm{d}b_1 \int_0^{L/4} K(a_1-b_0)\,\mathrm{d}a_1 \int_{7L/4}^{2L} K(a_3-b_2)\,\mathrm{d}b_2 \\ & \le \frac{C\sqrt{\varepsilon}}{\sqrt{L}}\int_0^{L/4} K(a_1-b_0)\,\mathrm{d}a_1 \int_{7L/4}^{2L} K(a_3-b_2)\,\mathrm{d}b_2, \end{aligned} \tag{6.16}$$

where we used the fact that $\int_L^{5L/4} K(a_2-b_1)\,\mathrm{d}a_2 \le \frac{C}{\sqrt{L-b_1}}$. Together with (6.13), this implies (6.12).

To prove (6.14), note that the bound therein certainly holds if we restrict integration to $a_2 \in [L, \frac{5L}{4}]$ and $b_2 \in [\frac{7L}{4}, 2L]$. If either of the constraints on a_2 and b_2 fails, without loss of generality, say $a_2 \in [\frac{5L}{4}, 2L]$, then because $K(t) \le \frac{C}{t^{3/2}}$ and $\int_0^t P(s)\,\mathrm{d}s \le C\sqrt{t}$, we have

$$\begin{aligned}\int_{5L/4}^{2L}\int_{a_2}^{2L} & K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)\,\mathrm{d}b_2\,\mathrm{d}a_2 \\ & \le \frac{C\sqrt{L}}{(5L/4-b_1)^{3/2}}\int_L^{2L} K(a_3-b_2)\,\mathrm{d}b_2 \\ & \le \frac{C\sqrt{L}}{(5L/4-b_1)^{3/2}}\int_{7L/4}^{2L} K(a_3-b_2)\,\mathrm{d}b_2 \\ & \le \frac{C}{\sqrt{L}}\int_L^{5L/4} K(a_2-b_1)\,\mathrm{d}a_2 \int_{7L/4}^{2L} K(a_3-b_2)\,\mathrm{d}b_2, \end{aligned}$$

since $\int_L^{5L/4} K(a_2-b_1)\,\mathrm{d}a_2 \ge \frac{CL}{4}(\frac{5L}{4}-b_1)^{-3/2}$. This proves (6.14). The proof of (6.15) is similar and will be omitted. This completes the proof of (6.10) as well as of Proposition 3.2.

7. Proof of Lemma 4.1

Note that (4.3) is obvious. For $s \in [0, L]$, let us denote

$$h_L(s,Y) := \int_{\substack{s<t<L \\ A_1<t-s<A_2}} \frac{1_{\{Y_s=Y_t\}}}{(\log(t-s))^{\xi}}\,\mathrm{d}t.$$

Then

$$
\begin{aligned}
&\operatorname{Var}\big(H_L(Y)\big)\\
&\quad = 2\int_{0<s_1<s_2<L} \big(\mathbb{E}_0^Y\big[h_L(s_1,Y)h_L(s_2,Y)\big] - \mathbb{E}_0^Y\big[h_L(s_1,Y)\big]\mathbb{E}_0^Y\big[h_L(s_2,Y)\big]\big)\,\mathrm{d}s_1\,\mathrm{d}s_2\\
&\quad \le 2\iiiint_{\substack{0<s_1<s_2<L,t_1,t_2<L\\ A_1<t_1-s_1,t_2-s_2<A_2}} \frac{|\mathbb{P}_0^Y(Y_{s_1}=Y_{t_1},Y_{s_2}=Y_{t_2}) - \mathbb{P}_0^Y(Y_{s_1}=Y_{t_1})\mathbb{P}_0^Y(Y_{s_2}=Y_{t_2})|}{(\log(t_1-s_1)\log(t_2-s_2))^{\xi}}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}s_1\,\mathrm{d}s_2\\
&\quad \le 2\int_{0<s_1<s_2<L}\phi(s_2-s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2 \le 2L\int_0^{\infty}\phi(w)\,\mathrm{d}w,
\end{aligned}
$$

where

$$
\phi(w) = \int_{A_1}^{\infty}\int_{w+A_1}^{\infty} \frac{|\mathbb{P}_0^Y(Y_0=Y_{s_1},Y_w=Y_{s_2}) - \mathbb{P}_0^Y(Y_0=Y_{s_1})\mathbb{P}_0^Y(Y_w=Y_{s_2})|}{(\log s_1\log(s_2-w))^{\xi}}\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{7.1}
$$

To prove (4.4), it suffices to show that $\int_0^{\infty}\phi(w)\,\mathrm{d}w \le C/\rho^3$.

Note that in (7.1), s_1, s_2 fall into three regions: (0) $0<s_1<w$; (1) $w<s_1<s_2$; (2) $w<s_2<s_1$. In case (0), the integrand in (7.1) is 0 by the independent increment properties of Y. In case (1), let $r_1 = s_1 - w$ and $r_2 = s_2 - s_1$, while in case (2) let $r_1 = s_2 - w$ and $r_2 = s_1 - s_2$, then

$$
\phi(w) = \mathrm{I}(w) + \mathrm{II}(w) \tag{7.2}
$$

with

$$
\begin{aligned}
\mathrm{I}(w) &= \iint_{[0,\infty)^2} 1_{\{w+r_1>A_1,r_1+r_2>A_1\}} \frac{|\mathbb{P}_0^Y(Y_0=Y_{w+r_1},Y_w=Y_{w+r_1+r_2}) - p_{\rho(w+r_1)}(0)p_{\rho(r_1+r_2)}(0)|}{(\log(w+r_1)\log(r_1+r_2))^{\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_2,\\
\mathrm{II}(w) &= \iint_{[0,\infty)^2} 1_{\{r_1>A_1\}} \frac{|\mathbb{P}_0^Y(Y_0=Y_{w+r_1+r_2},Y_w=Y_{w+r_1}) - p_{\rho(w+r_1+r_2)}(0)p_{\rho r_1}(0)|}{(\log(w+r_1+r_2)\log r_1)^{\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_2.
\end{aligned}
$$

Since $\xi > 1/2$, we establish (4.4) once we show that there exists $C>0$ such that

$$
\begin{aligned}
&\mathrm{I}(w),\mathrm{II}(w) \le \frac{C}{\rho^2} \quad \text{for all } w>0,\\
&\mathrm{I}(w),\mathrm{II}(w) \le \frac{C}{\rho^3 w(\log w)^{2\xi}} \quad \text{for all } w>A_1=\mathrm{e}.
\end{aligned} \tag{7.3}
$$

In $\mathrm{I}(w)$, by Lemmas B.1 and B.2,

$$
\begin{aligned}
&\mathbb{P}_0^Y(Y_0=Y_{w+r_1},Y_w=Y_{w+r_1+r_2})\\
&\quad = \sum_{x\in\mathbb{Z}^3} p_{\rho w}(x)p_{\rho r_1}(-x)p_{\rho r_2}(x) \le \min\left\{p_{\rho r_1}(0)p_{\rho r_2}(0), \frac{C}{\rho^3(wr_1+wr_2+r_1r_2)^{3/2}}\right\},
\end{aligned}
$$

from which we easily deduce that $\mathrm{I}(w) \le 2(\int_0^{\infty}p_{\rho r}(0)\,\mathrm{d}r)^2 = 2G^2\rho^{-2}$. Similarly, $\mathrm{II}(w) \le 2G^2\rho^{-2}$. On the other hand, by the local central limit theorem, Lemma B.1, we have

$$
\begin{aligned}
\mathrm{I}(w) \le{}& \frac{C}{\rho^3}\iint_{[0,\infty)^2} 1_{\{w+r_1>A_1,r_1+r_2>A_1\}}\\
&\times \frac{1/(wr_1+wr_2+r_1r_2)^{3/2} + 1/((w+r_1)^{3/2}(r_1+r_2)^{3/2})}{(\log(w+r_1)\log(r_1+r_2))^{\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_2.
\end{aligned} \tag{7.4}
$$

Let $r_1 = wt_1$ and $r_2 = wt_2$, and assume $w > A_1$, then (7.4) becomes

$$\begin{aligned}
\mathrm{I}(w) &\le \frac{C}{\rho^3 w} \iint_{[0,\infty)^2} 1_{\{1+t_1>A_1w^{-1}, t_1+t_2>A_1w^{-1}\}} \frac{1/(t_1+t_2+t_1t_2)^{3/2} + 1/((1+t_1)^{3/2}(t_1+t_2)^{3/2})}{(\log(w(1+t_1))\log(w(t_1+t_2)))^\xi}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&\le \frac{C}{\rho^3 w(\log w)^{2\xi}} \iint_{\substack{t_1,t_2\ge 0\\ t_1+t_2\ge 1}} \left(\frac{1}{(t_1+t_2+t_1t_2)^{3/2}} + \frac{1}{(1+t_1)^{3/2}(t_1+t_2)^{3/2}}\right)\mathrm{d}t_1\,\mathrm{d}t_2 \qquad (7.5)\\
&\quad + \frac{C}{\rho^3 w(\log w)^{\xi}} \iint_{\substack{t_1,t_2\ge 0\\ t_1+t_2\le 1}} 1_{\{t_1+t_2>A_1w^{-1}\}} \frac{2}{(t_1+t_2)^{3/2}(\log(w(t_1+t_2)))^\xi}\,\mathrm{d}t_1\,\mathrm{d}t_2. \qquad (7.6)
\end{aligned}$$

The integral in (7.5) is finite. Letting $y = t_1 + t_2$, the integral in (7.6) equals

$$\begin{aligned}
\int_{A_1/w}^{1} \frac{2}{\sqrt{y}(\log(wy))^\xi}\,\mathrm{d}y &= \frac{1}{\sqrt{w}} \int_{A_1}^{w} \frac{2}{\sqrt{x}(\log x)^\xi}\,\mathrm{d}x \qquad (7.7)\\
&= \frac{4\sqrt{x}}{\sqrt{w}(\log x)^\xi}\bigg|_{A_1}^{w} + \frac{4\xi}{\sqrt{w}} \int_{A_1}^{w} \frac{1}{\sqrt{x}(\log x)^{1+\xi}}\,\mathrm{d}x \le \frac{C}{(\log w)^\xi},
\end{aligned}$$

which proves that $\mathrm{I}(w) \le \frac{C}{\rho^3 w(\log w)^{2\xi}}$ for $w > A_1$.

In $\mathrm{II}(w)$,

$$\mathbb{P}_0^Y(Y_0 = Y_{w+r_1+r_2}, Y_w = Y_{w+r_1}) = \sum_{x\in\mathbb{Z}^3} p_{\rho w}(x)p_{\rho r_1}(0)p_{\rho r_2}(-x) = p_{\rho r_1}(0)p_{\rho(w+r_2)}(0).$$

Therefore

$$\mathrm{II}(w) = \iint_{[0,\infty)^2} 1_{\{r_1>A_1\}} \frac{p_{\rho r_1}(0)|p_{\rho(w+r_2)}(0) - p_{\rho(w+r_1+r_2)}(0)|}{(\log(w+r_1+r_2)\log r_1)^\xi}\,\mathrm{d}r_1\,\mathrm{d}r_2. \qquad (7.8)$$

We separate the integral in (7.8) according to whether $r_1 > w$ or $r_1 < w$. When $r_1 > w$, we have

$$\begin{aligned}
&\int_w^\infty \int_0^\infty 1_{\{r_1>A_1\}} \frac{p_{\rho r_1}(0)|p_{\rho(w+r_2)}(0) - p_{\rho(w+r_1+r_2)}(0)|}{(\log(w+r_1+r_2)\log r_1)^\xi}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&\le \frac{C\sqrt{w}}{\rho^{3/2}(\log w)^\xi} \int_1^\infty \int_0^\infty \frac{p_{\rho w(1+t_2)}(0) + p_{\rho w(1+t_1+t_2)}(0)}{t_1^{3/2}(\log(wt_1))^\xi}\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\le \frac{C\sqrt{w}}{\rho^{3}(\log w)^\xi} \int_1^\infty \int_0^\infty \frac{1/(w(1+t_2))^{3/2}}{t_1^{3/2}(\log w)^\xi}\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\le \frac{C}{\rho^3 w(\log w)^{2\xi}},
\end{aligned}$$

where we used the local central limit theorem to bound $p_s(0) \le Cs^{-3/2}$ and made the change of variables $r_1 = wt_1$ and $r_2 = wt_2$. When $0 < r_1 < w$ in (7.8), by Lemma B.3, we have

$$\begin{aligned}
&\int_0^w \int_0^\infty 1_{\{r_1>A_1\}} \frac{p_{\rho r_1}(0)|p_{\rho(w+r_2)}(0) - p_{\rho(w+r_1+r_2)}(0)|}{(\log(w+r_1+r_2)\log r_1)^\xi}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&\le \frac{C}{(\log w)^\xi} \int_0^w \int_0^\infty 1_{\{r_1>A_1\}} \frac{p_{\rho r_1}(0)(r_1/\rho^{3/2}(w+r_2)^{5/2})}{(\log r_1)^\xi}\,\mathrm{d}r_2\,\mathrm{d}r_1 \\
&\le \frac{C}{\rho^3(\log w)^\xi} \int_0^w \int_0^\infty 1_{\{r_1>A_1\}} \frac{1}{\sqrt{r_1}(w+r_2)^{5/2}(\log r_1)^\xi}\,\mathrm{d}r_2\,\mathrm{d}r_1
\end{aligned}$$

$$= \frac{C}{\rho^3 w(\log w)^\xi} \int_0^1 \int_0^\infty 1_{\{t_1 > A_1 w^{-1}\}} \frac{1}{\sqrt{t_1}(1+t_2)^{5/2}(\log(wt_1))^\xi} \, \mathrm{d}t_2 \, \mathrm{d}t_1$$
$$\le \frac{C}{\rho^3 w(\log w)^\xi} \int_{A_1/w}^1 \frac{1}{\sqrt{t_1}(\log(wt_1))^\xi} \, \mathrm{d}t_1 \le \frac{C}{\rho^3 w(\log w)^{2\xi}},$$

where the last inequality follows from the same calculation as in (7.7). This proves that $\mathrm{II}(w) \le \frac{C}{\rho^3 w(\log w)^{2\xi}}$ for $w > A_1$ and concludes the proof of Lemma 4.1.

8. Proof of Lemma 4.2

Proof of Lemma 4.2, (4.8)–(4.9). By definition, conditioned on σ,

$$\mathbb{E}_0^{Y^\sigma}\bigl[H^{\mathrm{ext}}_{[\sigma_i,\sigma_{i+1}]}(Y^\sigma)\bigr] - \mathbb{E}_0^{Y}\bigl[H^{\mathrm{ext}}_{[\sigma_i,\sigma_{i+1}]}(Y)\bigr] = \iint_{\substack{\sigma_i < s_1 < \sigma_{i+1} < s_2 \\ A_1 < s_2 - s_1 < A_2}} \frac{\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}) - \mathbb{P}(Y_{s_1} = Y_{s_2})}{\log(s_2 - s_1)} \, \mathrm{d}s_2 \, \mathrm{d}s_1.$$

To prove (4.8), it suffices to show that

$$\mathbb{P}\bigl(Y^\sigma_{s_1} = Y^\sigma_{s_2}\bigr) > \mathbb{P}(Y_{s_1} = Y_{s_2}).$$

This follows from Lemma B.5. Indeed, we can decompose $Y^\sigma_{s_2} - Y^\sigma_{s_1}$ as the sum of independent random variables $Z_1, Z_2, \ldots, Z_{n+1}$, where n is such that $\sigma_{i+n} \le s_2 < \sigma_{i+n+1}$, $Z_1 = Y^\sigma_{\sigma_{i+1}} - Y^\sigma_{s_1}$, $Z_j = Y^\sigma_{\sigma_{i+j}} - Y^\sigma_{\sigma_{i+j-1}}$ for $2 \le i \le n$, and $Z_{n+1} = Y^\sigma_{s_2} - Y^\sigma_{\sigma_{i+n}}$. From the definition of Y^σ,

$$\mathbb{P}(Z_1 = y) = \frac{\sum_x p^Y_{s_1-\sigma_i}(x) p^Y_{\sigma_{i+1}-s_1}(y) p^X_{\sigma_{i+1}-\sigma_i}(x+y)}{p^{X-Y}_{\sigma_{i+1}-\sigma_i}(0)} = \frac{p_{\rho(\sigma_{i+1}-s_1)}(y) p_{\sigma_{i+1}-\sigma_i+\rho(s_1-\sigma_i)}(y)}{p_{(1+\rho)(\sigma_{i+1}-\sigma_i)}(0)},$$

where we used the fact that X and Y have the same symmetric jump probability kernel with respective rates 1 and ρ. Therefore Z_1 is distributed as $X_{\rho(\sigma_{i+1}-s_1)}$ conditioned on $X_{(1+\rho)(\sigma_{i+1}-\sigma_i)} = 0$. Similarly, Z_j for $2 \le j \le n$ is distributed as $X_{\rho(\sigma_{i+j}-\sigma_{i+j-1})}$ conditioned on $X_{(1+\rho)(\sigma_{i+j}-\sigma_{i+j-1})} = 0$, and Z_{n+1} is distributed as $X_{\rho(s_2-\sigma_{i+n})}$ conditioned on $X_{(1+\rho)(\sigma_{i+n+1}-\sigma_{i+n})} = 0$. Therefore Lemma B.5 applies. The proof of (4.9) is analogous and simpler. □

Proof of Lemma 4.2, (4.10). Without loss of generality, assume that $\sigma_0 = a = 0$, and let $\sigma_1 = \Delta$. For $0 \le s_1 \le s_2 \le \Delta$, we have

$$\mathbb{P}\bigl(Y^\sigma_{s_1} = Y^\sigma_{s_2}\bigr) = \frac{\sum_{x,y\in\mathbb{Z}^3} p_{\rho s_1}(x) p_{\rho(s_2-s_1)}(0) p_{\rho(\Delta-s_2)}(y) p_\Delta(x+y)}{p_{(1+\rho)\Delta}(0)}$$
$$= \frac{p_{(1+\rho)\Delta-\rho(s_2-s_1)}(0) p_{\rho(s_2-s_1)}(0)}{p_{(1+\rho)\Delta}(0)}.$$

Therefore, conditioned on $\sigma_0 = 0$ and $\sigma_1 = \Delta$,

$$\mathbb{E}_0^{Y^\sigma}\bigl[H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}(Y^\sigma)\bigr] - \mathbb{E}_0^{Y}\bigl[H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}(Y)\bigr]$$
$$= \iint_{\substack{0 < s_1 < s_2 < \Delta \\ A_1 < s_2 - s_1 < A_2}} \frac{\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}) - \mathbb{P}(Y_{s_1} = Y_{s_2})}{(\log(s_2 - s_1))^\xi} \, \mathrm{d}s_2 \, \mathrm{d}s_1$$
$$= \iint_{\substack{0 < s_1 < s_2 < \Delta \\ A_1 < s_2 - s_1 < A_2}} \frac{p_{\rho(s_2-s_1)}(0)(p_{(1+\rho)\Delta-\rho(s_2-s_1)}(0) - p_{(1+\rho)\Delta}(0))}{p_{(1+\rho)\Delta}(0)(\log(s_2 - s_1))^\xi} \, \mathrm{d}s_2 \, \mathrm{d}s_1$$

$$\geq C \iint_{\substack{0<s_1<s_2<\Delta \\ A_1<s_2-s_1<A_2}} \frac{((1+\rho)\Delta)^{3/2}(\rho(s_2-s_1)/((1+\rho)\Delta)^{5/2})}{\rho^{3/2}(s_2-s_1)^{3/2}(\log(s_2-s_1))^{\xi}} \,\mathrm{d}s_2\,\mathrm{d}s_1$$

$$\geq \frac{C}{\Delta\sqrt{\rho}} \iint_{\substack{0<s_1<s_2<\Delta \\ A_1<s_2-s_1<A_2}} \frac{\mathrm{d}s_2\,\mathrm{d}s_1}{\sqrt{s_2-s_1}(\log(s_2-s_1))^{\xi}} \geq \frac{C\sqrt{\Delta}}{8\sqrt{\rho}(\log\Delta)^{\xi}} 1_{\{2A_1<\Delta<A_2\}}, \tag{8.1}$$

where we have applied Lemma B.3 and used the local central limit theorem to bound $p_t(0) \leq C_1 t^{-3/2}$ for all $t \geq 0$ and $p_t \geq C_2 t^{-3/2}$ for all $t \geq 1$. This proves (4.10). $\square$

Proof of Lemma 4.2, (4.11). Without loss of generality, let $\sigma_0 = a = 0$ and $\sigma_1 = \Delta$. We have

$$\begin{aligned}
&\operatorname{Var}\bigl(H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}\bigl(Y^{\sigma}\bigr)\big|\sigma\bigr) \\
&= \iint_{\substack{0<s_1<s_2<\Delta, A_1<s_2-s_1<A_2, \\ 0<s_1'<s_2'<\Delta, A_1<s_2'-s_1'<A_2}} \frac{\mathbb{P}(Y^{\sigma}_{s_1}=Y^{\sigma}_{s_2}, Y^{\sigma}_{s_1'}=Y^{\sigma}_{s_2'}) - \mathbb{P}(Y^{\sigma}_{s_1}=Y^{\sigma}_{s_2})\mathbb{P}(Y^{\sigma}_{s_1'}=Y^{\sigma}_{s_2'})}{(\log(s_2-s_1)\log(s_2'-s_1'))^{\xi}} \,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_1'\,\mathrm{d}s_2' \\
&\leq 2\iint_{\substack{0<s_1<s_2<\Delta, A_1<s_2-s_1, \\ s_1<s_1'<s_2'<\Delta, A_1<s_2'-s_1'}} \frac{|\mathbb{P}(Y^{\sigma}_{s_1}=Y^{\sigma}_{s_2}, Y^{\sigma}_{s_1'}=Y^{\sigma}_{s_2'}) - \mathbb{P}(Y^{\sigma}_{s_1}=Y^{\sigma}_{s_2})\mathbb{P}(Y^{\sigma}_{s_1'}=Y^{\sigma}_{s_2'})|}{(\log(s_2-s_1)\log(s_2'-s_1'))^{\xi}} \,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_1'\,\mathrm{d}s_2'.
\end{aligned} \tag{8.2}$$

In the integral above, s_1, s_2, s_1' and s_2' fall into three regions: (1) $s_1 < s_2 < s_1' < s_2'$; (2) $s_1 < s_1' < s_2 < s_2'$; (3) $s_1 < s_1' < s_2' < s_2$. In region (1), let $r_1 = s_2 - s_1$, $r_2 = s_1' - s_2$, $r_3 = s_2' - s_1'$, and similarly in regions (2) and (3), let r_1, r_2 and r_3 be the successive increments of the ordered variables. Let (1), (2) and (3) also denote the respective contributions to the integral in (8.2) from the three regions. Then for (1), we have

$$\begin{aligned}
&\mathbb{P}\bigl(Y^{\sigma}_{s_1}=Y^{\sigma}_{s_2}, Y^{\sigma}_{s_1'}=Y^{\sigma}_{s_2'}\bigr) \\
&= \frac{\sum_{x,y,z\in\mathbb{Z}^d} p_{\rho s_1}(x)p_{\rho r_1}(0)p_{\rho r_2}(y)p_{\rho r_3}(0)p_{\rho(\Delta-s_1-r_1-r_2-r_3)}(z)p_{\Delta}(x+y+z)}{p_{(1+\rho)\Delta}(0)} \\
&= \frac{p_{\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho(r_1+r_3)}(0)}{p_{(1+\rho)\Delta}(0)},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\bigl(Y^{\sigma}_{s_1}=Y^{\sigma}_{s_2}\bigr)\mathbb{P}\bigl(Y^{\sigma}_{s_1'}=Y^{\sigma}_{s_2'}\bigr) \\
&= \frac{p_{\rho r_1}(0)p_{(1+\rho)\Delta-\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho r_3}(0)}{p_{(1+\rho)\Delta}(0)^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(1) = \iint_{\substack{0<s_1,r_1,r_2,r_3<\Delta, A_1<r_1,r_3 \\ s_1+r_1+r_2+r_3<\Delta}} &\left| \frac{p_{\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho(r_1+r_3)}(0)}{p_{(1+\rho)\Delta}(0)} \right. \\
&\left. - \frac{p_{\rho r_1}(0)p_{(1+\rho)\Delta-\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho r_3}(0)}{p_{(1+\rho)\Delta}(0)^2} \right| \\
&\Big/ (\log r_1 \log r_3)^{\xi} \,\mathrm{d}s_1\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3.
\end{aligned} \tag{8.3}$$

By similar considerations, we have

$$(2) = \int_{\substack{A_1 < r_1+r_2, r_2+r_3, s_1+r_1+r_2+r_3<\Delta}} \int_{0<s_1,r_1,r_2,r_3<\Delta,} \frac{\mathrm{d}s_1\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3}{(\log(r_1+r_2)\log(r_2+r_3))^{\xi}}$$
$$\times \left| \frac{\sum_{x\in\mathbb{Z}^3} p_{\rho r_1}(x) p_{\rho r_2}(x) p_{\rho r_3}(x) p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(x)}{p_{(1+\rho)\Delta}(0)} \right.$$
$$\left. - \frac{p_{\rho(r_1+r_2)}(0) p_{(1+\rho)\Delta-\rho(r_1+r_2)}(0) p_{\rho(r_2+r_3)}(0) p_{(1+\rho)\Delta-\rho(r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)^2} \right|, \tag{8.4}$$

$$(3) = \int_{\substack{A_1<r_2, s_1+r_1+r_2+r_3<\Delta}} \int_{0<s_1,r_1,r_2,r_3<\Delta,} \frac{\mathrm{d}s_1\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3}{(\log(r_1+r_2+r_3)\log r_2)^{\xi}}$$
$$\times \left| \frac{p_{\rho(r_1+r_3)}(0) p_{\rho r_2}(0) p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)} \right.$$
$$\left. - \frac{p_{\rho(r_1+r_2+r_3)}(0) p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0) p_{\rho r_2}(0) p_{(1+\rho)\Delta-\rho r_2}(0)}{p_{(1+\rho)\Delta}(0)^2} \right|. \tag{8.5}$$

We will show that (1), (2), (3) are all bounded by $C\rho^{-3}\Delta$ for some C uniform in $\rho \in (0,1]$ and $\Delta > 0$.

For (1), we have

$$(1) \le \Delta^2 \iint_{\substack{A_1<r_1,r_3<\Delta \\ r_1+r_3<\Delta}} p_{\rho r_1}(0) p_{\rho r_3}(0) \big| p_{(1+\rho)\Delta}(0) p_{(1+\rho)\Delta-\rho(r_1+r_3)}(0)$$
$$- p_{(1+\rho)\Delta-\rho r_1}(0) p_{(1+\rho)\Delta-\rho r_3}(0) \big| / p_{(1+\rho)\Delta}(0)^2 (\log r_1 \log r_3)^{\xi}\,\mathrm{d}r_1\,\mathrm{d}r_3$$
$$\le \frac{C\Delta^5}{\rho^3} \iint_{A_1<r_1,r_3<\Delta} \frac{\rho^2 r_1 r_3/\Delta^5}{r_1^{3/2} r_3^{3/2} (\log r_1 \log r_3)^{\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_3$$
$$\le \frac{C\Delta}{\rho},$$

where we used the local central limit theorem to bound $p_s(0)$ and we applied Lemma B.4 with $t = (1+\rho)\Delta - \rho(r_1 + r_3)$.

We can bound (2) by passing the absolute value in (8.4) inside. By Lemma B.1, $p_s(0) \le \frac{C}{(1+s)^{3/2}}$ for some $C > 0$ for all $s \ge 0$, and for Δ large, we have

$$\frac{p_{\Delta+s}(x)}{p_{(1+\rho)\Delta}(0)} < (1+\rho)^2 \quad \text{for all } 0 \le s \le \rho\Delta \text{ and } x \in \mathbb{Z}^3. \tag{8.6}$$

Therefore

$$\int_{\substack{A_1<r_1+r_2,r_2+r_3,s_1+r_1+r_2+r_3<\Delta}} \int_{0<s_1,r_1,r_2,r_3<\Delta,} \big(p_{\rho(r_1+r_2)}(0) p_{(1+\rho)\Delta-\rho(r_1+r_2)}(0) p_{\rho(r_2+r_3)}(0) p_{(1+\rho)\Delta-\rho(r_2+r_3)}(0) / p_{(1+\rho)\Delta}(0)^2\big)$$
$$/\big(\log(r_1+r_2)\log(r_2+r_3)\big)^{\xi}\,\mathrm{d}s_1\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3$$
$$\le \frac{C\Delta}{\rho^3} \iint_{0\le r_1,r_2,r_3<\infty} \frac{1}{(1+r_1+r_2)^{3/2}(1+r_2+r_3)^{3/2}(\log(\mathrm{e}+r_1+r_2)\log(\mathrm{e}+r_2+r_3))^{\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3$$
$$\le \frac{C\Delta}{\rho^3}, \tag{8.7}$$

where the last integral is finite since integrating out r_1 and r_3 leads to a bound of the form $\int_0^\infty \frac{C}{(1+r_2)(\log(e+r_2))^{2\xi}}\,dr_2 < \infty$. For the remaining term in (8.4), we have

$$\begin{aligned}
&\int\int_{\substack{0<s_1,r_1,r_2,r_3<\Delta,\\ A_1<r_1+r_2,r_2+r_3,s_1+r_1+r_2+r_3<\Delta}} \frac{\sum_{x\in\mathbb{Z}^3} p_{\rho r_1}(x)p_{\rho r_2}(x)p_{\rho r_3}(x)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(x)}{p_{(1+\rho)\Delta}(0)(\log(r_1+r_2)\log(r_2+r_3))^{\xi}}\,ds_1\,dr_1\,dr_2\,dr_3\\
&\quad\le \frac{C\Delta}{\rho^3}\iint_{0\le r_1,r_2,r_3<\infty}\frac{dr_1\,dr_2\,dr_3}{(1+r_1r_2+r_1r_3+r_2r_3)^{3/2}(\log(e+r_1+r_2)\log(e+r_2+r_3))^{\xi}}\\
&\quad\le \frac{C\Delta}{\rho^3}\iint_{0\le r_1,r_2<\infty}\frac{dr_1\,dr_2}{(\log(e+r_2))^{2\xi}(r_1+r_2)\sqrt{1+r_1r_2}}\\
&\quad= \frac{C\Delta}{\rho^3}\iint_{0\le t,r_2<\infty}\frac{dt\,dr_2}{(\log(e+r_2))^{2\xi}(1+t)\sqrt{1+tr_2^2}},
\end{aligned}\tag{8.8}$$

where we used (8.6), applied Lemma B.2, and made a change of variable $r_1 = tr_2$. The integral in (8.8) is clearly finite when integrated over $r_2 > 1$, since we can bound $\frac{1}{\sqrt{1+tr_2^2}}$ by $\frac{1}{r_2\sqrt{t}}$. For $0 < r_2 < 1$, note that

$$\int_0^\infty \frac{dt}{(1+t)\sqrt{1+tr_2^2}} = \int_0^\infty \frac{dw}{(w+r_2^2)\sqrt{1+w}} \le C - 2\ln r_2,$$

which is integrable over $r_2 \in [0, 1]$. Therefore the integral in (8.8) is finite, and together with (8.7), this shows that $(2) \le C\rho^{-3}\Delta$.

For (3), we have

$$\begin{aligned}
(3) &\le \Delta\int\int_{\substack{0<r_1,r_3<\Delta\\ A_1<r_2,r_1+r_2+r_3<\Delta}} dr_1\,dr_2\,dr_3\,\frac{p_{\rho r_2}(0)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)^2(\log(r_2))^{2\xi}}\\
&\qquad\times\left|p_{\rho(r_1+r_3)}(0)p_{(1+\rho)\Delta}(0) - p_{\rho(r_1+r_2+r_3)}(0)p_{(1+\rho)\Delta-\rho r_2}(0)\right|\\
&\le \frac{C\Delta^{5/2}}{\rho^3/2}\iint_{\substack{0<r_1,r_3<\Delta\\ A_1<r_2<\Delta}} dr_1\,dr_2\,dr_3\left(\frac{p_{\rho(r_1+r_3)}(0)|p_{(1+\rho)\Delta}(0)-p_{(1+\rho)\Delta-\rho r_2}(0)|}{(1+r_2)^{3/2}(\log(r_2))^{2\xi}}\right.\\
&\qquad\left.+\frac{p_{(1+\rho)\Delta-\rho r_2}(0)|p_{\rho(r_1+r_3)}(0)-p_{\rho(r_1+r_2+r_3)}(0)|}{(1+r_2)^{3/2}(\log(r_2))^{2\xi}}\right),
\end{aligned}$$

where we applied (8.6) to $\frac{p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)}$. Using Lemma B.3, we have

$$\begin{aligned}
&\frac{C\Delta^{5/2}}{\rho^{3/2}}\iint_{\substack{0<r_1,r_3<\Delta\\ A_1<r_2<\Delta}}\frac{p_{\rho(r_1+r_3)}(0)|p_{(1+\rho)\Delta}(0)-p_{(1+\rho)\Delta-\rho r_2}(0)|}{(1+r_2)^{3/2}(\log(r_2))^{2\xi}}\,dr_1\,dr_2\,dr_3\\
&\quad\le \frac{C\Delta^{5/2}}{\rho^3}\iint_{0<r_1,r_2,r_3<\Delta}\frac{\rho r_2/\Delta^{5/2}}{(1+r_1+r_3)^{3/2}(1+r_2)^{3/2}(\log(e+r_2))^{2\xi}}\,dr_1\,dr_2\,dr_3\\
&\quad\le \frac{C}{\rho^2}\int_0^\Delta \frac{1}{\sqrt{r_2}}\,dr_2\iint_{0\le r_1+r_3\le 2\Delta}\frac{1}{(r_1+r_3)^{3/2}}\,dr_1\,dr_3\\
&\quad\le \frac{C\Delta}{\rho^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{C\Delta^{5/2}}{\rho^{3/2}} \iint_{\substack{0<r_1,r_3<\Delta\\ A_1<r_2<\Delta}} \frac{p_{(1+\rho)\Delta-\rho r_2}(0)|p_{\rho(r_1+r_3)}(0)-p_{\rho(r_1+r_2+r_3)}(0)|}{(1+r_2)^{3/2}(\log(r_2))^{2\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3\\
&\quad\le \frac{C\Delta}{\rho^{3/2}} \iint_{\substack{0<r_1,r_3<\infty\\ A_1<r_2<\infty}} \frac{r_2/(\rho^{3/2}(r_1+r_3)^{3/2}(r_1+r_2+r_3))}{(1+r_2)^{3/2}(\log(r_2))^{2\xi}}\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3\\
&\quad\le \frac{C\Delta}{\rho^3}\int_{A_1}^{\infty}\int_0^{\infty}\frac{1}{\sqrt{r_2}(\log(r_2))^{2\xi}}\frac{1}{\sqrt{w}(r_2+w)}\,\mathrm{d}w\,\mathrm{d}r_2\\
&\quad= \frac{C\Delta}{\rho^3}\int_{A_1}^{\infty}\int_0^{\infty}\frac{1}{r_2(\log(r_2))^{2\xi}}\frac{1}{\sqrt{t}(1+t)}\,\mathrm{d}t\,\mathrm{d}r_2\\
&\quad\le \frac{C\Delta}{\rho^3},
\end{aligned}$$

where we used the fact that $\iint_{[0,\infty)^2} f(r_1+r_3)\,\mathrm{d}r_1\,\mathrm{d}r_3=\int_0^\infty wf(w)\,\mathrm{d}w$, and made a change of variable $w=r_2t$. Thus we have proved (3) $\le C\rho^{-3}\Delta$, which concludes the proof of (4.11). □

Appendix A: Renewal estimates

Consider a renewal process $\sigma:=\{\sigma_0=0<\sigma_1<\cdots\}$ on $[0,\infty)$, where $(\sigma_i-\sigma_{i-1})_{i\in\mathbb{N}}$ are i.i.d. with distribution $K(t)\,\mathrm{d}t$ for a bounded density K on $\mathbb{R}_+$ satisfying

$$K(t)\sim c_K t^{-1-\alpha},\quad t\to\infty, \tag{A.1}$$

for some $\alpha\in(0,1)$ and $c_K\in(0,\infty)$. Let K^{*n} denote the n-fold convolution of K with itself, and let $P(t)=\sum_{n=1}^\infty K^{*n}(t)$, as defined in (3.13), be the corresponding renewal density.

We prove in Lemma A.1 a special case of the continuous time version of Doney's local limit theorem for renewal processes with infinite mean [5], Theorem 3. Note that [5] allows a general regularly varying function in the right-hand side of (A.1). We stick to the narrower class, which suffices for our purposes, for the sake of a less cumbersome proof.

Lemma A.1. *We have*

$$\lim_{t\to\infty} c_K t^{1-\alpha}P(t)=\frac{\alpha\sin(\alpha\pi)}{\pi}. \tag{A.2}$$

Lemma A.2. *There exists a positive stable random variable G with exponent α, such that*

$$\lim_{t\to\infty}\mathbb{P}\big(\big|\sigma\cap[0,t]\big|\ge at^\alpha\big)=\mathbb{P}\biggl(G\le\frac{1}{a^{1/\alpha}}\biggr)\quad \textit{for all } a>0. \tag{A.3}$$

It is well known that

$$\frac{\sigma_n}{n^{1/\alpha}}=\frac{(\sigma_1-\sigma_0)+\cdots+(\sigma_n-\sigma_{n-1})}{n^{1/\alpha}}\xrightarrow{d}G\quad \text{as } n\to\infty, \tag{A.4}$$

where G is a one-sided stable random variable of index α. Note that $\int_t^\infty K(s)\,\mathrm{d}s\sim(c_K/\alpha)t^{-\alpha}$, thus the normalisation is chosen here in such a way that $\mathbb{E}[\mathrm{e}^{-\lambda G}]=\exp(-\frac{c_K\Gamma(1-\alpha)}{\alpha}\lambda^\alpha)$, $\lambda\ge0$, i.e., G is $(c_K\Gamma(1-\alpha)/\alpha)^{1/\alpha}$ times a "standard" one-sided stable random variable of index α (see, e.g., [7], Theorem XIII.6.2). Since the characteristic function of G decays faster than any polynomial at infinity, G has a C^∞ density g, see, e.g., [12], p. 48. As G is a limit of non-negative random variables, we must have $g(x) = 0$ for $x < 0$, implying $g(0) = 0$ by continuity. Furthermore, $g(x) \sim c_G x^{-1-\alpha}$ for $x \to \infty$ with some $c_G \in (0, \infty)$, see, e.g., [12], Theorem 2.4.1. In particular, $x \mapsto x^{-\alpha} g(x)$ is bounded and uniformly continuous with $x^{-\alpha} g(x) \le c/(1 + x^{1+2\alpha})$ for some $c < \infty$. We have

$$\int_0^\infty x^{-\alpha} g(x)\,\mathrm{d}x = \frac{\alpha}{c_K \Gamma(1-\alpha)} \cdot \frac{1}{\Gamma(1+\alpha)} = \frac{1}{c_K \Gamma(1-\alpha)\Gamma(\alpha)} = \frac{\sin(\alpha\pi)}{c_K \pi}. \tag{A.5}$$

For the first equality, note that $G^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^\infty \lambda^{\alpha-1} \mathrm{e}^{-\lambda G}\,\mathrm{d}\lambda$. The second identity uses well-known facts about the Γ function.

Proof of Lemma A.2. By (A.4),

$$\lim_{t\to\infty} \mathbb{P}\big(\big|\sigma \cap [0,t]\big| \ge at^\alpha\big) = \lim_{t\to\infty} \mathbb{P}\biggl(\frac{\sigma_{\lfloor at^\alpha \rfloor}}{\lfloor at^\alpha \rfloor^{1/\alpha}} \le \frac{t}{\lfloor at^\alpha \rfloor^{1/\alpha}}\biggr) = \mathbb{P}\biggl(G \le \frac{1}{a^{1/\alpha}}\biggr),$$

since the distribution of G contains no atoms. □

We will need the following uniform one-sided large deviation estimate.

Lemma A.3. *We have for any sequence $c_n \to \infty$*

$$\lim_{n\to\infty} \sup_{t \ge c_n n^{1/\alpha}} \left| \frac{K^{*n}(t)}{nK(t)} - 1 \right| = 0. \tag{A.6}$$

Proof. This follows from [17], Theorem 1, by specialising to the one-dimensional asymmetric case. Note that Zaigraev [17] attributes the result in the present case (one-dimensional situation, K in the normal domain of attraction of a stable law) to Tkačuk [15], which the authors unfortunately could not access. □

Proof of Lemma A.1. Our proof follows more or less the scheme of [5], Theorem 3, with [5], Theorem 2, replaced by Lemma A.3. Even though we use Lemma A.1 in this paper only for $\alpha = 1/2$, the proof is the same for all $\alpha \in (0,1)$.

By a local limit theorem for sums of random variables in the domain of attraction of a stable law, e.g., [12], Theorem 4.3.1, we have

$$\sup_{t\in\mathbb{R}_+} \big| n^{1/\alpha} K^{*n}\big(n^{1/\alpha} t\big) - g(t) \big| \to 0 \quad \text{as } n \to \infty, \tag{A.7}$$

where g is the density of the one-sided stable random variable appearing as the limit in (A.4). Thus, we can find a continuous, strictly decreasing function $\rho : [0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} \rho(t) = 0$ such that

$$\sup_{t\in\mathbb{R}_+} \big| n^{1/\alpha} K^{*n}\big(n^{1/\alpha} t\big) - g(t) \big| \le \rho(n) \quad \text{for } n \in \mathbb{N}. \tag{A.8}$$

Obviously, $\rho^{-1} : (0, \rho(0)] \to [0,\infty)$ is continuous and strictly decreasing with $\lim_{y\to 0+} \rho^{-1}(y) = \infty$. Note that the function $\psi : (0, \rho(0)^{1/(2-\alpha)}] \to [0,\infty)$ with $\psi(y) = (\rho^{-1}(y^{2-\alpha}))^{1/\alpha}/y$ is strictly decreasing, and $\lim_{y\to 0+} \psi(y) = \infty$. Define $\delta(t) := \psi^{-1}(t)$ for $t \ge 0$. Observe that then $t \mapsto \delta(t)$ is strictly decreasing and satisfies $\lim_{t\to\infty} \delta(t) = 0$. Furthermore,

$$\rho\big(\big(\delta(t)t\big)^\alpha\big) = \rho\big(\big(\delta(t)\psi\big(\delta(t)\big)\big)^\alpha\big) = \rho\big(\rho^{-1}\big(\delta(t)^{2-\alpha}\big)\big) = \delta(t)^{2-\alpha}, \tag{A.9}$$

proving that $t\delta(t) \to \infty$ as $t \to \infty$, and

$$\frac{\rho((\delta(t)t)^\alpha)}{\delta(t)^{1-\alpha}} = \delta(t) \to 0 \quad \text{as } t \to \infty. \tag{A.10}$$

Decompose

$$t^{1-\alpha}\sum_{n\geq 1}K^{*n}(t)=t^{1-\alpha}\sum_{n>(\delta(t)t)^{\alpha}}K^{*n}(t)+t^{1-\alpha}\sum_{n=1}^{[(\delta(t)t)^{\alpha}]}K^{*n}(t)=:S_1+S_2. \tag{A.11}$$

We have

$$S_1=t^{1-\alpha}\sum_{n>(\delta(t)t)^{\alpha}}\frac{1}{n^{1/\alpha}}g\left(\frac{t}{n^{1/\alpha}}\right)+t^{1-\alpha}\sum_{n>(\delta(t)t)^{\alpha}}\frac{1}{n^{1/\alpha}}\left(n^{1/\alpha}K^{*n}\left(n^{1/\alpha}\frac{t}{n^{1/\alpha}}\right)-g\left(\frac{t}{n^{1/\alpha}}\right)\right)=:S_1'+R_1,$$

where

$$|R_1|\leq\rho\big(\big(\delta(t)t\big)^{\alpha}\big)t^{1-\alpha}\sum_{n>(\delta(t)t)^{\alpha}}\frac{1}{n^{1/\alpha}}.$$

Since $\sum_{n>(\delta(t)t)^{\alpha}}\frac{1}{n^{1/\alpha}}\sim\int_{(\delta(t)t)^{\alpha}}^{\infty}x^{-1/\alpha}\,\mathrm{d}x\sim\frac{\alpha}{1-\alpha}(\delta(t)t)^{\alpha-1}$, we obtain from (A.10) that $R_1\to 0$ as $t\to\infty$.

Put $x_n^{(t)}:=t/n^{1/\alpha}$, then we have

$$\frac{t}{n^{1/\alpha}}\sim\alpha n\big(x_n^{(t)}-x_{n+1}^{(t)}\big)=\alpha\big(t/x_n^{(t)}\big)^{\alpha}\big(x_n^{(t)}-x_{n+1}^{(t)}\big)$$

since $\alpha n n^{1/\alpha}(x_n^{(t)}-x_{n+1}^{(t)})/t=\alpha n(1-\frac{n^{1/\alpha}}{(n+1)^{1/\alpha}})\sim\alpha n(1-(1-\frac{1}{n+1})^{1/\alpha})\to 1$, and hence

$$S_1'\sim\alpha\sum_{n>(\delta(t)t)^{\alpha}}\frac{(t/x_n^{(t)})^{\alpha}}{t^{\alpha}}\big(x_n^{(t)}-x_{n+1}^{(t)}\big)g\big(x_n^{(t)}\big)\sim\alpha\sum_{n\,:\,n^{1/\alpha}>\delta(t)t}\big(x_n^{(t)}-x_{n+1}^{(t)}\big)\big(x_n^{(t)}\big)^{-\alpha}g\big(x_n^{(t)}\big).$$

The term on the right is an approximating Riemann sum and $n^{1/\alpha}>\delta(t)t$ means $x_n^{(t)}<1/\delta(t)$, which tends to ∞ as $t\to\infty$. Thus, recalling (A.5) and the discussion above it, we have

$$S_1\to\alpha\int_0^{\infty}x^{-\alpha}g(x)\,\mathrm{d}x=\frac{\alpha\sin(\alpha\pi)}{c_K\pi}\quad\text{as } t\to\infty.$$

To bound S_2, note that $n\leq(\delta(t)t)^{\alpha}$ implies $t\geq n^{1/\alpha}/\delta(t)\geq n^{1/\alpha}$ for t sufficiently large. In particular, for such t and n, $\delta(t)\leq\delta(n^{1/\alpha})$, so $t\geq(\delta(n^{1/\alpha}))^{-1}n^{1/\alpha}$. Applying Lemma A.3 with $c_n:=1/\delta(n^{1/\alpha})\to\infty$, we see that there exists $n_0\in\mathbb{N}$, $t_0<\infty$ and $C<\infty$ such that

$$K^{*n}(t)\leq CnK(t)\quad\text{for all } n\geq n_0,\, t\geq\frac{n^{1/\alpha}}{\delta(t)}\vee t_0. \tag{A.12}$$

Note that

$$K^{*n}(t)\leq 2nc_K(t/n)^{-1-\alpha}\leq 4n^{2+\alpha}K(t)\quad\text{for } t \text{ sufficiently large}, \tag{A.13}$$

which follows from (A.1) and the observation that $K^{*n}(t)$ is bounded from above by

$$\begin{aligned}&\sum_{j=1}^{n}\int\cdots\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=t}1_{\{\sigma_j-\sigma_{j-1}\geq t/n\}}\prod_{i=1}^{m}K(\sigma_i-\sigma_{i-1})\prod_{i=1}^{m-1}\mathrm{d}\sigma_i\\&\quad\leq 2c_K\left(\frac{t}{n}\right)^{-1-\alpha}\sum_{j=1}^{n}\int\cdots\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=t}\prod_{i=1,i\neq j}^{m}K(\sigma_i-\sigma_{i-1})\prod_{i=1}^{m-1}\mathrm{d}\sigma_i\\&\quad\leq 2nc_K\left(\frac{t}{n}\right)^{-1-\alpha}\left(\int_0^{\infty}K(\sigma)\,\mathrm{d}\sigma\right)^{n-1}.\end{aligned}$$

Therefore if the constant C appearing in (A.12) is suitably increased, (A.12) holds for all $n \in \mathbb{N}$. Thus for t sufficiently large, we have

$$S_2 = t^{1-\alpha} \sum_{n=1}^{[(\delta(t)t)^\alpha]} K^{*n}(t) \le 2Cc_K t^{-2\alpha} \sum_{n=1}^{[(\delta(t)t)^\alpha]} n \le 2Cc_K t^{-2\alpha}\big(\delta(t)t\big)^{2\alpha} = 2Cc_K \delta(t)^{2\alpha},$$

which converges to 0 as $t \to \infty$. □

Appendix B: Random walk estimates

Lemma B.1 (Local central limit theorem). *Let $(X_t)_{t\ge 0}$ with $X_0 = 0$ be a continuous time random walk on $\mathbb{Z}^d$ with jump rate* 1 *and jump probability kernel $(q(x))_{x\in\mathbb{Z}^d}$, which is irreducible and symmetric with finite covariance matrix $Q_{ij} = \sum_{x\in\mathbb{Z}^d} x_i x_j q(x)$, $1 \le i, j \le d$. Let $p_t(\cdot)$ denote the transition probability kernel of X at time t. Then*

$$p_t(x) \le p_t(0) \quad \textit{for all } x \in \mathbb{Z}^d \textit{ and } t \ge 0 \tag{B.1}$$

and

$$\lim_{t\to\infty} (2\pi t)^{d/2}\sqrt{\det Q}\, p_t(0) = 1. \tag{B.2}$$

Proof. Since $\hat{p}(k) := \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\langle k,x\rangle} p_t(x) = \mathrm{e}^{-t(1-\phi(k))}$, where $\phi(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\langle k,x\rangle} q(x)$ is real by the symmetry of q, by inverse Fourier transform,

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \mathrm{e}^{-t(1-\phi(k))}\, \mathrm{d}k \le \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{e}^{-t(1-\phi(k))}\, \mathrm{d}k = p_t(0).$$

For (B.2), see, e.g., [14], Proposition 7.9, Chapter II, where a discrete time version was proved. The proof for the continuous time version is identical. □

Lemma B.2. *Let X, $q(\cdot)$ and $p_t(\cdot)$ be as in Lemma* B.1 *without the symmetry assumption on q. Then for any $a, b, c >$* 0, *there exists some $C > 0$ depending only on q such that*

$$\sum_{x\in\mathbb{Z}^d} p_a(x)p_b(x)p_c(x) \le \frac{C}{(1+ab+bc+ca)^{d/2}}. \tag{B.3}$$

Proof. Without loss of generality, assume that $a \ge b \ge c$. By the local central limit theorem, there exists $C_1 > 0$ such that uniformly in $t > 0$ and $x \in \mathbb{Z}^d$, we have $p_t(x) \le \frac{C_1}{(1+t)^{d/2}}$. Then

$$\sum_{x\in\mathbb{Z}^d} p_a(x)p_b(x)p_c(x) \le \frac{C_1^2}{(1+ab)^{d/2}} \sum_{x\in\mathbb{Z}^d} p_c(x) = \frac{C_1^2}{(1+ab)^{d/2}} \le \frac{C}{(1+ab+bc+ca)^{d/2}}.$$

□

Lemma B.3. *Let X, $q(\cdot)$, Q and $p_t(\cdot)$ be as in Lemma* B.1 *so that q is symmetric. Then there exist $C_1, C_2 > 0$ depending on q, such that*

$$\frac{C_1 r}{t^{d/2}(t+r)} \le p_t(0) - p_{t+r}(0) \le \frac{C_2 r}{t^{d/2}(t+r)}, \tag{B.4}$$

where the first inequality holds for all $r > 0$, $t > 1$, and the second inequality holds for all $r, t > 0$.

Proof. By the symmetry of q, $\phi(k) := \sum_x \mathrm{e}^{\mathrm{i}\langle k,x\rangle} q(x) \in [-1,1]$, and $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle k,X_t\rangle}] = \mathrm{e}^{-t(1-\phi(k))}$. Therefore,

$$p_t(0) - p_{t+r}(0) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \bigl(\mathrm{e}^{-t(1-\phi(k))} - \mathrm{e}^{-(t+r)(1-\phi(k))}\bigr)\,\mathrm{d}k.$$

By irreducibility of $q(\cdot)$, $\phi(k) = 1$ only at $k = 0$, and hence $c := \inf_{|k|\ge\varepsilon, k\in[-\pi,\pi]^d}(1-\phi(k)) > 0$ for any $\varepsilon > 0$. By Taylor expansion, if $\varepsilon > 0$ is sufficiently small, then

$$\frac{1}{4}\langle k, Qk\rangle \le \bigl(1-\phi(k)\bigr) \le \langle k, Qk\rangle \quad \forall |k| < \varepsilon.$$

Therefore

$$\begin{aligned}
(2\pi)^d\bigl(p_t(0) - p_{t+r}(0)\bigr) &= \int_{[-\pi,\pi]^d} \mathrm{e}^{-t(1-\phi(k))}\bigl(1 - \mathrm{e}^{-r(1-\phi(k))}\bigr)\,\mathrm{d}k \\
&\le r \int_{[-\pi,\pi]^d} \bigl(1-\phi(k)\bigr)\mathrm{e}^{-t(1-\phi(k))}\,\mathrm{d}k \\
&\le 2r \int_{|k|>\varepsilon, k\in[-\pi,\pi]^d} \mathrm{e}^{-t(1-\phi(k))}\,\mathrm{d}k + r\int_{|k|\le\varepsilon} \langle k, Qk\rangle \mathrm{e}^{-t\langle k,Qk\rangle/4}\,\mathrm{d}k \\
&\le 2(2\pi)^d \mathrm{e}^{-ct} r + \frac{r}{t^{d/2+1}} \int_{\mathbb{R}^d} \langle k, Qk\rangle \mathrm{e}^{-\langle k,Qk\rangle/4}\,\mathrm{d}k \\
&\le \frac{Cr}{t^{d/2+1}},
\end{aligned} \tag{B.5}$$

which implies that $p_t(0) - p_{t+r}(0) \le \frac{C_2 r}{t^{d/2}(t+r)}$ for $r < t$. When $r \ge t$, the same bound follows from the local central limit theorem.

Similarly,

$$\begin{aligned}
(2\pi)^d\bigl(p_t(0) - p_{t+r}(0)\bigr) &= \int_{[-\pi,\pi]^d} \mathrm{e}^{-(t+r)(1-\phi(k))}\bigl(\mathrm{e}^{r(1-\phi(k))} - 1\bigr)\,\mathrm{d}k \\
&\ge r \int_{|k|\le\varepsilon, k\in[-\pi,\pi]^d} \bigl(1-\phi(k)\bigr)\mathrm{e}^{-(t+r)(1-\phi(k))}\,\mathrm{d}k \\
&\ge r \int_{|k|\le\varepsilon, k\in[-\pi,\pi]^d} \frac{\langle k, Qk\rangle}{4}\mathrm{e}^{-(t+r)\langle k,Qk\rangle}\,\mathrm{d}k \\
&\ge \frac{Cr}{(t+r)^{d/2+1}},
\end{aligned}$$

which follows by a change of variable for k and the fact that $t + r > 1$. This implies $p_t(0) - p_{t+r}(0) \ge \frac{C_1 r}{t^{d/2}(t+r)}$ for $r < t$. When $r > t$, the same bound follows from the local central limit theorem. □

Lemma B.4. *Let X, $q(\cdot)$ and $p_t(\cdot)$ be as in Lemma B.1 so that q is symmetric. Then there exist $C > 0$ depending only on q such that, for all $a, b > 0$ and $t > 0$,*

$$\bigl|p_t(0)p_{t+a+b}(0) - p_{t+a}(0)p_{t+b}(0)\bigr| \le \frac{Cab}{t^d(t+a)(t+b)}. \tag{B.6}$$

Proof. Note that

$$\begin{aligned}
&p_t(0)p_{t+a+b}(0) - p_{t+a}(0)p_{t+b}(0) \\
&\quad = p_{t+a+b}(0)\bigl(p_t(0) - p_{t+a}(0)\bigr) - p_{t+a}(0)\bigl(p_{t+b}(0) - p_{t+a+b}(0)\bigr) \\
&\quad = \bigl(p_{t+a+b}(0) - p_{t+a}(0)\bigr)\bigl(p_t(0) - p_{t+a}(0)\bigr) + p_{t+a}(0)\bigl(p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0)\bigr).
\end{aligned} \tag{B.7}$$

By Lemma B.3, the first term in (B.7) is bounded in absolute value by

$$\frac{Cb}{(t+a)^{d/2}(t+a+b)} \cdot \frac{Ca}{t^{d/2}(t+a)},$$

which is clearly bounded by the right-hand side of (B.6).

For the second term in (B.7), we claim that

$$0 \le p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \le \frac{Cab}{t^{d/2}(t+a)(t+b)}, \tag{B.8}$$

which together with the fact that $p_{t+a}(0) \le Ct^{-d/2}$ imply (B.6). Note that

$$\begin{aligned}(2\pi)^d\big(p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0)\big) &= \int_{[-\pi,\pi]^d} \mathrm{e}^{-t(1-\phi(k))}\big(1-\mathrm{e}^{-a(1-\phi(k))}\big)\big(1-\mathrm{e}^{-b(1-\phi(k))}\big)\,\mathrm{d}k \\ &\le ab\int_{[-\pi,\pi]^d}\big(1-\phi(k)\big)^2\mathrm{e}^{-t(1-\phi(k))}\,\mathrm{d}k.\end{aligned}$$

Clearly $p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \ge 0$. For the upper bound, exactly as in (B.5), we can Taylor expand $\phi(k)$ around $k=0$ for $|k| \le \varepsilon$ and bound $|\phi(k)|$ uniformly for $|k| > \varepsilon$, which gives

$$p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \le \frac{Cab}{t^{d/2+2}}.$$

When $a, b < t$, this implies (B.8). If $b > t$, then (B.8) follows from the bound

$$p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \le \frac{Ca}{t^{d/2}(t+a)} + \frac{Ca}{(t+b)^{d/2}(t+a+b)}$$

by Lemma B.3. The same argument applies when $a > t$. □

Lemma B.5 (Comparison of return probabilities). *Let X, $q(\cdot)$ and $p_t(\cdot)$ be as in Lemma B.1 so that q is symmetric. For $1 \le i \le n$, let $a_i, b_i > 0$, and let Z_i be an independent random variable distributed as X_{a_i} conditioned on $X_{a_i+b_i} = 0$. Then*

$$\mathbb{P}(Z_1 + \cdots + Z_n = 0) > \mathbb{P}(X_{a_1+\cdots+a_n} = 0). \tag{B.9}$$

Proof. Let $\phi(k) = \sum_x \mathrm{e}^{\mathrm{i}\langle k,x\rangle}q(x)$ and $\psi_i(k) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle k,Z_i\rangle}]$. Since $\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle k,X_t\rangle}] = \mathrm{e}^{-t(1-\phi(k))}$, by Fourier transform, (B.9) is equivalent to

$$\int_{[-\pi,\pi]^d} \psi_1(k)\cdots\psi_n(k)\,\mathrm{d}k > \int_{[-\pi,\pi]^d} \mathrm{e}^{-\sum_{i=1}^n a_i(1-\phi(k))}\,\mathrm{d}k. \tag{B.10}$$

By symmetry of q, $\phi(k) \in [-1,1]$ and $\mathrm{e}^{-a_i(1-\phi(k))} \in (0,1]$. Therefore to verify (B.10), it suffices to show that for each $1 \le i \le n$,

$$\psi_i(k) \ge \mathrm{e}^{-a_i(1-\phi(k))} \tag{B.11}$$

for all $k \in [-\pi,\pi]^d$, with strict inequality for some $k \in [-\pi,\pi]^d$.

Note that $\hat{p}_s(k) := \sum_x \mathrm{e}^{\mathrm{i}\langle k,x\rangle}p_s(x) = \mathrm{e}^{-s(1-\phi(k))}$. By definition, $\mathbb{P}(Z_i = x) = \frac{p_{a_i}(x)p_{b_i}(x)}{p_{a_i+b_i}(0)}$, and hence

$$\psi_i(k) = \frac{(\hat{p}_{a_i} * \hat{p}_{b_i})(k)}{p_{a_i+b_i}(0)} = \frac{\int_{[-\pi,\pi]^d}\mathrm{e}^{-a_i(1-\phi(k-u))-b_i(1-\phi(u))}\,\mathrm{d}u}{\int_{[-\pi,\pi]^d}\mathrm{e}^{-(a_i+b_i)(1-\phi(u))}\,\mathrm{d}u}.$$

By symmetry, $\psi_i(k) = \psi_i(-k)$, and hence

$$\psi_i(k) = \frac{\int_{[-\pi,\pi]^d} \mathrm{e}^{-b_i(1-\phi(u))}((\mathrm{e}^{-a_i(1-\phi(k-u))} + \mathrm{e}^{-a_i(1-\phi(-k-u))})/2)\,\mathrm{d}u}{\int_{[-\pi,\pi]^d} \mathrm{e}^{-(a_i+b_i)(1-\phi(u))}\,\mathrm{d}u}$$
$$\geq \frac{\int_{[-\pi,\pi]^d} \mathrm{e}^{-b_i(1-\phi(u))}\mathrm{e}^{-a_i(1-(\phi(k-u)+\phi(-k-u))/2)}\,\mathrm{d}u}{\int_{[-\pi,\pi]^d} \mathrm{e}^{-(a_i+b_i)(1-\phi(u))}\,\mathrm{d}u}, \tag{B.12}$$

where we applied Jensen's inequality. Note that since $\phi(x)$ is not identically equal to 1, for some choice of k and u, we have $\phi(k-u) \neq \phi(-k-u)$ so that there is strict inequality in (B.12) for some k. By symmetry,

$$\begin{aligned}
\phi(k-u) + \phi(-k-u) &= \sum_x q(x)\big(\mathrm{e}^{\mathrm{i}\langle k-u,x\rangle} + \mathrm{e}^{\mathrm{i}\langle -k-u,x\rangle}\big) \\
&= \sum_x q(x)\big(\cos\langle k-u,x\rangle + \cos\langle -k-u,x\rangle\big) \\
&= 2\sum_x q(x)\cos\langle k,x\rangle\cos\langle u,x\rangle \\
&\geq 2\sum_x q(x)\big(\cos\langle k,x\rangle + \cos\langle u,x\rangle - 1\big) \\
&= 2\big(\phi(k) + \phi(u) - 1\big),
\end{aligned} \tag{B.13}$$

where we used $(1-\cos\alpha)(1-\cos\beta) \geq 0$. Plugging this bound into (B.12) then yields (B.11). □

Appendix C: Proof of Theorem 1.3

Let $\rho' > \rho \geq 0$. Let $X, Y, Y^{(1)}, Y^{(2)}$ be independent random walks on $\mathbb{Z}^d$ with the same symmetric jump kernel with finite second moments and with respective jump rates 1, ρ, $\frac{1+\rho'}{1+\rho}\rho$ and $\frac{\rho'-\rho}{1+\rho}$. Then $Y' := Y^{(1)} + Y^{(2)}$ and $X' := X - Y^{(2)}$ are random walks with the same jump kernel and jump rates ρ' and $\frac{1+\rho'}{1+\rho}$, where for X' we used the symmetry of the kernel. The key observation is that

$$\big(\mathbb{E}^{Y^{(2)}}\big[Z^{\beta}_{t,Y'}\big]\big)_{t>0} \overset{\text{law}}{=} \big(Z^{\beta(1+\rho)/(1+\rho')}_{t(1+\rho')/(1+\rho),Y}\big)_{t>0}, \tag{C.1}$$

which is a simple consequence of the fact that

$$\mathbb{E}^{Y^{(2)}}\big[Z^{\beta}_{t,Y'}\big] = \mathbb{E}^{Y^{(2)},X}\big[\mathrm{e}^{\beta\int_0^t 1_{\{X_s=Y^{(1)}_s+Y^{(2)}_s\}}\,\mathrm{d}s}\big] = \mathbb{E}^{X'}\big[\mathrm{e}^{\beta\int_0^t 1_{\{X'_s=Y^{(1)}_s\}}\,\mathrm{d}s}\big],$$
$$Z^{\beta(1+\rho)/(1+\rho')}_{t(1+\rho')/(1+\rho),Y} = \mathbb{E}^X\big[\mathrm{e}^{\beta(1+\rho)/(1+\rho')\int_0^{t(1+\rho')/(1+\rho)} 1_{\{X_s=Y_s\}}\,\mathrm{d}s}\big],$$

and the fact that

$$(X_{(1+\rho')/(1+\rho)s}, Y_{(1+\rho')/(1+\rho)s})_{s\geq 0} \overset{\text{law}}{=} \big(X'_s, Y^{(1)}_s\big)_{s\geq 0}.$$

Note that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{Y^{(2)}}\big[Z^{\beta}_{t,Y'}\big] \geq \lim_{t\to\infty}\frac{1}{t}\log Z^{\beta}_{t,Y'} = F\big(\beta,\rho'\big) \quad \text{a.s.}$$

On the other hand, by (C.1),

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{Y^{(2)}}\big[Z^{\beta}_{t,Y'}\big] = \lim_{t\to\infty}\frac{1}{t}\log Z^{\beta(1+\rho)/(1+\rho')}_{t(1+\rho')/(1+\rho),Y} = \frac{1+\rho'}{1+\rho}F\left(\beta\frac{1+\rho}{1+\rho'},\rho\right) \quad \text{a.s.}$$

Therefore

$$F(\beta, \rho') \le \frac{1+\rho'}{1+\rho} F\left(\beta \frac{1+\rho}{1+\rho'}, \rho\right) \quad \text{for all } \rho' > \rho \ge 0, \tag{C.2}$$

which implies the first inequality in (1.8).

Similarly, by (C.1),

$$\sup_{t>0} Z^{\beta(1+\rho)/(1+\rho')}_{t(1+\rho')/(1+\rho),Y} < \infty \quad \text{a.s.} \quad \Longleftrightarrow \quad \sup_{t>0} \mathbb{E}^{Y^{(2)}}\left[Z^{\beta}_{t,Y'}\right] < \infty \quad \text{a.s.} \quad \Longrightarrow \quad \sup_{t>0} Z^{\beta}_{t,Y'} < \infty \quad \text{a.s.},$$

which implies the second inequality in (1.8).

To prove the first inequality in (1.9), let $\phi(\rho) := \frac{\beta_c(\rho)}{1+\rho}$, recall that $\beta_c^{\mathrm{ann}}(\rho) = (1+\rho)/G$, and note that

$$\begin{aligned} \beta_c(\rho') - \beta_c^{\mathrm{ann}}(\rho') - \left(\beta_c(\rho) - \beta_c^{\mathrm{ann}}(\rho)\right) &= (1+\rho')\left(\phi(\rho') - G^{-1}\right) - (1+\rho)\left(\phi(\rho) - G^{-1}\right) \\ &= (\rho' - \rho)\left(\phi(\rho') - G^{-1}\right) + (1+\rho)\left(\phi(\rho') - \phi(\rho)\right) > 0, \end{aligned}$$

since $\phi(\rho)$ is non-decreasing in ρ, and $(1+\rho')(\phi(\rho') - G^{-1}) = \beta_c(\rho') - \beta_c^{\mathrm{ann}}(\rho') > 0$ by Theorem 1.1 and its analogue in dimensions $d \ge 4$ shown in [3]. The proof of the second inequality in (1.9) is identical.

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