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# Almost sure absolute continuity of Bernoulli convolutions

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**Abstract.** We prove an extension of a result by Peres and Solomyak on almost sure absolute continuity in a class of symmetric Bernoulli convolutions.

**Résumé.** La continuité absolue, presque sûrement, est démontrée dans une classe de convolutions de Bernoulli symétrique, étendant un résultat de Peres et Solomyak.

*MSC:* 60Exx; 60G50

*Keywords:* Bernoulli convolutions; Absolute continuity

## 1. Introduction

For $\lambda \in (0, 1)$, define the random series

$$Y_\lambda = \sum_{n\geq 1} \pm\lambda^n,$$

where the signs are chosen independently with probability $1/2$. It is easy to see that the distribution $\nu_\lambda$ of $Y_\lambda$ is singular for $\lambda < 1/2$, see Kershner and Wintner [2]. Wintner [7] noted that $\nu_{1/2}$ is uniform on $[-1, 1]$. For Lebesgue almost every $1/2 < \lambda < 1$, Erdös conjectured that $\nu_\lambda$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. This conjecture has attracted a lot of attention during the years, and was finally settled by Solomyak [4] in 1995, who also proved that the densities are in $L^2(\mathbb{R})$. A simpler proof was later given by Peres and Solomyak in [3].

In this paper we discuss one of the many possible applications of the techniques developed in the paper of Peres and Solomyak. We will show absolute continuity statements for the distribution of the random series

$$Y_\lambda = \sum_{n\geq 1} \pm\lambda^{\varphi(n)},$$

where, as above, the signs $\pm$ are chosen independently and with probability $1/2$, and where the function $\varphi : \mathbb{N} \to \mathbb{R}$ is assumed to satisfy

$$0 \leq \lim_{n\to\infty} \frac{\varphi(n)}{n} < \infty, \tag{1}$$

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and some minor technical conditions. In particular, we treat the cases when $\varphi(n) = n + r(n)$, where $r$ is the logarithm of a slowly varying function, and $\varphi(n) = n^\alpha$ for $0 < \alpha < 1$ (see Example 1 and Example 3). If the limit in (1) is infinite it follows that the measure $\nu_\lambda$ is singular, see e.g. [2], Criteria (10).

## 2. Bernoulli convolutions and examples

For a function $\varphi : \mathbb{N} \to \mathbb{R}$ and $0 < \lambda < 1$ we consider the infinite convolution product of $(\delta_{-\lambda^{\varphi(n)}} + \delta_{\lambda^{\varphi(n)}})/2$ for $n \geq 1$. This convolution product converges to a measure $\nu_\lambda$ if and only if

$$\sum_{n\geq 1} \lambda^{2\varphi(n)} < \infty, \tag{2}$$

and the finiteness of (2) implies furthermore that this infinite convolution converges absolutely, i.e. the order of the terms in the convolution is interchangeable (see e.g. Jessen and Wintner [1], Theorem 5 and Theorem 6). Let $\Omega = \{-1, 1\}^{\mathbb{N}}$ be the sequence space equipped with the product topology and $\mu$ the Bernoulli measure on $\Omega$ with the weights $(1/2, 1/2)$. The measure $\nu_\lambda$ can be written as the push-forward of $\mu$ by the random series

$$Y_\lambda(\omega) = \sum_{n\geq 1} \omega_n \lambda^{\varphi(n)}, \tag{3}$$

where $\omega_n$ denotes the $n$th coordinate of an element $\omega$ in $\Omega$. We are interested in the set of $\lambda$ in the interval $(0, 1)$ for which the measure $\nu_\lambda$ is absolutely continuous with respect to the Lebesgue measure $m$ on $\mathbb{R}$. Our first result deals with the class of random series where

$$\lim_{n\to\infty} \varphi(n+1) - \varphi(n) = 0. \tag{4}$$

Observe that for functions $\varphi$ with the property (4), it follows that

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = 0.$$

We begin by stating the following theorem.

**Theorem 2.1.** *If $\varphi : \mathbb{N} \to \mathbb{R}$ satisfies property (4) and if there is a $\lambda_1 \in (0, 1]$ such that, for all $\lambda \in (0, \lambda_1)$, condition (2) is fulfilled then, for a.e. $\lambda \in (0, \lambda_1)$, the measure $\nu_\lambda$ induced by the random series (3) is absolutely continuous and has an $L^2$-density.*

***Example 1.*** *If $\varphi(n) = n^\alpha$, $0 < \alpha < 1$, it follows immediately that the distribution of*

$$Y_\lambda = \sum_{n\geq 1} \pm\lambda^{n^\alpha}$$

*is absolutely continuous for a.e. $\lambda \in (0, 1)$, and that the density is in $L^2$.*

***Example 2.*** *Observe that the function $\varphi(n) = n/\log n$ fulfills (4) and hence the distribution of*

$$Y_\lambda = \sum_{n\geq 1} \pm\lambda^{n/\log n}$$

*is absolutely continuous for a.e. $\lambda \in (0, 1)$, and the density is in $L^2$.*

The method used by Wintner in [5,6] and [7] gives a better result in Example 1 in the case when $0 < \alpha < 1/2$. In fact, if $0 < \alpha < 1/2$, then the distribution of $Y_\lambda$ is absolutely continuous for *all* $\lambda \in (0, 1)$ and, furthermore, the density is smooth. Wintner considered the Fourier transform of the measure $\nu_\lambda$ which can be represented as a convergent infinite product: $\hat{\nu}_\lambda(t) = \prod_{n=1}^{\infty} \cos(\lambda^{\varphi(n)} t)$. Since $\cos(\lambda^{\varphi(n)} t) \le 2/3$ if $1 \le \lambda^{\varphi(n)} t \le 2$, it follows that

$$|\hat{\nu}_\lambda(t)| \le (2/3)^{K(t)},$$

where $K(t) = \#\{n;\, 1 \le \lambda^{\varphi(n)} t \le 2\}$. In Example 1 a minor calculation yields that, for $0 < \alpha < 1/2$, $(2/3)^{K(t)}$ decreases faster than polynomially and thus, $\nu_\lambda$ is absolutely continuous and the density is smooth. To guarantee a sufficiently fast growing of $K(t)$, the function $\varphi(n)$ cannot grow too fast. The method seems to break down at $\alpha = 1/2$. However, by taking the slowly growing function $\varphi(n) = \log n$, Wintner's method applies and we see that the distribution of

$$Y_\lambda = \sum_{n \ge 1} \pm \frac{1}{n^\alpha}$$

is absolutely continuous for *all* $\alpha > 1/2$ and the density is smooth.

## 3. Absolute continuity of Bernoulli convolutions

Theorem 2.1 will be derived from the following result.

**Theorem 3.1.** *Suppose $\tau : \mathbb{N} \to \mathbb{R}$ is of the form $\tau(n) = \beta n + r(n)$, where the function $r(n)$ satisfies* (4). *Then the measure $\eta_\lambda$ induced by the random series $Z_\lambda = \sum_{n \ge 1} \pm \lambda^{\tau(n)}$, is absolutely continuous and has an $L^2$-density, for a.e. $\lambda \in (2^{-1/\beta}, 2^{-2/3\beta})$.*

***Example 3.*** *If $\tau(n) = n + n^\alpha$, $0 < \alpha < 1$, it follows from Theorem 3.1 that, for a.e. $\lambda \in (2^{-1}, 2^{-2/3})$, the distribution of*

$$Z_\lambda = \sum_{n \ge 1} \pm \lambda^{n + n^\alpha}$$

*is absolutely continuous and the density is in $L^2$.*

**Proof of Theorem 2.1.** Let $\{n_j;\, j \ge 1\}$, be a subset of $\mathbb{N}$ such that $\varphi(n_{j+1}) < \varphi(n_j)$, $j \ge 1$, and such that for every $n \ge 1$ there is an $j \ge 1$ with $\varphi(n) = \varphi(n_j)$, i.e. the sequence $n_j$ should be thought of as the times when $\varphi$ makes a jump. Observe that we still have

$$\lim_{j \to \infty} \varphi(n_{j+1}) - \varphi(n_j) = 0. \tag{5}$$

Let $\tilde{\varphi} : [1, \infty) \to \mathbb{R}$ be the continuous function which satisfies $\tilde{\varphi}(j) = \varphi(n_j)$ and which is linear on $[j, j+1]$, $j \ge 1$. Fix $0 < \beta < \infty$ and set $\psi(x) = \tilde{\varphi}^{-1}(\beta x)$. Since $\tilde{\varphi}(x+1) - \tilde{\varphi}(x) \to 0$ as $x \to \infty$, we can choose $N_0$ such that $\psi(x+1) - \psi(x) > 1$, for $x \ge N_0$. Let $[x]$ denote the integer part of the real number $x$. We split the random series $Y_\lambda$ into two parts:

$$\begin{aligned} Y_\lambda(\omega) &= \sum_{j \ge N_0} \omega_{n_{[\psi(j)]}} \lambda^{\tilde{\varphi}([\psi(j)])} + \sum_{\substack{n \ge 1 \\ n \notin \{n_{[\psi(j)]};\, j \ge N_0\}}} \omega_n \lambda^{\varphi(n)} \\ &=: Z_\lambda(\omega) + R_\lambda(\omega). \end{aligned}$$

Note that this is possible since the infinite convolution $Y_\lambda$ is absolutely convergent. We want to apply Theorem 3.1 to the function $\tau(n) = \tilde{\varphi}([\psi(n)])$. Let $r(n) = \tilde{\varphi}([\psi(n)]) - \beta n$. By the definition of $\psi$, $r(n) = \tilde{\varphi}([\psi(n)]) - \tilde{\varphi}(\psi(n))$ which, by (5) tends to 0 as $n \to \infty$. Hence, $r(n)$ satisfies trivially condition (4). Let $\eta_\lambda$ be the measure induced by the random series $Z_\lambda$. It follows from Theorem 3.1, that, for a.e. $\lambda \in (2^{-1/\beta}, 2^{-2/3\beta})$, $\eta_\lambda$ is absolutely continuous and has an $L^2$-density. The random variables $Z_\lambda$ and $R_\lambda$ are independent. Hence, for a.e. $\lambda \in (2^{-1/\beta}, 2^{-2/3\beta}) \cap (0, \lambda_1)$, we can write the measure $\nu_\lambda$ as a convolution of two measures where one of them is an absolutely continuous measure having an $L^2$-density. Thus, the measure $\nu_\lambda$ itself is absolutely continuous and the density of $\nu_\lambda$ is in $L^2(\mathbb{R})$. Since $0 < \beta < \infty$ was arbitrary we can fill out the whole interval $(0, \lambda_1)$, which concludes the proof of Theorem 2.1. □

***Remark 1.*** *We have already noted that a function* $r : \mathbb{N} \to \mathbb{R}$ *satisfying* (4) *also fulfills* $\lim_{n\to\infty} r(n)/n = 0$. *Observe furthermore that property* (4) *implies*:

$$\lim_{k\to\infty} r(k+j) - r(k) = 0 \quad \textit{for all } j \geq 1$$

*and*

$$\lim_{k\to\infty} \frac{\sup_{j\geq 1} |r(k+j) - r(k)|}{k} = 0.$$

## 4. Proof of Theorem 3.1

In [3], Peres and Solomyak studied power series of the form,

$$g(\lambda) = 1 + \sum_{j\geq 1} b_j \lambda^j, \quad b_j \in \{-1, 0, 1\},$$

for $\lambda \in (0, 1)$, and proved the following lemma:

**Lemma 4.1.** *Suppose g is of the above form. There is a* $\delta > 0$, *such that, if* $g(\lambda) < \delta$, *for some* $\lambda$ *in the interval* $[0, 2^{-2/3}]$, *then* $g'(\lambda) < -\delta$.

We will study slight modifications of these series. Let $r_{k,j}$, $k, j \geq 1$ be any sequence of real numbers, such that, for every $j \geq 1$,

$$r_{k,j} \to 0 \quad \text{as } k \to \infty \tag{6}$$

and

$$\lim_{k\to\infty} \frac{\log^+(\sup_{j\geq 1} |r_{k,j}|)}{k} = 0. \tag{7}$$

Define

$$g_k(\lambda) = 1 + \sum_{j\geq 1} b_j \lambda^{j + r_{k,j}} = g(\lambda) + \sum_{j\geq 1} b_j \lambda^j (\lambda^{r_{k,j}} - 1), \tag{8}$$

where $g(\lambda) = 1 + \sum_{j\geq 1} b_j \lambda^j$. Using Lemma 4.1, we can prove:

**Lemma 4.2.** *There is a positive constant* $\delta'$ *and a positive integer* $K$, *such that, if* $k \geq K$ *and* $g_k(\lambda) < \delta'$ *for some* $\lambda$ *in* $[0, 2^{-2/3}]$, *then* $g_k'(\lambda) < -\delta'$.

**Proof.** We have

$$\begin{aligned} g_k'(\lambda) &= \sum_{j\geq 1} (j + r_{k,j}) b_j \lambda^{j-1+r_{k,j}} \\ &= g'(\lambda) + \sum_{j\geq 1} (j + r_{k,j}) b_j \lambda^{j-1} (\lambda^{r_{k,j}} - 1) + \sum_{j\geq 1} r_{k,j} b_j \lambda^{j-1}. \end{aligned}$$

Let $\delta$ be the constant in Lemma 4.1. Set $\delta' = \delta/2$ and pick $0 < \varepsilon < \delta/8$. Since $\lambda \leq 2^{-2/3} < 1$ and because of (7), we can choose $j_\varepsilon \geq 1$ such that

$$\left|\sum_{j\geq j_\varepsilon}(j+r_{k,j})b_j\lambda^{j-1}(\lambda^{r_{k,j}}-1)\right| \leq \varepsilon \quad \text{and} \quad \left|\sum_{j\geq j_\varepsilon} r_{k,j}b_j\lambda^{j-1}\right| \leq \varepsilon,$$

and

$$\left|\sum_{j\geq j_\varepsilon} b_j\lambda^j(\lambda^{r_{k,j}}-1)\right| \leq \varepsilon,$$

for all $k \geq 1$. Furthermore, by condition (6), we can choose $K_\varepsilon \geq 1$ such that

$$\left|\sum_{j=1}^{j_\varepsilon}(j+r_{k,j})b_j\lambda^{j-1}(\lambda^{r_{k,j}}-1)\right| \leq \varepsilon \quad \text{and} \quad \left|\sum_{j=1}^{j_\varepsilon} r_{k,j}b_j\lambda^{j-1}\right| \leq \varepsilon,$$

and

$$\left|\sum_{j=1}^{j_\varepsilon} b_j\lambda^j(\lambda^{r_{k,j}}-1)\right| \leq \varepsilon,$$

for all $k \geq K_\varepsilon$. Note that $g_k(\lambda) < \delta'$ and $k \geq K_\varepsilon$ implies, by (8),

$$g(\lambda) \leq \delta' + \left|\sum_{j\geq 1} b_j\lambda^j(\lambda^{r_{k,j}}-1)\right| \leq \delta' + 2\varepsilon < \delta.$$

Hence, if $g_k(\lambda) < \delta'$ and $k \geq K_\varepsilon$, by Lemma 4.1,

$$\begin{aligned} g_k'(\lambda) &\leq -\delta + \left|\sum_{j\geq 1}(j+r_{k,j})b_j\lambda^{j-1}(\lambda^{r_{k,j}}-1)\right| + \left|\sum_{j\geq 1} r_{k,j}b_j\lambda^{k-1}\right| \\ &\leq -\delta + 4\varepsilon < -\delta'. \end{aligned}$$

□

We can now finish the proof of Theorem 3.1. Let $\tau : \mathbb{N} \to \mathbb{R}$ be as in Theorem 3.1. We can without loss of generality assume that $\beta = 1$. The case for general $\beta$ follows immediately from a simple scaling argument. The proof closely follows the ideas outlined in [3]. Suppose $\eta_\lambda$ is the push–forward of the Bernoulli measure on $\Omega$ under the map

$$Z_\lambda(\omega) = \sum_{n\geq 1} \omega_n \lambda^{\tau(n)}.$$

Setting $r(n) = \tau(n) - n$, we note that, by Remark 1, the sequence $r_{k,j} = r(k+j) - r(k)$ satisfies condition (6) and (7). Let $I$ denote the interval $[\lambda_0, 2^{-2/3}]$, where $2^{-1} < \lambda_0 < 2^{-2/3}$, and let $K$ and $\delta'$ be the constants in Lemma 4.2. It is enough to show that the distribution of the random series

$$\tilde{Z}_\lambda(\omega) = \sum_{n\geq K} \omega_n \lambda^{\tau(n)}$$

is absolutely continuous, for a.e. $\lambda \in I$, and has an $L^2$-density. Let

$$\Omega_K = \{(\omega_K, \omega_{K+1}, \ldots); \omega \in \Omega\},$$

and denote by $\mu_K$ the Bernoulli measure on $\Omega_K$. Following [3], we need to prove that

$$S = \liminf_{r\to 0^+} \frac{1}{r} \int_{\Omega_K} \int_{\Omega_K} m\left(\left\{\lambda \in I; \left|\sum_{n\geq K} (\omega_n - \omega_n')\lambda^{\tau(n)}\right| < r\right\}\right) \mathrm{d}\mu_K(\omega)\,\mathrm{d}\mu_K(\omega') < +\infty.$$

For $k \geq K$, let $\tilde{\Omega}_k$ denote the subset of elements $(\omega, \omega')$ in $\Omega_K \times \Omega_K$ such that $\omega_j = \omega_j'$ for all $j \leq k-1$, and $\omega_k \neq \omega_k'$. Note that

$$(\mu_K \times \mu_K)(\tilde{\Omega}_k) = 2^{-(k+1)+K} \quad \text{and} \quad \tau(k+j) - \tau(k) = j + r_{k,j}.$$

We obtain

$$S \leq \liminf_{r\to 0^+} \frac{1}{r} \sum_{k\geq K} 2^{-(k+1)+K} \int_{\tilde{\Omega}_k} m\left(\left\{\lambda \in I; |g_k(\lambda;\omega,\omega')| < r2^{-1}\lambda_0^{-\tau(k)}\right\}\right) \mathrm{d}\mu_K(\omega)\,\mathrm{d}\mu_K(\omega'),$$

where

$$g_k(\lambda;\omega,\omega') = 1 + \sum_{j\geq 1} b_j(k;\omega,\omega')\lambda^{j+r_{k,j}}, \qquad b_j(k;\omega,\omega') \in \{-1,0,1\},$$

for $(\omega,\omega') \in \tilde{\Omega}_k$. By Lemma 4.2, the functions $g_k$ satisfy a transversality condition on the interval $I$, and thus,

$$m\left(\left\{\lambda \in I; |g_k(\lambda;\omega,\omega')| \leq r2^{-1}\lambda_0^{-\tau(k)}\right\}\right) \leq \delta'^{-1} r \lambda_0^{-\tau(k)}.$$

It follows that

$$S \leq \delta'^{-1} 2^{K-1} \sum_{k\geq K} 2^{-k} \lambda_0^{-\tau(k)}.$$

Note now that the right-hand side is finite since $\lambda_0 > 1/2$ and $\tau(k)/k \to 1$ as $k \to \infty$. Hence we have proved Theorem 3.1.

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